

Lecture (2) Plan:

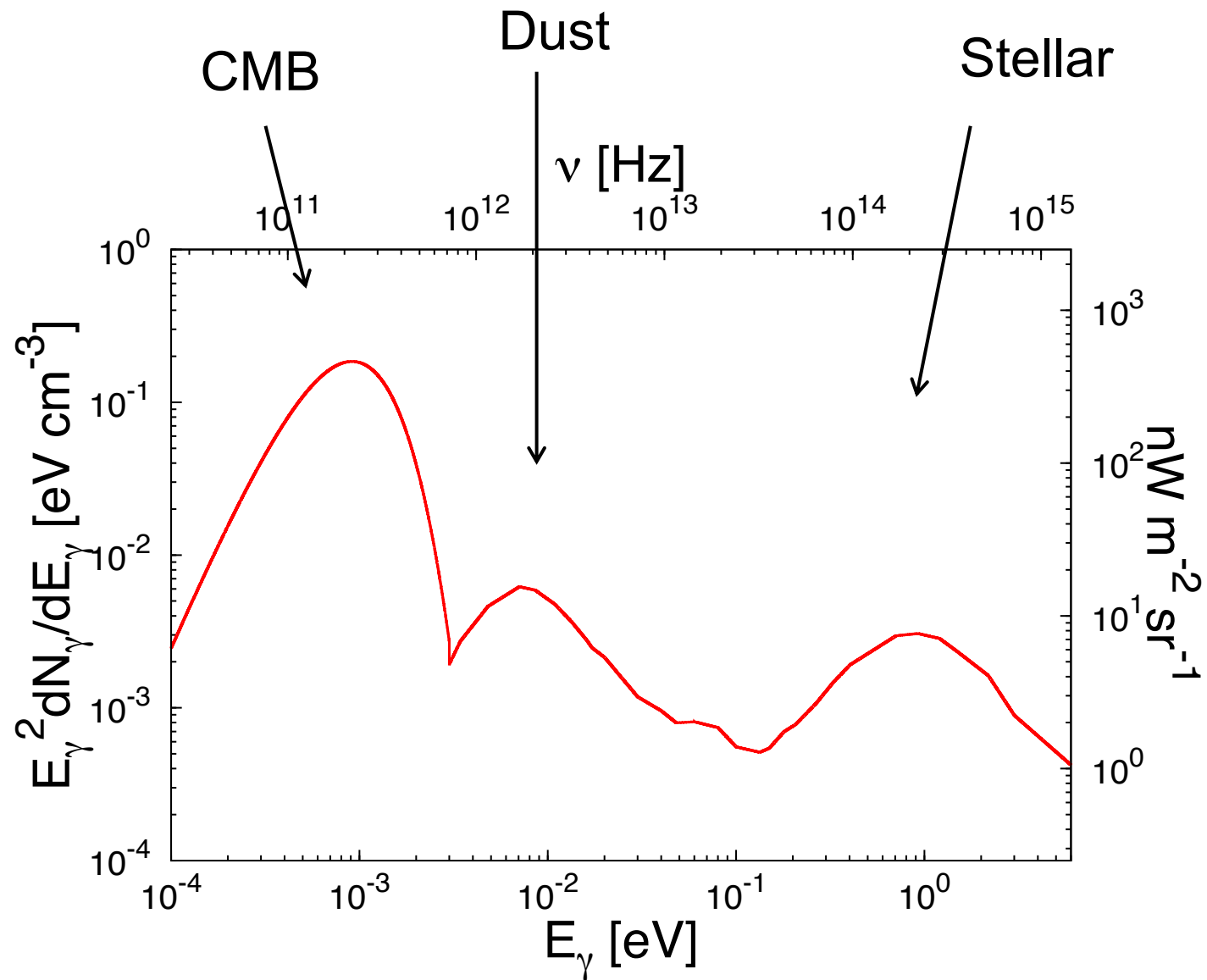
- **Extragalactic radiation fields**
- **Cosmic ray proton interaction rates with extragalactic radiation fields**
- **Cosmic ray nuclei interaction rates with extragalactic radiation fields**
- **High Energy electron and photon interaction rates with radiation fields**
- **Cosmic ray composition and gamma-rays as a probe of the source distribution**

Cosmic Radiation Fields- Energy Density

$$\begin{aligned} U_\gamma &= \int_0^\infty E_\gamma \frac{dn}{dE_\gamma} dE_\gamma \\ &= \int_0^\infty E_\gamma^2 \frac{dn}{dE_\gamma} d\ln E_\gamma \end{aligned}$$

Note- this amounts to a visual inspection version of Laplace's integral method

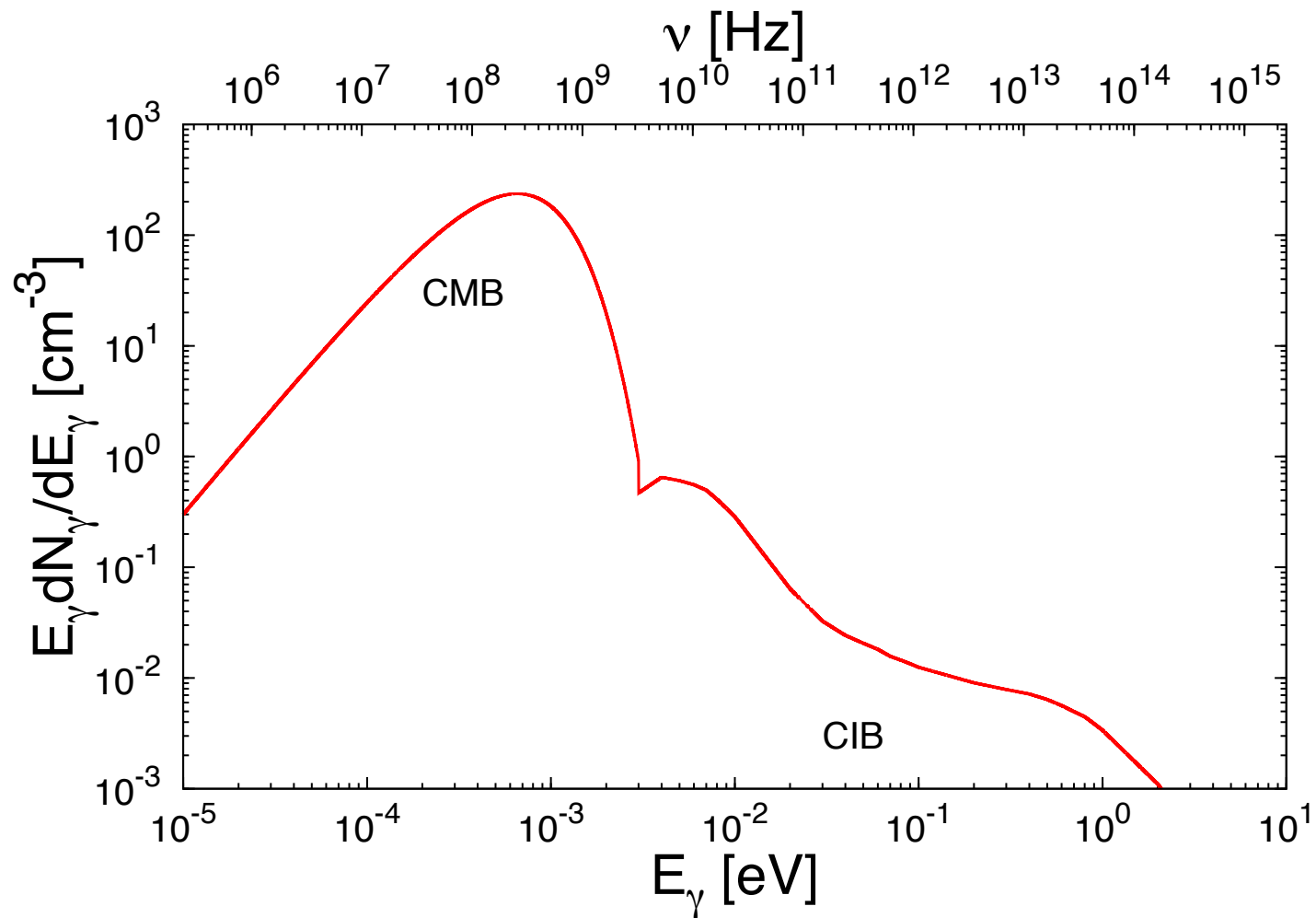
Cosmic Radiation Fields- Energy Density



Cosmic Radiation Fields- Number Density

$$\mathbf{n}_\gamma = \int_0^\infty \frac{d\mathbf{n}}{d\mathbf{E}_\gamma} d\mathbf{E}_\gamma$$
$$= \int_0^\infty \mathbf{E}_\gamma \frac{d\mathbf{n}}{d\mathbf{E}_\gamma} d\ln\mathbf{E}_\gamma$$

Cosmic Radiation Fields- Number Density



Cosmic Ray Proton Energy Losses

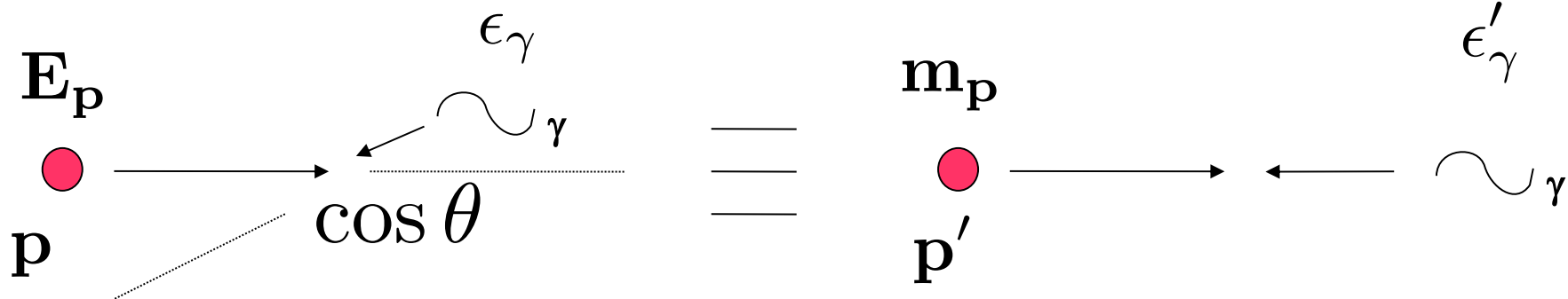
The Interaction Rate

radiation field

cross-section

$$\mathbf{R} = \int_0^\infty d\epsilon_\gamma \frac{dn}{d\epsilon_\gamma} \int_{-1}^1 \frac{1}{2} d(\cos \theta) \sigma(\cos \theta) (1 + \beta \cos \theta)$$

All values above in lab frame



The Interaction Rate

$$\mathbf{R} = \int_0^\infty d\epsilon_\gamma \frac{dn}{d\epsilon_\gamma} \int_{-1}^1 \frac{1}{2} d(\cos \theta) \sigma(\cos \theta) (1 + \beta \cos \theta)$$

Since, $\epsilon'_\gamma \mathbf{m}_p = \epsilon_\gamma \mathbf{E}_p (1 + \beta \cos \theta)$

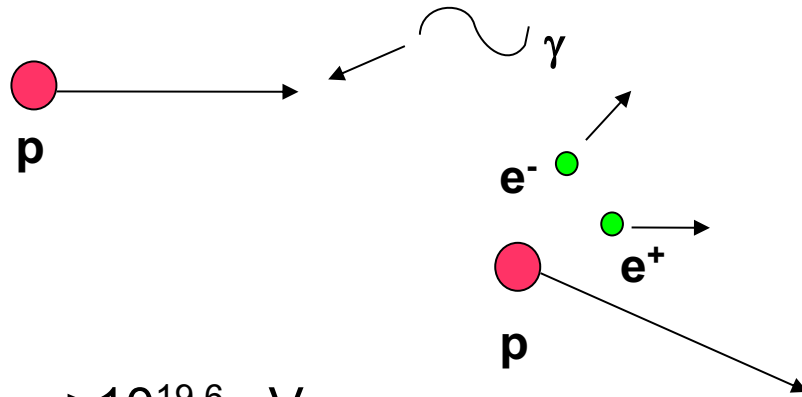
$$(1 + \beta \cos \theta) d \cos \theta = \frac{\epsilon'_\gamma \mathbf{m}_p}{\epsilon_\gamma \mathbf{E}_p} \frac{d(\epsilon'_\gamma \mathbf{m}_p)}{\epsilon_\gamma \mathbf{E}_p}$$

$$\mathbf{R} = \frac{1}{2} \int_0^\infty d\epsilon_\gamma \frac{dn}{d\epsilon_\gamma} \int_0^{2\epsilon_\gamma \mathbf{E}_p} d(\epsilon'_\gamma \mathbf{m}_p) \frac{\epsilon'_\gamma \mathbf{m}_p}{\epsilon_\gamma^2 \mathbf{E}_p^2} \sigma(\epsilon')$$

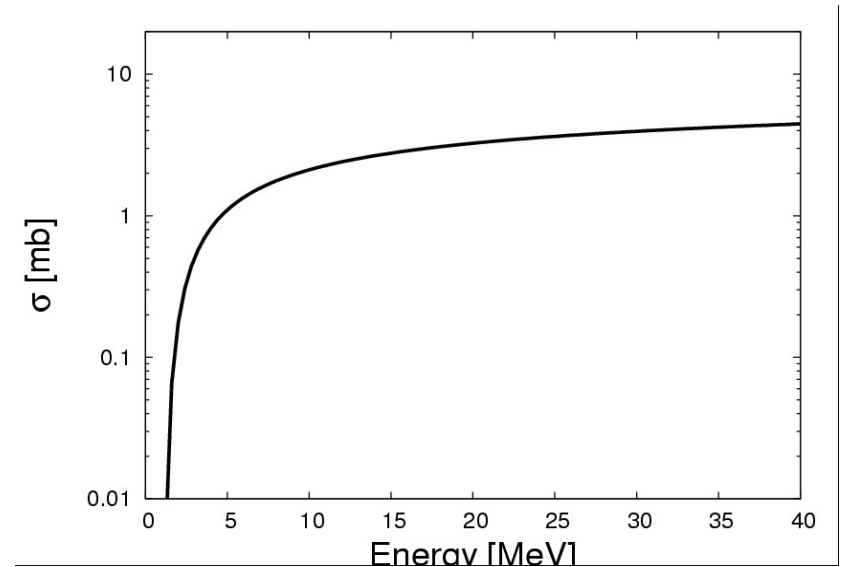
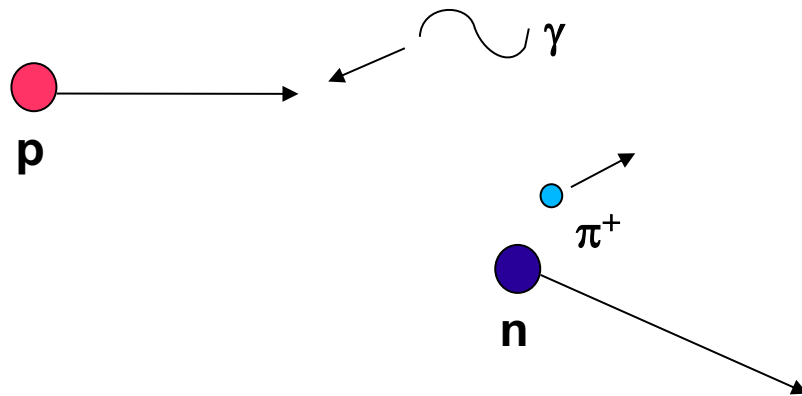
$$= \frac{m_p^2}{2\mathbf{E}_p^2} \int_0^\infty d\epsilon_\gamma \frac{1}{\epsilon_\gamma^2} \frac{dn}{d\epsilon_\gamma} \int_0^{2\epsilon_\gamma \frac{\mathbf{E}_p}{m_p}} d\epsilon'_\gamma \epsilon'_\gamma \sigma(\epsilon'_\gamma)$$

Cosmic Ray Proton Interactions

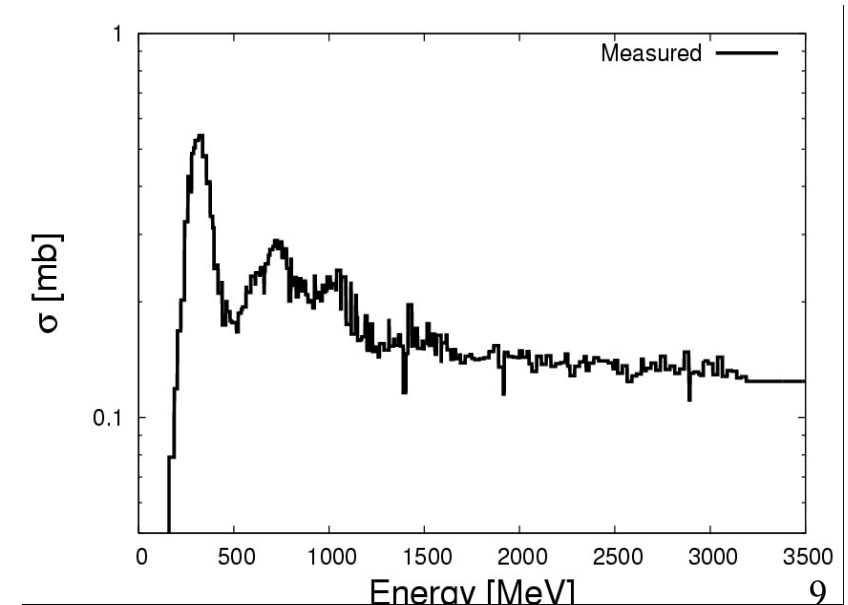
For $E_{\text{proton}} < 10^{19.6}$ eV



For $E_{\text{proton}} > 10^{19.6}$ eV



$E_{\gamma}^{\text{th}} \sim 1 \text{ MeV}$



$E_{\gamma}^{\text{th}} \sim 140 \text{ MeV}$



Threshold Energy- Proton Pair Production

$$(\mathbf{E}_p + \mathbf{E}_\gamma)^2 - (\mathbf{p}_p - \mathbf{E}_\gamma)^2 = (m_p + 2m_e)^2$$

$$m_p^2 + 2\mathbf{E}_p\mathbf{E}_\gamma + 2\mathbf{p}_p\mathbf{E}_\gamma \approx m_p^2 + 4m_p m_e$$

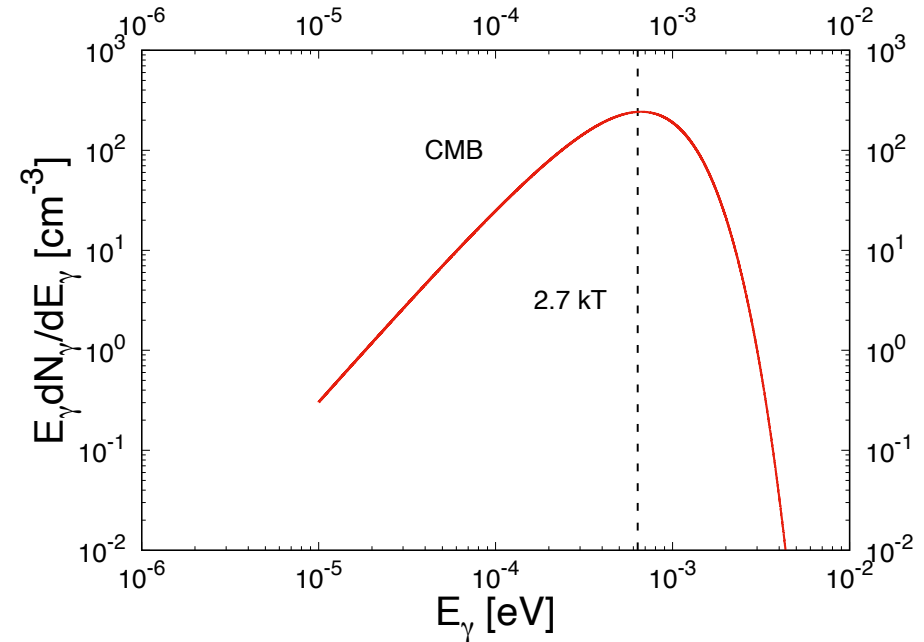
$$\mathbf{E}_p \approx \frac{m_e}{\mathbf{E}_\gamma} m_p \approx \left(\frac{0.5 \times 10^6}{6 \times 10^{-4}} \right) 0.9 \times 10^9 = 8 \times 10^{17} \text{ eV}$$

Repeat this calculation for pion production

CMB- Total Number Density

$$\frac{dn}{d\epsilon_\gamma} = \frac{8\pi}{(hc)^3} \frac{\epsilon_\gamma^2}{e^{\epsilon_\gamma/kT} - 1}$$

$$n_\gamma^{\text{BB}} = \frac{8\pi(kT)^3}{(hc)^3} \int_0^\infty \frac{x^2}{e^x - 1} dx$$



$$\frac{8\pi(kT_{\text{CMB}})^3}{(hc)^3} \approx 170 \text{ cm}^{-3}$$

$$\zeta(\mathbf{x}) = \sum_{n=1}^{\infty} \frac{1}{n^{\mathbf{x}}}$$

$$n_\gamma^{\text{CMB}} = 8\pi \frac{(kT_{\text{CMB}})^3}{(hc)^3} \gamma(\mathbf{3}) \zeta(\mathbf{3}) \approx 400 \text{ cm}^{-3}$$



CMB- Total Number Density

For a blackbody radiation field distribution, with temperature T,

$$n_{\gamma} = 8\pi \frac{(kT)^3}{(hc)^3} \gamma(3)\zeta(3) \approx 400 \text{ cm}^{-3}$$

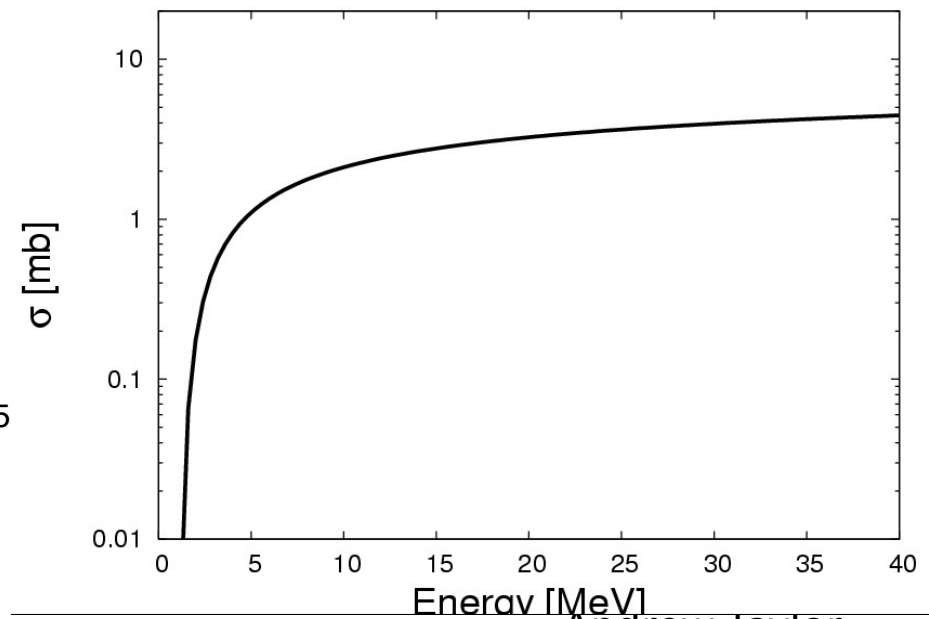
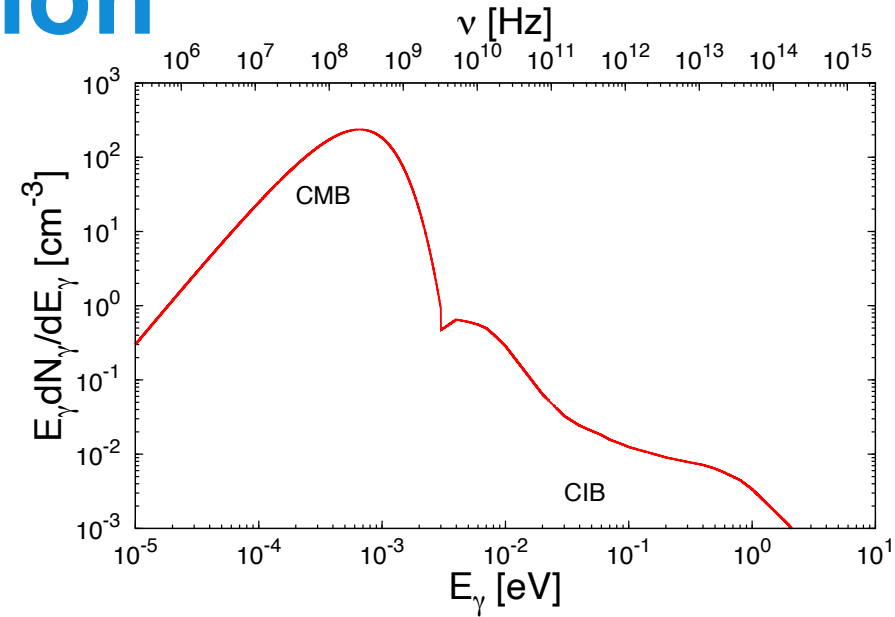
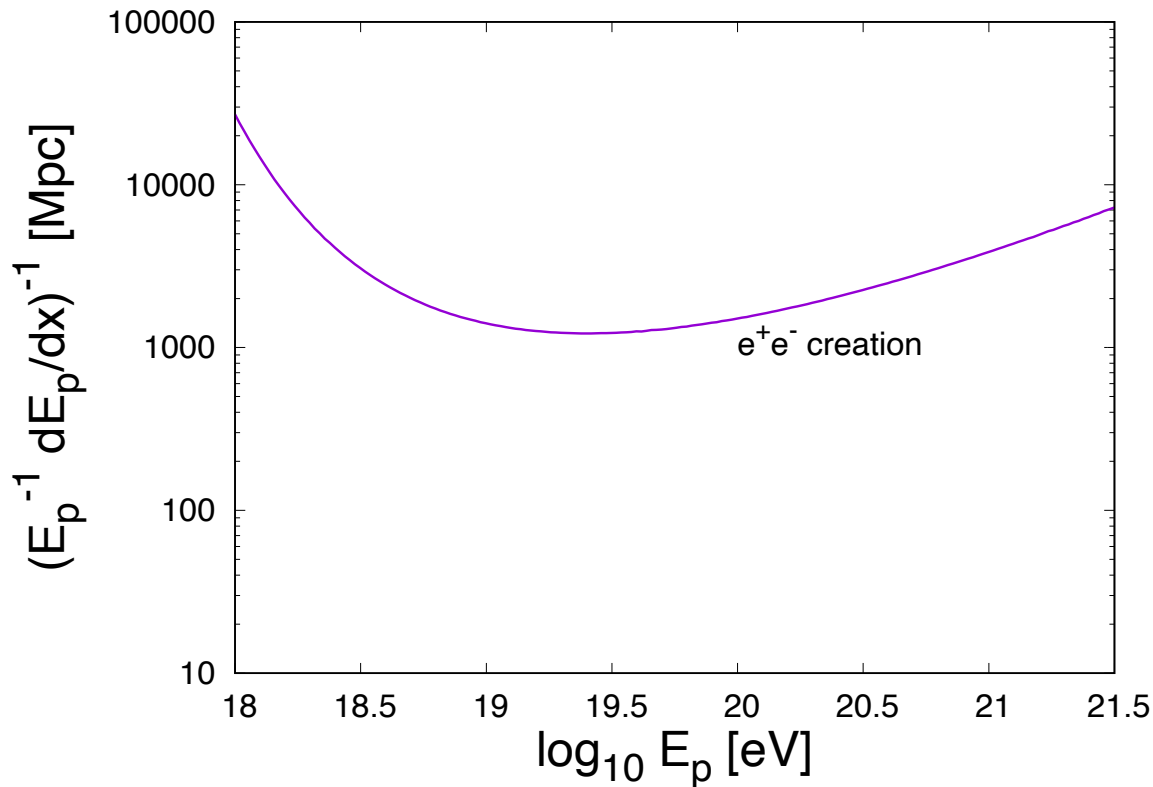
$$U_{\gamma} = 8\pi \frac{(kT)^4}{(hc)^3} \gamma(4)\zeta(4) = 0.25 \text{ eV cm}^{-3}$$

$$\int_0^{\infty} \frac{x^n}{e^x - 1} dx = \gamma(n+1)\zeta(n+1)$$

Have a go at demonstrating this to yourself!

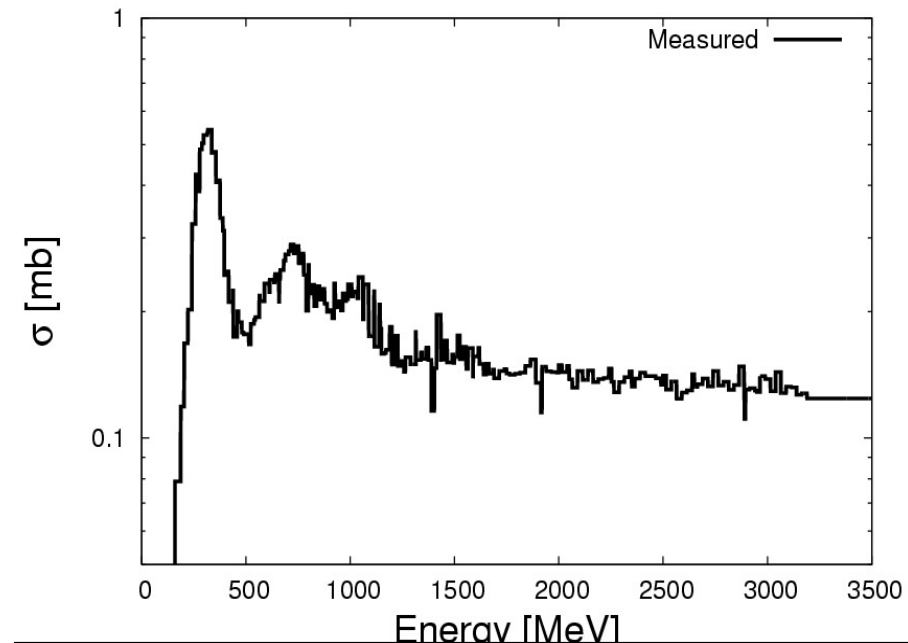
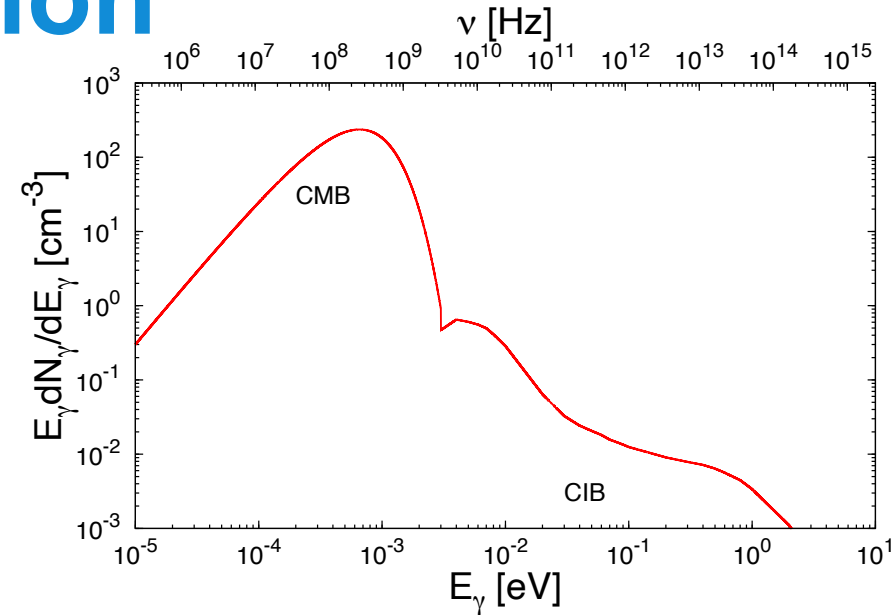
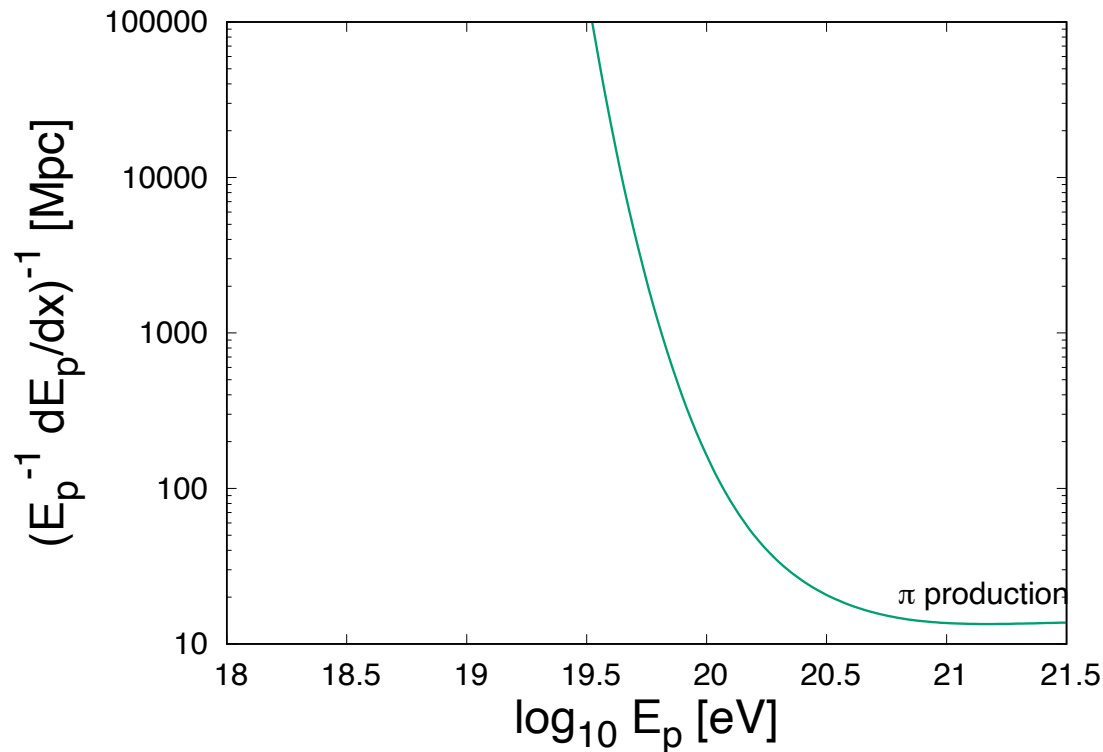
Energy Loss Rate- Pair Production

$$R = \frac{m_p^2 c^4}{2E^2} \int_0^\infty d\epsilon_\gamma \frac{1}{\epsilon_\gamma^2} \frac{dn}{d\epsilon_\gamma} \int_0^{2E\epsilon_\gamma/(m_p c^2)} d\epsilon'_\gamma \epsilon'_\gamma \sigma_{p\gamma}(\epsilon'_\gamma) K_p$$



Energy Loss Rate- Pion Production

$$R = \frac{m_p^2 c^4}{2E^2} \int_0^\infty d\epsilon_\gamma \frac{1}{\epsilon_\gamma^2} \frac{dn}{d\epsilon_\gamma} \int_0^{2E\epsilon_\gamma/(m_p c^2)} d\epsilon'_\gamma \epsilon'_\gamma \sigma_{p\gamma}(\epsilon'_\gamma) K_p$$

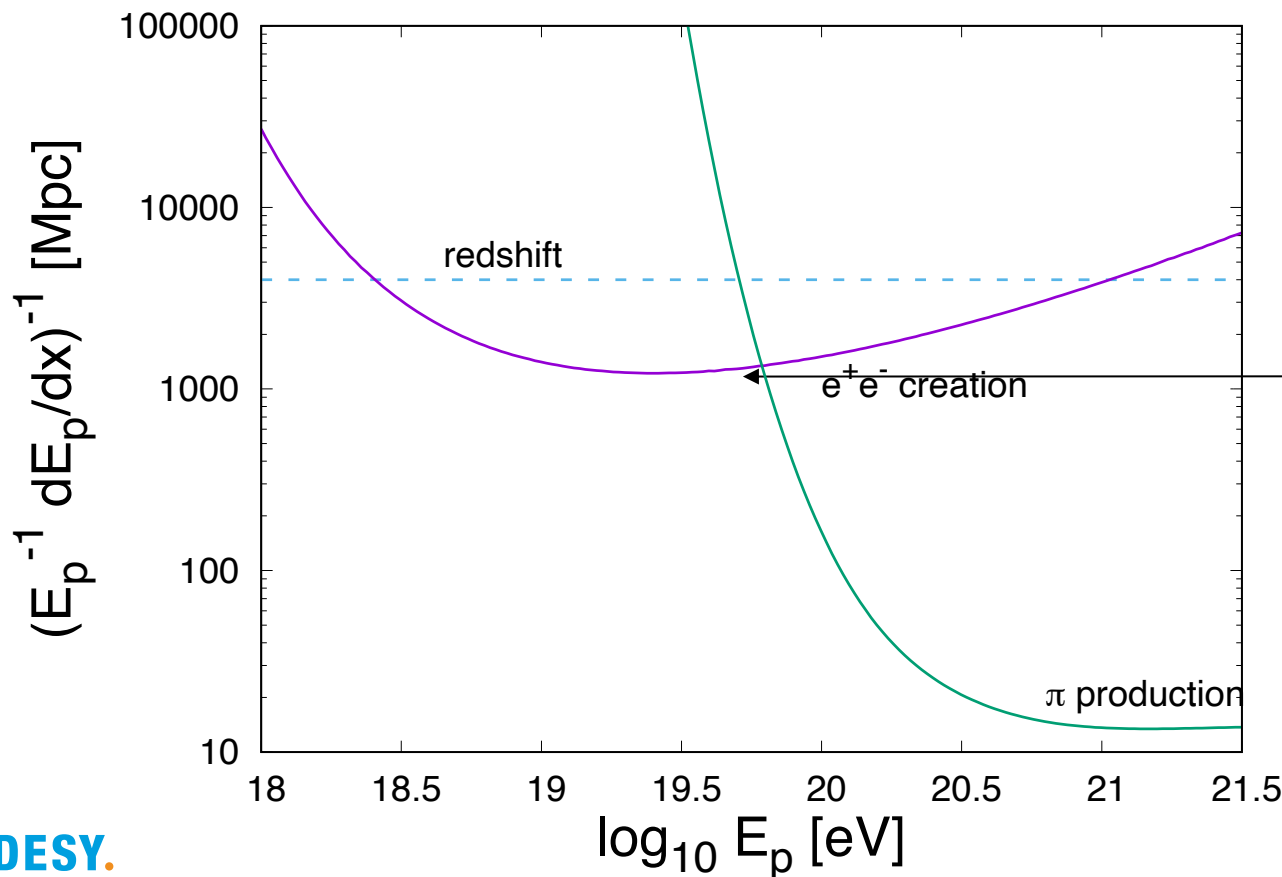


Energy Loss Rates due to Proton Interactions

$$R = \frac{m_p^2 c^4}{2E^2} \int_0^\infty d\epsilon_\gamma \frac{1}{\epsilon_\gamma^2} \frac{dn}{d\epsilon_\gamma} \int_0^{2E\epsilon_\gamma/(m_p c^2)} d\epsilon'_\gamma \epsilon'_\gamma \sigma_{p\gamma}(\epsilon'_\gamma) K_p$$

where R is the energy loss rate

where K_p is the inelasticity



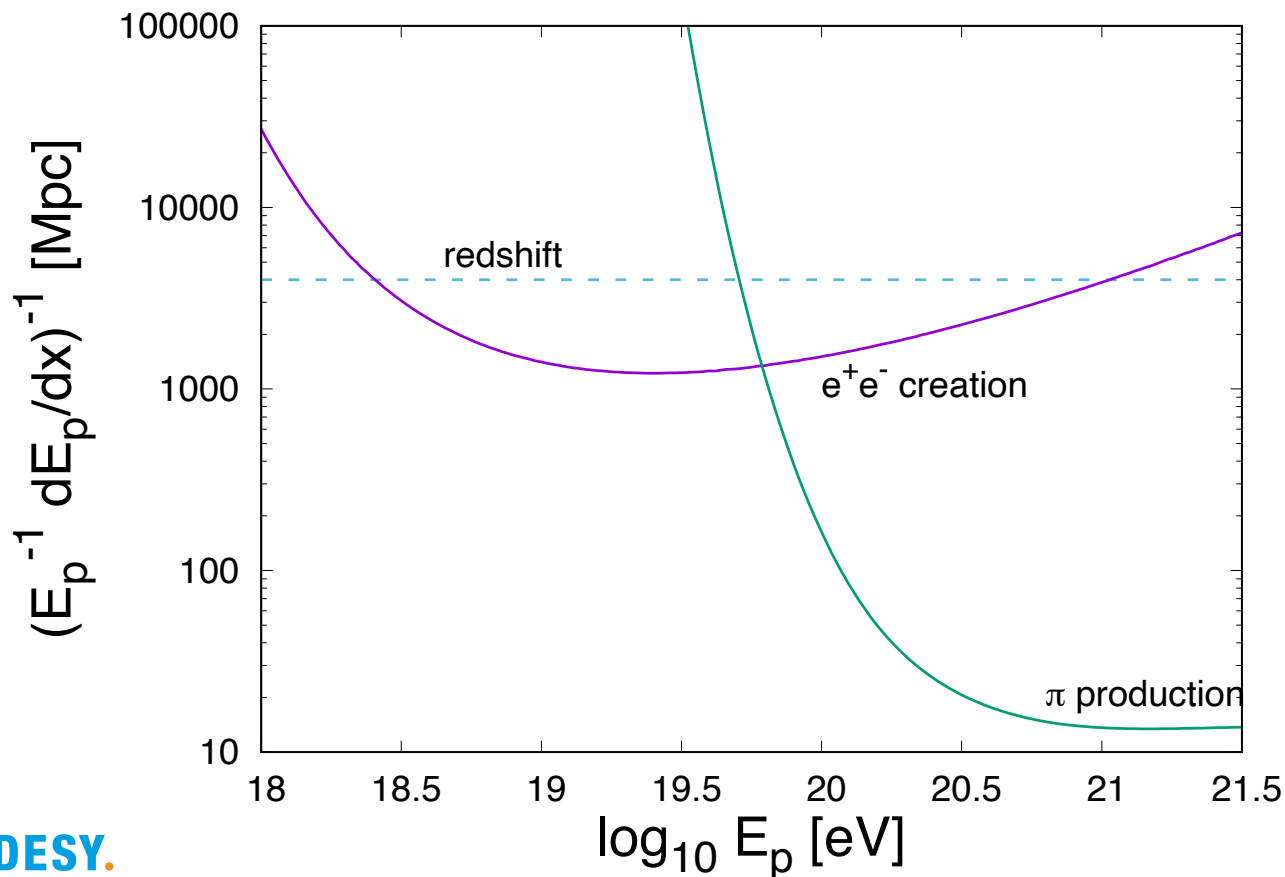
$$\approx \frac{m_p}{m_e} \frac{1}{n_{\text{CMB}} \sigma_{p\gamma}}$$

Energy Loss Rates due to Proton Interactions

$$R = \frac{m_p^2 c^4}{2E^2} \int_0^\infty d\epsilon_\gamma \frac{1}{\epsilon_\gamma^2} \frac{dn}{d\epsilon_\gamma} \int_0^{2E\epsilon_\gamma/(m_p c^2)} d\epsilon'_\gamma \epsilon'_\gamma \sigma_{p\gamma}(\epsilon'_\gamma) K_p$$

where R is the energy loss rate

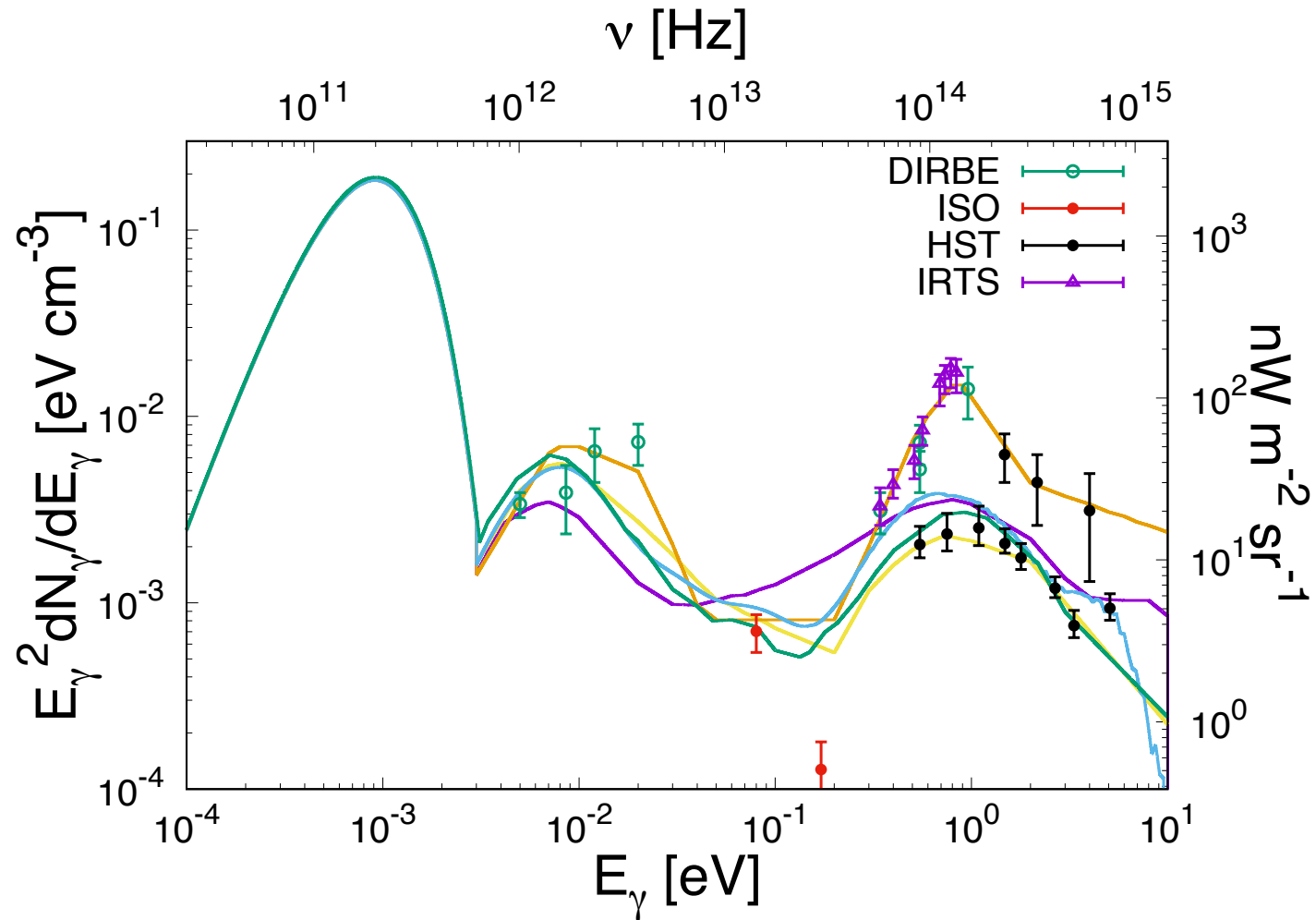
where K_p is the inelasticity



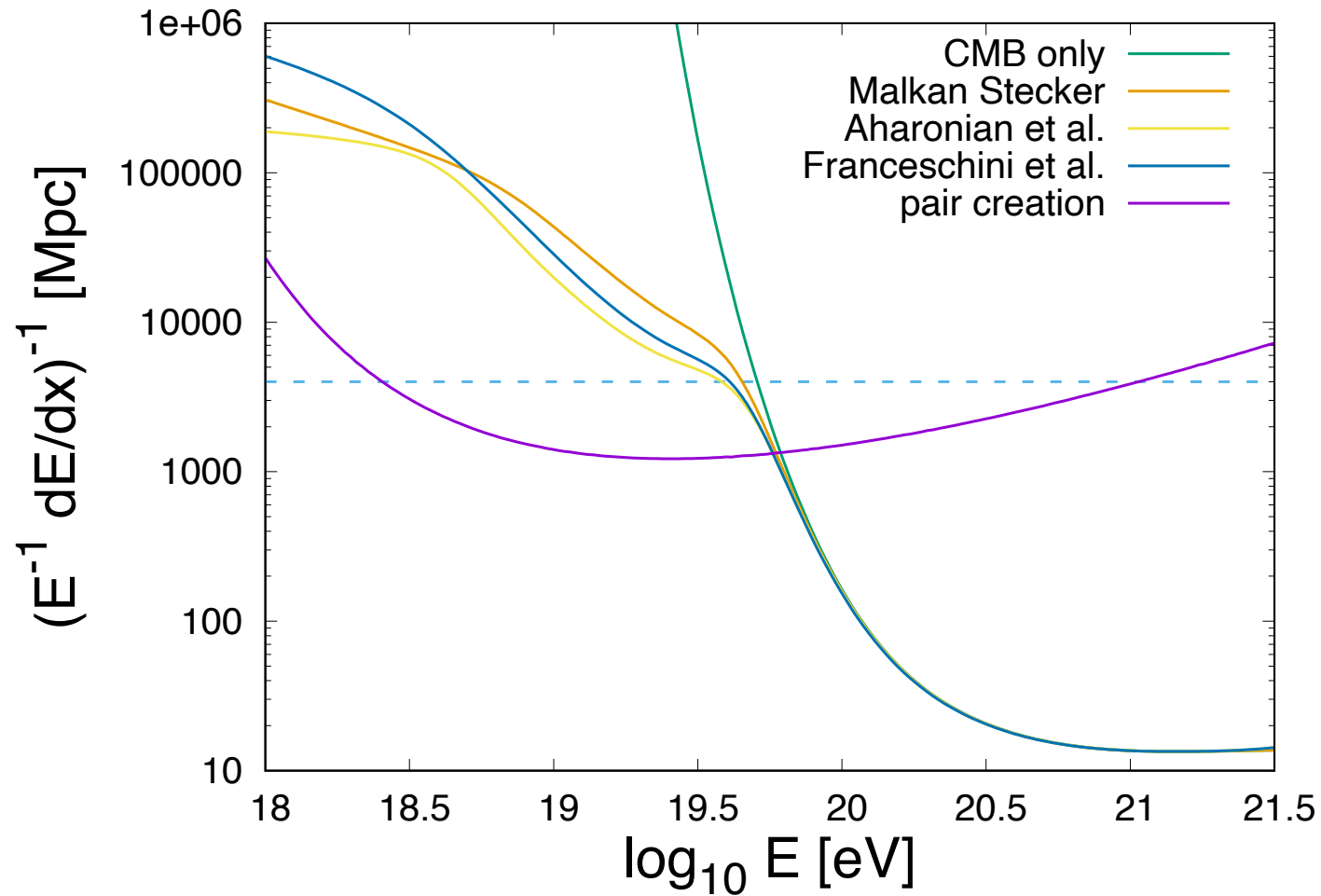
$$\approx \frac{m_p}{m_\pi} \frac{1}{n_{\text{CMB}} \sigma_{p\gamma}} \frac{1}{16}$$

Andrew Taylor

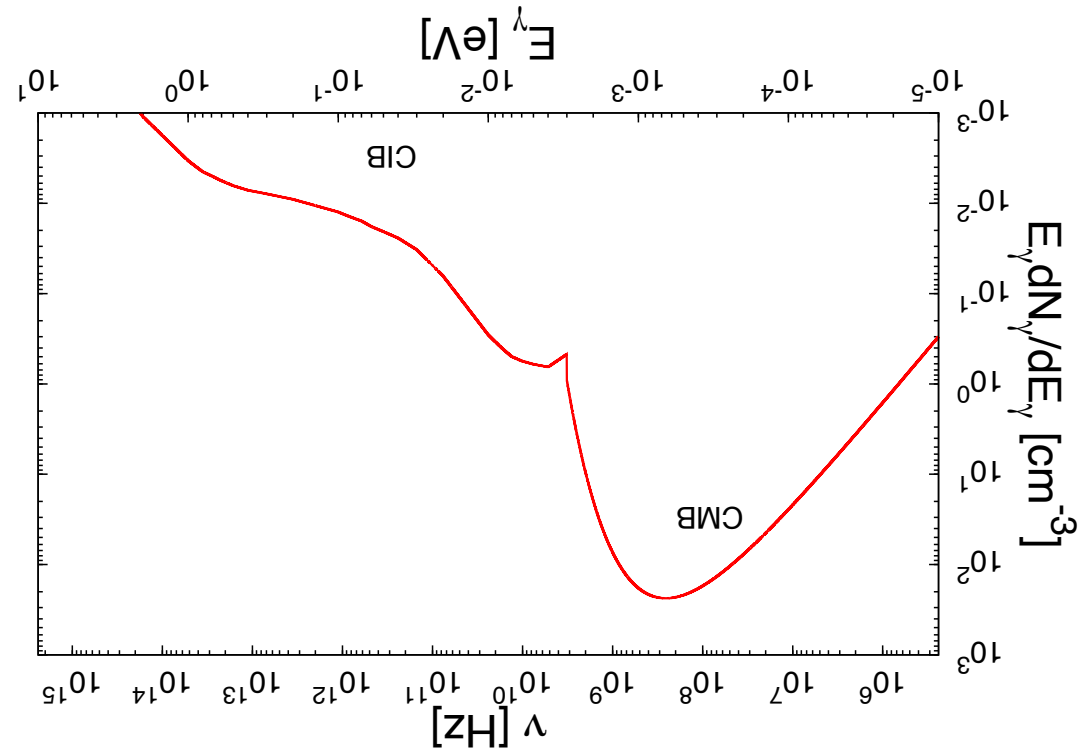
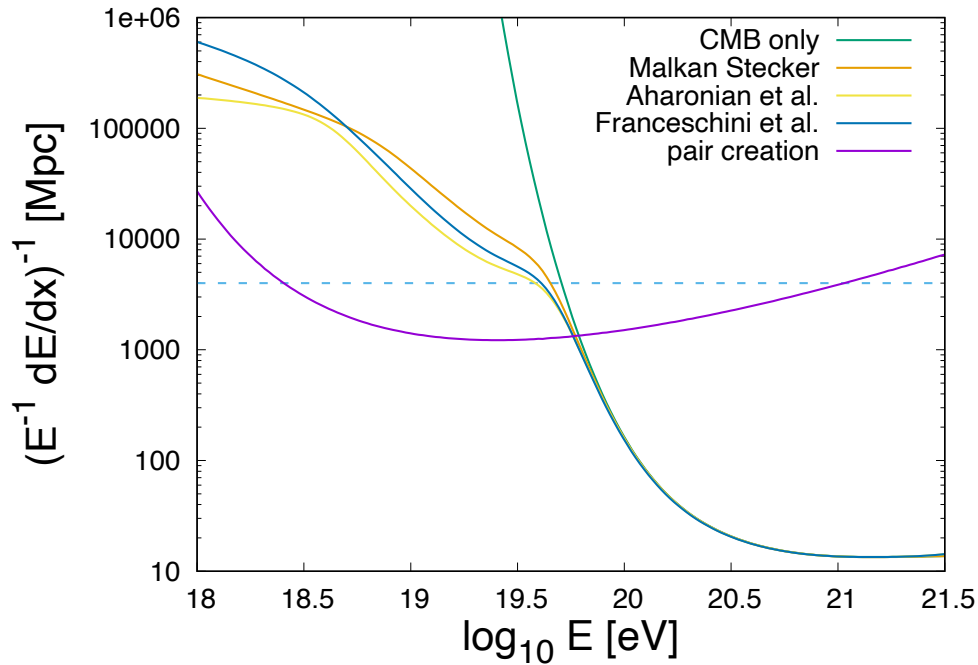
EBL Radiation Field Models



....with Different IR Backgrounds



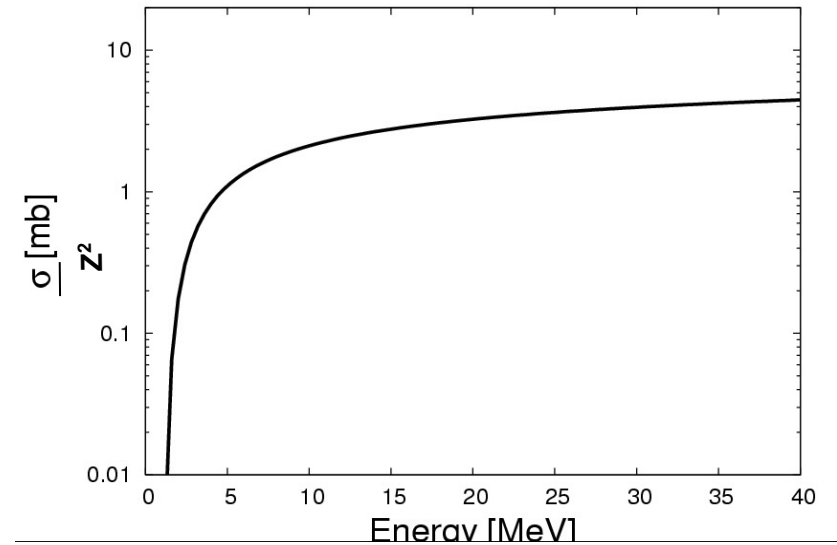
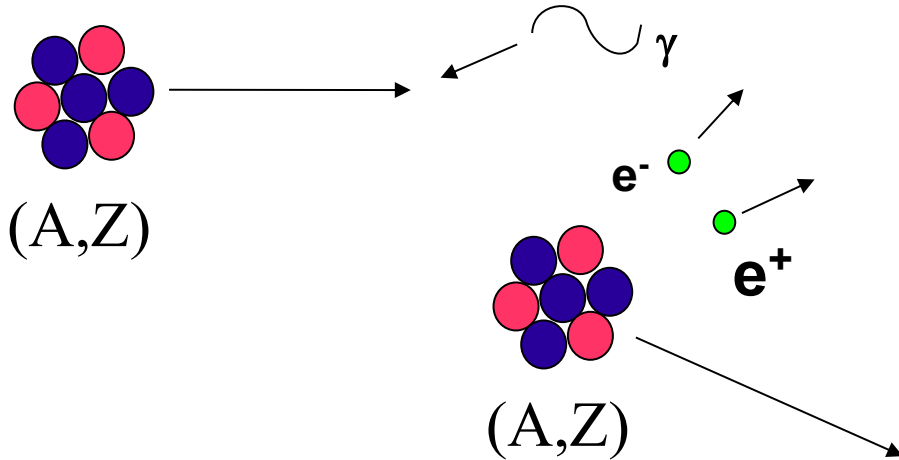
....with Different IR Backgrounds



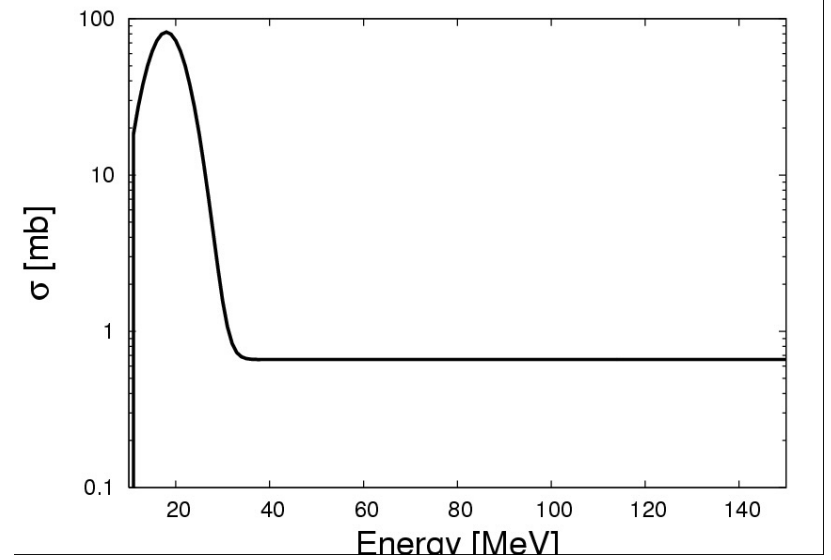
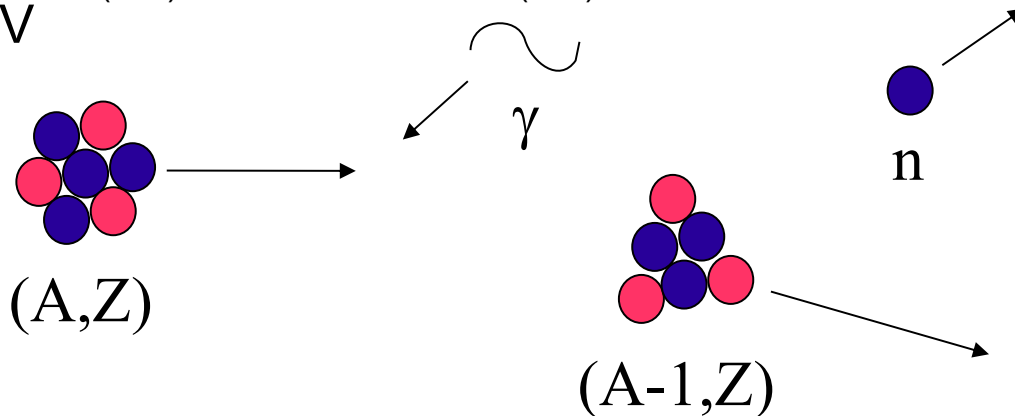
Cosmic Ray Nuclei Energy Losses

Cosmic Ray Nuclei Interactions

For $10^{19.7} < E_{(A,Z)} < 10^{20.2}$
eV



For $E_{(A,Z)} < 10^{19.7}$ and $E_{(A,Z)} < 10^{20.2}$
eV



Cosmic Ray Nuclei Interactions

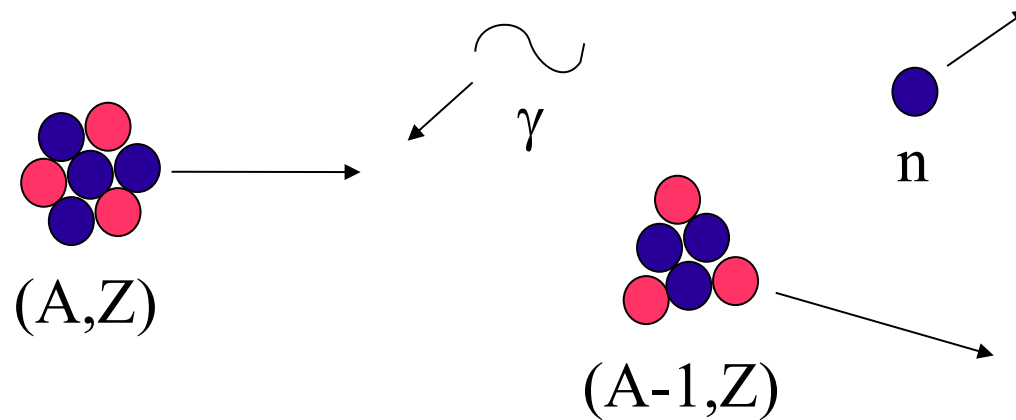
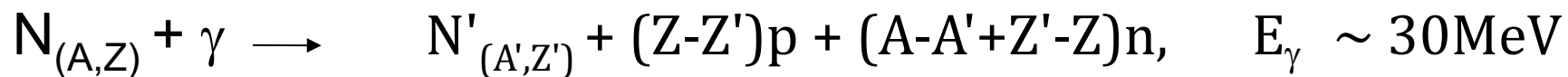


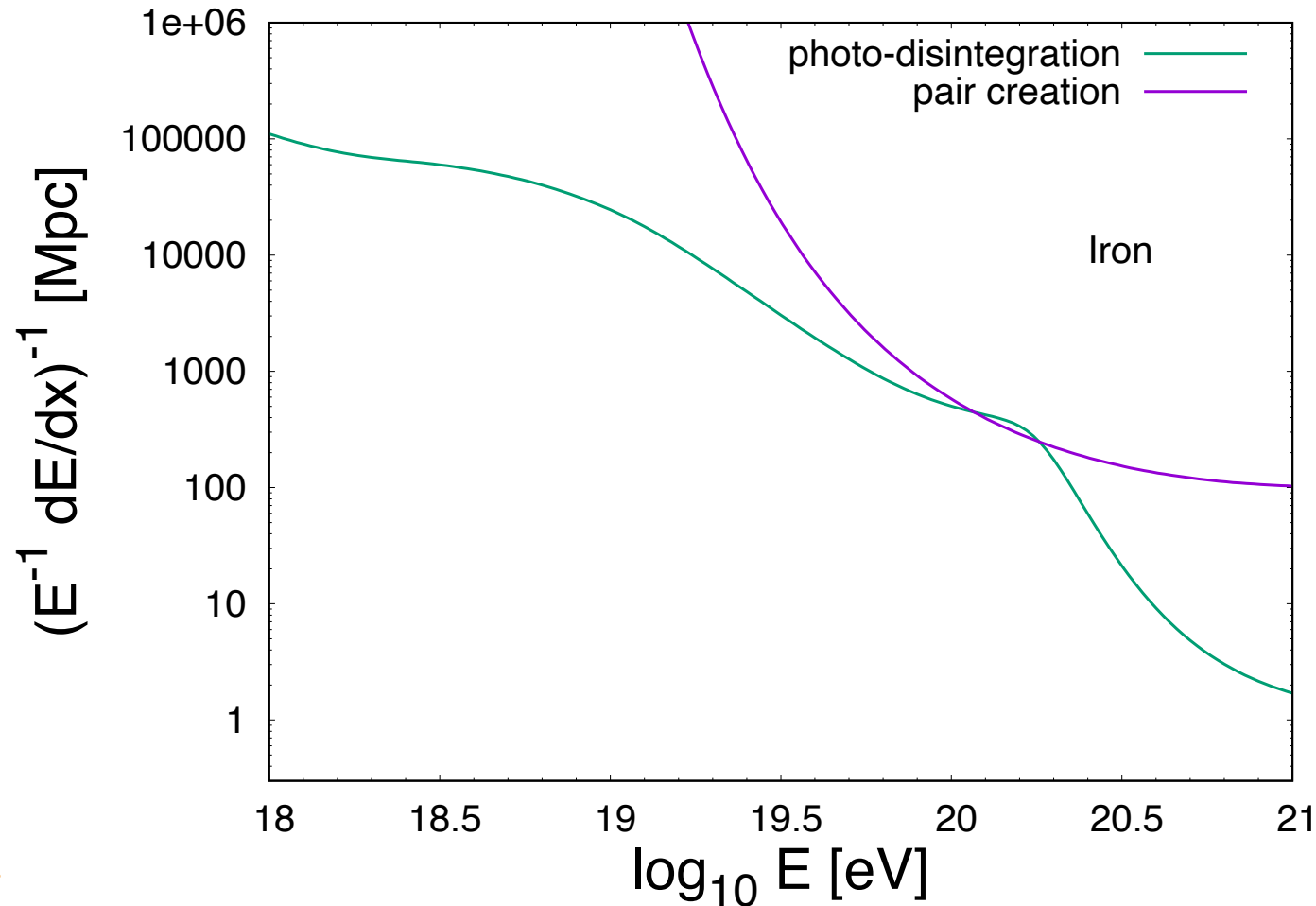
Photo-disintegration-



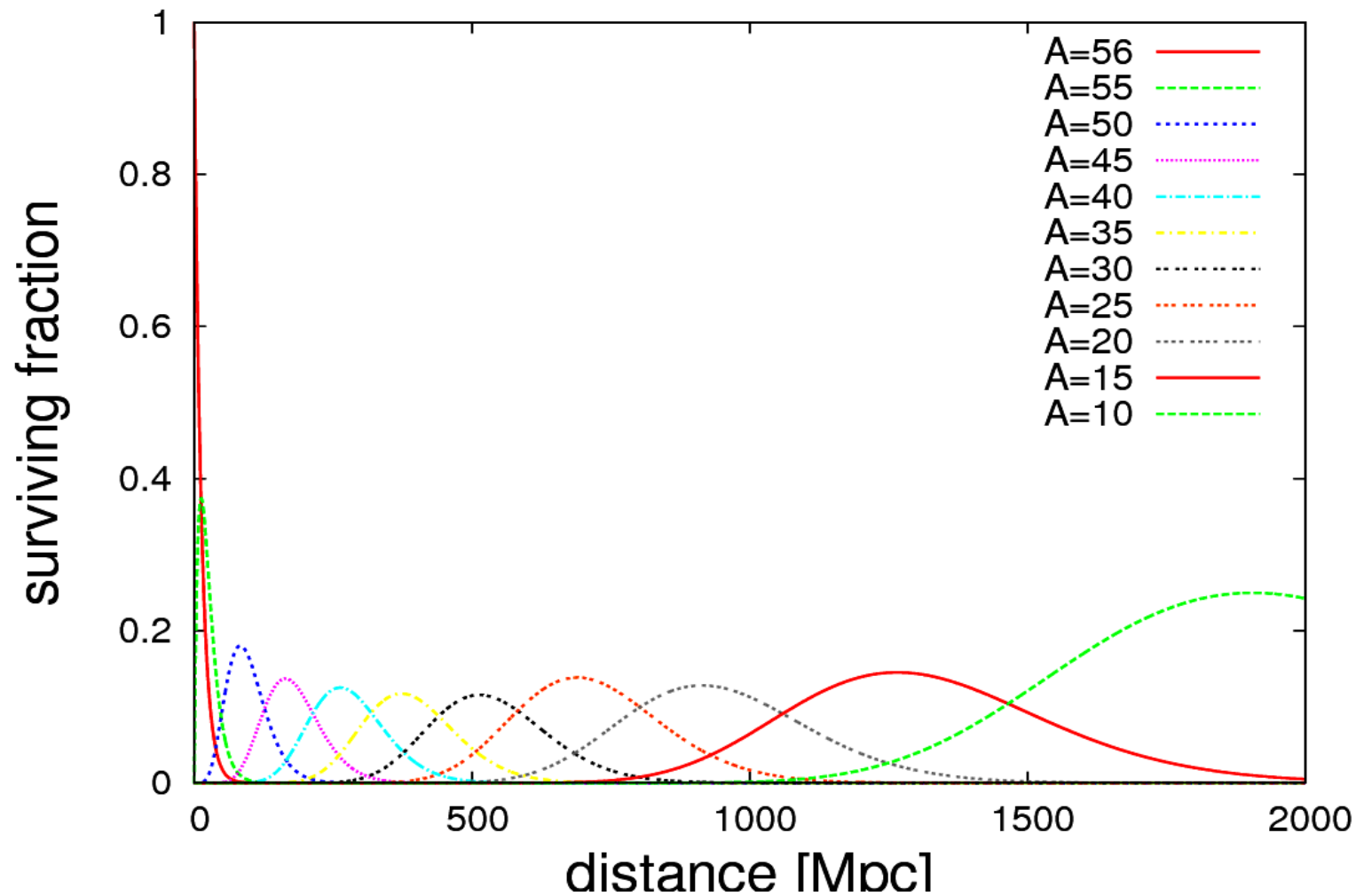
Energy Loss Rates due to Nuclei Interactions

$$R = \frac{A^2 m_p^2 c^4}{2E^2} \int_0^\infty d\epsilon_\gamma \frac{1}{\epsilon_\gamma^2} \frac{dn}{d\epsilon_\gamma} \int_0^{2E\epsilon_\gamma / (Am_p c^2)} d\epsilon'_\gamma \epsilon'_\gamma \sigma_{N\gamma}(\epsilon'_\gamma) K_p$$

where R is the energy loss rate



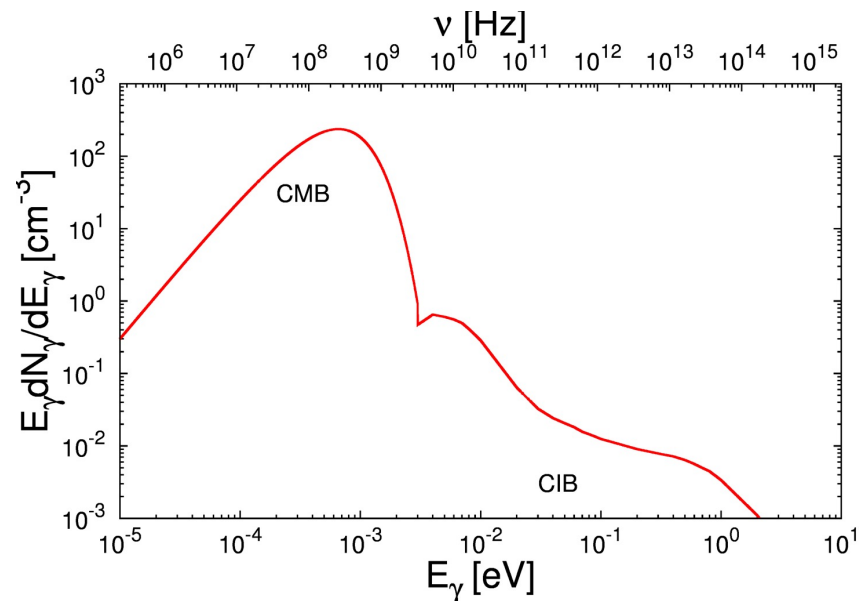
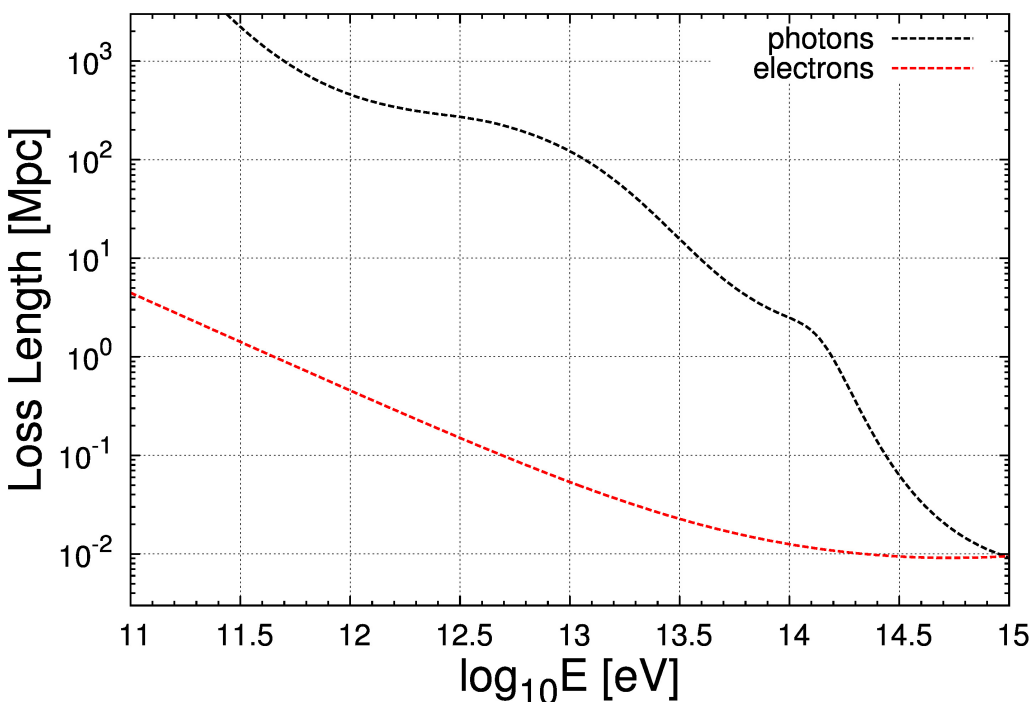
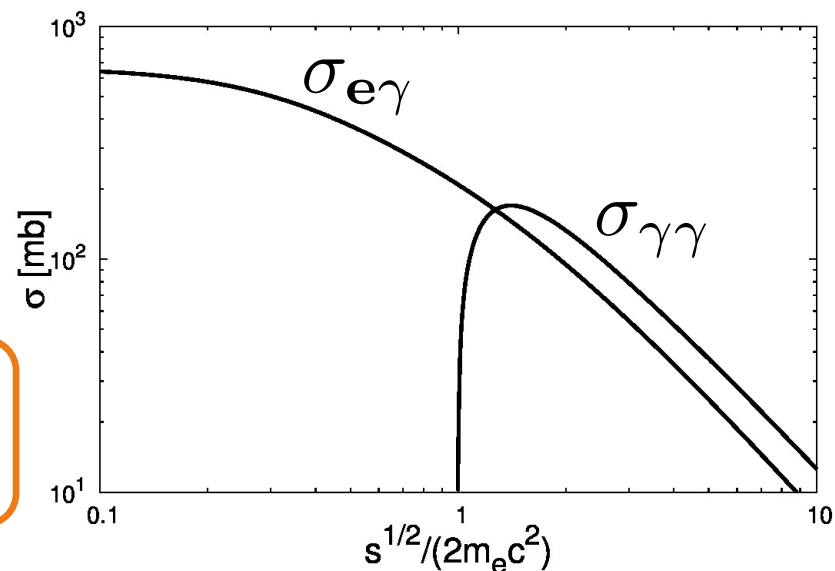
Cosmic Ray Disintegration During Propagation



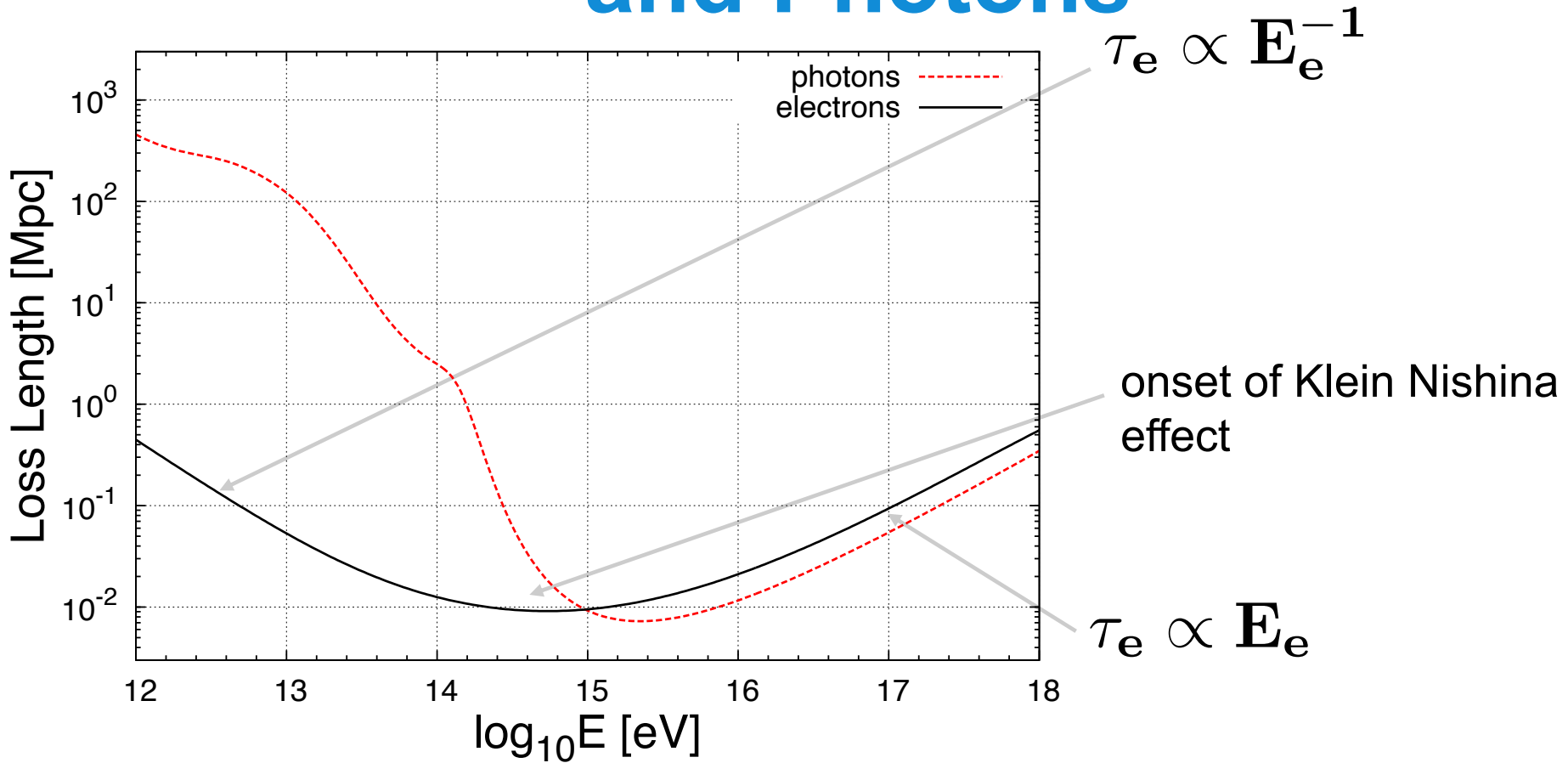
High Energy Gamma-Ray and Electron Propagation

Energy Loss Rates of Electrons and Photons

$$R = \frac{m_e^2 c^4}{2E_e^2} \int_0^\infty d\epsilon_\gamma \frac{1}{\epsilon_\gamma^2} \frac{dn}{d\epsilon_\gamma} K_e \int_0^{2E_e \epsilon_\gamma / (m_e c^2)} d\epsilon'_\gamma \epsilon'_\gamma \sigma(\epsilon'_\gamma)$$



Energy Loss Rates of Electrons and Photons



Thomson regime electron cooling:

$$\mathbf{K_e} = \frac{\mathbf{E_\gamma}}{\mathbf{E_e}} = \mathbf{b}$$



$$\mathbf{b} = \frac{\mathbf{E_e \epsilon_\gamma}}{(\mathbf{m_e c^2})^2}$$

Cosmic Ray Spectra

Assumptions on Source Population

Spatial Distribution

$$\frac{dN}{dV_C} \propto (1+z)^n \quad z < z_{\max}$$

$$n = -6, -3, 0, 3$$

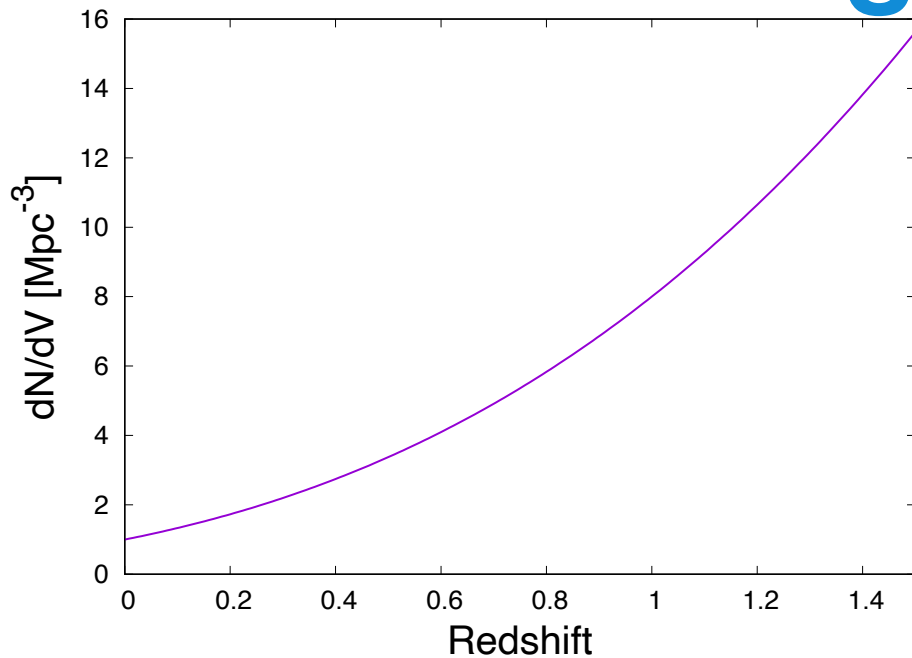
Energy Distribution

$$\frac{dN}{dE} \propto E^{-\alpha} \exp[-E/E_{Z,\max}]$$

$$E_{Z,\max} = (Z/26) \times E_{\text{Fe},\max}$$

Note- magnetic field horizon effects are neglected in the following. This amounts to assuming: $d_s < (ct_H \lambda_{\text{scat}})^{1/2}$ ie. the source distribution may be approximated to be spatially continuous (also note, presence of t_H term comes from temporally continuous assumption)

A Cosmological Distribution of Sources

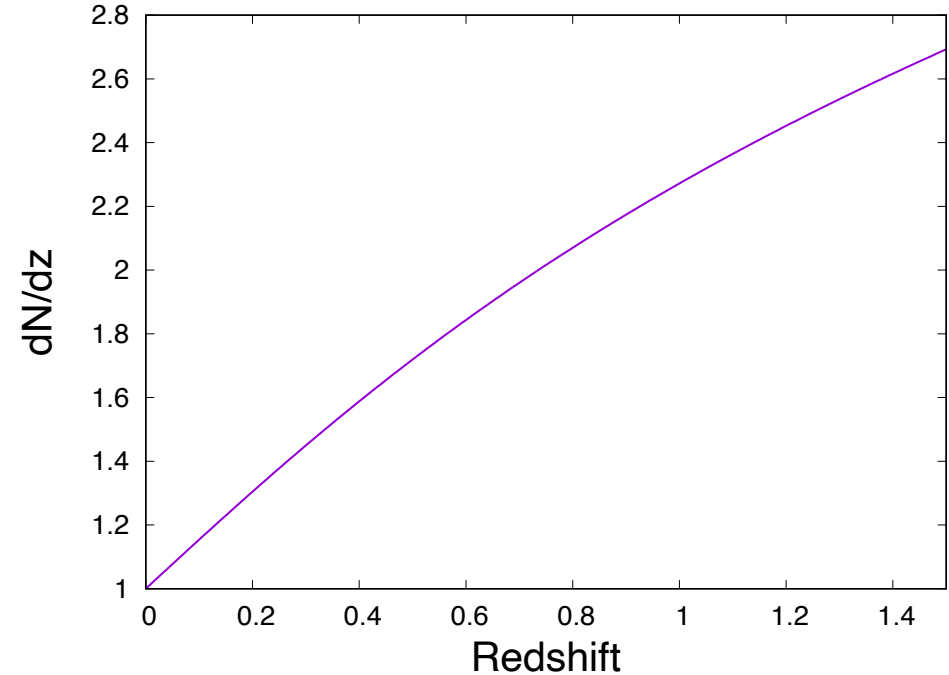


Distribution of sources in a comoving volume

$$dV_c = 4\pi\chi^2 d\chi$$

$$d\chi = \frac{dz}{H}$$

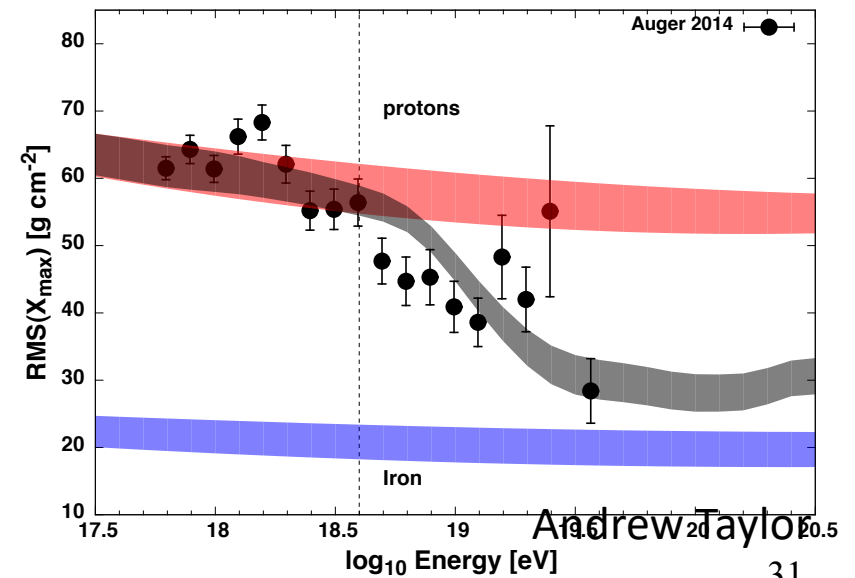
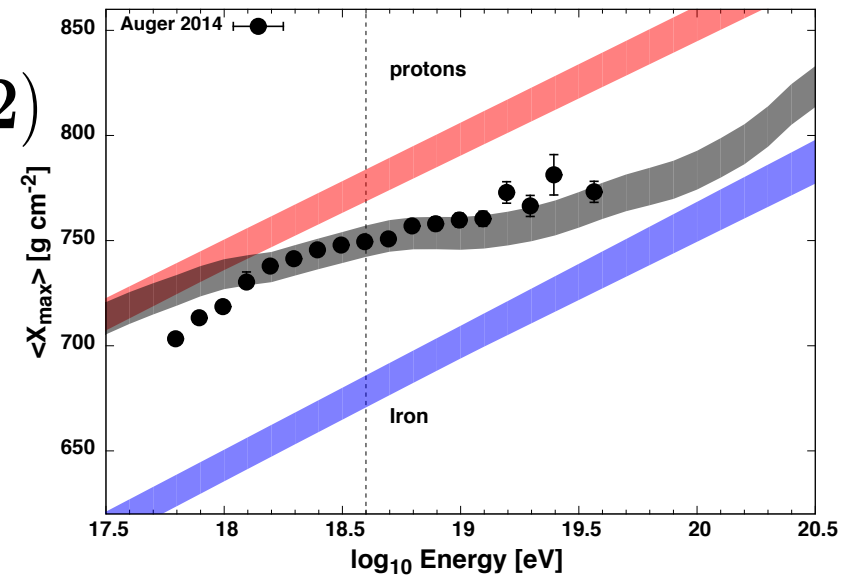
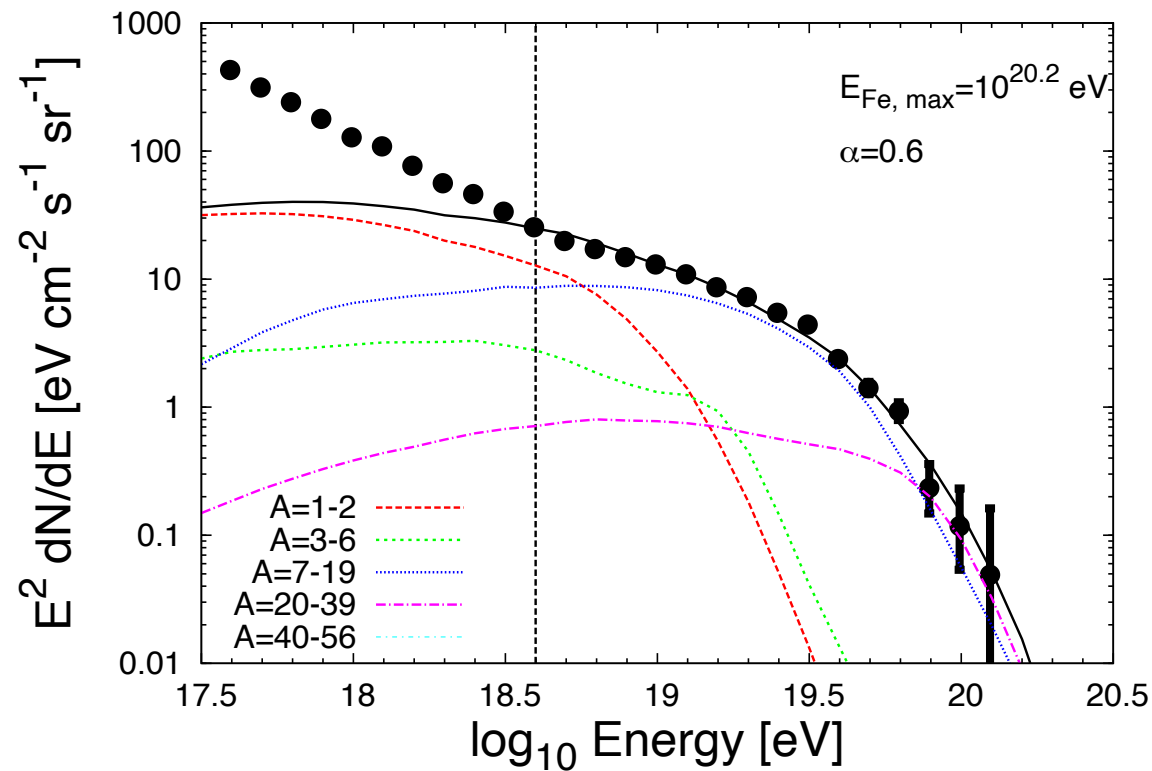
$$\approx \frac{dz}{H_0(\Omega_M(1+z)^3 + \Omega_\Lambda)^{1/2}}$$



MCMC Likelihood Scan: Spectral + Composition Fits

$$\mathcal{L}(f_p, f_{\text{He}}, f_{\text{N}}, f_{\text{Si}}, E_{\text{max}}, \alpha) \propto \exp(-\chi^2/2)$$

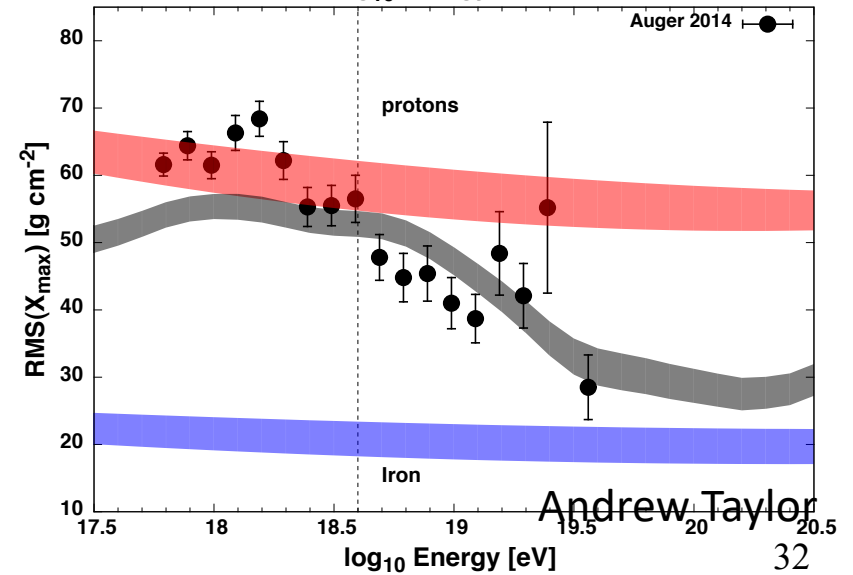
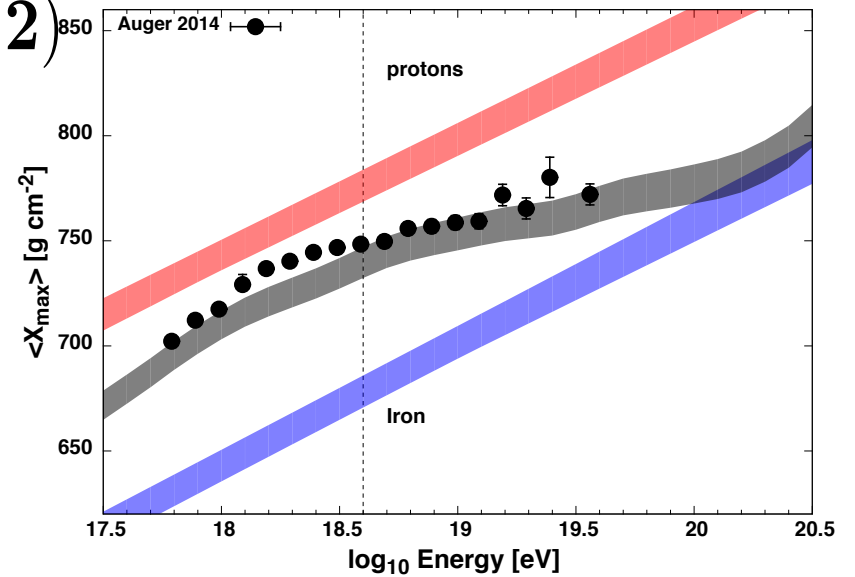
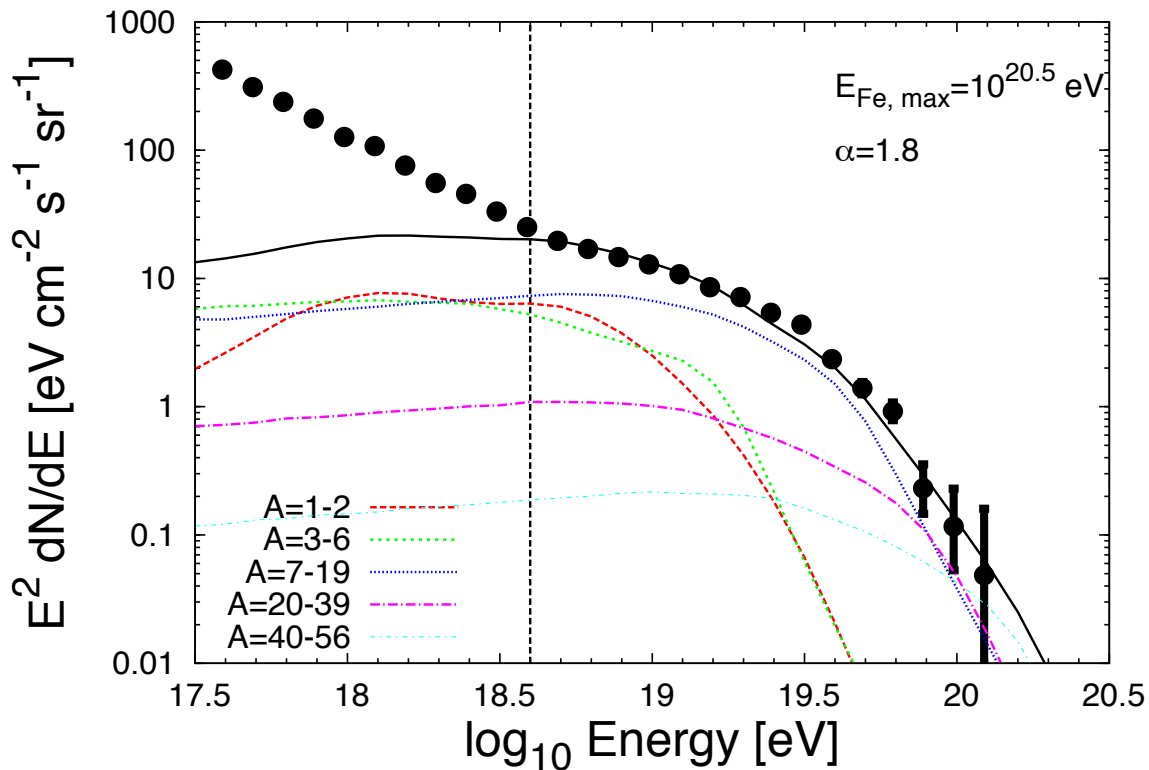
n=3 evolution result



MCMC Likelihood Scan: “Soft” Spectra Solutions

$$L(f_p, f_{\text{He}}, f_{\text{N}}, f_{\text{Si}}, E_{\text{max}}, \alpha) \propto \exp(-\chi^2/2)$$

n=-6 evolution result



MCMC Results Table

From astro-ph/1505.06090 (Taylor et al. 2015)

Similar conclusion arrived at by others too: astro-ph/1612.07155 (Aab et al. 2017)

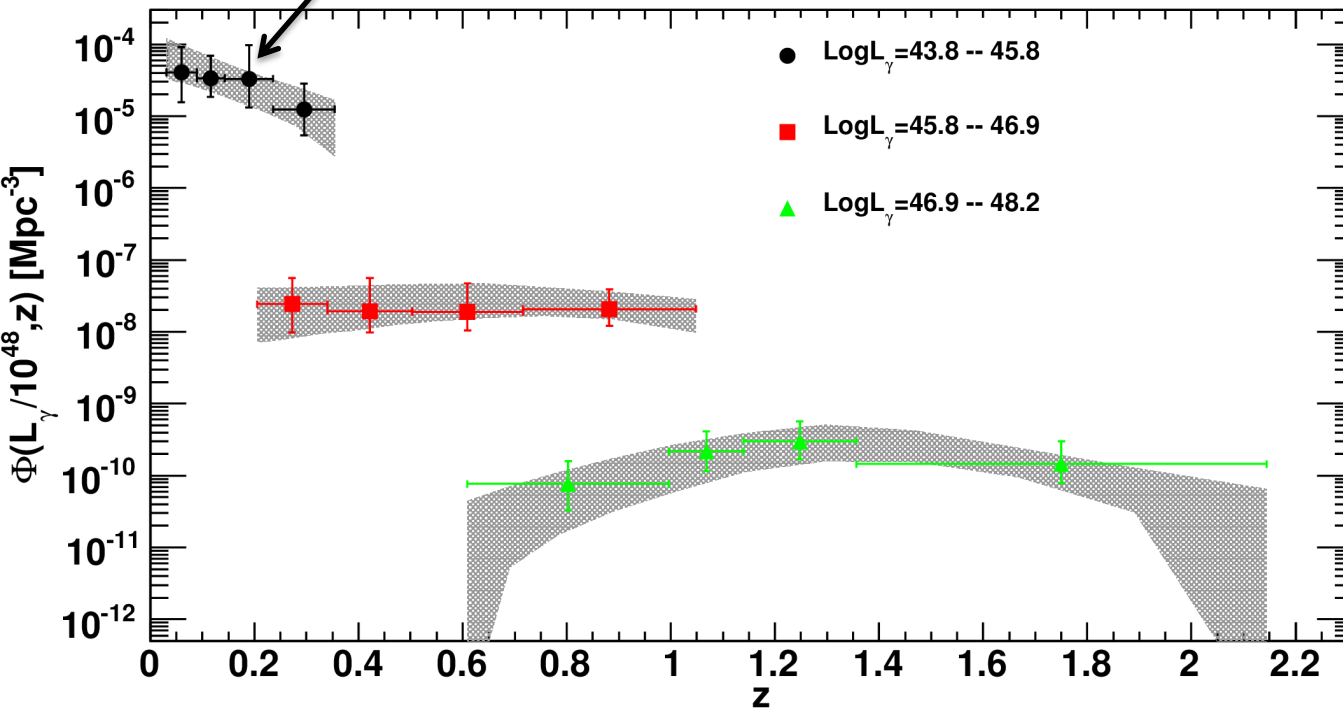
Parameter	$n = -6$		$n = -3$		$n = 0$		$n = 3$	
	Best-fit Value	Posterior Mean & Standard Deviation	Best-fit Value	Posterior Mean & Standard Deviation	Best-fit Value	Posterior Mean & Standard Deviation	Best-fit Value	Posterior Mean & Standard Deviation
f_p	0.03	0.14 ± 0.12	0.08	0.15 ± 0.13	0.17	0.17 ± 0.16	0.19	0.20 ± 0.16
f_{He}	0.50	0.21 ± 0.17	0.42	0.17 ± 0.16	0.53	0.20 ± 0.17	0.32	0.23 ± 0.20
f_{N}	0.40	0.50 ± 0.18	0.42	0.51 ± 0.19	0.29	0.47 ± 0.19	0.43	0.45 ± 0.21
f_{Si}	0.06	0.11 ± 0.12	0.08	0.12 ± 0.13	0.0	0.11 ± 0.12	0.06	0.078 ± 0.086
f_{Fe}	0.01	0.052 ± 0.039	0.0	0.053 ± 0.042	0.01	0.050 ± 0.038	0.0	0.044 ± 0.034
α	1.8	1.83 ± 0.31	1.6	1.67 ± 0.36	1.1	1.33 ± 0.41	0.6	0.64 ± 0.44
$\log_{10}\left(\frac{E_{\text{Fe,max}}}{\text{eV}}\right)$	20.5	20.55 ± 0.26	20.5	20.52 ± 0.27	20.2	20.38 ± 0.25	20.2	20.16 ± 0.18

Flatter spectra preferred for negative source evolution

Hard spectra preferred for source evolution following that of the SFR

High Spectral Peaked Blazar Evolution

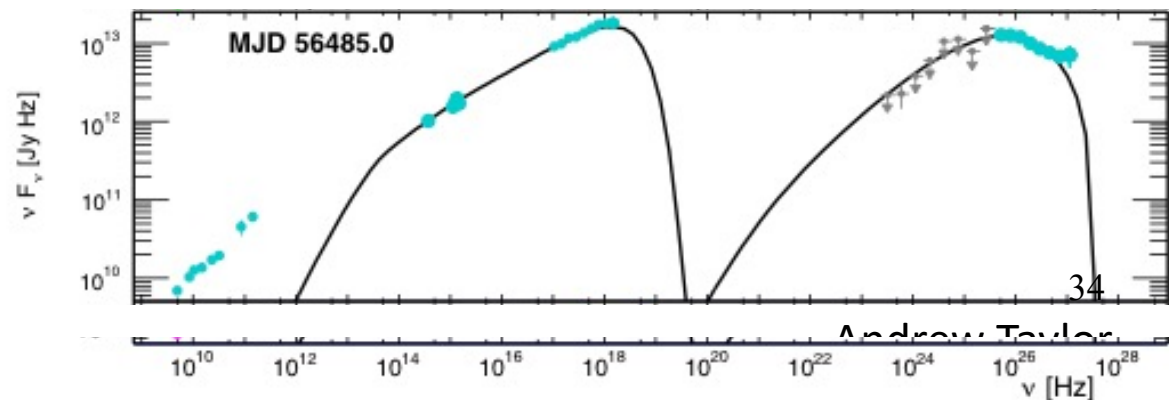
n=-6 evolution result



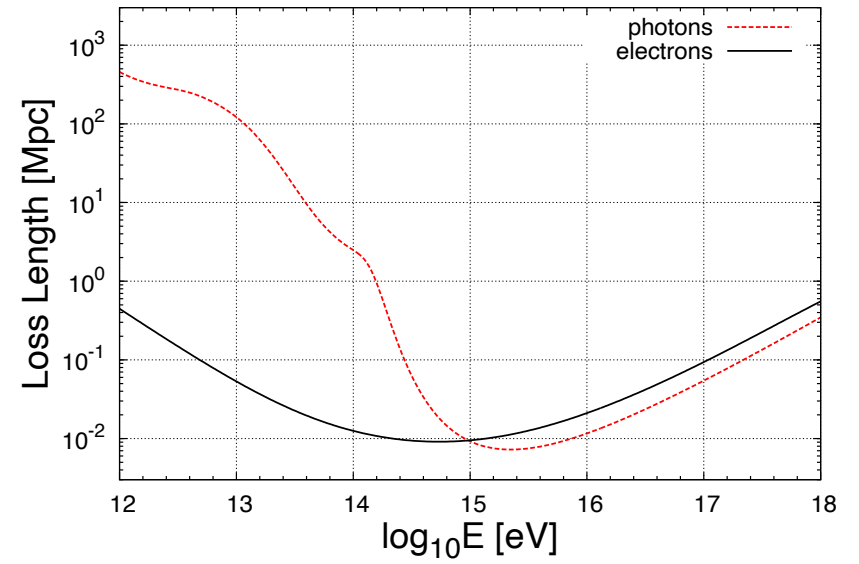
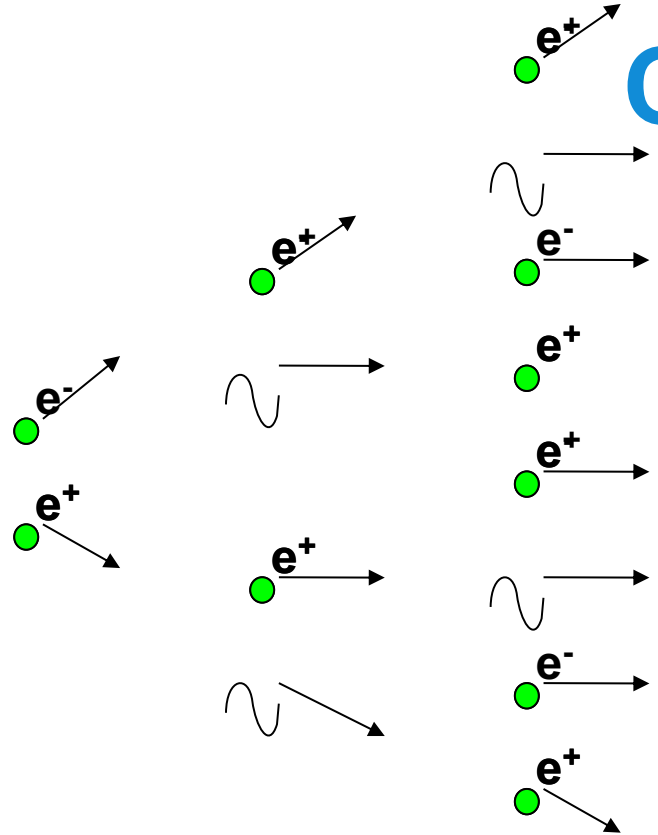
- Reminder:
Blazar -> BL Lac (FR1) -> HSP
- Supports idea that FSRQ (gas accreting) AGN evolve into BL Lac (gas starved) AGN

From astro-ph/1310.0006 (Ajello et al. 2014)

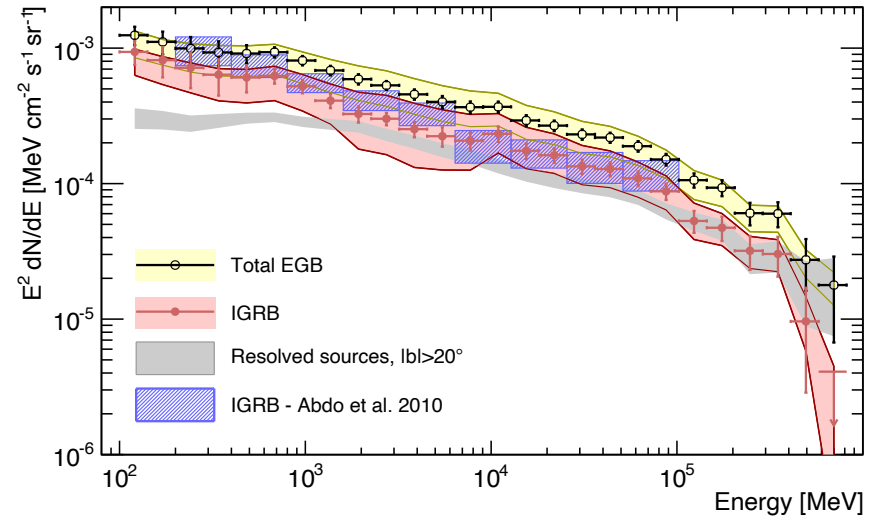
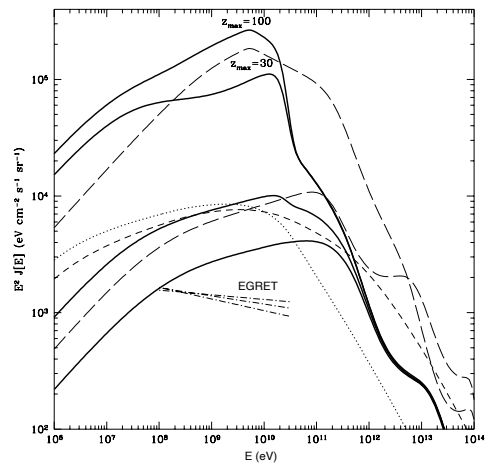
Archetypal HSP
example Mrk 501



Cascade Spectra + the IGRB



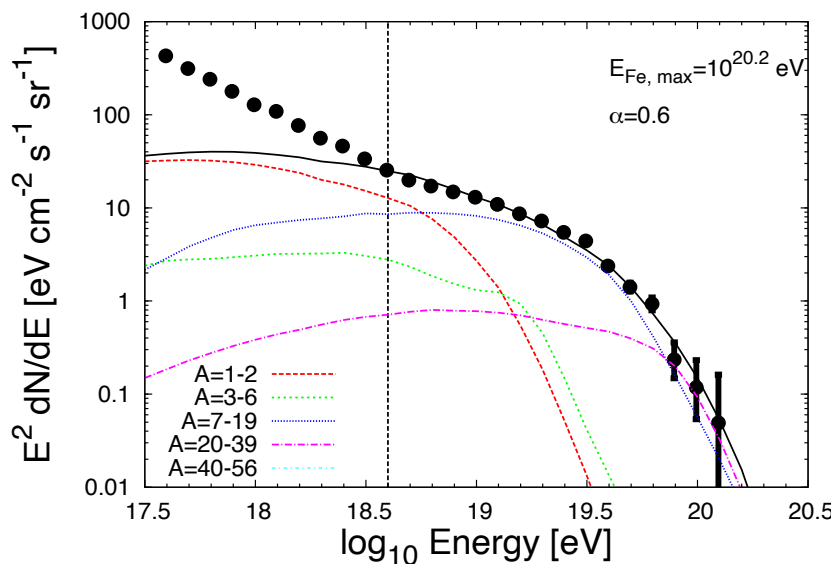
From Coppi et al. astro-ph/9610176



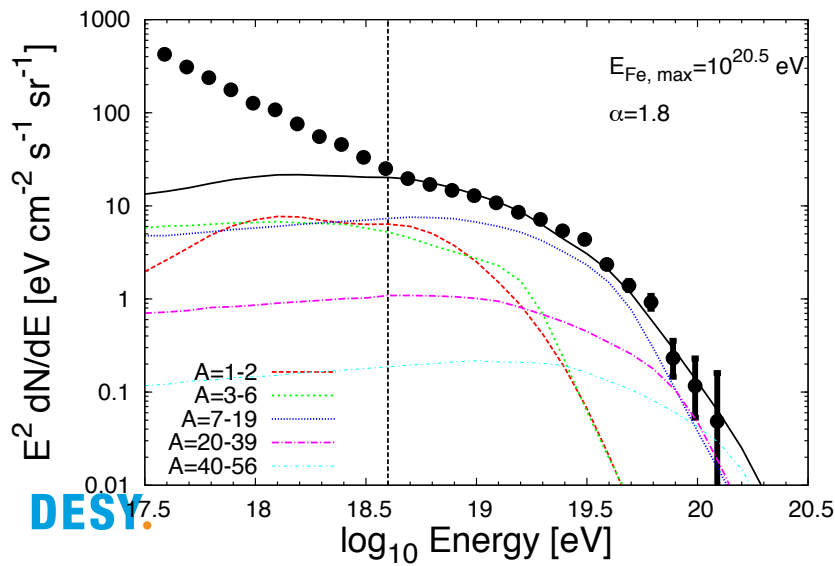
Regardless of where the energy is injected (ie independent of source z), the arriving flux possesses a \sim universal shape

Secondary (Guaranteed) Gamma-Ray Fluxes From $>10^{18.6}$ eV UHECR

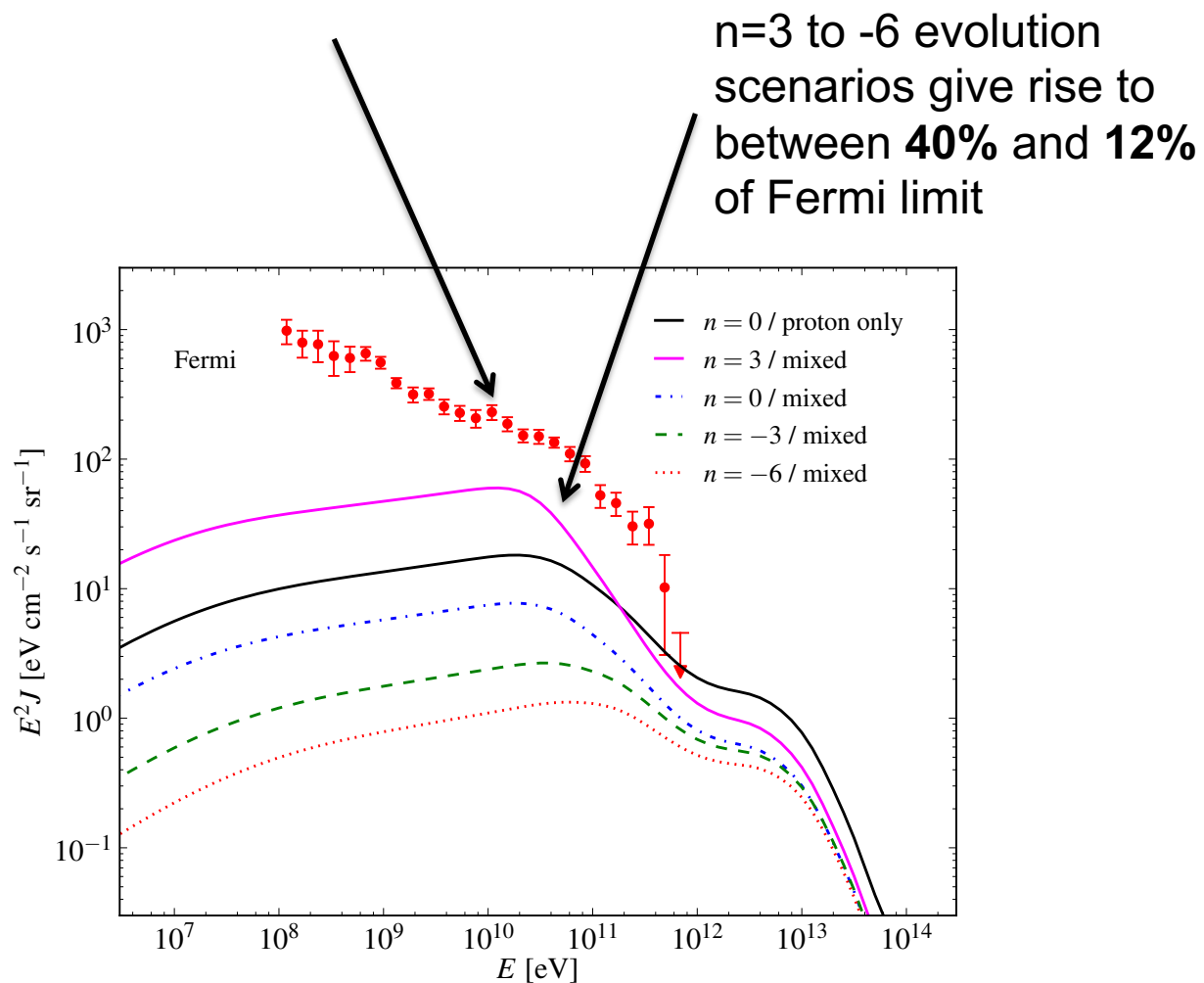
$n=3$ evolution result



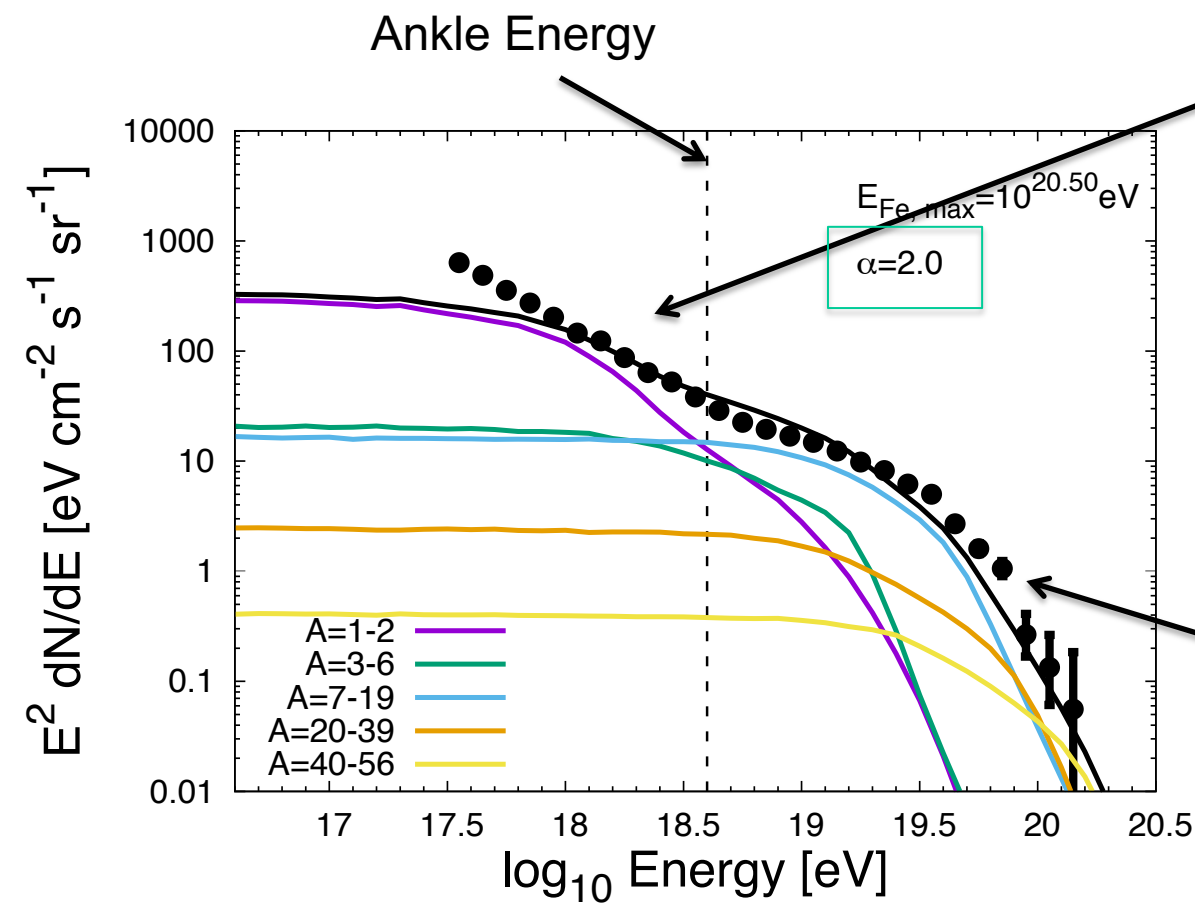
$n=-6$ evolution result



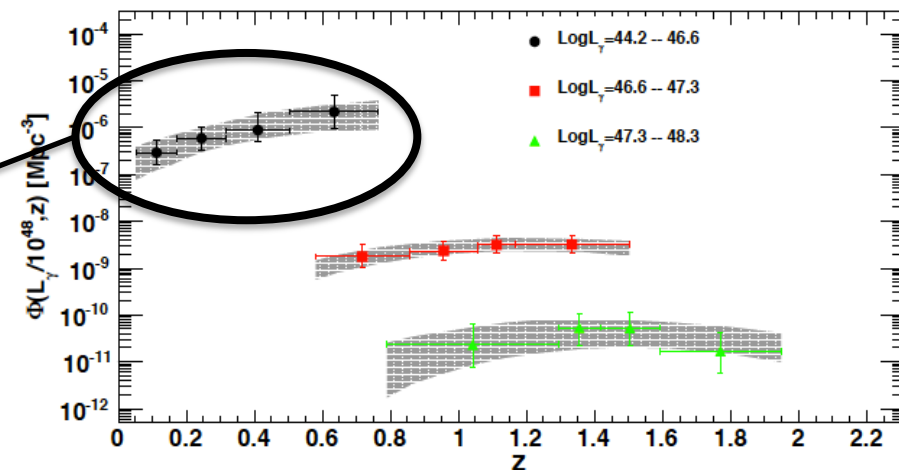
IGRB (EGB with resolved points sources removed)



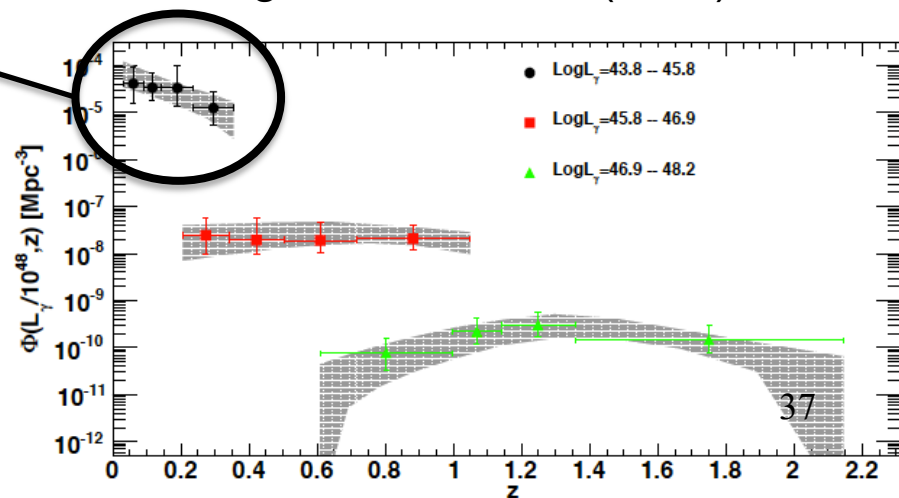
Does a Separate Class of Extragalactic Source Dominate at Sub-Ankle Energies?



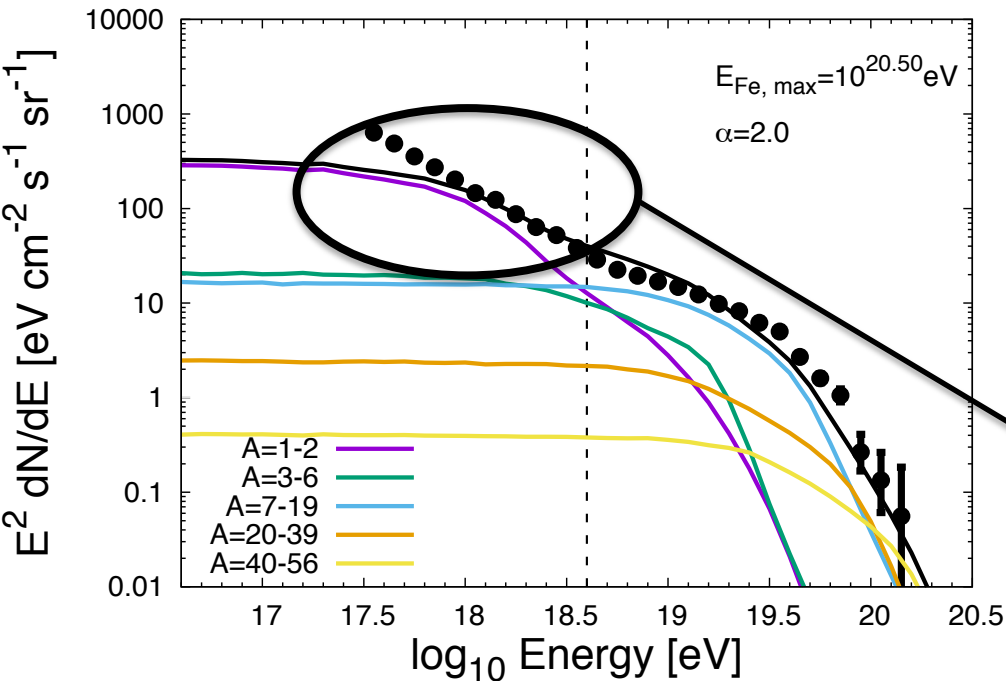
Positive evolution (ISP + LSP)



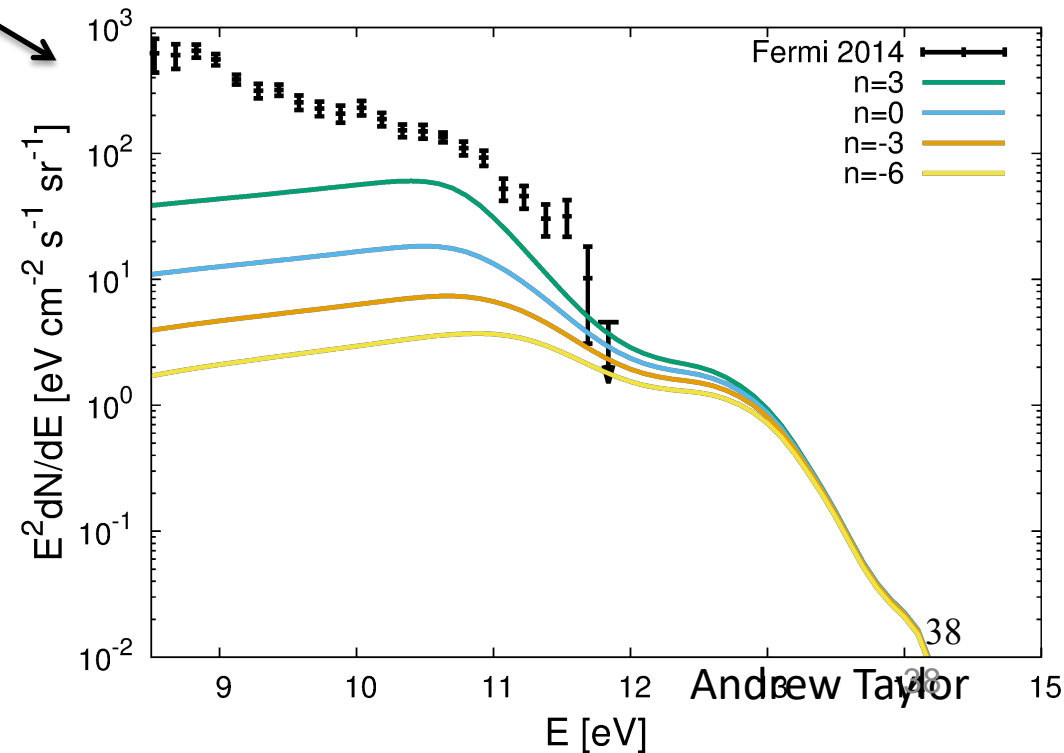
Negative evolution (HSP)



Cascade Contribution from Second Source Population

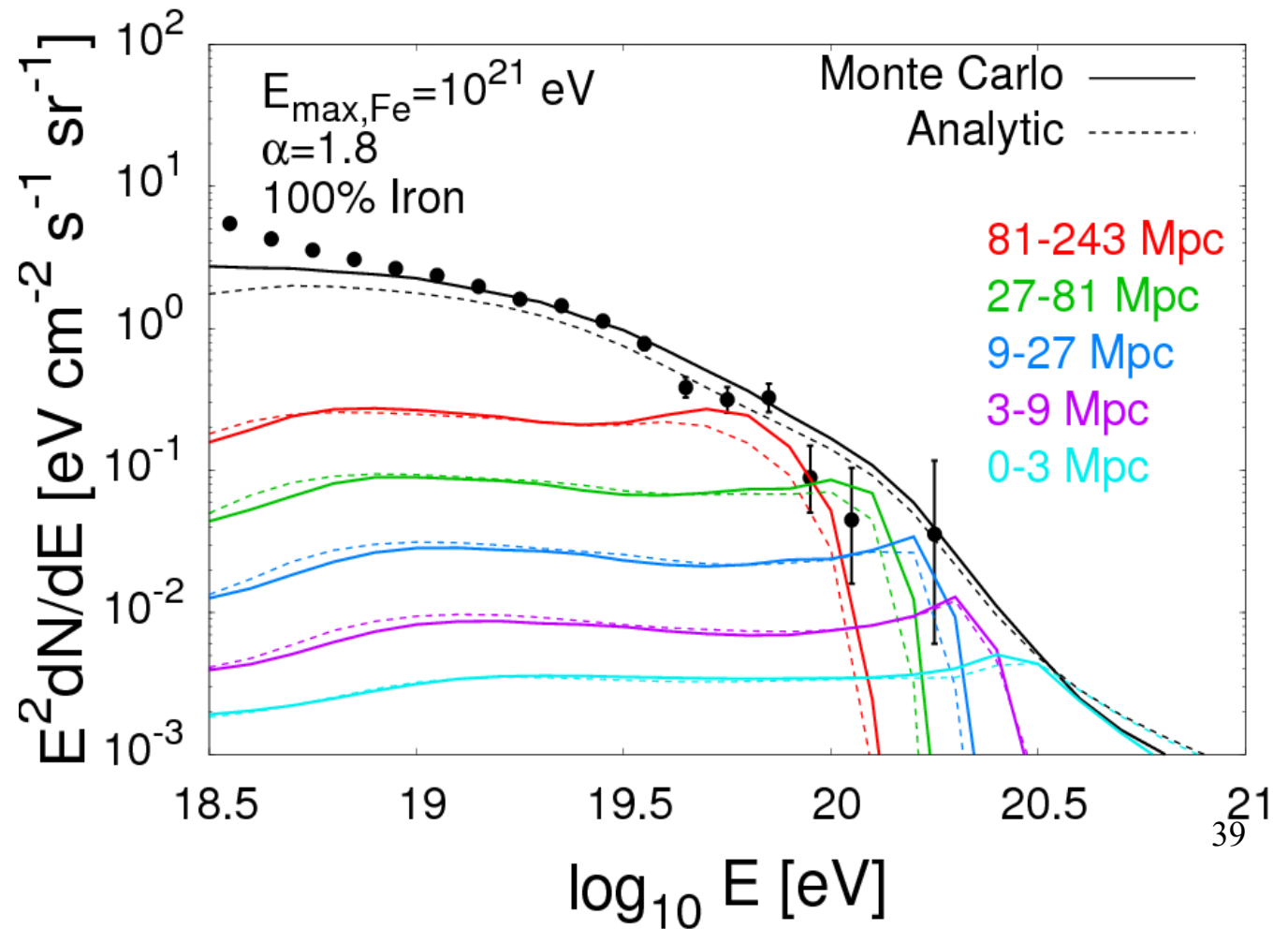
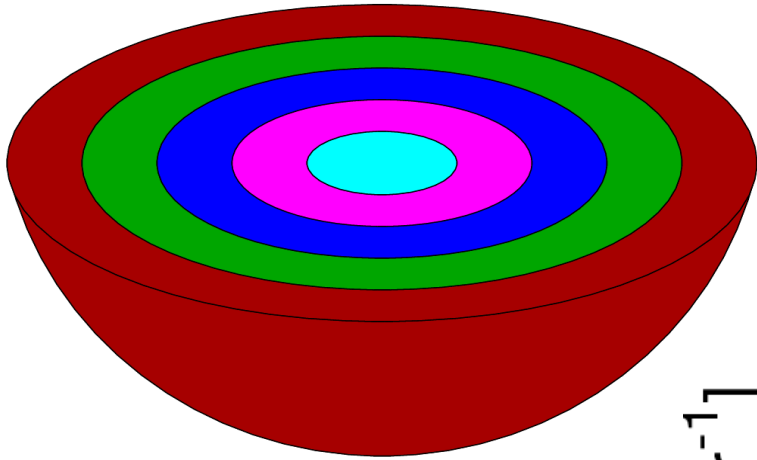


$n=3$ to -6 evolution scenarios give rise to between **100%** and **40%** of Fermi limit



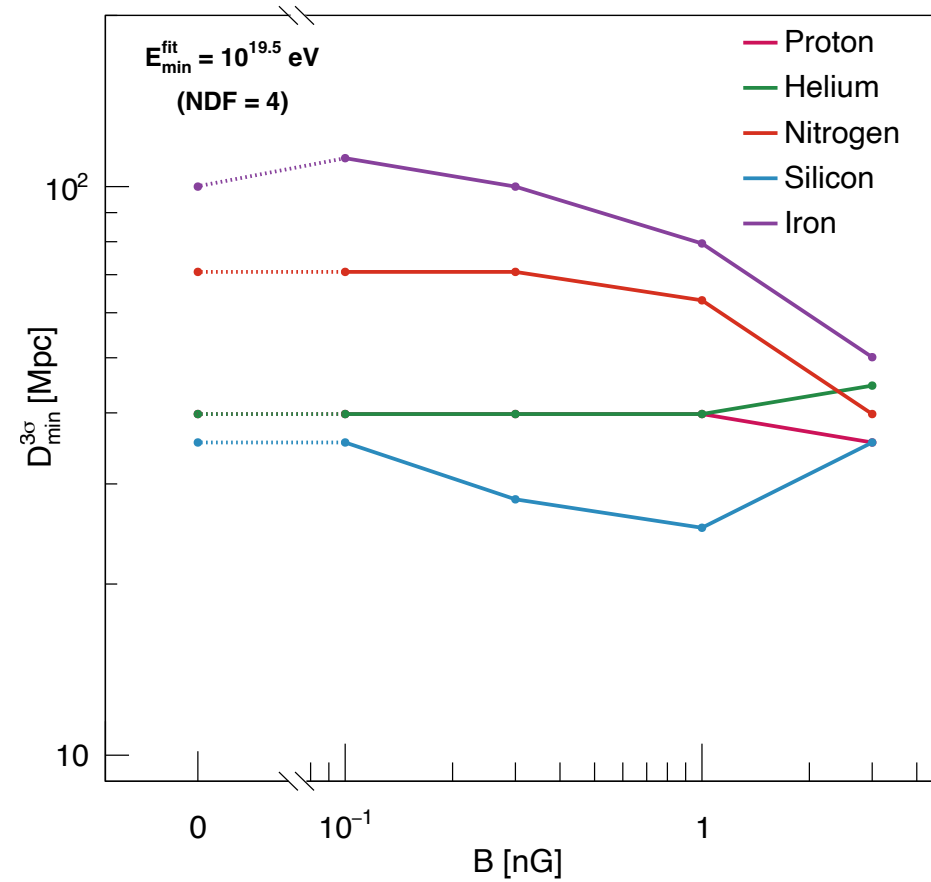
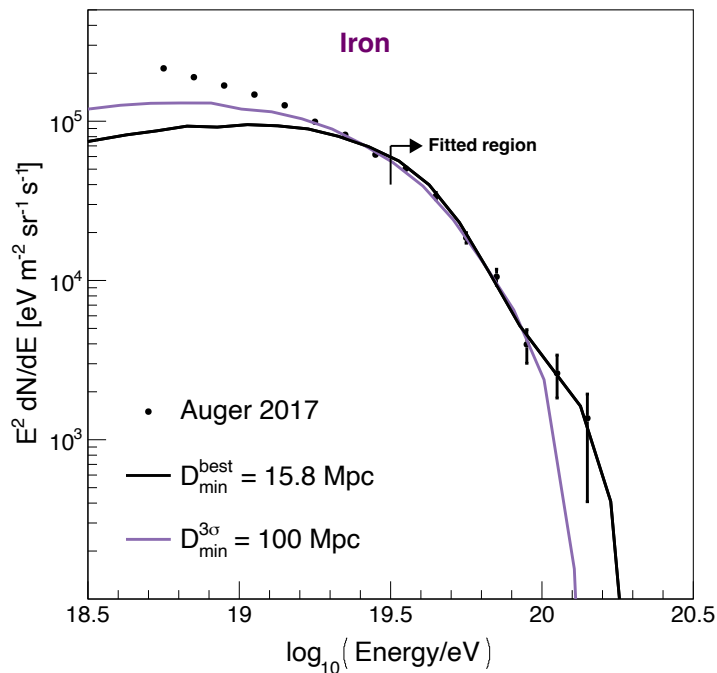
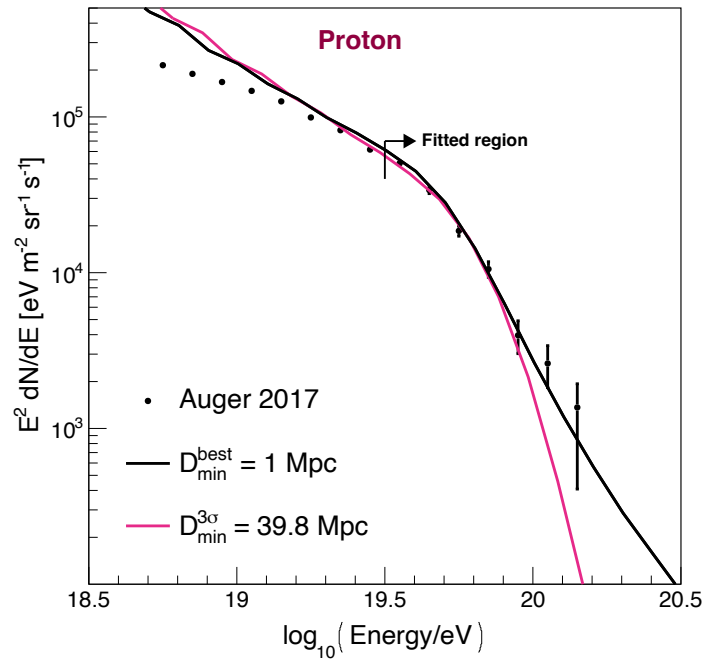
Local Scales Effect Highest Energies (logarithmic scale)

0 3 9 27 81 243 Mpc
|-----|-----|-----|-----|-----|----->



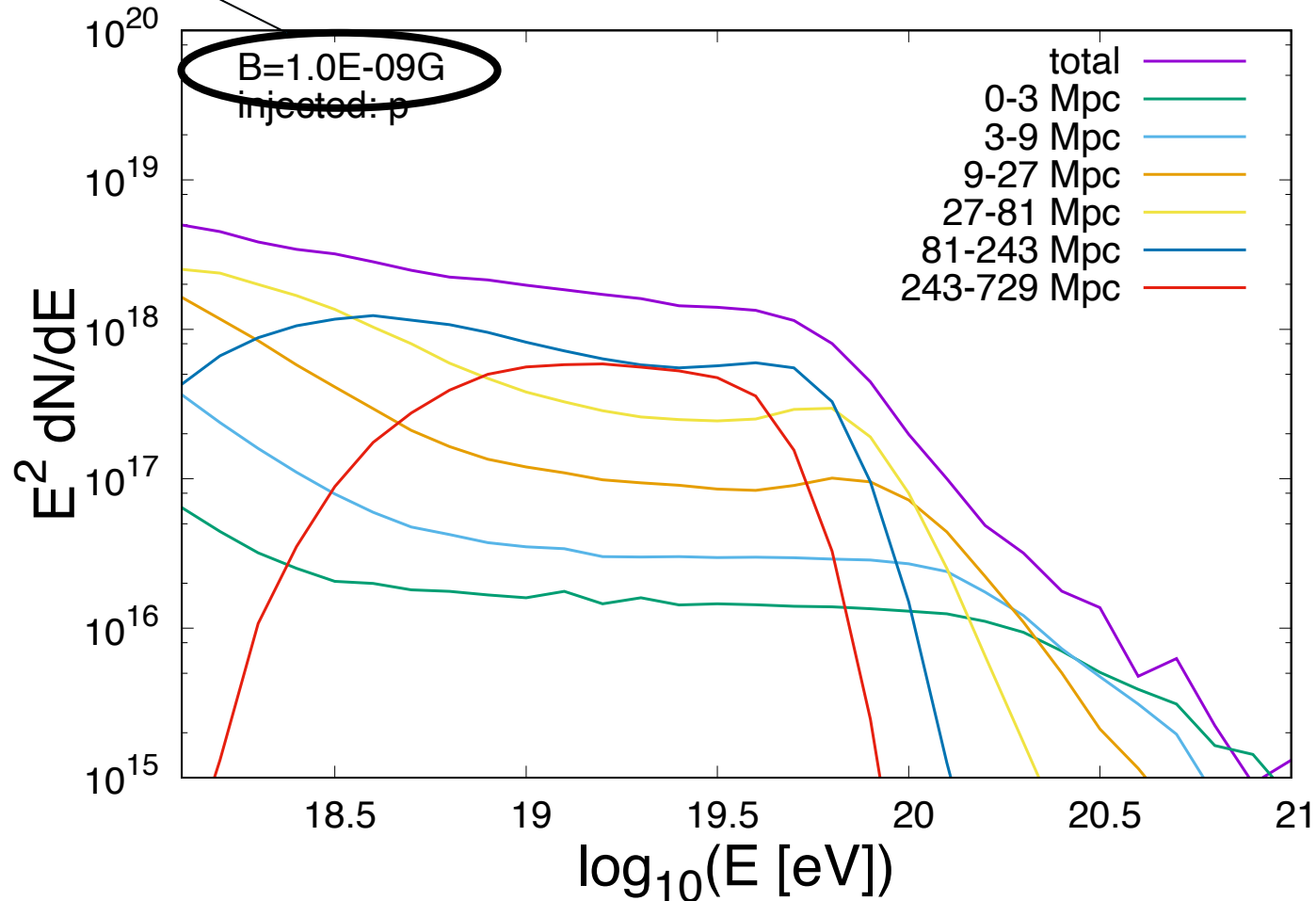
Proximity of Local Sources?

From astro-ph/2005.14275 (Lang et al. 2020)



Magnetic Horizon Effect

Note strong B-field strength considered



Conclusions

- The attenuation of cosmic ray protons/nuclei/photons/electrons due to the presence of background radiation fields is well understood
- The largest limitation presently is the EBL (dust and stellar emission components)
- Despite these limitations, calculations for the propagation ultra high energy cosmic rays in these background radiation fields are predictive
- A negative evolution of sources allows for softer source injection spectra (more consistent with the Fermi acceleration model)
- A negative evolution of sources gives rise to a reduced level of diffuse gamma-ray background contribution
- The current cosmic ray data at the highest energies is suggestive that the sources should be no further than a few 10s of Mpc

End of Lecture



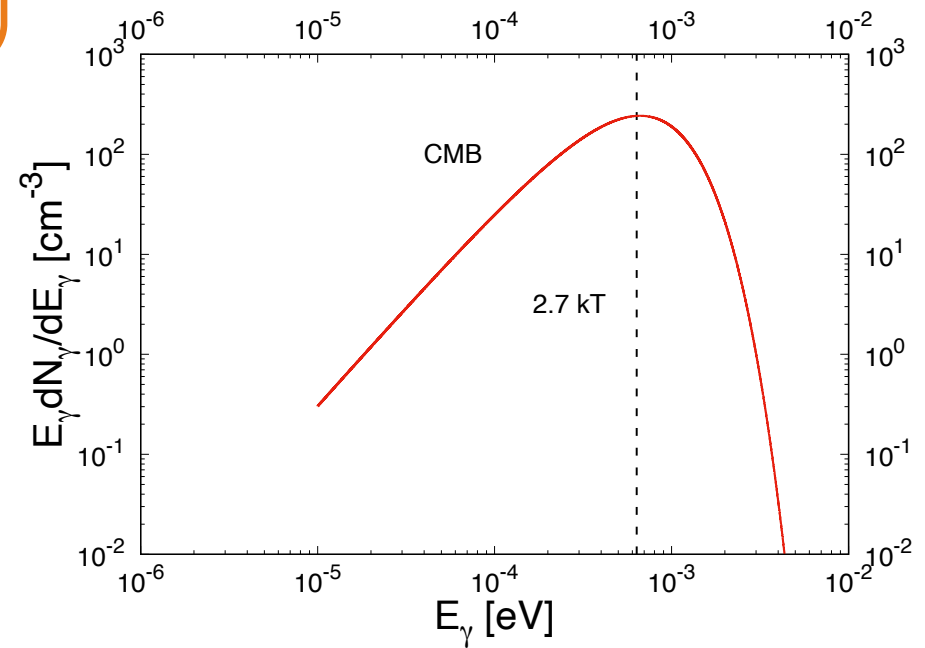
Blackbody- Total Number Density

$$n_{\gamma}^{\text{BB}} = 8\pi \frac{(kT)^3}{(hc)^3} \gamma(3)\zeta(3)$$

$$n_{\gamma}^{\text{BB}} = \frac{8\pi(kT)^3}{(hc)^3} \int_0^{\infty} \frac{x^2}{e^x - 1} dx$$

$$\int_0^{\infty} x^2 e^{-x} dx = \gamma(3)$$

$$\frac{x^n}{e^x - 1} = \frac{e^{-x} x^n}{1 - e^{-x}}$$





CMB- Total Number Density

$$n_{\gamma}^{\text{BB}} = 8\pi \frac{(kT)^3}{(hc)^3} \gamma(3)\zeta(3)$$

$$\frac{x^n}{e^x - 1} = \frac{e^{-x} x^n}{1 - e^{-x}}$$

$$= \sum_{m=0}^{\infty} e^{-mx} e^{-x} x^n$$

$$= \sum_{m=1}^{\infty} e^{-mx} x^n$$



CMB- Total Number Density

$$n_{\gamma}^{\text{BB}} = 8\pi \frac{(kT)^3}{(hc)^3} \gamma(3)\zeta(3)$$

$$\int \frac{x^n}{e^x - 1} dx = \sum_{m=1}^{\infty} \int e^{-mx} x^n dx$$

Let $y = mx$

$$\int \frac{x^n}{e^x - 1} dx = \sum_{m=1}^{\infty} \int e^{-y} \left(\frac{y}{m}\right)^n d\left(\frac{y}{m}\right)$$

$$\int \frac{x^n}{e^x - 1} dx = \sum_{m=1}^{\infty} \frac{1}{m^{n+1}} \int y^n e^{-y} dy = \gamma(n+1)\zeta(n+1)$$



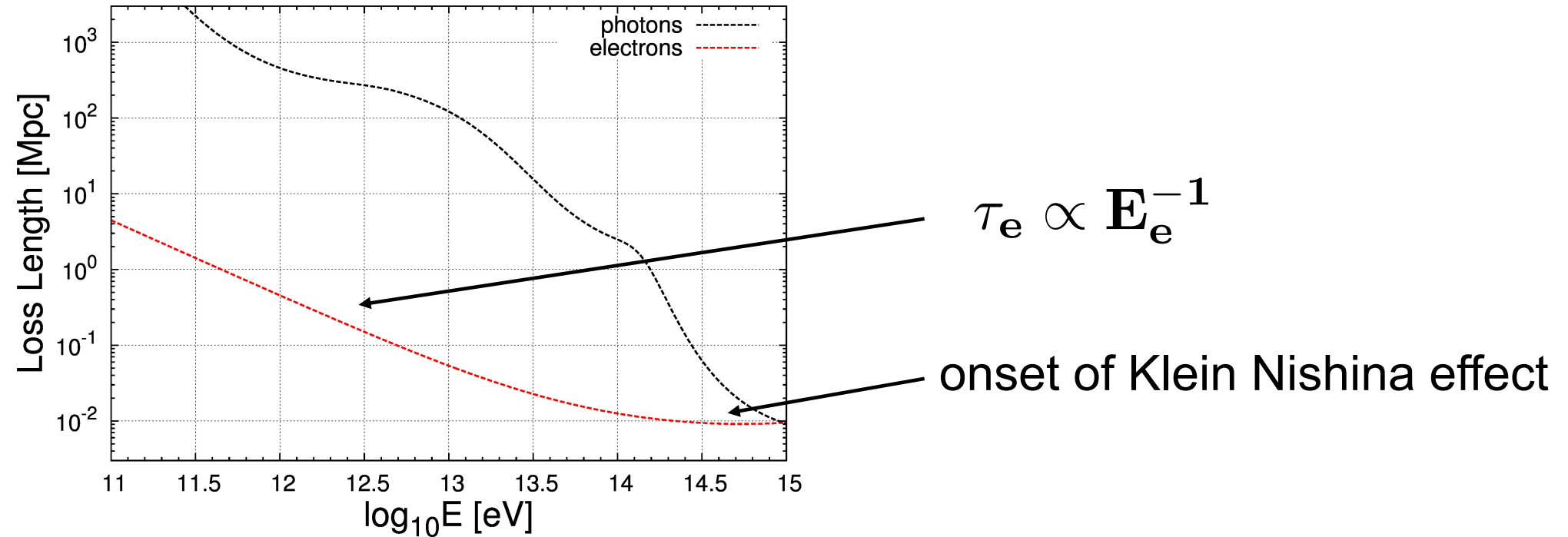
Threshold Energy- Proton Pion Production

$$(\mathbf{E}_p + \mathbf{E}_\gamma)^2 - (\mathbf{p}_p - \mathbf{E}_\gamma)^2 = (m_p + m_\pi)^2$$

$$m_p^2 + 2\mathbf{E}_p\mathbf{E}_\gamma + 2\mathbf{p}_p\mathbf{E}_\gamma \approx m_p^2 + 2m_p m_\pi$$

$$\mathbf{E}_p \approx \frac{m_\pi}{2\mathbf{E}_\gamma} m_p \approx \left(\frac{135 \times 10^6}{2 \times 6 \times 10^{-4}} \right) 0.9 \times 10^9 = 10^{20} \text{ eV}$$

Energy Loss Rates of Electrons and Photons



Thomson regime electron cooling:

$$\tau_e^{-1} = \frac{m_e^2 c^4}{2E_e^2} \int_0^\infty d\epsilon_\gamma \frac{1}{\epsilon_\gamma^2} \frac{dn}{d\epsilon_\gamma} K_e \int_0^{2E_e \epsilon_\gamma / (m_e c^2)} d\epsilon'_\gamma \epsilon'_\gamma \sigma(\epsilon'_\gamma)$$

$$\approx \sigma_T \int_0^\infty b \frac{dn}{d\epsilon_\gamma} d\epsilon_\gamma = \sigma_T \int_0^\infty \frac{E_e \epsilon_\gamma}{(m_e c^2)^2} \frac{dn}{d\epsilon_\gamma} d\epsilon_\gamma = \sigma_T \frac{E_e}{(m_e c^2)^2} U_\gamma$$

Comparison of Analytic and Monte Carlo Results

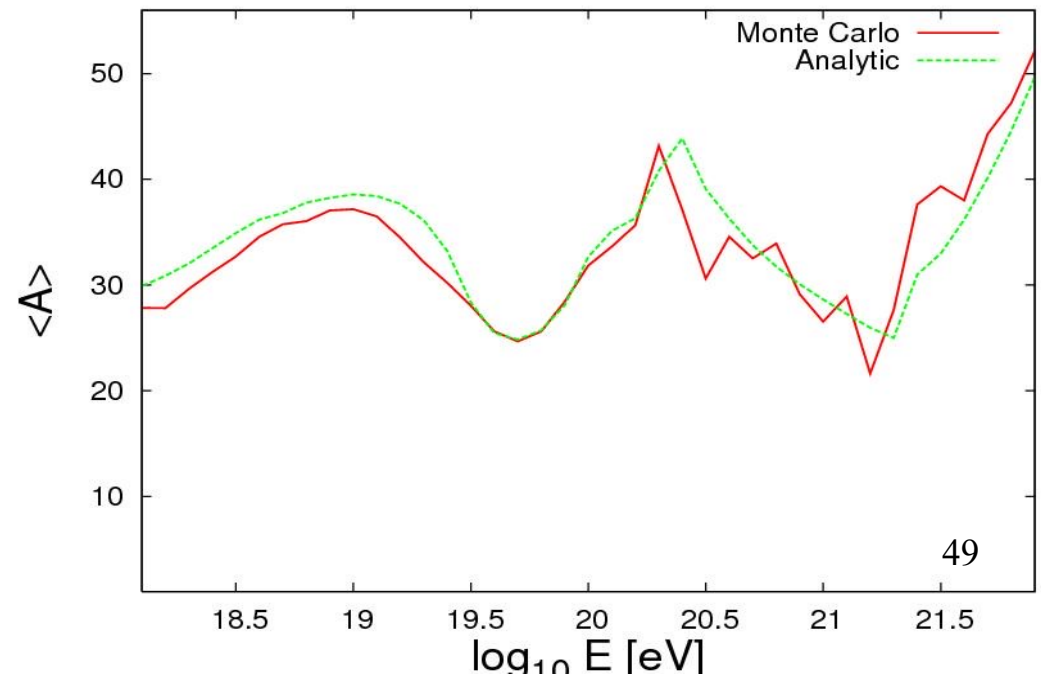
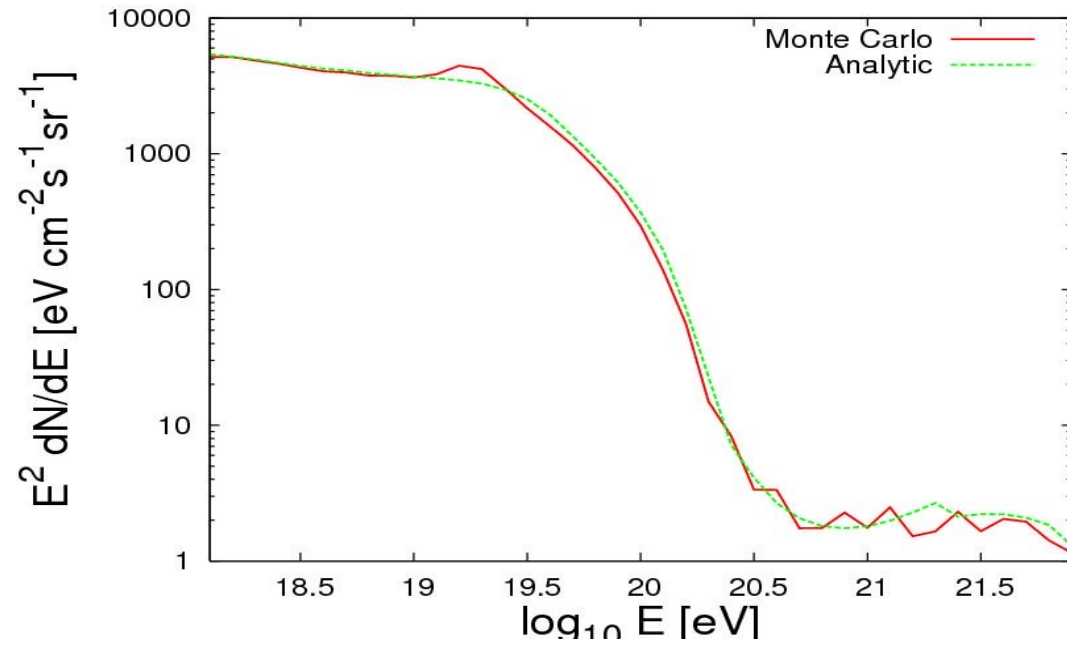


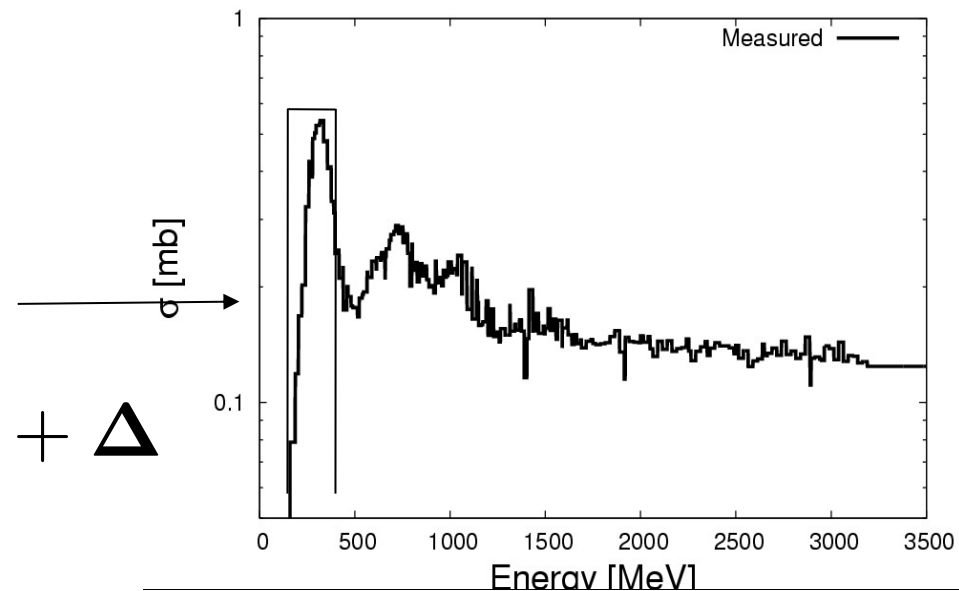
Photo-Pion Production Rate

$$R = \frac{m_p^2 c^4}{2E^2} \int_0^\infty d\epsilon_\gamma \frac{1}{\epsilon_\gamma^2} \frac{dn}{d\epsilon_\gamma} \int_0^{2E\epsilon_\gamma/(m_p c^2)} d\epsilon'_\gamma \epsilon'_\gamma \sigma_{p\gamma}(\epsilon'_\gamma) K_p$$

Assuming the cross-section is approximately:

$$\sigma_{p\gamma}(\epsilon_\gamma) = 0 \quad \begin{array}{l} \epsilon_\gamma < E - \Delta \\ \epsilon_\gamma > E + \Delta \end{array}$$

$$\sigma_{p\gamma}(\epsilon_\gamma) = \sigma_{p\gamma} \quad E - \Delta < \epsilon_\gamma < E + \Delta$$



Where $\sigma_{p\gamma} = 0.5 \text{ mb}$, $E = 300 \text{ MeV}$, $\Delta = 100 \text{ MeV}$

Photo-Pion Production Rate

$$R(\Gamma) \approx \sigma_0 \int_{(\mathbf{E}_0 - \Delta_0)/2\Gamma}^{(\mathbf{E}_0 + \Delta_0)/2\Gamma} \left(\frac{\epsilon^2 - [(\mathbf{E}_0 - \Delta_0)/2\Gamma]^2}{\epsilon^2} \right) \frac{dn}{d\epsilon} d\epsilon +$$

$$\sigma_0 \int_{(\mathbf{E}_0 + \Delta_0)/2\Gamma}^{\infty} \left(\frac{[(\mathbf{E}_0 + \Delta_0)/2\Gamma]^2 - [(\mathbf{E}_0 - \Delta_0)/2\Gamma]^2}{\epsilon^2} \right) \frac{dn}{d\epsilon} d\epsilon$$

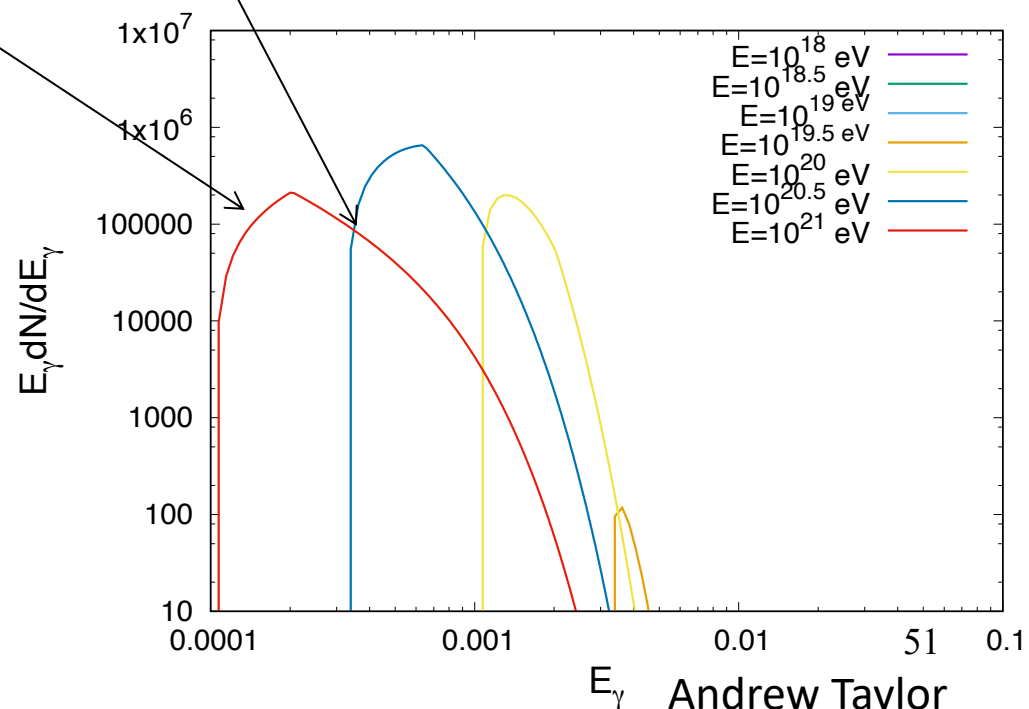
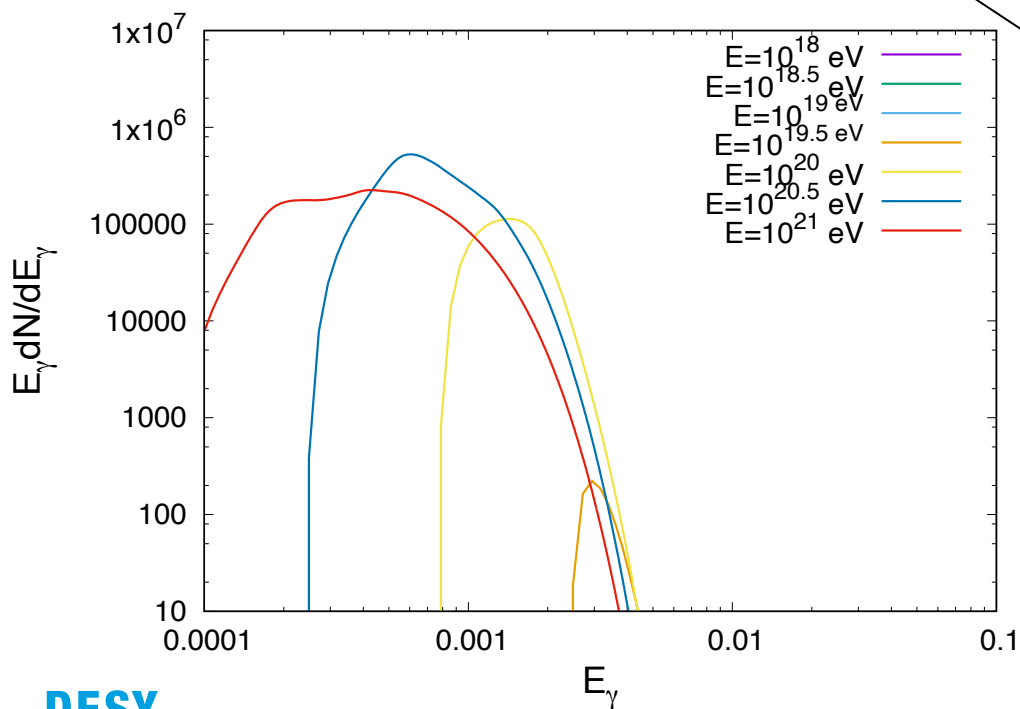


Photo-Pion Production Rate

$$\mathbf{R}(\Gamma) \approx \mathbf{n}_0 \sigma_0 \int_{\mathbf{x}_1(\Gamma)}^{\mathbf{x}_2(\Gamma)} \frac{(\mathbf{x}^2 - \mathbf{x}_1(\Gamma)^2)}{e^{\mathbf{x}} - 1} d\mathbf{x} +$$

$$\mathbf{n}_0 \sigma_0 \int_{\mathbf{x}_2(\Gamma)}^{\infty} \frac{(\mathbf{x}_2^2(\Gamma) - \mathbf{x}_1^2(\Gamma))}{e^{\mathbf{x}} - 1}$$

$$\mathbf{R}(\Gamma) \approx \frac{1}{l_0} [(\gamma_i(\mathbf{3}, \mathbf{x}_2(\Gamma)) - \gamma_i(\mathbf{3}, \mathbf{x}_1(\Gamma))) - \mathbf{x}_1(\Gamma)^2 (\gamma_i(\mathbf{1}, \mathbf{x}_2(\Gamma)) - \gamma_i(\mathbf{1}, \mathbf{x}_1(\Gamma))) + \mathbf{x}_2(\Gamma)^2 (1 - \gamma_i(\mathbf{1}, \mathbf{x}_2(\Gamma))) - \mathbf{x}_1(\Gamma)^2 (1 - \gamma_i(\mathbf{1}, \mathbf{x}_2(\Gamma)))]$$

$$\gamma_i(\mathbf{3}, \mathbf{x}) = \mathbf{2} - (\mathbf{2} + \mathbf{2x} + \mathbf{x}^2) \exp(-\mathbf{x}) \quad \gamma_i(\mathbf{1}, \mathbf{x}) = \mathbf{1} - \exp(-\mathbf{x})$$

$$\mathbf{R}(\Gamma) \approx \frac{\mathbf{2}}{l_0} [e^{-\mathbf{x}_1} (1 - e^{-\mathbf{x}_1} + \mathbf{x}_1 (1 - 2e^{-\mathbf{x}_1}))]]$$

Photo-Pion Production Rate: Blackbody Interactions

$$\mathbf{R}(\Gamma) \approx n_0 \sigma_0 \int_{x_1(\Gamma)}^{x_2(\Gamma)} \frac{(x^2 - x_1(\Gamma)^2)}{e^x - 1} dx +$$

$$n_0 \sigma_0 \int_{x_2(\Gamma)}^{\infty} \frac{(x_2^2(\Gamma) - x_1^2(\Gamma))}{e^x - 1}$$

$$\mathbf{R}(\Gamma) \approx \frac{2}{l_0} \left[e^{-x_1} (1 - e^{-x_1} + x_1 (1 - 2e^{-x_1})) \right]$$

Where, $l_0 = 10 \text{ Mpc}$ $x_1 = \frac{(\mathbf{E} - \Delta)m_p}{2kT_{\text{CMB}}\mathbf{E}_p} = \frac{10^{20.5} \text{ eV}}{\mathbf{E}_p}$

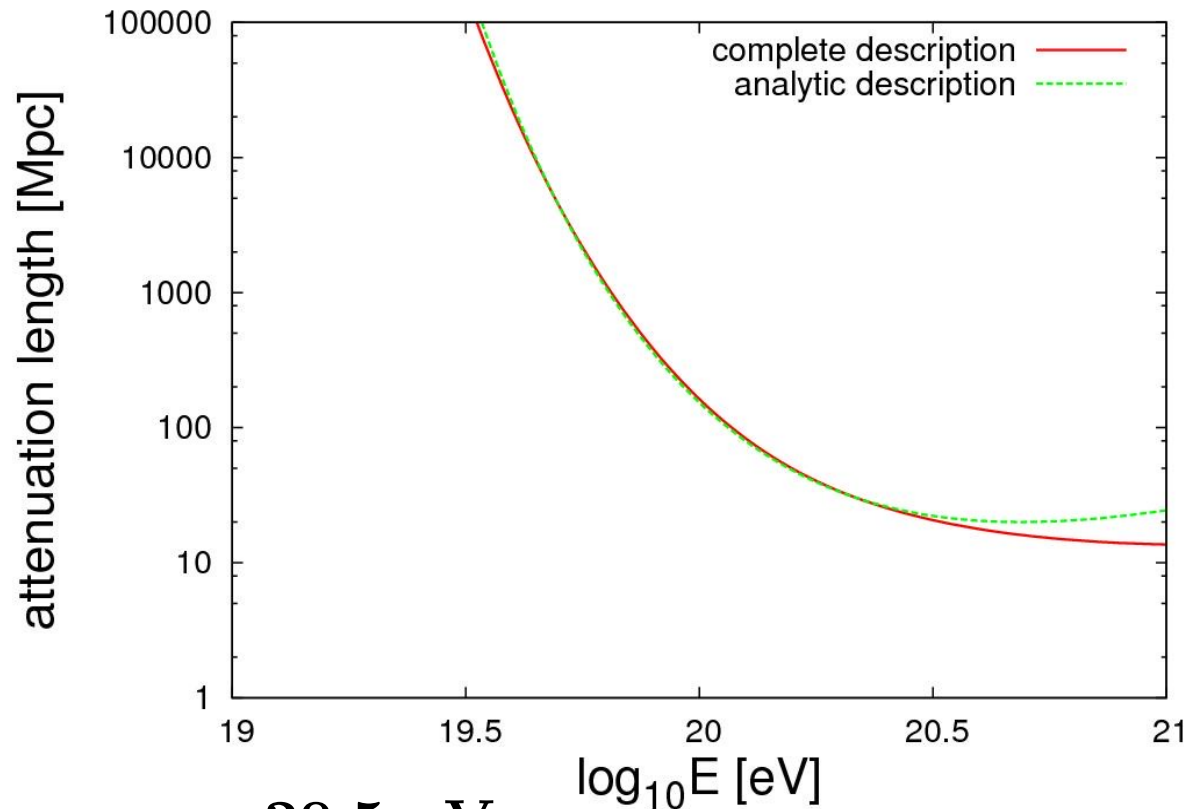
Photo-Pion Production Rate: Blackbody Interactions

With, $kT_{\text{CMB}} \approx 2 \times 10^{-4} \text{ eV}$

$$\mathbf{R} \approx 0.2 \sigma_{\text{p}\gamma} \int_{\frac{\mathbf{E}-\Delta}{2\Gamma}}^{\frac{\mathbf{E}+\Delta}{2\Gamma}} d\epsilon_{\gamma} \frac{dn}{d\epsilon_{\gamma}}$$

$$\approx \left(\frac{l_0}{e^{-x_1} (1 - e^{-x_1})} \right)^{-1}$$

Where l_0 is 5 Mpc and $x_1 = \frac{10^{20.5} \text{ eV}}{E_p}$

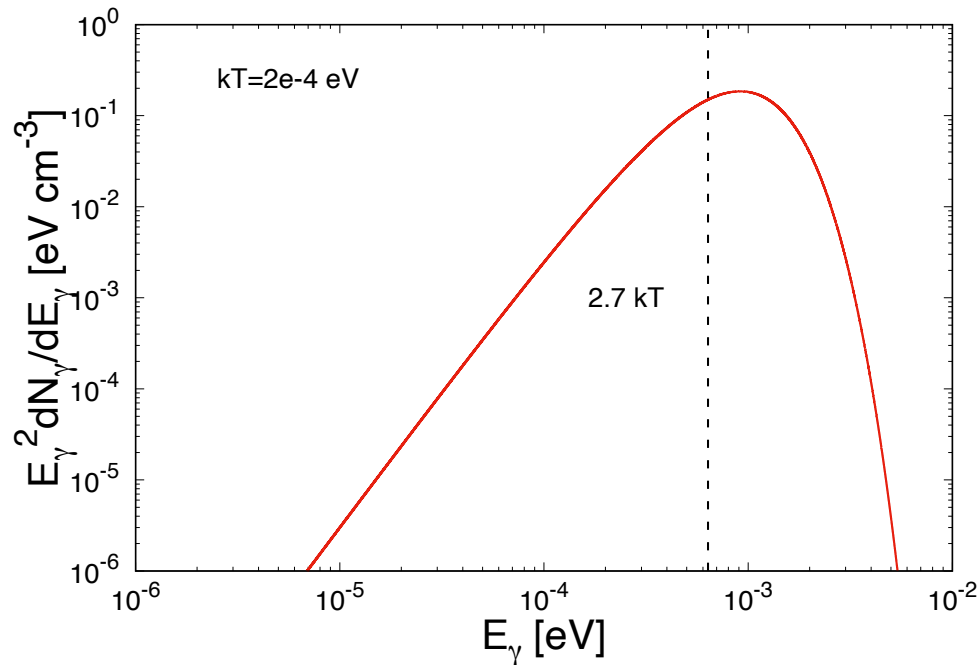




CMB- Total Energy Density

$$\rho_{\gamma}^{\text{BB}} = \frac{8\pi(\mathbf{kT})^4}{(\mathbf{hc})^3} \int_0^{\infty} \frac{x^3}{e^x - 1} dx$$

$$\rho_{\gamma}^{\text{CMB}} = 8\pi \frac{(\mathbf{kT}_{\text{CMB}})^4}{(\mathbf{hc})^3} \gamma(4)\zeta(4) \approx \mathbf{0.25 \text{ eV cm}^{-3}}$$



An Analytic Description of these Results

Differential Equation Describing System State

$$\frac{d}{dt} \begin{pmatrix} f_{56} \\ f_{55} \\ f_{54} \end{pmatrix} = \Lambda \begin{pmatrix} f_{56} \\ f_{55} \\ f_{54} \end{pmatrix}$$

$$\Lambda = \begin{pmatrix} -\left(\frac{1}{\tau_{56 \rightarrow 55}} + \frac{1}{\tau_{56 \rightarrow 54}} + \dots\right) & 0 & 0 \\ \frac{1}{\tau_{56 \rightarrow 55}} & -\left(\frac{1}{\tau_{55 \rightarrow 54}} + \frac{1}{\tau_{55 \rightarrow 53}} + \dots\right) & 0 \\ \frac{1}{\tau_{56 \rightarrow 54}} & \frac{1}{\tau_{55 \rightarrow 54}} & -\left(\frac{1}{\tau_{54 \rightarrow 53}} + \frac{1}{\tau_{54 \rightarrow 52}} + \dots\right) \end{pmatrix}$$

by
$$\mathbf{f}_q(t) = \sum_{n=q}^{56} \mathbf{A}_n \mathbf{f}_n(t)$$

then
$$\mathbf{f}_q(t) = \sum_{n=q}^{56} \mathbf{A}_n e^{-\lambda_n t} \mathbf{f}_n(0)$$

(where A_n values are set by the initial conditions)

Only Considering Single Nucleon Losses

$$\Lambda = \begin{pmatrix} -\frac{1}{\tau_{56 \rightarrow 55}} & 0 & 0 \\ \frac{1}{\tau_{56 \rightarrow 55}} & -\frac{1}{\tau_{55 \rightarrow 54}} & 0 \\ 0 & \frac{1}{\tau_{55 \rightarrow 54}} & -\frac{1}{\tau_{54 \rightarrow 53}} \end{pmatrix}$$

and

$$\mathbf{f}_q(t) = \sum_{n=q}^{56} \mathbf{f}_{56}(0) \frac{\tau_q \tau_n^{56-q-1}}{\prod_{p=q}^{56} (\tau_n - \tau_p)} e^{-\frac{t}{\tau_n}}$$

Nuclear Cascade Description

Consider

$$\frac{df_q}{dt} + \frac{f_q}{\tau_q} = \frac{f_{q+1}}{\tau_{q+1}}$$

$$e^{\left(\frac{-t}{\tau_q}\right)} \frac{d}{dt} \left[e^{\left(\frac{t}{\tau_q}\right)} f_q \right] = \frac{f_{q+1}}{\tau_{q+1}}$$

$$f_q = e^{\left(\frac{-t}{\tau_q}\right)} \int e^{\left(\frac{t}{\tau_q}\right)} \frac{f_{q+1}}{\tau_{q+1}} dt$$

Assume solution is true for q , apply to $q+1$

$$\frac{f_{q+1}(t)}{f_{56}(0)} = \sum_{n=q+1}^{56} \frac{\tau_{q+1} \tau_n^{56-q-2}}{\prod_{p=q+1}^{56} (\tau_n - \tau_p)} e^{-\frac{t}{\tau_n}}$$

Nuclear Cascade Description

Assume solution is true

$$\frac{\mathbf{f}_{q+1}(\mathbf{t})}{\mathbf{f}_{56}(\mathbf{0})} = \sum_{n=q+1}^{56} \frac{\tau_{q+1} \tau_n^{56-q-2}}{\prod_{p=q+1}^{56} (\tau_n - \tau_p)} e^{-\frac{t}{\tau_n}}$$

$$\mathbf{f}_q = e^{\left(\frac{-t}{\tau_q}\right)} \int e^{\left(\frac{t}{\tau_q}\right)} \frac{\mathbf{f}_{q+1}}{\tau_{q+1}} dt$$

$$\frac{\mathbf{f}_q(\mathbf{t})}{\mathbf{f}_{56}(\mathbf{0})} = \sum_{n=q+1}^{56} \frac{\tau_n^{56-q-2}}{\prod_{p=q+1}^{56} (\tau_n - \tau_p)} \left[\left(\frac{1}{\tau_q} - \frac{1}{\tau_n} \right)^{-1} e^{\frac{-t}{\tau_n}} \right] - \mathbf{c} e^{\frac{-t}{\tau_q}}$$

Since $\mathbf{f}_q(\mathbf{0}) = \mathbf{0}$

$$\mathbf{c} = \sum_{n=q+1}^{56} \frac{\tau_q \tau_n^{56-q-1}}{\prod_{p=q}^{56} (\tau_n - \tau_p)}$$

Nuclear Cascade Description

$$\frac{f_q(t)}{f_{56}(0)} = \sum_{n=q+1}^{56} \frac{\tau_n^{56-q-2}}{\prod_{p=q}^{56} (\tau_n - \tau_p)} e^{-\frac{t}{\tau_n}} - \sum_{n=q+1}^{56} \frac{\tau_q \tau_n^{56-q-1}}{\prod_{p=q}^{56} (\tau_n - \tau_p)} e^{-\frac{t}{\tau_q}}$$

$$\frac{f_q(t)}{f_{56}(0)} = \sum_{n=q}^{56} \frac{\tau_q \tau_n^{56-q-1}}{\prod_{p=q}^{56} (\tau_n - \tau_p)} e^{-\frac{t}{\tau_n}}$$

These are equivalent if:

$$\sum_{n=q+1}^{56} \frac{\tau_q \tau_n^{56-q-1}}{\prod_{p=q}^{56} (\tau_n - \tau_p)} = \frac{\tau_q \tau_q^{56-q-1}}{\prod_{p=q}^{56} (\tau_q - \tau_p)}$$

Consider:

$$\frac{w^2}{(w-x)(w-y)(w-z)} + \frac{x^2}{(x-w)(x-y)(x-z)} + \frac{y^2}{(y-w)(y-x)(y-z)} = -\frac{z^2}{(z-w)(z-x)(z-y)}$$

Nuclear Cascade Description

$$\sum_{n=q+1}^{56} \frac{\tau_q \tau_n^{56-q-1}}{\prod_{p=q}^{56} (\tau_n - \tau_p)} = \frac{\tau_q \tau_q^{56-q-1}}{\prod_{p=q}^{56} (\tau_q - \tau_p)}$$

Consider the case

$$\frac{w^2}{(w-x)(w-y)(w-z)} + \frac{x^2}{(x-w)(x-y)(x-z)} + \frac{y^2}{(y-w)(y-x)(y-z)} = -\frac{z^2}{(z-w)(z-x)(z-y)}$$

$$\begin{vmatrix} 1 & w & w^2 & w^2 \\ 1 & x & x^2 & x^2 \\ 1 & y & y^2 & y^2 \\ 1 & z & z^2 & z^2 \end{vmatrix} = 0$$

Cascade of Nuclei Through Species- single nucleon loss

Since nuclei Lorentz factor remains
~conserved, and cross-section varies mildly
with A (nuclear mass)

$$\tau_{56 \rightarrow 55} \approx \tau_{55 \rightarrow 54} \dots$$

For the case $\tau_{56 \rightarrow 55} = \tau_{55 \rightarrow 54} \dots$

$$f_q = \frac{t^{(q_{max}-q)}}{\tau_q (q_{max}-q)!} e^{-t/\tau_q}$$

ie. Gaisser-Hillas
type function!

(used to describe air showers)

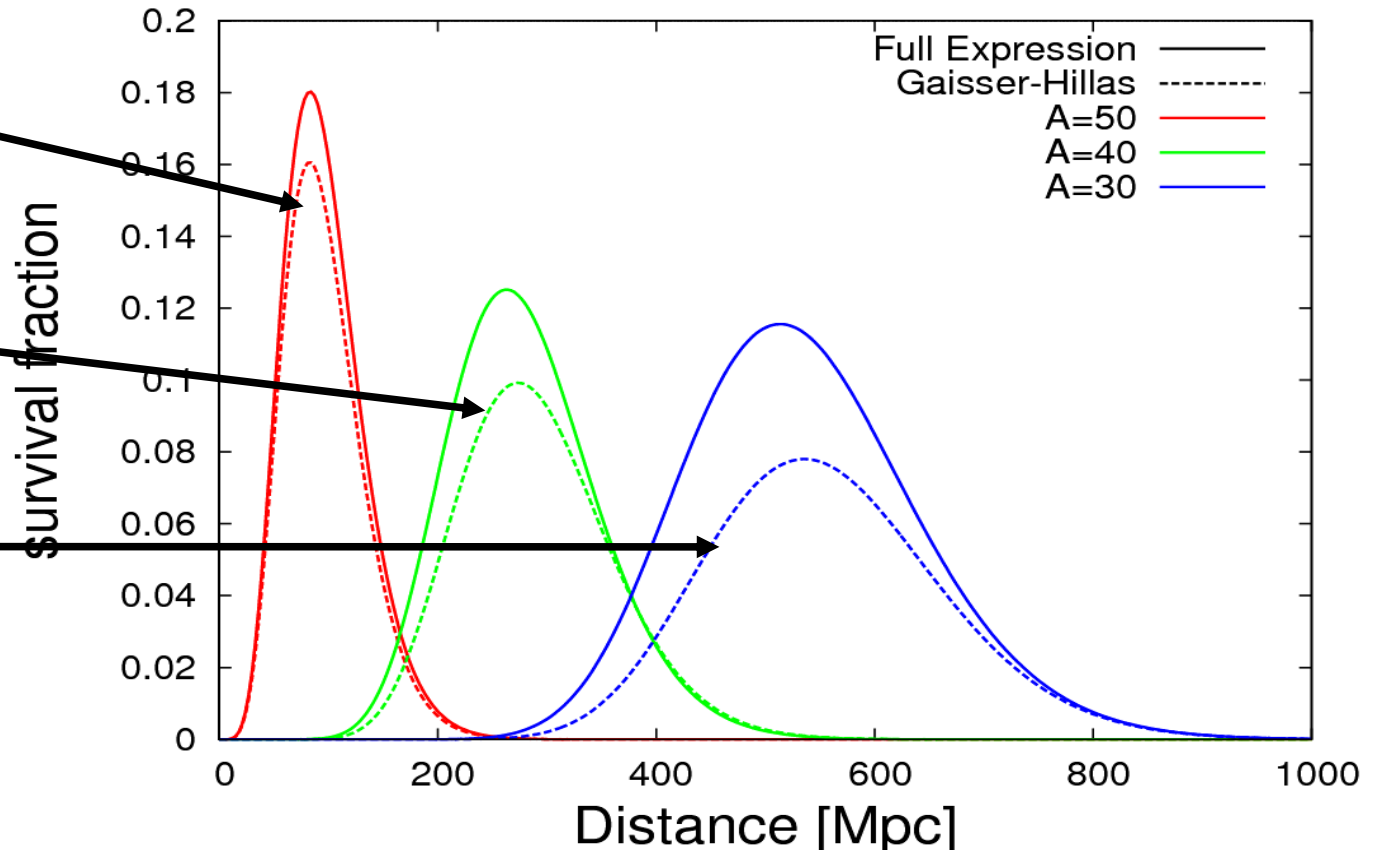
Cascade of Nuclei Through Species- Comparison of Approximation

Starting with Fe, $q_{\max} = 56$

$$f_{50} = \frac{t^6}{6!} e^{-\frac{t}{\tau_{50}}}$$

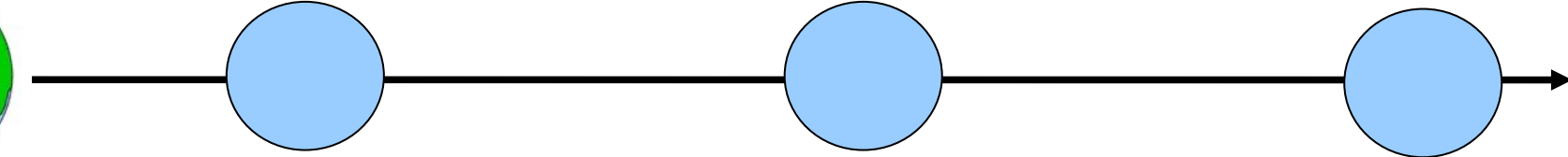
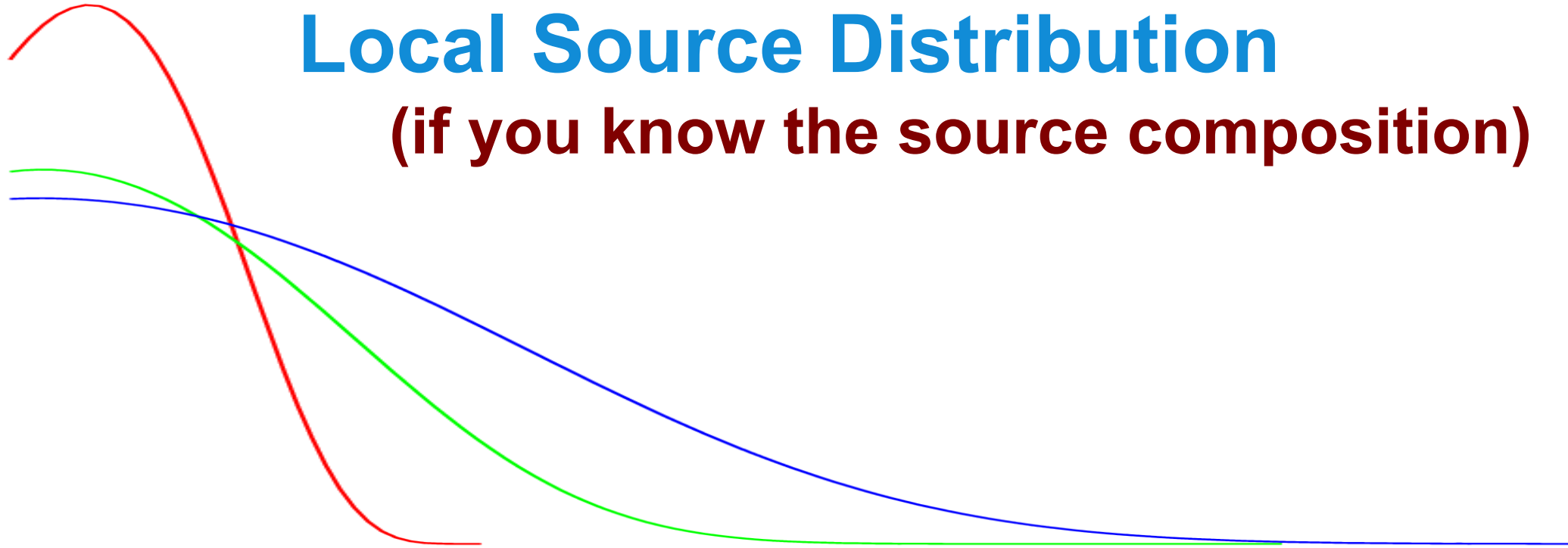
$$f_{40} = \frac{t^{16}}{16!} e^{-\frac{t}{\tau_{40}}}$$

$$f_{30} = \frac{t^{26}}{26!} e^{-\frac{t}{\tau_{30}}}$$



Composition – an Excellent Probe of the Local Source Distribution

(if you know the source composition)



100 Mpc
→

300 Mpc
→

500 Mpc
→

Assumptions on Source Population

Spatial Distribution

motivated by star formation rate evolution

$$\frac{dN}{dV_C} \propto (1+z)^3 \quad z < 1.9$$

$$\frac{dN}{dV_C} \propto (1+1.9)^3 \quad 1.9 < z < 2.7$$

$$\frac{dN}{dV_C} \propto (1+1.9)^3 e^{-z/1.7} \quad z > 2.7$$

Energy Distribution

motivated by Fermi acceleration theory

$$\frac{dN}{dE} \propto E^{-\alpha} \exp[-E/E_{Z,\max}]$$

$$E_{Z,\max} = (Z/26) \times E_{\text{Fe},\max}$$

Note- magnetic field horizon effects are neglected in the following. This amounts to assuming: $d_s < (ct_H \lambda_{\text{scat}})^{1/2}$ ie. the source distribution may be approximated to be spatially continuous (also note, presence of t_H term comes from temporally continuous assumption)