Lecture 2 Plan:

1) Cosmic Ray proton + nuclei interaction rates in extragalactic radiation fields

2) Analytic guidance on functional form of interaction lengths

3) Results from propagating CR from an ensemble of sources and how results depend upon the source distribution assumption.

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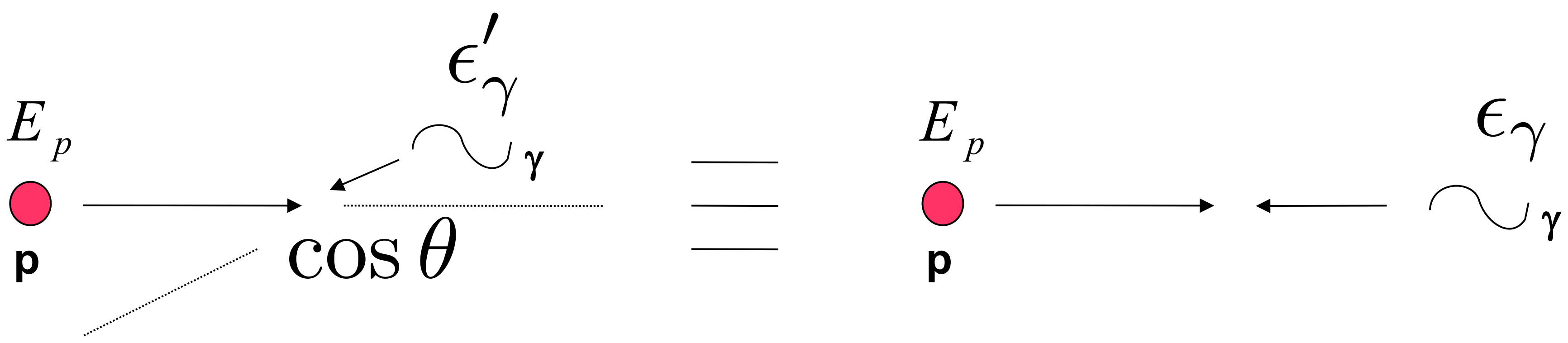
Cosmic Ray Proton Energy Losses

The Interaction Rate

$$\mathbf{R} = \int_{\mathbf{0}}^{\infty} \mathbf{d}\epsilon_\gamma \frac{\mathbf{dn}}{\mathbf{d}\epsilon_\gamma} \int_{-\mathbf{1}}^{\mathbf{1}} \frac{\mathbf{1}}{\mathbf{2}} \mathbf{d}(\cos\theta) \frac{\mathbf{d}\sigma}{\mathbf{d}\cos\theta} (\mathbf{1} + \beta\cos\theta)$$

All values above in lab frame



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The Interaction Rate

$$\mathbf{R} = \int_0^\infty \mathbf{d}\epsilon_\gamma \frac{\mathbf{dn}}{\mathbf{d}\epsilon_\gamma} \int_{-1}^{1} \frac{1}{2}\mathbf{d}(\cos\theta) \frac{\mathbf{d}\sigma}{\mathbf{d}\cos\theta}(\mathbf{1} + \beta\cos\theta)$$

Since, $\epsilon_\gamma \mathbf{E_p} = \epsilon'_\gamma \mathbf{E_p}(\mathbf{1} + \beta\cos\theta)$

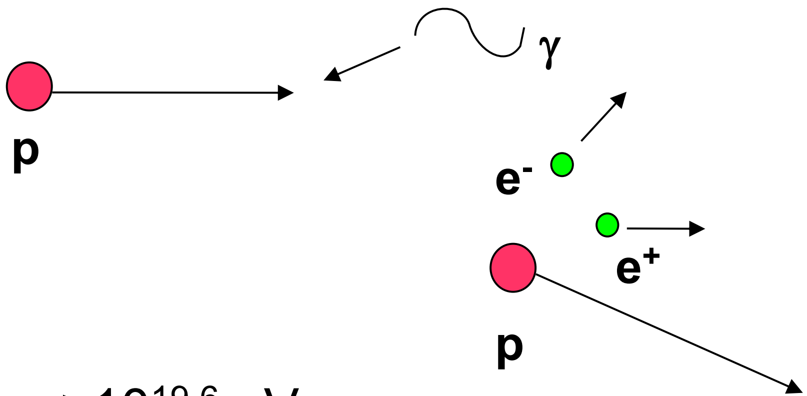
$$(\mathbf{1} + \beta\cos\theta)\mathbf{d}\cos\theta = \frac{\epsilon_\gamma \mathbf{E_p}}{\epsilon'_\gamma \mathbf{E_p}} \frac{\mathbf{d}(\epsilon_\gamma \mathbf{E_p})}{\epsilon'_\gamma \mathbf{E_p}}$$

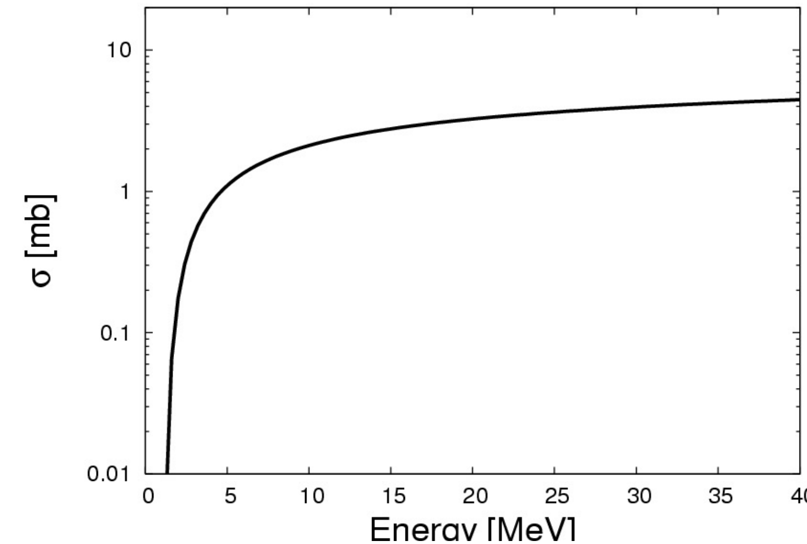
$$\mathbf{R} = \int_0^\infty \mathbf{d}\epsilon_\gamma \frac{\mathbf{dn}}{\mathbf{d}\epsilon_\gamma} \int_0^{\mathbf{2}\epsilon_\gamma \mathbf{E_p}} \mathbf{d}(\epsilon_\gamma \mathbf{E_p}) \frac{\epsilon_\gamma \mathbf{E_p}}{\epsilon'^{\mathbf{2}}_\gamma \mathbf{E_p^2}} \frac{\mathbf{d}\sigma}{\mathbf{d}(\epsilon_\gamma \mathbf{E_p})}$$

$$= \frac{\mathbf{m_p^2}}{\mathbf{2E_p^2}} \int_0^\infty \mathbf{d}\epsilon'_\gamma \frac{\mathbf{1}}{\epsilon'^{\mathbf{2}}_\gamma} \frac{\mathbf{dn}}{\mathbf{d}\epsilon'_\gamma} \int_0^{\mathbf{2}\epsilon'_\gamma \frac{\mathbf{E_p}}{\mathbf{m_p}}} \mathbf{d}\epsilon_\gamma \epsilon_\gamma \frac{\mathbf{d}\sigma}{\mathbf{d}\epsilon_\gamma}$$

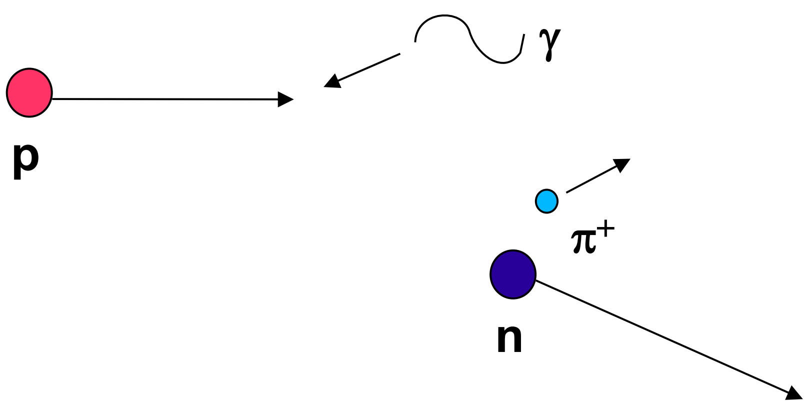
Cosmic Ray Proton Interactions

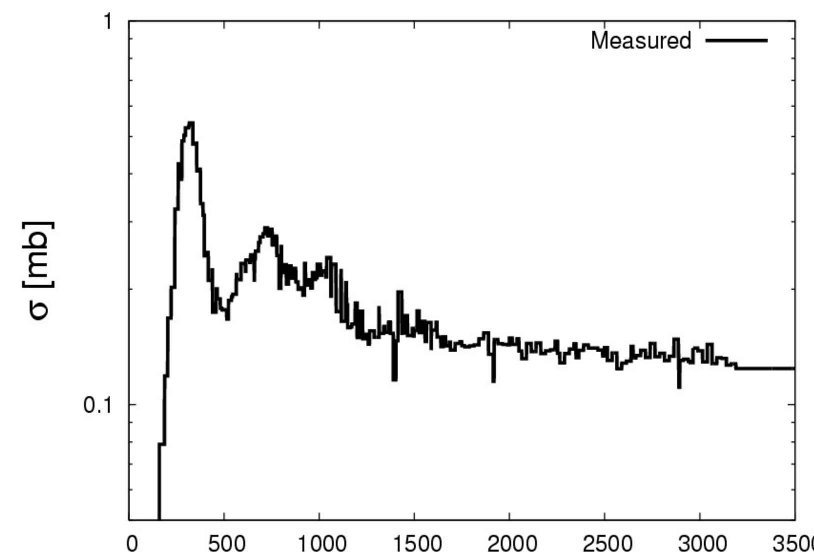
For $E_{proton} < 10^{19.6}$ eV





For $E_{proton} > 10^{19.6}$ eV





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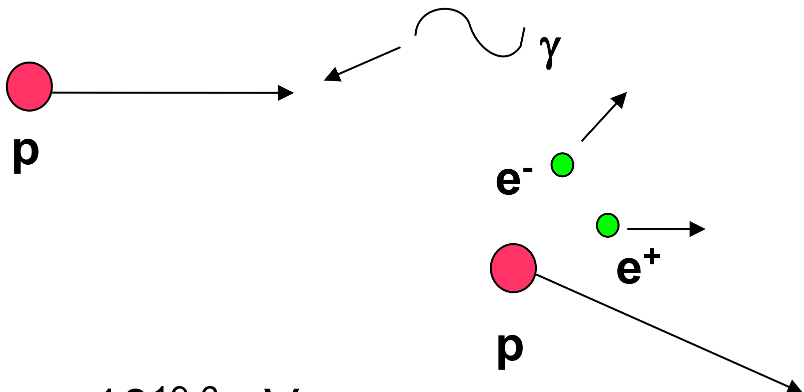
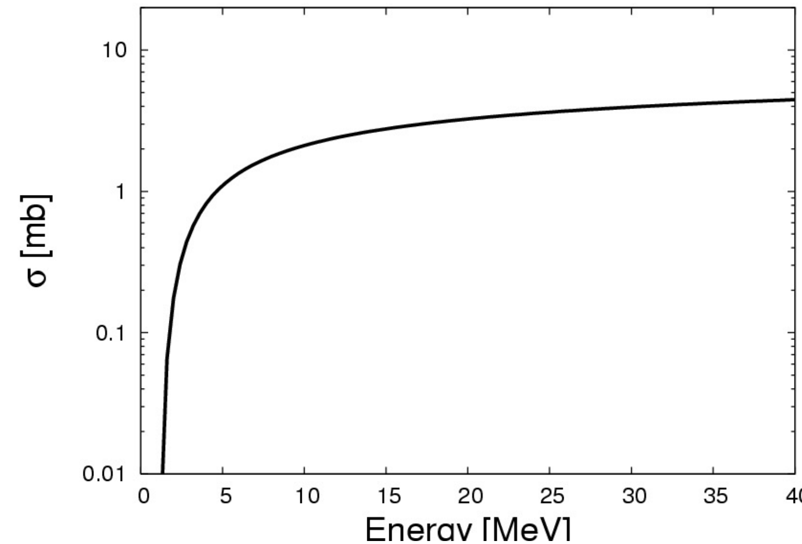
Cosmic Ray Proton Interactions

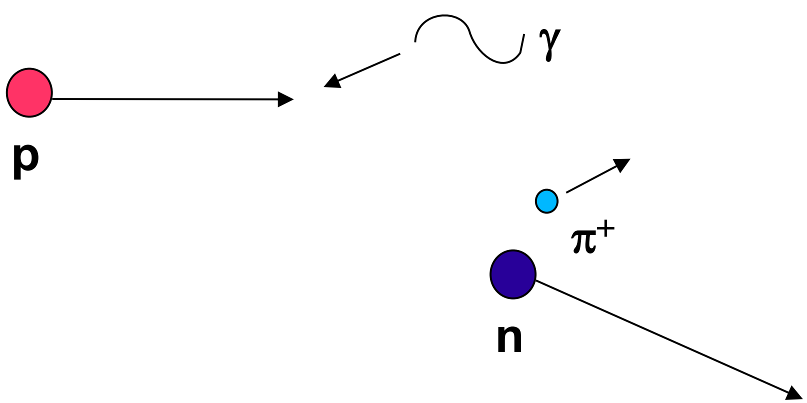
For $E_{proton} < 10^{19.6}$ eV

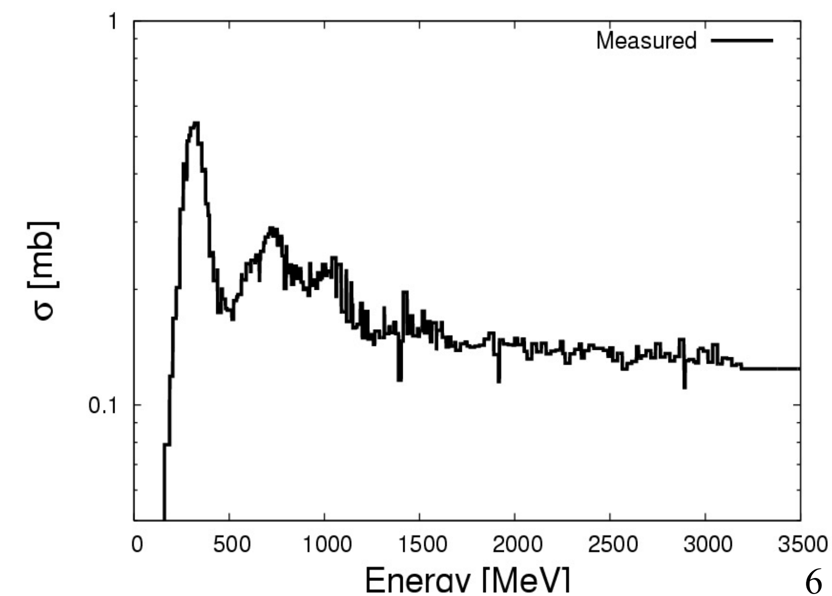




$E_\gamma^{th} \sim 1$MeV

For $E_{proton} > 10^{19.6}$ eV





$E_\gamma^{th} \sim 140$ MeV

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Threshold Energy- Proton Pair Production

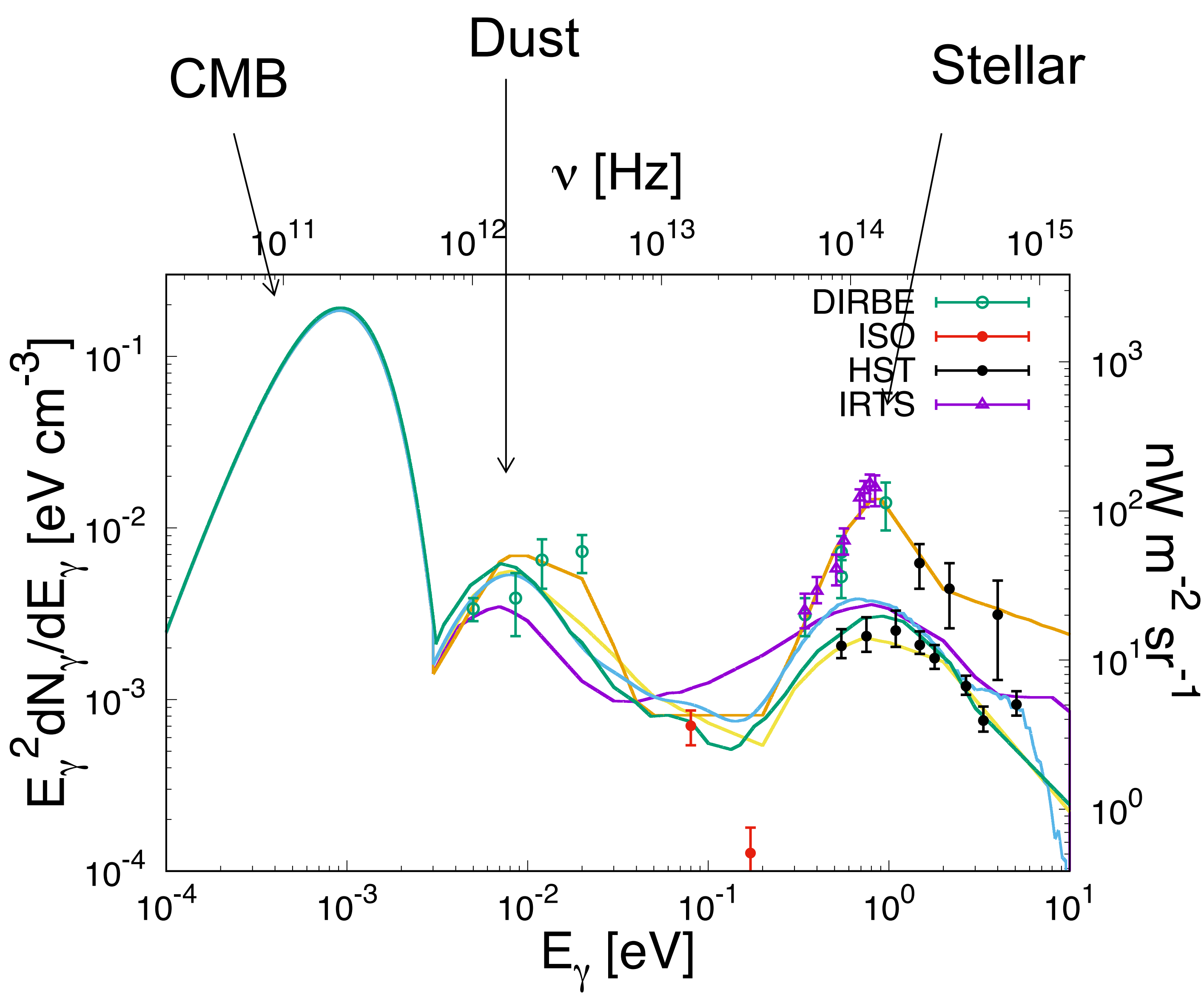
$$(\mathbf{E_p} + \mathbf{E_\gamma})^2 - (\mathbf{p_p} - \mathbf{E_\gamma})^2 = (\mathbf{m_p} + 2\mathbf{m_e})^2$$

$$\mathbf{m_p^2} + 2\mathbf{E_p}\mathbf{E_\gamma} + 2\mathbf{p_p}\mathbf{E_\gamma} \approx \mathbf{m_p^2} + 4\mathbf{m_p}\mathbf{m_e}$$

$$\boxed{\mathbf{E_p} \approx \frac{\mathbf{m_e}}{\mathbf{E_\gamma}}\mathbf{m_p}} \approx \left(\frac{0.5 \times 10^6}{6 \times 10^{-4}}\right) 0.9 \times 10^9 = 8 \times 10^{17}\ \mathbf{eV}$$

Repeat this calculation for pion production

Cosmic Radiation Fields- Energy Density



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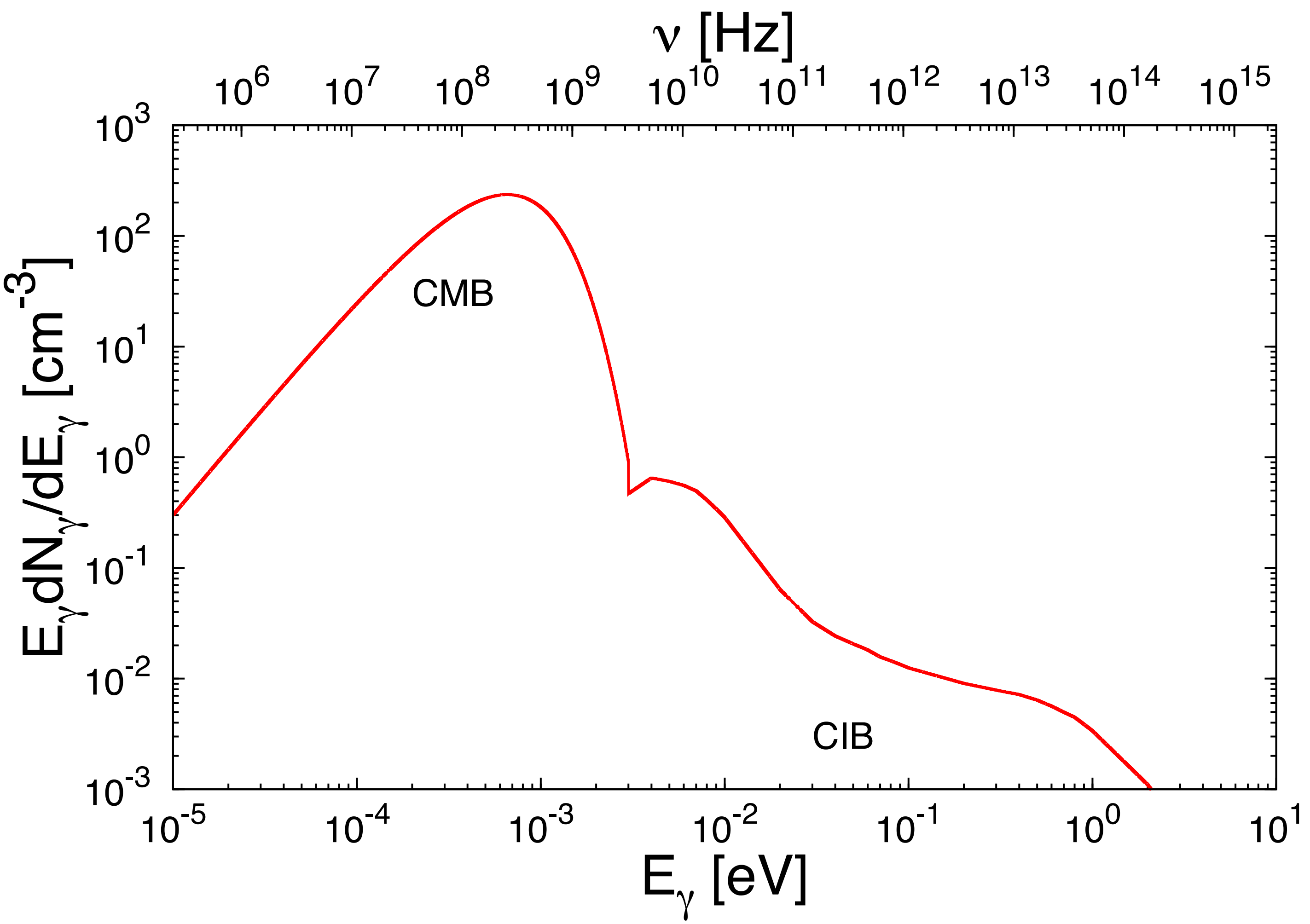
Cosmic Radiation Fields- Number Density



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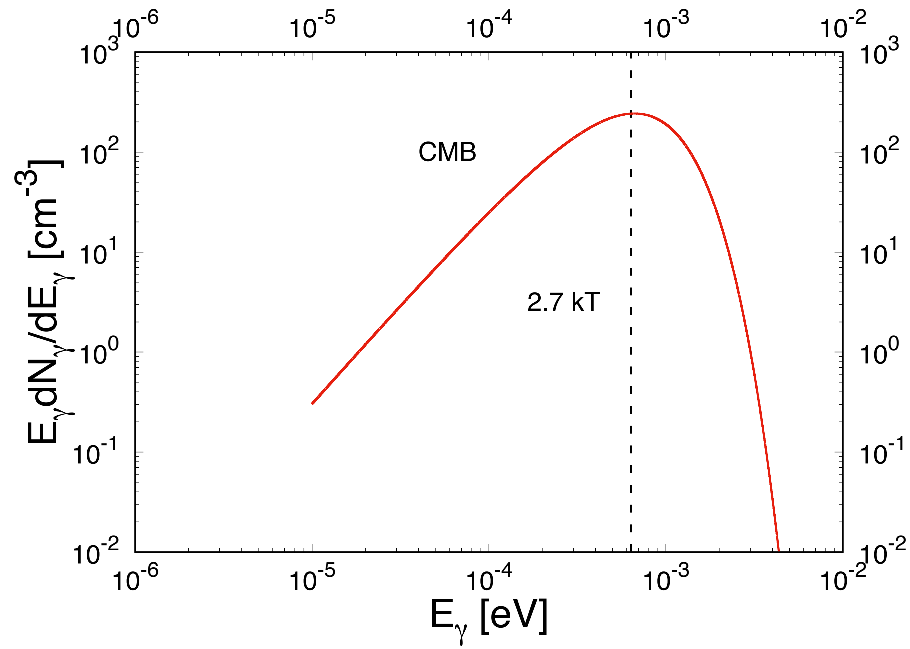
CMB- Total Number Density

$$\frac{\mathbf{dn}}{\mathbf{d}\epsilon_\gamma} = \frac{8\pi}{(\mathbf{hc})^3}\frac{\epsilon_\gamma^2}{\mathbf{e}^{\epsilon_\gamma/\mathbf{kT}}-1}$$



$$\mathbf{n}_\gamma^{\mathbf{BB}} = \frac{8\pi(\mathbf{kT})^3}{(\mathbf{hc})^3}\int_0^\infty \frac{\mathbf{x}^2}{\mathbf{e}^{\mathbf{x}}-1}\mathbf{dx}$$

$$\frac{8\pi(\mathbf{kT_{CMB}})^3}{(\mathbf{hc})^3} \approx 170\ \mathbf{cm}^{-3}$$

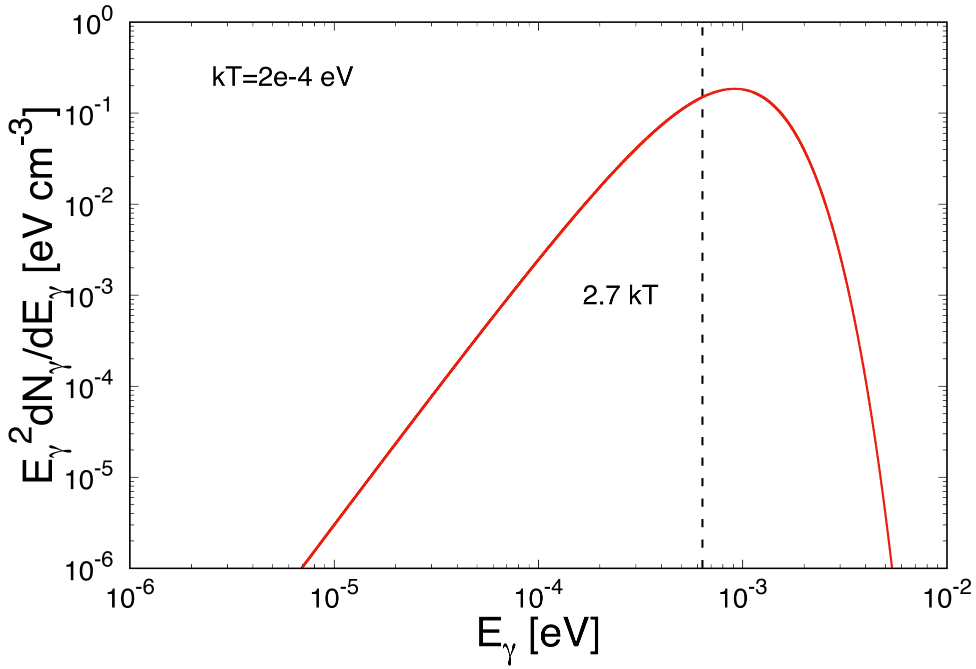
$$\zeta(\mathbf{x}) = \sum_{\mathbf{n}=1}^{\infty}\frac{1}{\mathbf{n}^{\mathbf{x}}}$$

$$\mathbf{n}_\gamma^{\mathbf{CMB}} = 8\pi\frac{(\mathbf{kT_{CMB}})^3}{(\mathbf{hc})^3}\gamma(3)\zeta(3) \approx 400\ \mathbf{cm}^{-3}$$

CMB- Total Energy Density

$$\rho_\gamma^{\mathbf{BB}} = \frac{\mathbf{8}\pi(\mathbf{kT})^{\mathbf{4}}}{(\mathbf{hc})^{\mathbf{3}}} \int_{\mathbf{0}}^{\infty} \frac{\mathbf{x^3}}{\mathbf{e^x - 1}} \mathbf{dx}$$

$$\rho_\gamma^{\mathbf{CMB}} = \mathbf{8}\pi \frac{(\mathbf{kT_{CMB}})^{\mathbf{4}}}{(\mathbf{hc})^{\mathbf{3}}} \gamma(4)\zeta(4) \approx \mathbf{0.25\ eV\ cm^{-3}}$$

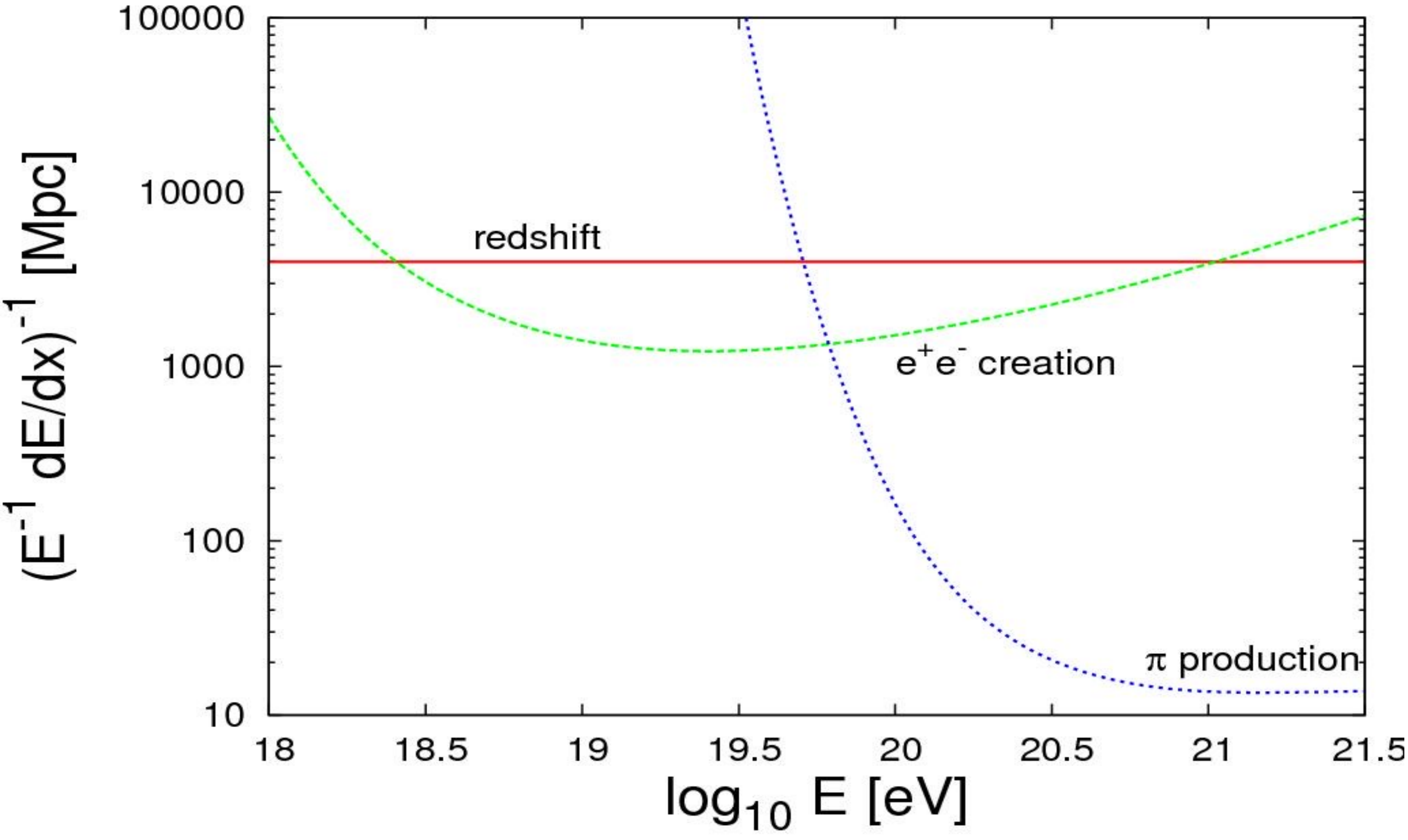


Energy Loss Rates due to Proton Interactions

$$\mathbf{R} = \frac{\mathbf{m_p^2 c^4}}{\mathbf{2E^2}} \int_0^\infty \mathbf{d}\epsilon_\gamma \frac{\mathbf{1}}{\epsilon_\gamma^{\mathbf{2}}} \frac{\mathbf{dn}}{\mathbf{d}\epsilon_\gamma} \int_0^{\mathbf{2E}\epsilon_\gamma/(\mathbf{m_p c^2})} \mathbf{d}\epsilon'_\gamma \epsilon'_\gamma \sigma_{\mathbf{p}\gamma}(\epsilon'_\gamma) \mathbf{K_p}$$

where R is the energy loss rate

where K_p is the inelasticity



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Energy Loss Rates due to Proton Interactions

$$\mathbf{R} = \frac{\mathbf{m_p^2 c^4}}{\mathbf{2E^2}} \int_0^\infty \mathbf{d}\epsilon_\gamma \frac{\mathbf{1}}{\epsilon_\gamma^2} \frac{\mathbf{dn}}{\mathbf{d}\epsilon_\gamma} \int_0^{\mathbf{2E}\epsilon_\gamma/(\mathbf{m_p c^2})} \mathbf{d}\epsilon'_\gamma \epsilon'_\gamma \sigma_{\mathbf{p}\gamma}(\epsilon'_\gamma) \mathbf{K_p}$$

where R is the energy loss rate

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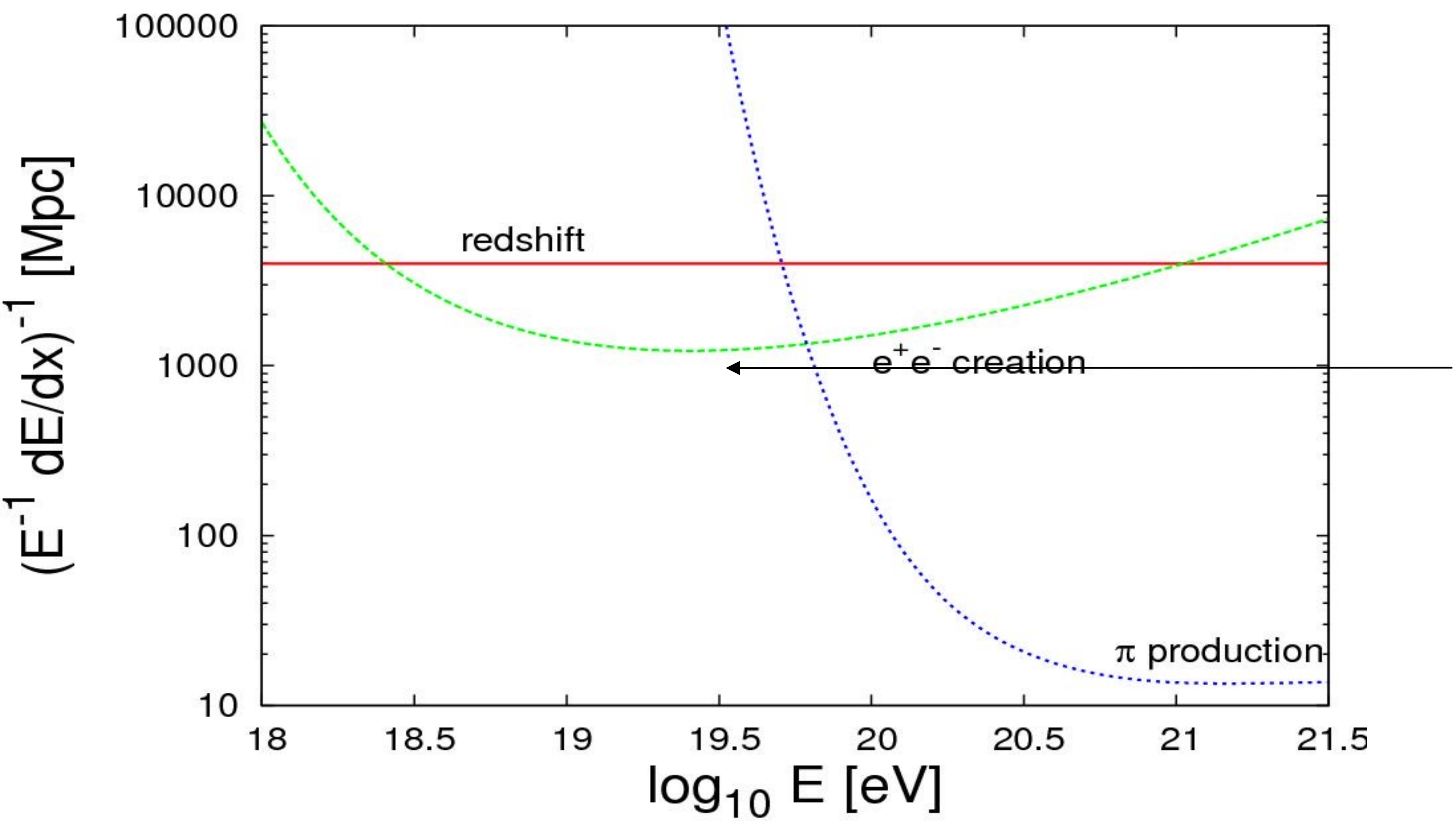

$$\approx \frac{\mathbf{m_p}}{\mathbf{m_e}} \frac{\mathbf{1}}{\mathbf{n_{CMB}}\sigma_{\mathbf{p}\gamma}}$$

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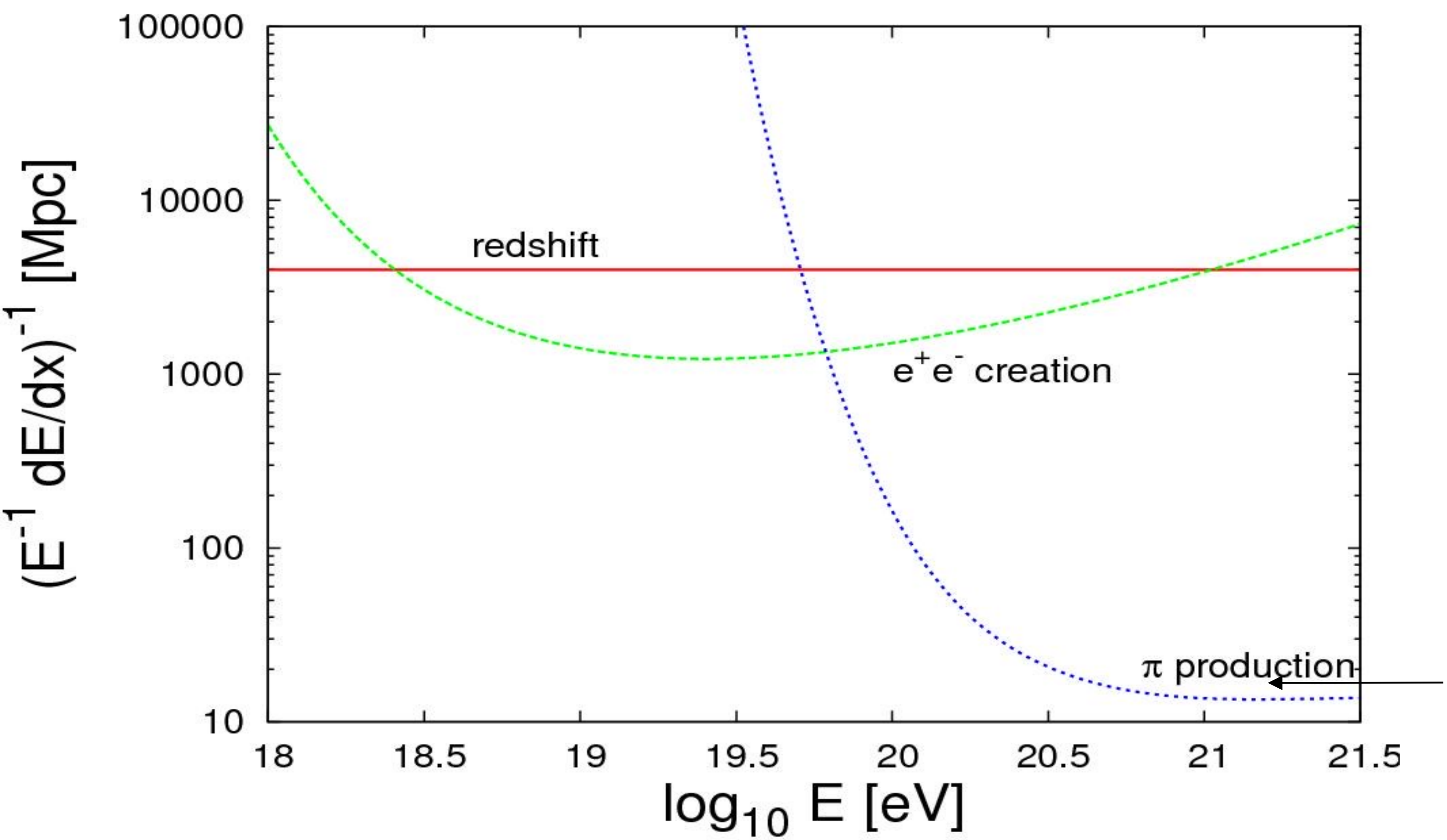
Energy Loss Rates due to Proton Interactions

$$\mathbf{R} = \frac{\mathbf{m_p^2 c^4}}{\mathbf{2E^2}} \int_0^\infty \mathbf{d}\epsilon_\gamma \frac{\mathbf{1}}{\epsilon_\gamma^{\mathbf{2}}} \frac{\mathbf{dn}}{\mathbf{d}\epsilon_\gamma} \int_0^{\mathbf{2E}\epsilon_\gamma/(\mathbf{m_p c^2})} \mathbf{d}\epsilon'_\gamma \epsilon'_\gamma \sigma_{\mathbf{p}\gamma}(\epsilon'_\gamma) \mathbf{K_p}$$

where R is the energy loss rate

where K_p is the inelasticity



$$\approx \frac{\mathbf{m_p}}{\mathbf{m_\pi}} \frac{\mathbf{1}}{\mathbf{n_{CMB}}\sigma_{\mathbf{p}\gamma}}$$

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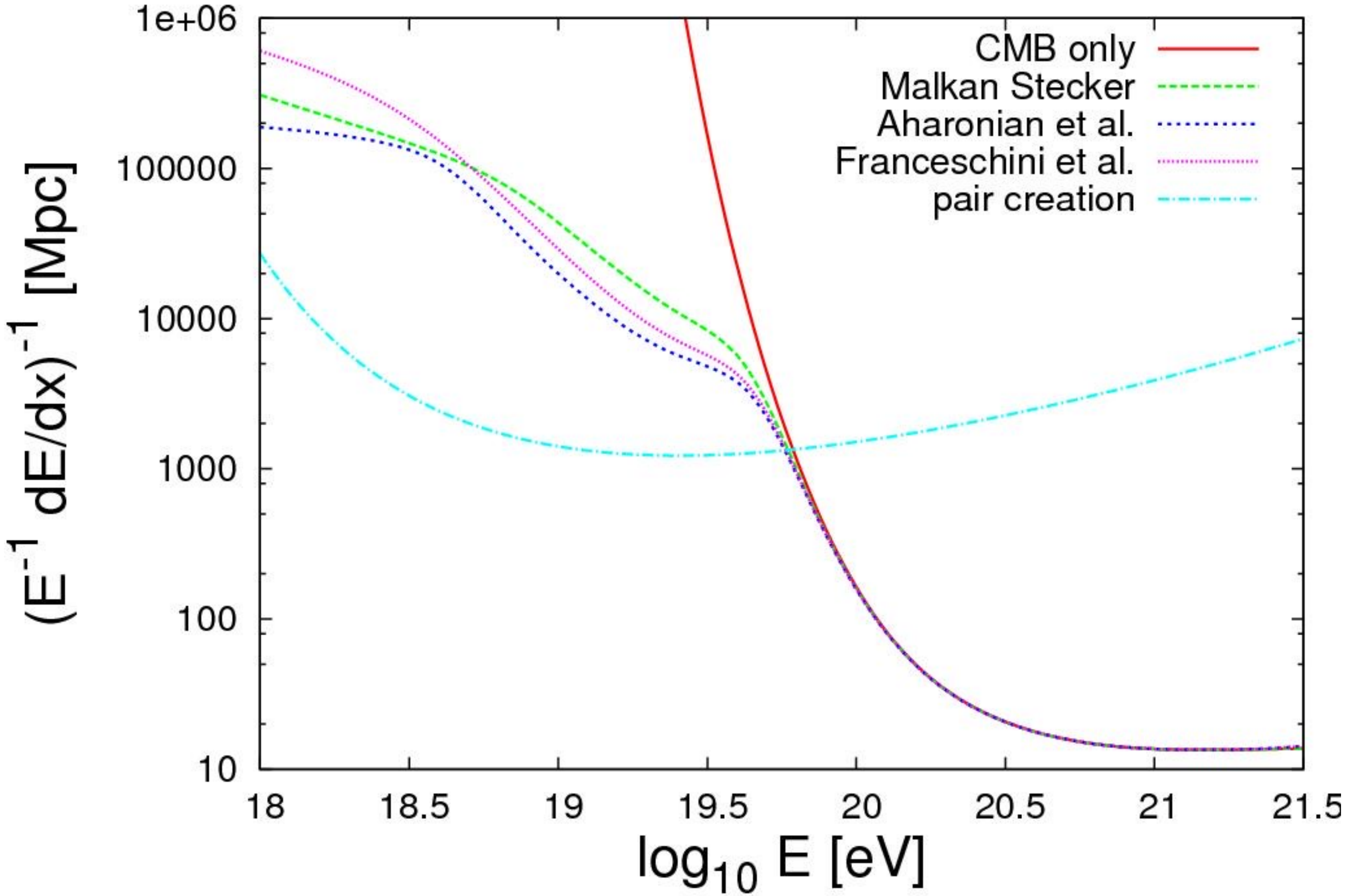
....with Different IR Backgrounds



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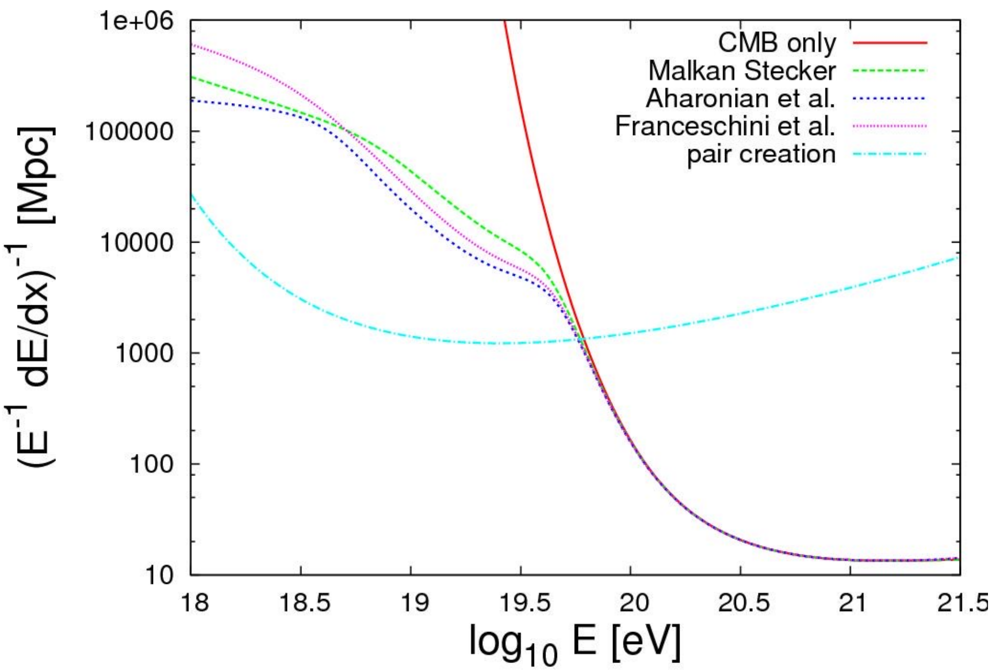
....with Different IR Backgrounds



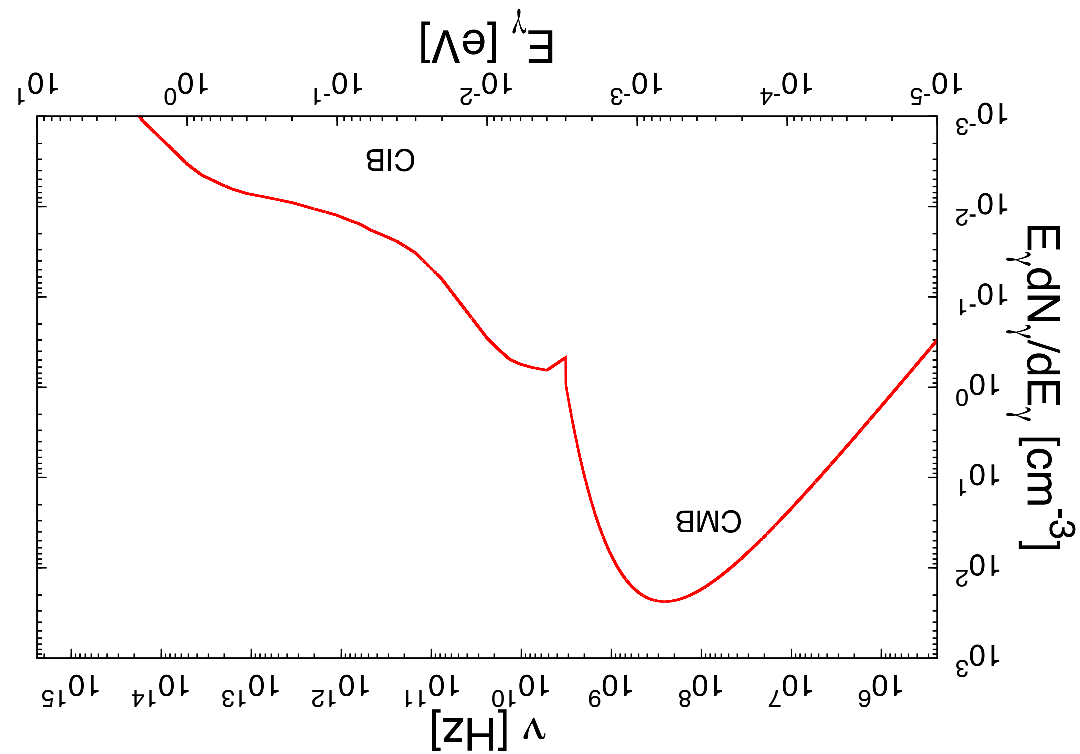


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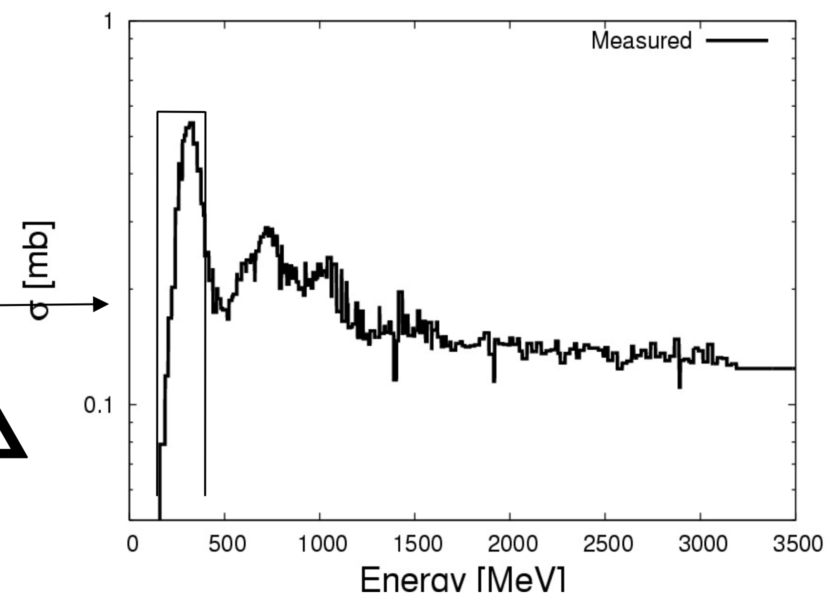
Photo-Pion Production Rate

$$\mathbf{R} = \frac{\mathbf{m_p^2 c^4}}{\mathbf{2E^2}} \int_0^\infty \mathbf{d}\epsilon_\gamma \frac{\mathbf{1}}{\epsilon_\gamma^{\mathbf{2}}} \frac{\mathbf{dn}}{\mathbf{d}\epsilon_\gamma} \int_0^{\mathbf{2E}\epsilon_\gamma/(\mathbf{m_p c^2})} \mathbf{d}\epsilon'_\gamma \epsilon'_\gamma \sigma_{\mathbf{p}\gamma}(\epsilon'_\gamma) \mathbf{K_p}$$

Assuming the cross-section is approximately:

$$\sigma_{\mathbf{p}\gamma}(\epsilon_\gamma) = \mathbf{0} \qquad \begin{array}{l} \epsilon_\gamma < \mathbf{E} - \mathbf{\Delta} \\ \epsilon_\gamma > \mathbf{E} + \mathbf{\Delta} \end{array}$$

$$\sigma_{\mathbf{p}\gamma}(\epsilon_\gamma) = \sigma_{\mathbf{p}\gamma} \qquad \mathbf{E} - \mathbf{\Delta} < \epsilon_\gamma < \mathbf{E} + \mathbf{\Delta}$$

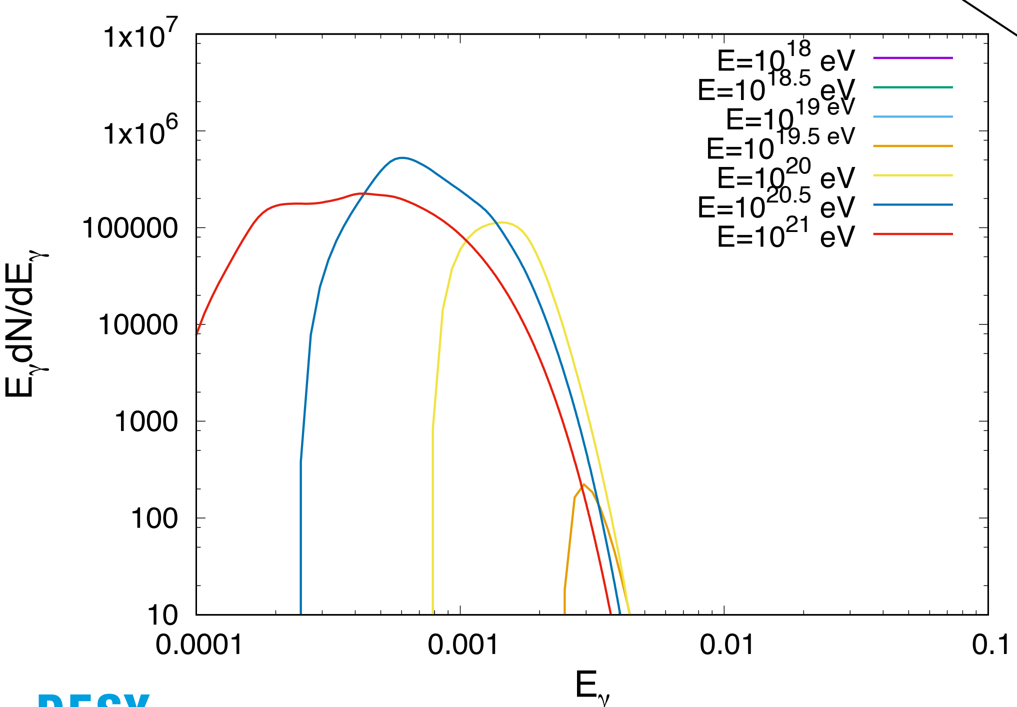


Where $\sigma_{\mathbf{p}\gamma} = \mathbf{0.5\ mb}$, $\mathbf{E} = \mathbf{300\ MeV}$, $\mathbf{\Delta} = \mathbf{100\ MeV}$

Photo-Pion Production Rate

$$\mathbf{R}(\boldsymbol{\Gamma}) \approx \sigma_\mathbf{0} \int_{(\mathbf{E_0}-\boldsymbol{\Delta}_\mathbf{0})/\mathbf{2}\boldsymbol{\Gamma}}^{(\mathbf{E_0}+\boldsymbol{\Delta}_\mathbf{0})/\mathbf{2}\boldsymbol{\Gamma}} \left(\frac{\epsilon^\mathbf{2} - \left[(\mathbf{E_0} - \boldsymbol{\Delta}_\mathbf{0})/\mathbf{2}\boldsymbol{\Gamma}\right]^\mathbf{2}}{\epsilon^\mathbf{2}} \right) \frac{\mathbf{dn}}{\mathbf{d}\epsilon} \mathbf{d}\epsilon +$$

$$\sigma_\mathbf{0} \int_{(\mathbf{E_0}+\boldsymbol{\Delta}_\mathbf{0})/\mathbf{2}\boldsymbol{\Gamma}}^{\infty} \left(\frac{\left[(\mathbf{E_0} + \boldsymbol{\Delta}_\mathbf{0})/\mathbf{2}\boldsymbol{\Gamma}\right]^\mathbf{2} - \left[(\mathbf{E_0} - \boldsymbol{\Delta}_\mathbf{0})/\mathbf{2}\boldsymbol{\Gamma}\right]^\mathbf{2}}{\epsilon^\mathbf{2}} \right) \frac{\mathbf{dn}}{\mathbf{d}\epsilon} \mathbf{d}\epsilon$$



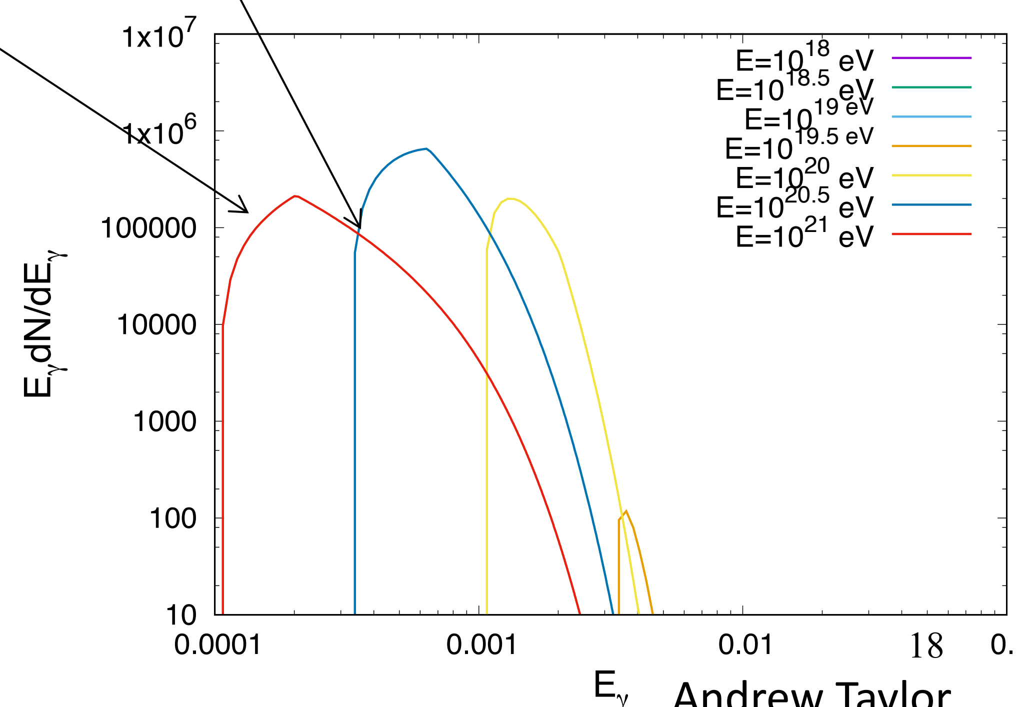


Photo-Pion Production Rate

$$\mathbf{R(\Gamma) \approx n_0\sigma_0 \int_{x_1(\Gamma)}^{x_2(\Gamma)} \frac{\left(x^2 - x_1(\Gamma)^2\right)}{e^x - 1} dx +}$$

$$\mathbf{n_0\sigma_0 \int_{x_2(\Gamma)}^{\infty} \frac{\left(x_2^2(\Gamma) - x_1^2(\Gamma)\right)}{e^x - 1}}$$

$$\mathbf{R(\Gamma) \approx \frac{1}{l_0} \left[\ (\gamma_i(3, x_2(\Gamma)) - \gamma_i(3, x_1(\Gamma))) - x_1(\Gamma)^2(\gamma_i(1, x_2(\Gamma)) - \gamma_i(1, x_1(\Gamma))) + \right.}$$
$$\mathbf{\left. x_2(\Gamma)^2(1 - \gamma_i(1, x_2(\Gamma))) - x_1(\Gamma)^2(1 - \gamma_i(1, x_2(\Gamma)))\ \right]}$$

$$\mathbf{\gamma_i(3, x) = 2 - (2 + 2x + x^2)\exp(-x) \qquad \gamma_i(1, x) = 1 - \exp(-x)}$$

$$\mathbf{R(\Gamma) \approx \frac{2}{l_0}\left[e^{-x_1}(1 - e^{-x_1} + x_1(1 - 2e^{-x_1}))\right]}$$

Photo-Pion Production Rate: Blackbody Interactions

$$R(\Gamma) \approx n_0\sigma_0 \int_{x_1(\Gamma)}^{x_2(\Gamma)} \frac{\left(x^2 - x_1(\Gamma)^2\right)}{e^x - 1} dx +$$

$$n_0\sigma_0 \int_{x_2(\Gamma)}^{\infty} \frac{\left(x_2^2(\Gamma) - x_1^2(\Gamma)\right)}{e^x - 1}$$

$$R(\Gamma) \approx \frac{2}{l_0}\left[e^{-x_1}(1 - e^{-x_1} + x_1(1 - 2e^{-x_1}))\right]$$

Where, $l_0 = 10\ \mathrm{Mpc}$ $\quad x_1 = \frac{(E - \Delta)m_p}{2kT_{CMB}E_p} = \frac{10^{20.5}\ \mathrm{eV}}{E_p}$

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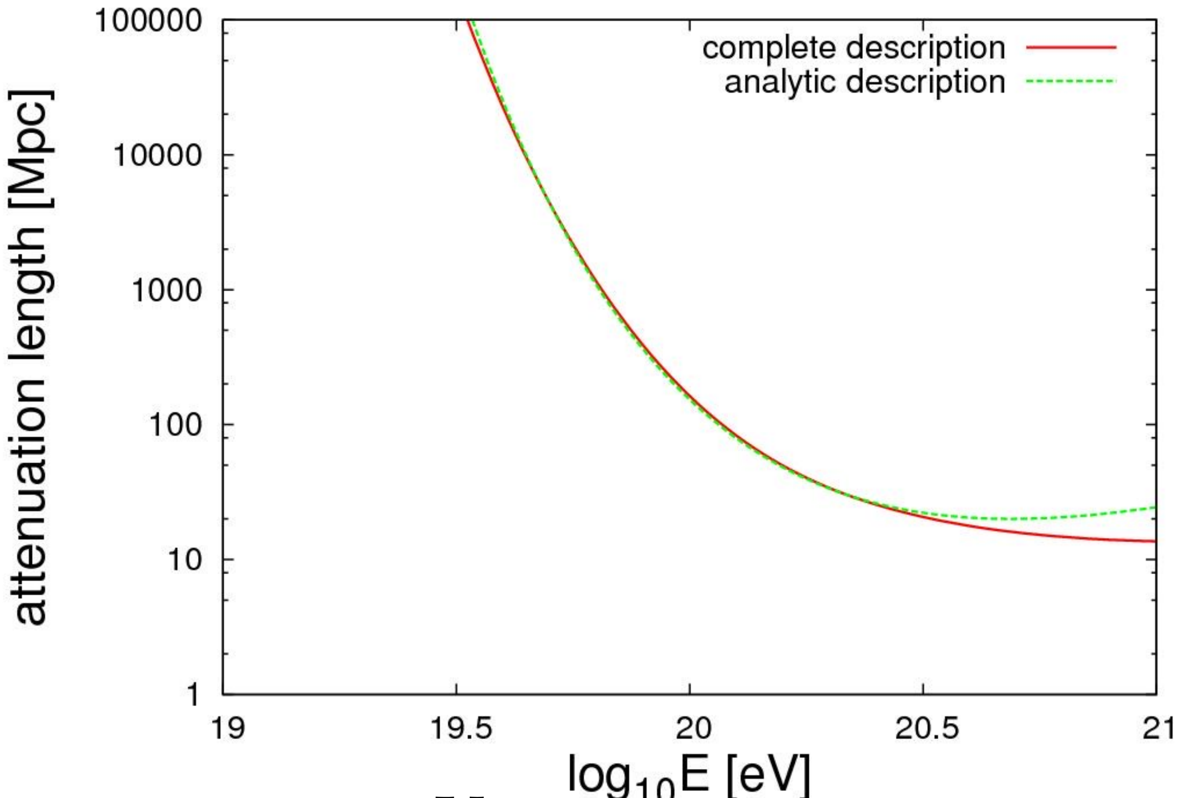
Photo-Pion Production Rate: Blackbody Interactions

With, $\mathbf{kT_{CMB} \approx 2 \times 10^{-4}\ eV}$

$$\mathbf{R \approx 0.2\sigma_{p\gamma} \int_{\frac{E-\Delta}{2\Gamma}}^{\frac{E+\Delta}{2\Gamma}} d\epsilon_\gamma \frac{dn}{d\epsilon_\gamma}}$$

$$\mathbf{\approx \left(\frac{l_0}{e^{-x_1}(1 - e^{-x_1})} \right)^{-1}}$$



Where $\mathbf{l_0}$ is 5 Mpc and $\mathbf{x_1 = \frac{10^{20.5}\ eV}{E_p}}$

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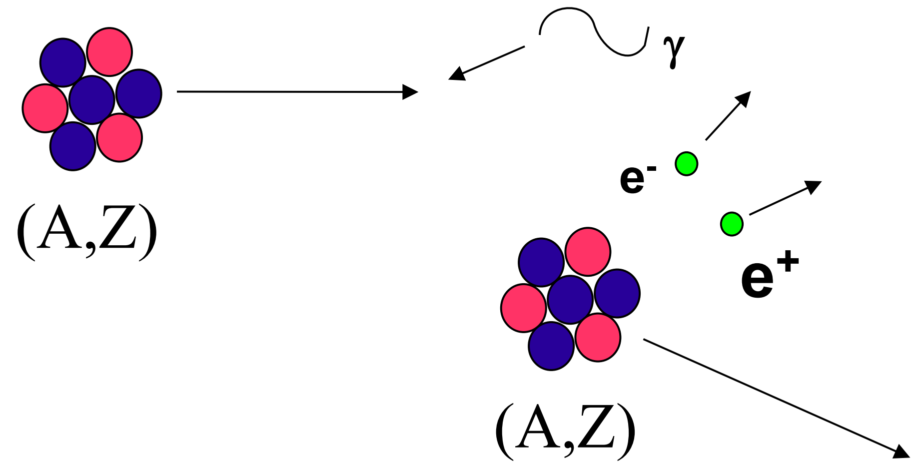
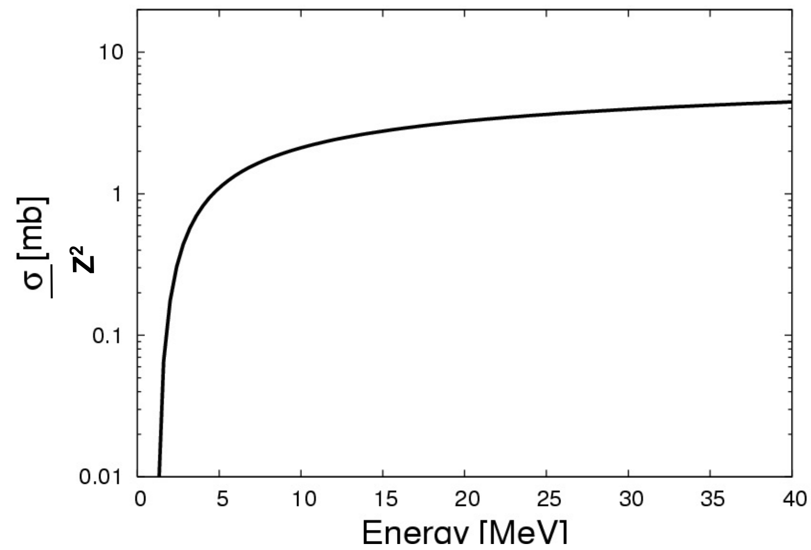
Cosmic Ray Nuclei Energy Losses

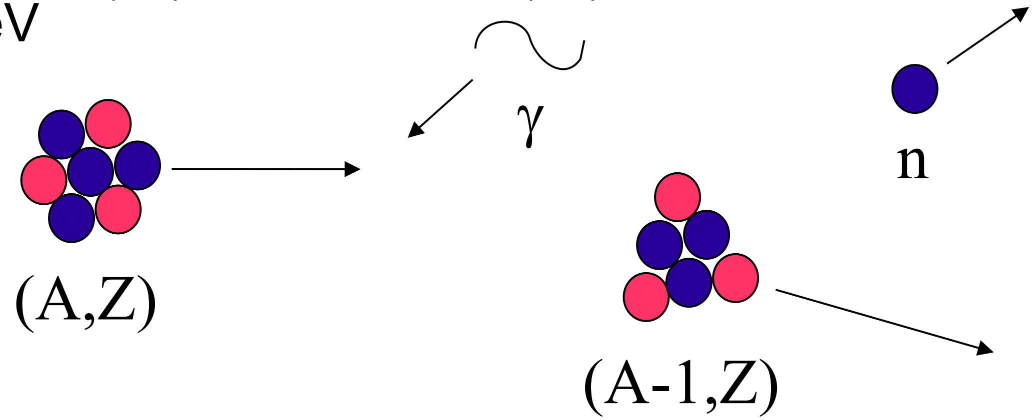
Cosmic Ray Nuclei Interactions

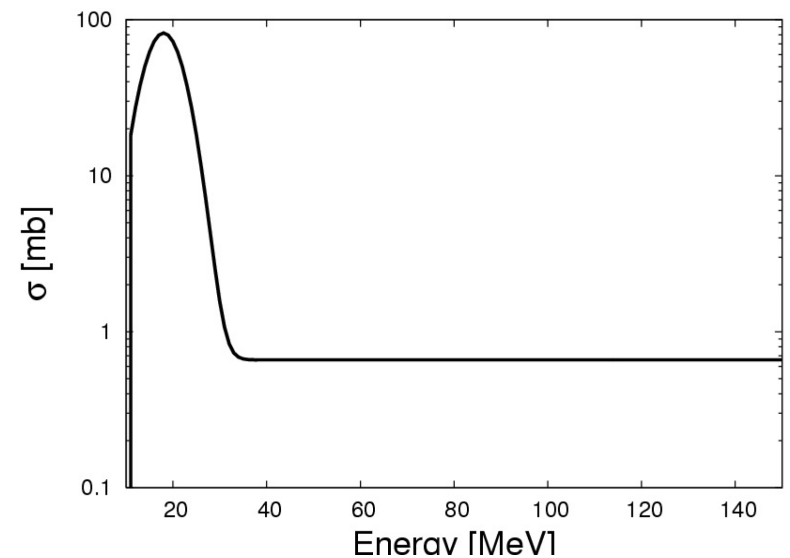
For $10^{19.7} < E_{(A,Z)} < 10^{20.2}$ eV





For $E_{(A,Z)} < 10^{19.7}$ and $E_{(A,Z)} < 10^{20.2}$ eV





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Cosmic Ray Nuclei Interactions

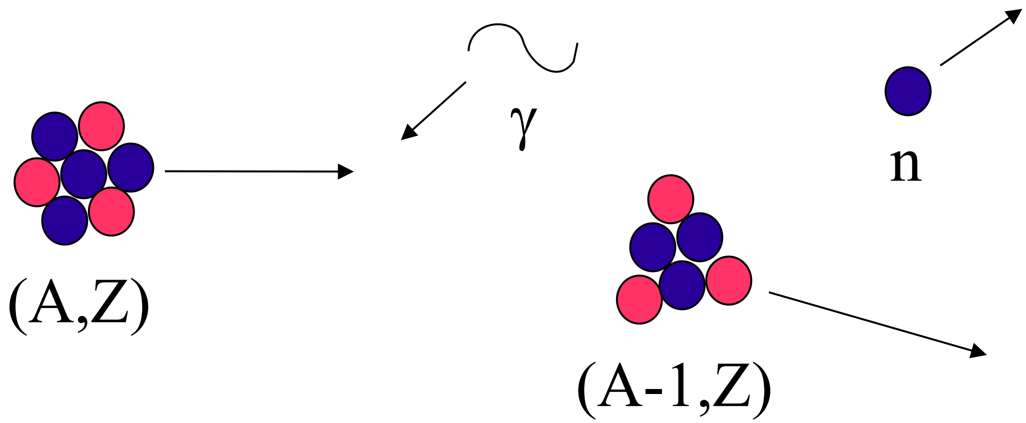

Photo-disintegration-

$$N_{(A,Z)} + \gamma \longrightarrow N'_{(A',Z')} + (Z-Z')p + (A-A'+Z'-Z)n, E_{\gamma} \sim 30\,\text{MeV}$$

$$n \longrightarrow p + e^{-} + \bar{\nu}_e$$

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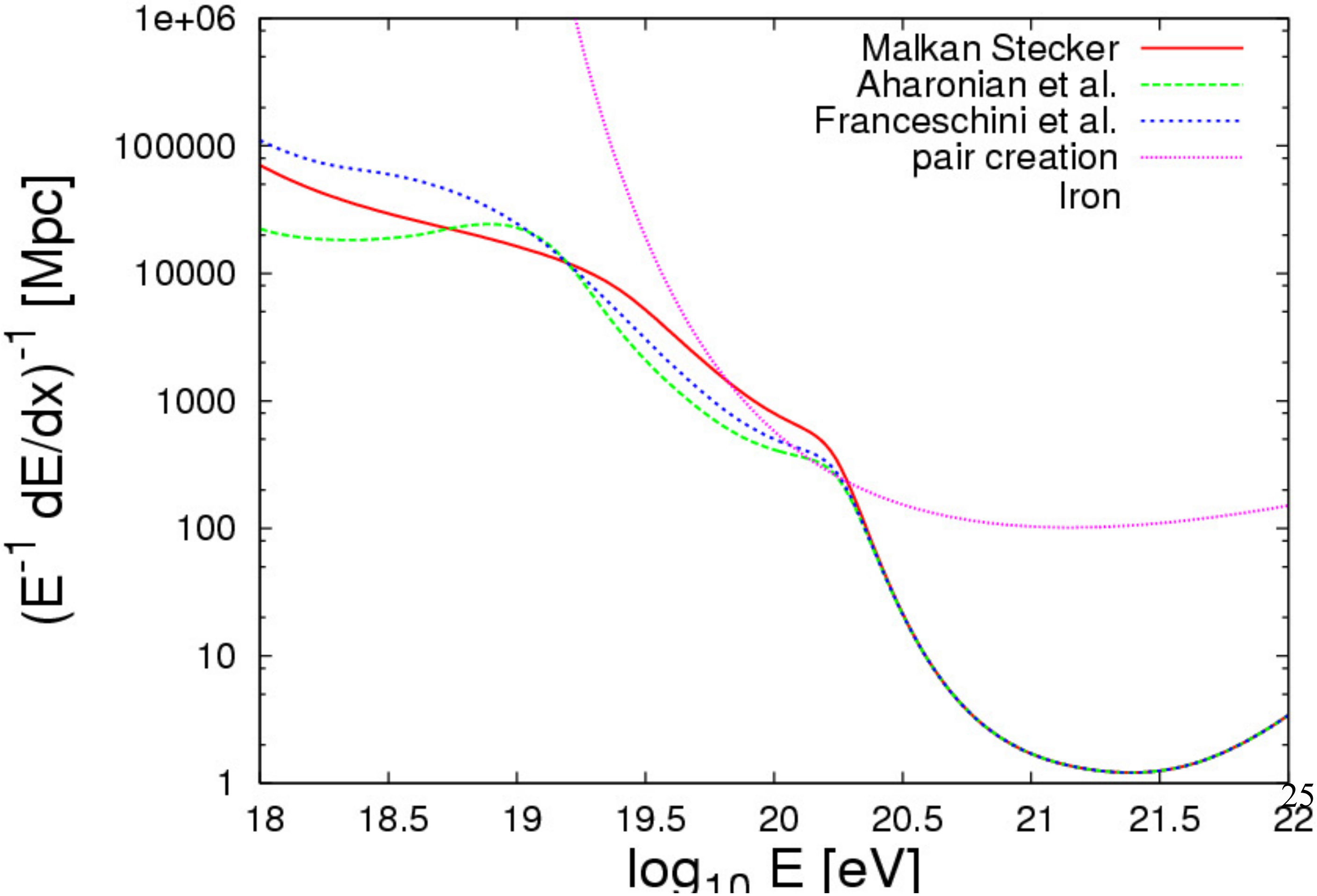
Energy Loss Rates due to Nuclei Interactions

$$\mathbf{R} = \frac{\mathbf{A^2 m_p^2 c^4}}{\mathbf{2E^2}} \int_0^\infty \mathbf{d}\epsilon_\gamma \frac{\mathbf{1}}{\epsilon_\gamma^\mathbf{2}} \frac{\mathbf{dn}}{\mathbf{d}\epsilon_\gamma} \int_0^{\mathbf{2E}\epsilon_\gamma/(\mathbf{Am_p c^2})} \mathbf{d}\epsilon'_\gamma \epsilon'_\gamma \sigma_{\mathbf{N}\gamma}(\epsilon'_\gamma) \mathbf{K_p}$$

where R is the energy loss rate



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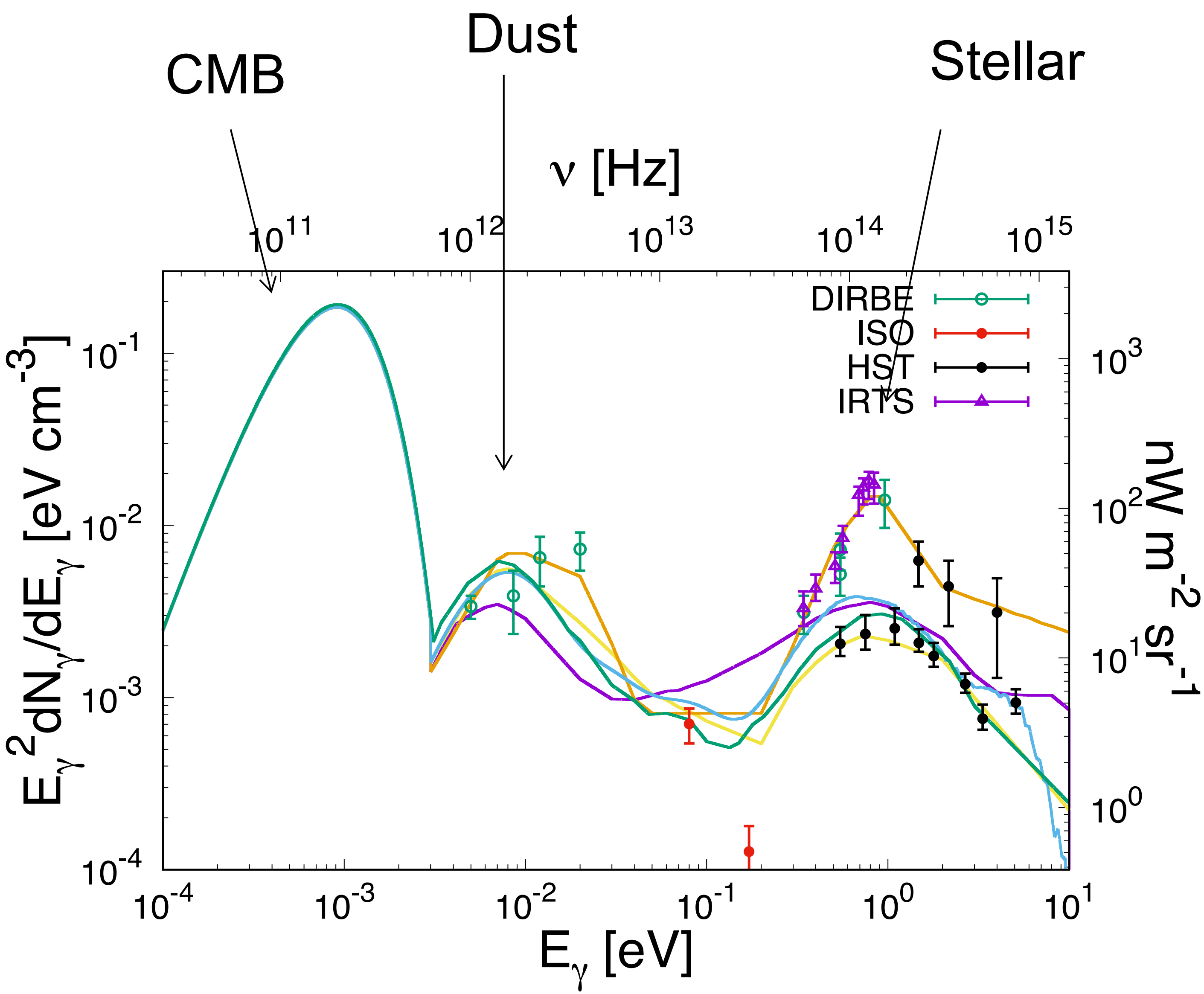
Cosmic Radiation Fields



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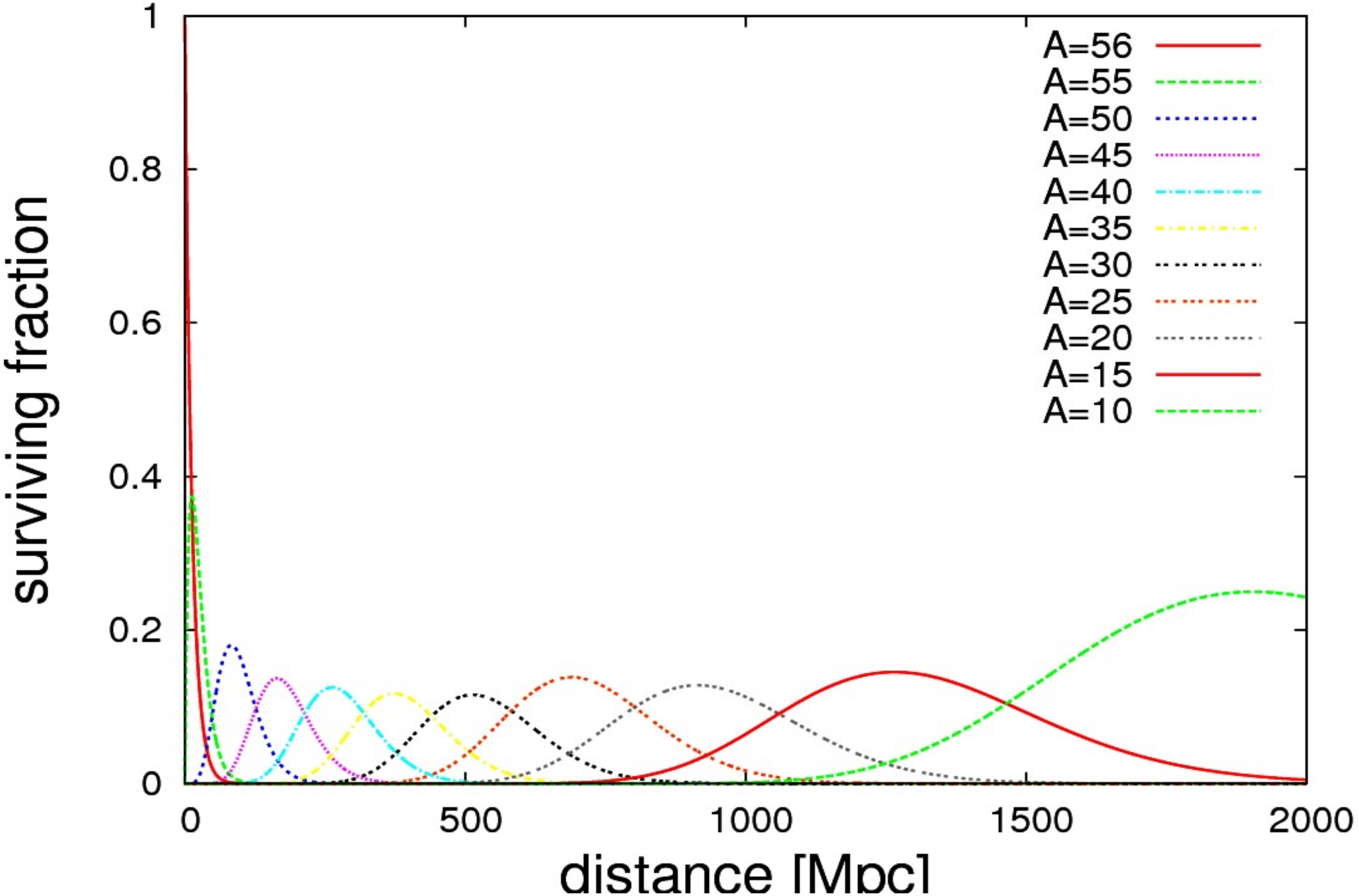
Cosmic Ray Disintegration During Propagation



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Cosmic Ray Spectra

Assumptions on Source Population

Spatial Distribution

motivated by star formation rate evolution

$$\frac{dN}{dV_C} \propto (1+z)^3 \qquad z < 1.9$$

$$\frac{dN}{dV_C} \propto (1+1.9)^3 \qquad 1.9 < z < 2.7$$

$$\frac{dN}{dV_C} \propto (1+1.9)^3 e^{-z/1.7} \qquad z > 2.7$$

Energy Distribution

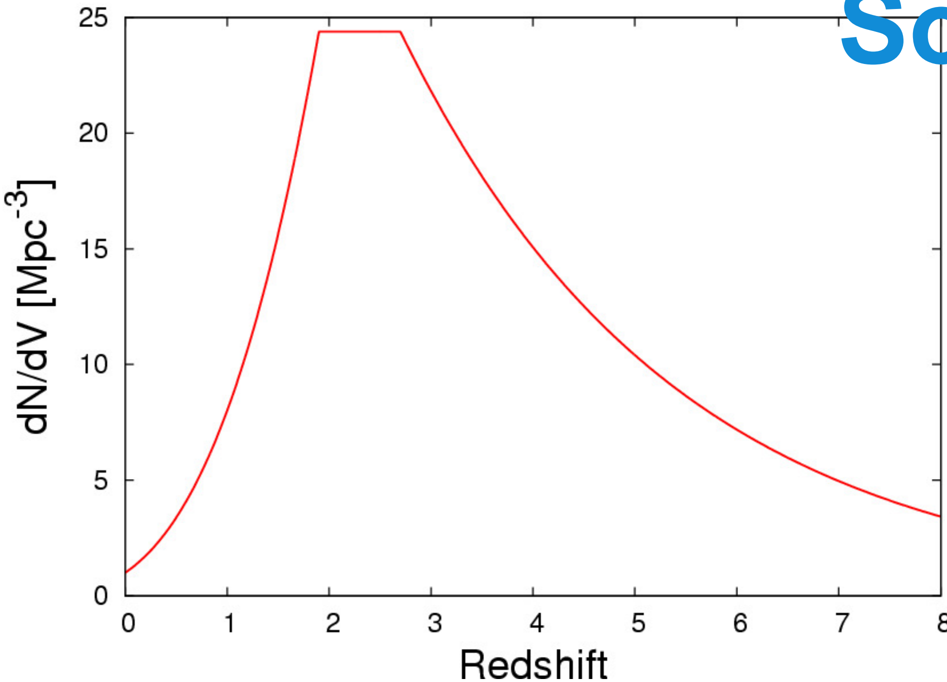
motivated by Fermi acceleration theory

$$\frac{dN}{dE} \propto E^{-\alpha} \exp[-E/E_{Z,max}]$$

$$E_{Z,max} = (Z/26) \times E_{Fe,max}$$

Note- magnetic field horizon effects are neglected in the following.
This amounts to assuming: $d_s < (c t_H \lambda_{scat})^{1/2}$
ie. the source distribution may be approximated to be spatially continuous (also note, presence of t_H term comes from temporally continuous assumption)

A Cosmological Distribution of Sources

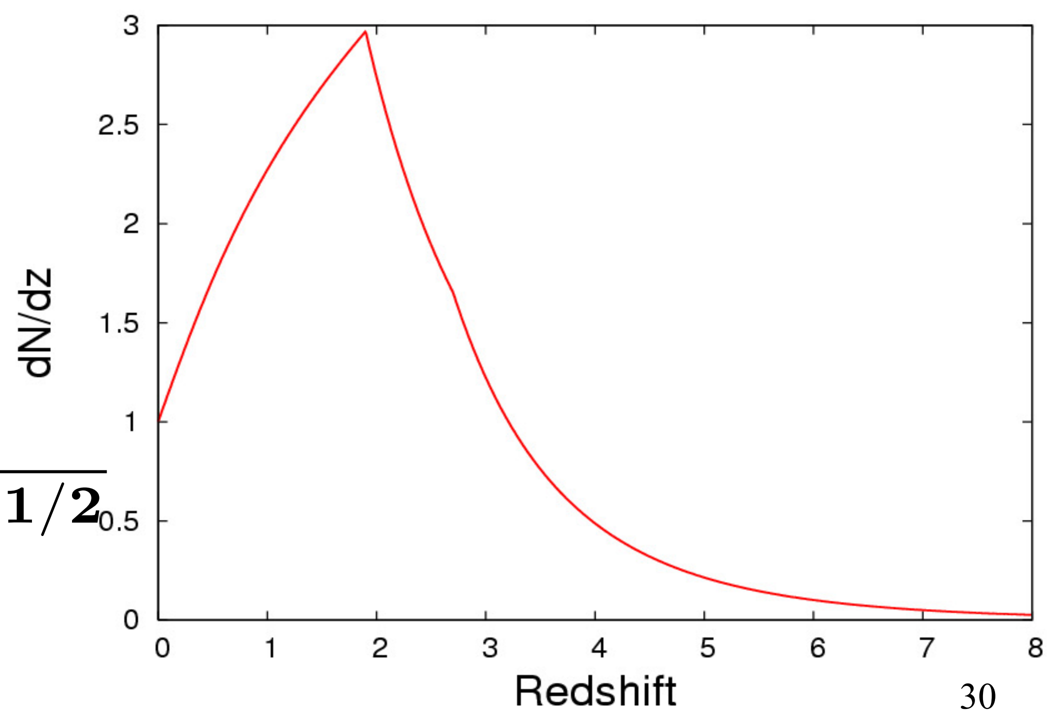


Distribution of sources in a comoving volume

$$\mathbf{dV_c = 4\pi\chi^2 d\chi}$$

$$\mathbf{d\chi = \frac{dz}{H}}$$

$$\approx \mathbf{\frac{dz}{H_0(\Omega_M(1+z)^3 + \Omega_\Lambda)^{1/2}}}$$



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Assumptions on Source Population

Spatial Distribution

$$\frac{dN}{dV_C} \propto (1+z)^n \qquad z < z_{max}$$

$$n = -6,\ -3,\ 0,\ 3$$

Energy Distribution

$$\frac{dN}{dE} \propto E^{-\alpha} \exp[-E/E_{Z,max}]$$

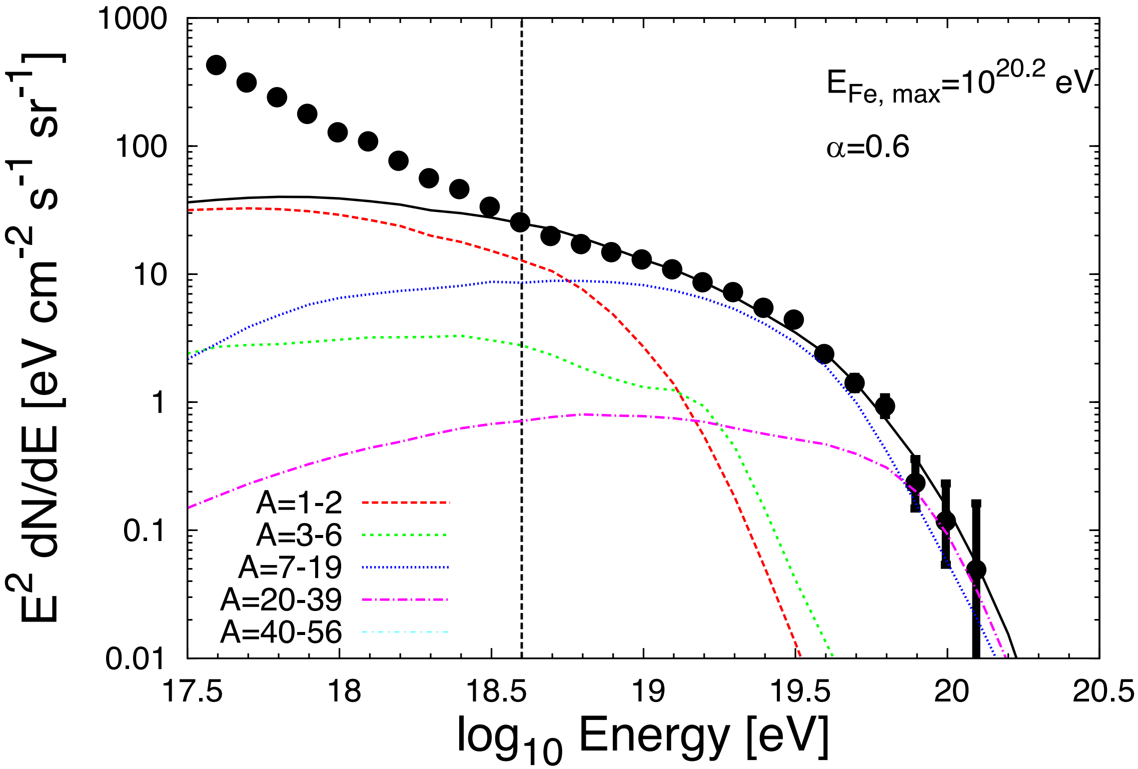
$$E_{Z,max} = (Z/26) \times E_{Fe,max}$$

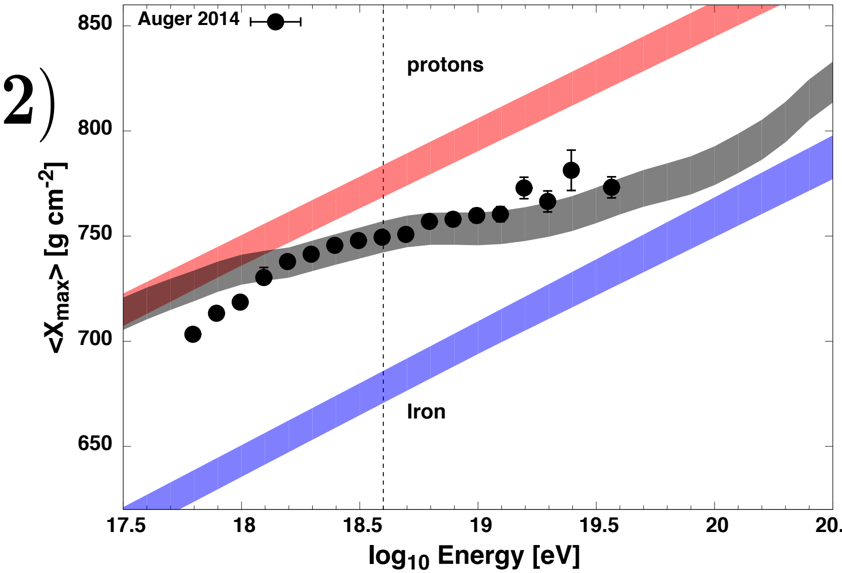
Note- magnetic field horizon effects are neglected in the following. This amounts to assuming: $d_s < (ct_H \lambda_{scat})^{1/2}$ ie. the source distribution may be approximated to be spatially continuous (also note, presence of t_H term comes from temporally continuous assumption)

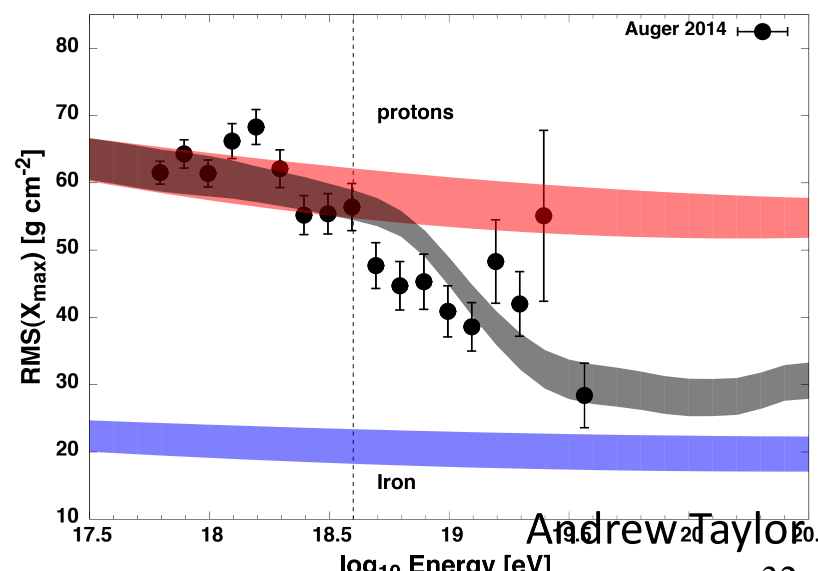
MCMC Likelihood Scan: Spectral + Composition Fits

$$\mathbf{L}(\mathbf{f_p}, \mathbf{f_{He}}, \mathbf{f_N}, \mathbf{f_{Si}}, \mathbf{E_{max}}, \alpha) \propto \exp(-\chi^2/2)$$

n=3 evolution result







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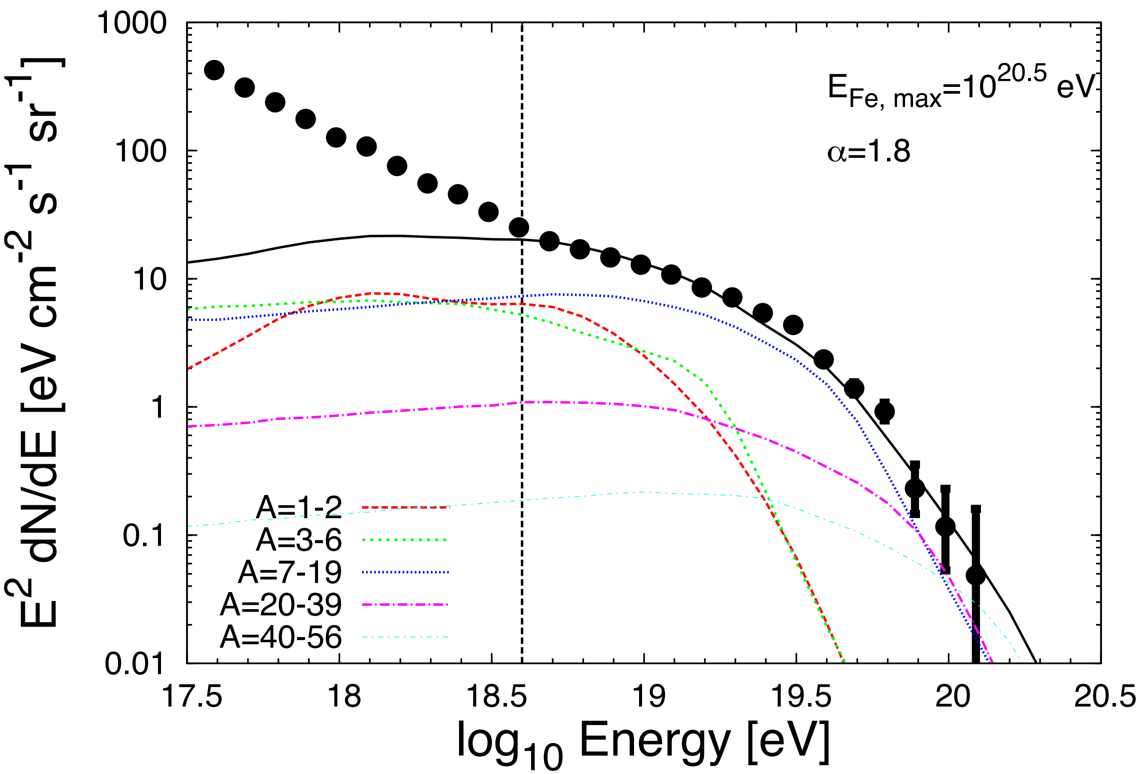
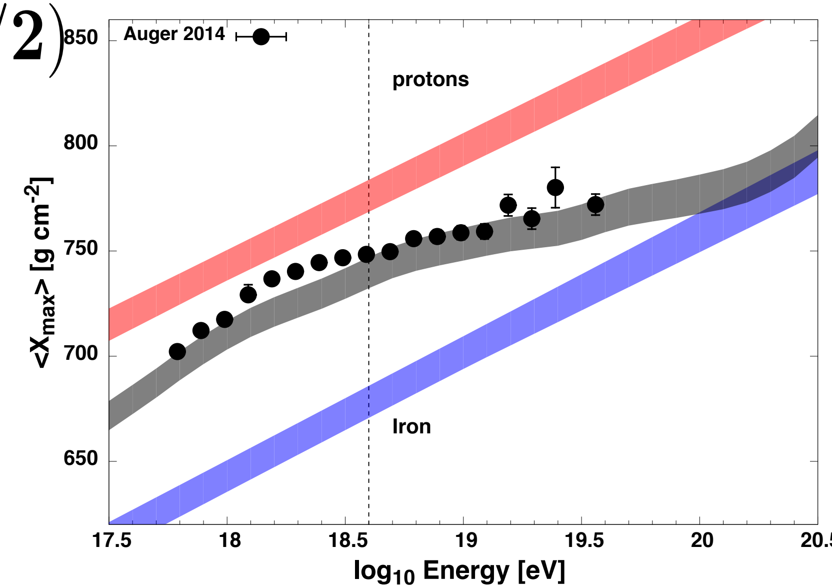
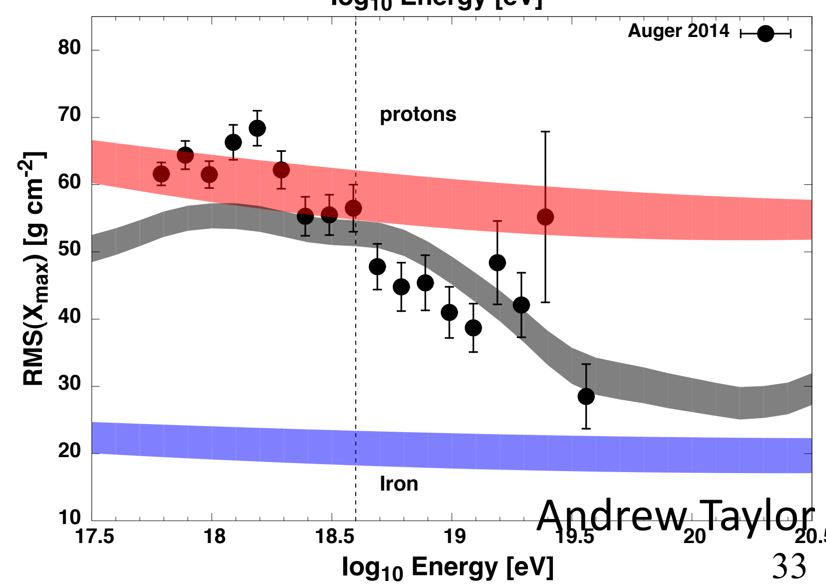
MCMC Likelihood Scan: "Soft" Spectra Solutions

$$\mathbf{L}(\mathbf{f_p}, \mathbf{f_{He}}, \mathbf{f_N}, \mathbf{f_{Si}}, \mathbf{E_{max}}, \alpha) \propto \exp(-\chi^2/2)$$

n=-6 evolution result







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MCMC Results Table

Similar conclusion arrives to by others (eg. ADD REF. TO KAMPERT ET AL.)

Parameter	$n=-6$		$n=-3$		$n=0$		$n=3$	
	Best-fit Value	Posterior Mean & Standard Deviation	Best-fit Value	Posterior Mean & Standard Deviation	Best-fit Value	Posterior Mean & Standard Deviation	Best-fit Value	Posterior Mean & Standard Deviation
f_p	0.03	0.14 ± 0.12	0.08	0.15 ± 0.13	0.17	0.17 ± 0.16	0.19	0.20 ± 0.16
f_{He}	0.50	0.21 ± 0.17	0.42	0.17 ± 0.16	0.53	0.20 ± 0.17	0.32	0.23 ± 0.20
f_{N}	0.40	0.50 ± 0.18	0.42	0.51 ± 0.19	0.29	0.47 ± 0.19	0.43	0.45 ± 0.21
f_{Si}	0.06	0.11 ± 0.12	0.08	0.12 ± 0.13	0.0	0.11 ± 0.12	0.06	0.078 ± 0.086
f_{Fe}	0.01	0.052 ± 0.039	0.0	0.053 ± 0.042	0.01	0.050 ± 0.038	0.0	0.044 ± 0.034
α	1.8	1.83 ± 0.31	1.6	1.67 ± 0.36	1.1	1.33 ± 0.41	0.6	0.64 ± 0.44
$\log_{10}\left(\frac{E_{\mathrm{Fe,max}}}{\mathrm{eV}}\right)$	20.5	20.55 ± 0.26	20.5	20.52 ± 0.27	20.2	20.38 ± 0.25	20.2	20.16 ± 0.18

Flatter spectra preferred for negative source evolution

Hard spectra preferred for source evolution following that of the SFR

An Analytic Description of these Results

Differential Equation Describing System State

$$\frac{\mathbf{d}}{\mathbf{dt}}\begin{pmatrix} f_{56} \\ f_{55} \\ f_{54} \end{pmatrix} = \mathbf{\Lambda}\begin{pmatrix} f_{56} \\ f_{55} \\ f_{54} \end{pmatrix}$$

$$\mathbf{\Lambda} = \begin{pmatrix} -\left(\frac{1}{\tau_{56\to55}} + \frac{1}{\tau_{56\to54}} + \ldots\right) & 0 & 0 \\ \frac{1}{\tau_{56\to55}} & -\left(\frac{1}{\tau_{55\to54}} + \frac{1}{\tau_{55\to53}} + \ldots\right) & 0 \\ \frac{1}{\tau_{56\to54}} & \frac{1}{\tau_{55\to54}} & -\left(\frac{1}{\tau_{54\to53}} + \frac{1}{\tau_{54\to52}} + \ldots\right) \end{pmatrix}$$

by $\mathbf{f_q(t) = \sum_{n=q}^{56} A_n f_n(t)}$

then $\mathbf{f_q(t) = \sum_{n=q}^{56} A_n e^{-\lambda_n t} f_n(0)}$

(where A_n values are set by the initial conditions)

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Only Considering Single Nucleon Losses

$$\mathbf{\Lambda} = \begin{pmatrix} -\frac{1}{\tau_{56\to55}} & 0 & 0 \\ \frac{1}{\tau_{56\to55}} & -\frac{1}{\tau_{55\to54}} & 0 \\ 0 & \frac{1}{\tau_{55\to54}} & -\frac{1}{\tau_{54\to53}} \end{pmatrix}$$

and

$$\mathbf{f_q(t) = \sum_{n=q}^{56} f_{56}(0) \frac{\tau_q \tau_n^{56-q-1}}{\prod_{p=q}^{56}(\tau_n - \tau_p)} e^{-\frac{t}{\tau_n}}}$$

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Nuclear Cascade Description

Consider

$$\frac{df_q}{dt} + \frac{f_q}{\tau_q} = \frac{f_{q+1}}{\tau_{q+1}}$$

$$e^{\left(\frac{-t}{\tau_q}\right)} \frac{d}{dt}\left[e^{\left(\frac{t}{\tau_q}\right)} f_q\right] = \frac{f_{q+1}}{\tau_{q+1}}$$

$$f_q = e^{\left(\frac{-t}{\tau_q}\right)} \int e^{\left(\frac{t}{\tau_q}\right)} \frac{f_{q+1}}{\tau_{q+1}} dt$$

Assume solution is true for q, apply to q+1

$$\frac{f_{q+1}(t)}{f_{56}(0)} = \sum_{n=q+1}^{56} \frac{\tau_{q+1}\tau_n^{56-q-2}}{\prod_{p=q+1}^{56}(\tau_n - \tau_p)} e^{-\frac{t}{\tau_n}}$$

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Nuclear Cascade Description

Assume solution is true

$$\frac{f_{q+1}(t)}{f_{56}(0)} = \sum_{n=q+1}^{56} \frac{\tau_{q+1}\tau_n^{56-q-2}}{\prod_{p=q+1}^{56}(\tau_n - \tau_p)} e^{-\frac{t}{\tau_n}}$$

$$f_q = e^{\left(\frac{-t}{\tau_q}\right)} \int e^{\left(\frac{t}{\tau_q}\right)} \frac{f_{q+1}}{\tau_{q+1}} dt$$

$$\frac{f_q(t)}{f_{56}(0)} = \sum_{n=q+1}^{56} \frac{\tau_n^{56-q-2}}{\prod_{p=q+1}^{56}(\tau_n - \tau_p)} \left[\left(\frac{1}{\tau_q} - \frac{1}{\tau_n}\right)^{-1} e^{\frac{-t}{\tau_n}}\right] - ce^{\frac{-t}{\tau_q}}$$

Since $f_q(0) = 0$

$$c = \sum_{n=q+1}^{56} \frac{\tau_q \tau_n^{56-q-1}}{\prod_{p=q}^{56}(\tau_n - \tau_p)}$$

Nuclear Cascade Description

$$\frac{f_q(t)}{f_{56}(0)} = \sum_{n=q+1}^{56} \frac{\tau_n^{56-q-2}}{\prod_{p=q}^{56}(\tau_n - \tau_p)} e^{\frac{-t}{\tau_n}} - \sum_{n=q+1}^{56} \frac{\tau_q \tau_n^{56-q-1}}{\prod_{p=q}^{56}(\tau_n - \tau_p)} e^{\frac{-t}{\tau_q}}$$

$$\frac{f_q(t)}{f_{56}(0)} = \sum_{n=q}^{56} \frac{\tau_q \tau_n^{56-q-1}}{\prod_{p=q}^{56}(\tau_n - \tau_p)} e^{-\frac{t}{\tau_n}}$$

These are equivalent if:

$$\sum_{n=q+1}^{56} \frac{\tau_q \tau_n^{56-q-1}}{\prod_{p=q}^{56}(\tau_n - \tau_p)} = \frac{\tau_q \tau_q^{56-q-1}}{\prod_{p=q}^{56}(\tau_q - \tau_p)}$$

Consider:

$$\frac{w^2}{(w-x)(w-y)(w-z)} + \frac{x^2}{(x-w)(x-y)(x-z)} + \frac{y^2}{(y-w)(y-x)(y-z)} = -\frac{z^2}{(z-w)(z-x)(z-y)}$$

Nuclear Cascade Description

$$\sum_{\mathbf{n=q+1}}^{\mathbf{56}} \frac{\tau_{\mathbf{q}}\tau_{\mathbf{n}}^{\mathbf{56-q-1}}}{\prod_{\mathbf{p=q}}^{\mathbf{56}}(\tau_{\mathbf{n}}-\tau_{\mathbf{p}})} = \frac{\tau_{\mathbf{q}}\tau_{\mathbf{q}}^{\mathbf{56-q-1}}}{\prod_{\mathbf{p=q}}^{\mathbf{56}}(\tau_{\mathbf{q}}-\tau_{\mathbf{p}})}$$

Consider the case

$$\frac{\mathbf{w^2}}{(\mathbf{w-x})(\mathbf{w-y})(\mathbf{w-z})} + \frac{\mathbf{x^2}}{(\mathbf{x-w})(\mathbf{x-y})(\mathbf{x-z})} + \frac{\mathbf{y^2}}{(\mathbf{y-w})(\mathbf{y-x})(\mathbf{y-z})} = -\frac{\mathbf{z^2}}{(\mathbf{z-w})(\mathbf{z-x})(\mathbf{z-y})}$$

$$\begin{vmatrix} 1 & w & w^2 & w^2 \\ 1 & x & x^2 & x^2 \\ 1 & y & y^2 & y^2 \\ 1 & z & z^2 & z^2 \end{vmatrix} = \mathbf{0}$$

Cascade of Nuclei Through Species- single nucleon loss

Since nuclei Lorentz factor remains ~conserved, and cross-section varies mildly with A (nuclear mass)

$$\tau_{56\rightarrow 55} \approx \tau_{55\rightarrow 54}\cdots$$

For the case $\tau_{56\rightarrow 55} = \tau_{55\rightarrow 54}\cdots$

$$f_q = \frac{t^{(q_{max}-q)}}{\tau_q (q_{max}-q)!} e^{-t/\tau_q}$$

ie. Gaisser-Hillas type function!
(used to describe air showers)

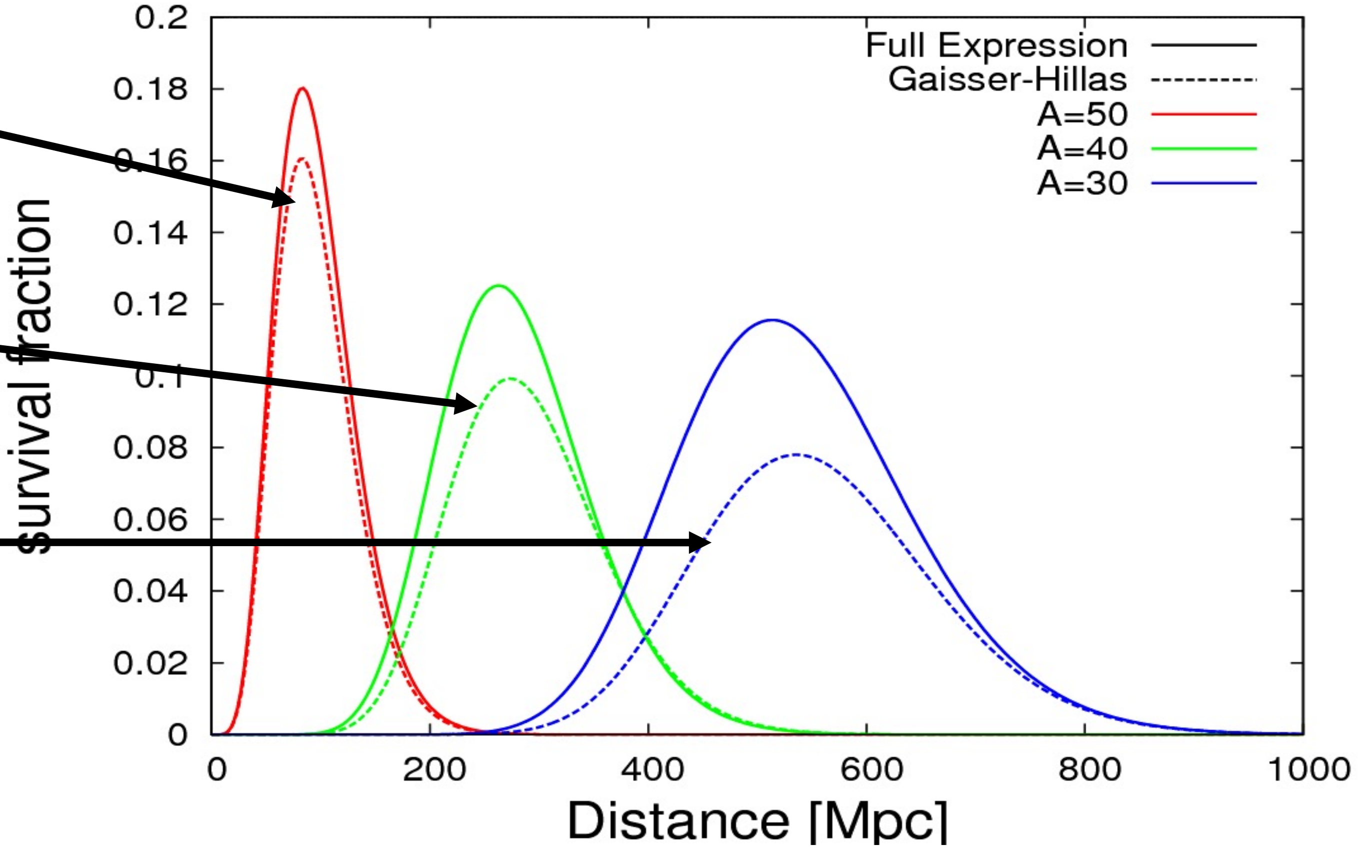
Cascade of Nuclei Through Species-Comparison of Approximation

Starting with Fe, $q_{\mathbf{max}} = \mathbf{56}$

$$f_{50} = \frac{t^6}{6!}e^{-\frac{t}{\tau_{50}}}$$

$$f_{40} = \frac{t^{16}}{16!}e^{-\frac{t}{\tau_{40}}}$$

$$f_{30} = \frac{t^{26}}{26!}e^{-\frac{t}{\tau_{30}}}$$



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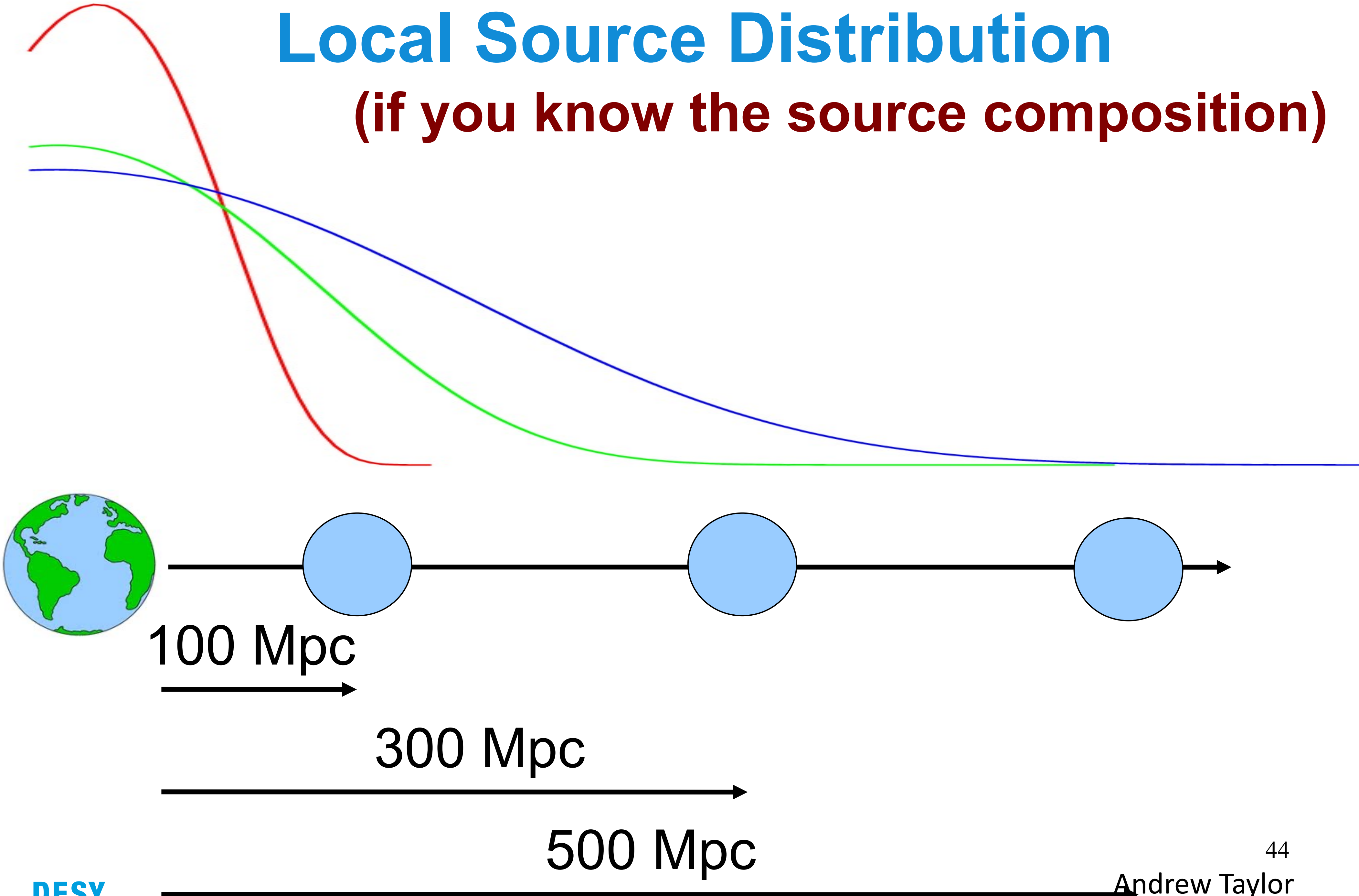
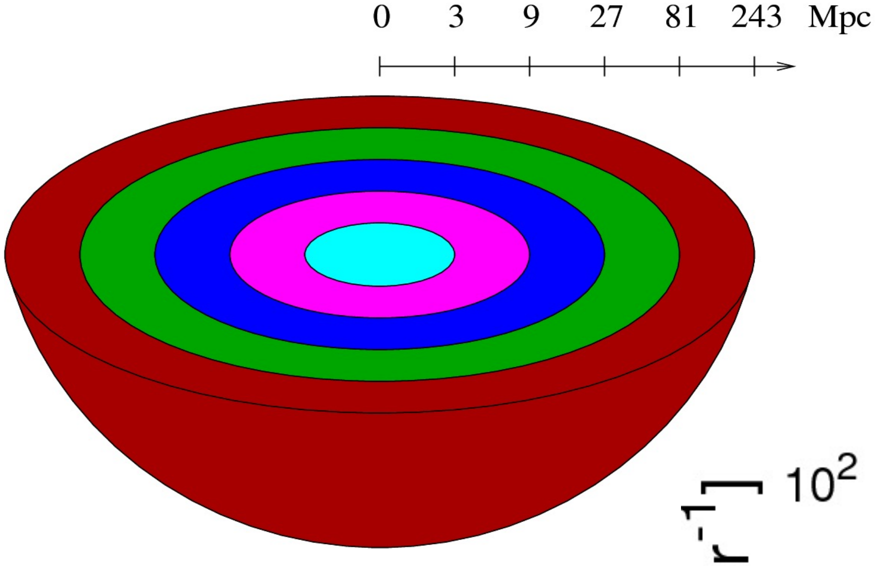
Composition – an Excellent Probe of the Local Source Distribution

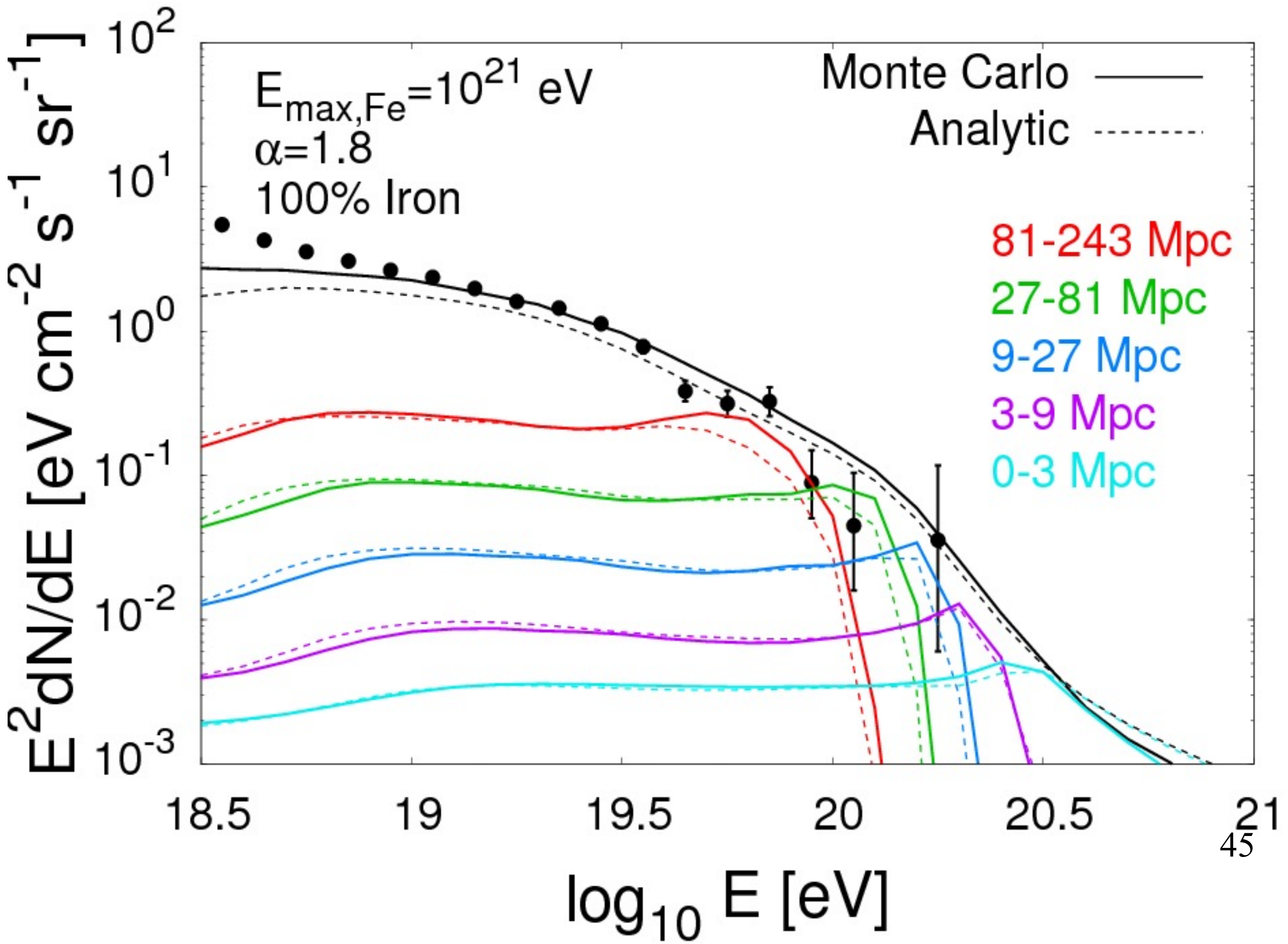
(if you know the source composition)



Local Scales Effect Highest Energies

(logarithmic scale)





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End of Second Lecture

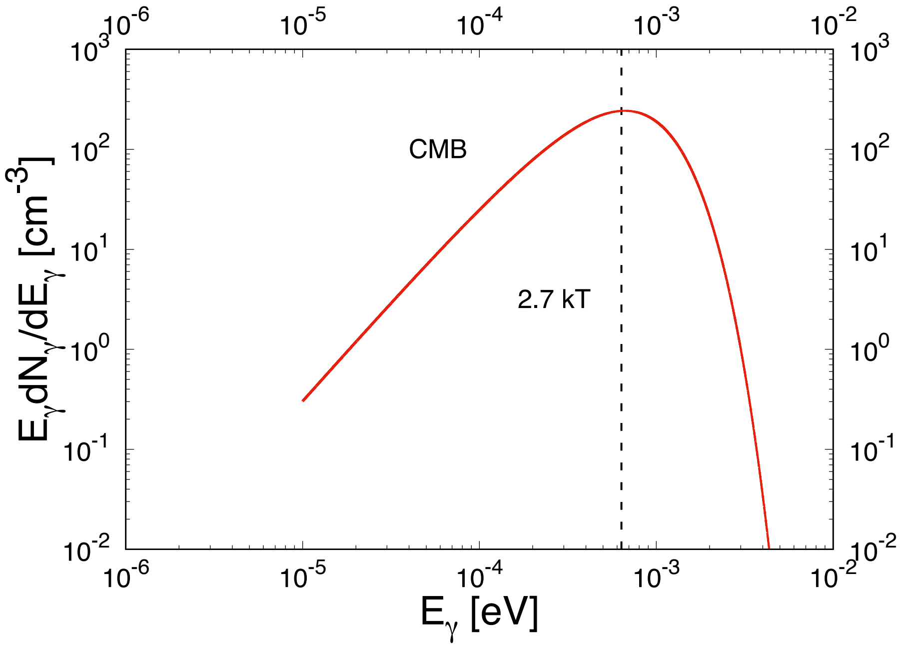
CMB- Total Number Density

$$\mathbf{n_\gamma^{BB}} = \mathbf{8}\pi \frac{\mathbf{(kT)^3}}{\mathbf{(hc)^3}} \gamma(\mathbf{3})\zeta(\mathbf{3})$$

$$\mathbf{n_\gamma^{BB}} = \frac{\mathbf{8}\pi\mathbf{(kT)^3}}{\mathbf{(hc)^3}} \int_{\mathbf{0}}^{\infty} \frac{\mathbf{x^2}}{\mathbf{e^x - 1}} \mathbf{dx}$$

$$\int_{\mathbf{0}}^{\infty} \mathbf{x^2 e^{-x} dx} = \gamma(\mathbf{3})$$

$$\frac{\mathbf{x^n}}{\mathbf{e^x - 1}} = \frac{\mathbf{e^{-x} x^n}}{\mathbf{1 - e^{-x}}}$$



CMB- Total Number Density

$$\mathbf{n}_\gamma^{\mathbf{BB}} = \mathbf{8}\pi \frac{(\mathbf{kT})^\mathbf{3}}{(\mathbf{hc})^\mathbf{3}} \gamma(\mathbf{3})\zeta(\mathbf{3})$$

$$\frac{\mathbf{x}^\mathbf{n}}{\mathbf{e}^\mathbf{x} - \mathbf{1}} = \frac{\mathbf{e}^{-\mathbf{x}}\mathbf{x}^\mathbf{n}}{\mathbf{1} - \mathbf{e}^{-\mathbf{x}}}$$

$$= \sum_{\mathbf{m}=\mathbf{0}}^{\infty} \mathbf{e}^{-\mathbf{mx}}\mathbf{e}^{-\mathbf{x}}\mathbf{x}^\mathbf{n}$$

$$= \sum_{\mathbf{m}=\mathbf{1}}^{\infty} \mathbf{e}^{-\mathbf{mx}}\mathbf{x}^\mathbf{n}$$

CMB- Total Number Density

$$n_\gamma^{BB} = 8\pi \frac{(kT)^3}{(hc)^3} \gamma(3)\zeta(3)$$

$$\int \frac{x^n}{e^x - 1} dx = \sum_{m=1}^{\infty} \int e^{-mx} x^n dx$$

Let $y = mx$

$$\int \frac{x^n}{e^x - 1} dx = \sum_{m=1}^{\infty} \int e^{-y} \left(\frac{y}{m}\right)^n d\left(\frac{y}{m}\right)$$

$$\int \frac{x^n}{e^x - 1} dx = \sum_{m=1}^{\infty} \frac{1}{m^{n+1}} \int y^n e^{-y} dy = \gamma(n+1)\zeta(n+1)$$

Threshold Energy- Proton Pion Production

$$(\mathbf{E_p} + \mathbf{E}_\gamma)^\mathbf{2} - (\mathbf{p_p} - \mathbf{E}_\gamma)^\mathbf{2} = (\mathbf{m_p} + \mathbf{m}_\pi)^\mathbf{2}$$

$$\mathbf{m_p^2} + \mathbf{2E_pE}_\gamma + \mathbf{2p_pE}_\gamma \approx \mathbf{m_p^2} + \mathbf{2m_pm}_\pi$$

$$\mathbf{E_p} \approx \frac{\mathbf{m}_\pi}{\mathbf{2E}_\gamma}\mathbf{m_p} \approx \left(\frac{\mathbf{135 \times 10^6}}{\mathbf{2 \times 6 \times 10^{-4}}}\right)\mathbf{0.9 \times 10^9} = \mathbf{10^{20}\ eV}$$

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Comparison of Analytic and Monte Carlo Results