A new treatment of mixed virtual and real IR-singularities

Tord Riemann, DESY, Zeuthen

based on work with:

J. Fleischer (U.Bielefeld) and J. Gluza (U.Silesia, Katowice),

Project B1 – SFB/TRR 9, HUB, Berlin, 30 Oct. 2007

Mini-WS on Massive Particle Production at the LHC

- **Introduction: IR-singularities of massive n-point functions**
- **Mellin-Barnes representations for Feynman diagrams**
- **Mixed IR-singularities from loops and soft real emission**
- **Summary**

Introduction: IR-singularities of massive n-point functions

- We collected some experience in using Mellin-Barnes (MB) representations for massive loop diagrams
- They have proven very useful for the separation – and also evaluation – of the poles in $\epsilon = (4-d)/2$ even for very complicated diagrams
  often quoted: V. Smirnov (and G. Heinrich) and B. Tausk, planar and non-planar massive double boxes.
- An interesting simpler application – with a potential of automatization – is demonstrated here:
  One-loop n-point functions with both virtual and real massless particles.
  They produce both $1/\epsilon$-poles from the virtual massless lines and the so-called end-point singularities from the phase space integrals with $\int dE/E \to \infty$ from $E=0$
- The MB-approach might be an ideal tools for the treatment of that at the amplitude level.
- The mathematica packages MB.m (Czakon, CPC 2005) and AMBRE.m (Gluza, Kajda, Riemann, arXiv:0704.2423, CPC) are well-suited for that.
- The result is not only numerical.
  We present here a representation in terms of inverse binomial sums and HPL's.

Example: Massive one-loop 5-point function

Radiative loop diagrams contribute to the NNLO corrections by interfering with radiative Born diagrams:

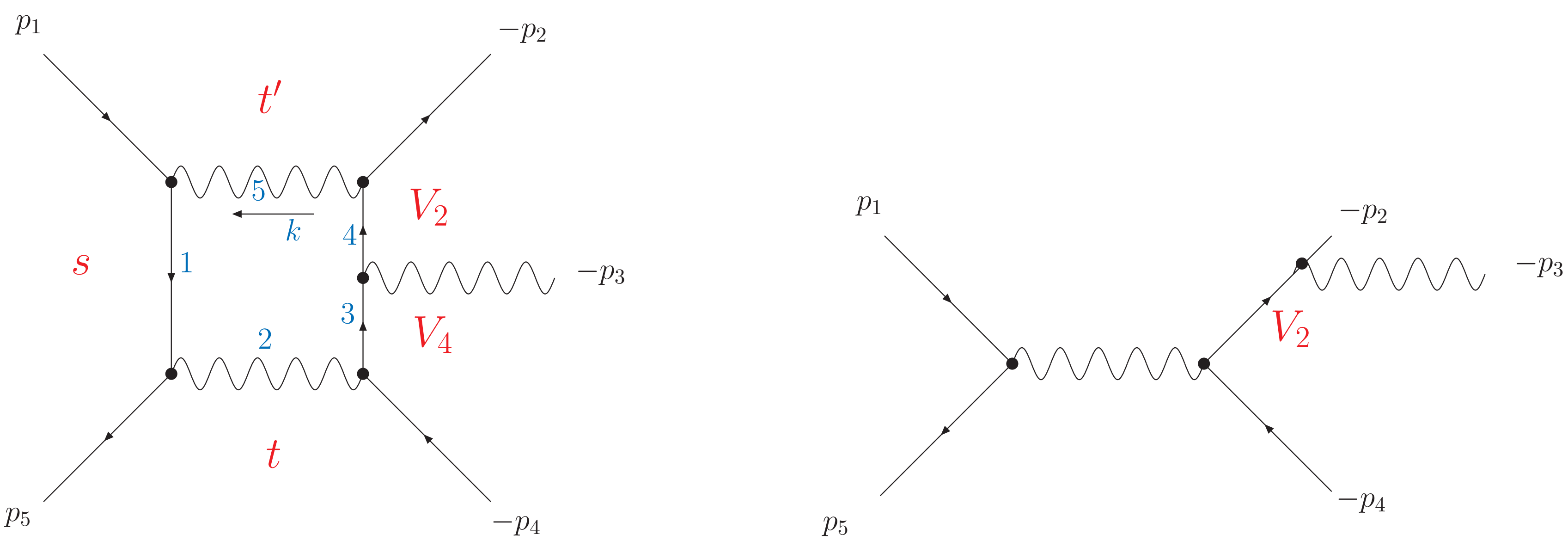


Figure 1: A pentagon topology and a Born topology

Five of the invariants are independent, e.g.:

$$
\begin{aligned}
s &= (p_1 + p_5)^2, \\
t &= (p_4 + p_5)^2, && (1) \\
t' &= (p_1 + p_2)^2, && (2) \\
V_2 &= 2p_2p_3 \quad \sim E_3, && (3) \\
V_4 &= 2p_4p_3 \quad \sim E_3 && (4)
\end{aligned}
$$

The invariants $V_i = 2p_ip_3$ appear also in the Born diagrams and produce the so-called endpoint singularities:

$$
\frac{1}{(p_2+p_3)^2 - m^2} = \frac{1}{2p_2p_3 + [p_2^2 - m^2] + [p_3^2 - 0]} = \frac{1}{V_2} = \frac{1}{2E_3E_2(1-\beta_2\cos\vartheta)} \sim \frac{1}{E_3}
$$

The massless particle's phase space integral is typically:

$$
\begin{aligned}
\int \frac{d^3p_3}{2E_3}\frac{1}{V_2V_4} \sim \int_0^\omega dE_3/E_3 &= \ln(E_3)|_0^\omega = \ln(\omega) - \ln(0) = divergent \\
\to \int_0^\omega dE_3/E_3^{5-d} &= \frac{1}{d-4}E_3^{d-4}|_0^\omega = \frac{\omega^{2\epsilon} - 0}{2\epsilon} = finite && (5)
\end{aligned}
$$

We have to safely control the dependence on V_2, V_4 as part of the mixed infrared problem due to the common existence of virtual and real IR-sources.

Consider now only the scalar 5-point function.
the massless propagators are $d_5 = k^2$ and $d_2 = (k + p_1 + p_5)^2$.
The leading singularity is easily found algebraically:

$$\frac{1}{d_1 d_2 d_3 d_4 d_5} = \frac{-1}{s}\left[\frac{2k(k+p_1+p_5)}{d_1 d_2 d_3 d_4 d_5} - \frac{1}{d_1 d_2 d_3 d_4} - \frac{1}{d_1 d_3 d_4 d_5}\right]$$

The two IR-divergent 4-point functions trace to one IR-div. 3-point f. each, e.g.

$$\frac{1}{d_1 d_3 d_4 d_5} = \frac{-1}{V_2}\left[\frac{2k(k+p_1+p_4+p_5)}{d_1 d_3 d_4 d_5} - \frac{1}{d_1 d_3 d_4} - \frac{1}{d_1 d_4 d_5}\right]$$

and the resulting IR-part is:

$$\begin{aligned} \int \frac{d^d k}{d_1 d_2 d_3 d_4 d_5} &= \frac{1}{sV_2}\int \frac{d^d k}{d_1 d_4 d_5} + \frac{1}{sV_4}\int \frac{d^d k}{d_1 d_2 d_3} + \cdots \\ &= \frac{1}{\epsilon}\left[\frac{F(t')}{sV_2} + \frac{F(t)}{sV_4}\right] + \cdots \end{aligned} \qquad (6)$$

Evidently, one separates only a leading singularity, while we expect an expression like

$$\int \frac{d^d k}{d_1 d_2 d_3 d_4 d_5} = \frac{A_2}{sV_2\epsilon} + \frac{A_4}{sV_4\epsilon} + \frac{B_2}{sV_2}\ln(V_2) + \frac{B_4}{sV_4}\ln(V_4) + \frac{C_2}{sV_2} + \frac{C_4}{sV_4} + \cdots$$

Mellin-Barnes representation for the massive pentagon

The chords q_i are defined from the propagators: $d_i = [(k-q_i)^2 - m_i^2]$

$$I_5[A(q)] = -e^{\epsilon\gamma_E} \int_0^1 \prod_{j=1}^{5} dx_j \; \delta\left(1 - \sum_{i=1}^{5} x_i\right) \frac{\Gamma(3+\epsilon)}{F(x)^{3+\epsilon}} \; B(q),$$

with $B(1)=1, B(q^\mu) = Q^\mu, B(q^\mu q^\nu) = Q^\mu Q^\nu - \frac{1}{2} g^{\mu\nu} F(x)/(2+\epsilon)$**, and** $Q^\mu = \sum x_i q_i^\mu$**.**

The diagram depends on five variables and the F**-form is:**

$$F(x) = m_0^2(x_2+x_4+x_5)^2 + [-s]x_1x_3 + [-V_4]x_3x_5 + [-t]x_2x_4 + [-t']x_2x_5 + [-V_2]x_1x_4. \quad (7)$$

Henceforth, $m_0 = 1$**. Massless particle's momentum is** p_3**.**

The Mellin-Barnes (MB)-representation,

$$\frac{1}{[A(x) + Bx_ix_j]^R} = \frac{1}{2\pi i} \int_{\mathcal{C}} dz [A(x)]^z [Bx_ix_j]^{-R-z} \frac{\Gamma(R+z)\Gamma(-z)}{\Gamma(R)},$$

is used several times for replacing in $F(x)$ **the sum over** x_ix_j **by products of monomials in the** x_ix_j**, thus allowing the subsequent** x**-integrations in a simple manner.**

Why the Mellin-Barnes integrals?

We want to apply a simple formula for integrating over the x_i:

$$\int_0^1 \prod_{j=1}^{N} dx_j \; x_j^{\alpha_j-1} \; \delta\left(1-x_1-\cdots-x_N\right) \quad = \quad \frac{\Gamma(\alpha_1)\cdots\Gamma(\alpha_N)}{\Gamma\left(\alpha_1+\cdots+\alpha_N\right)}$$

with coefficients α_i dependent on F

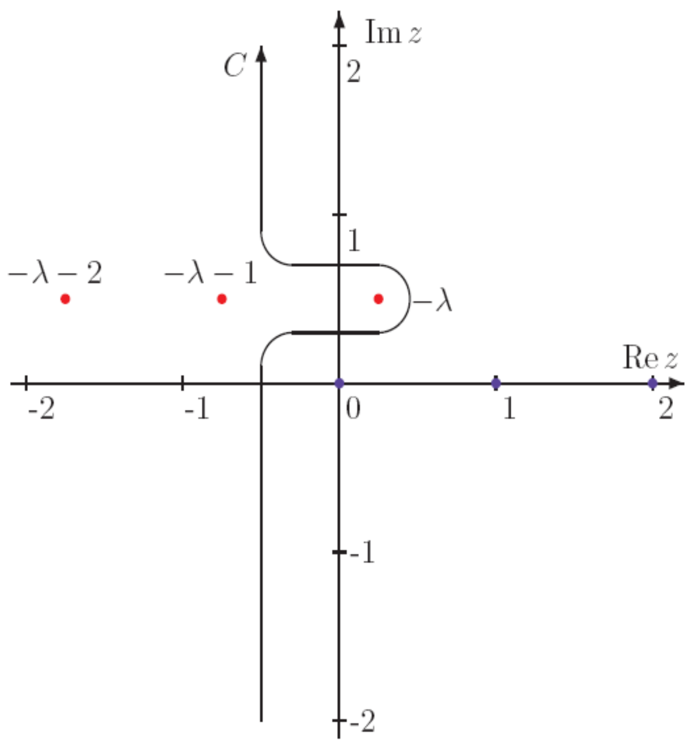
For this, we have to apply several MB-integrals here:

$$F(x) \quad = \quad m_0^2(x_2+x_4+x_5)^2+[-s]x_1x_3+[-V_4]x_3x_5+[-t]x_2x_4+[-t']x_2x_5+[-V_2]x_1x_4. \tag{8}$$

For each of the $+$-sign one MB-integral , so arrive at a 7-dimensional path integral.

$$\frac{1}{[A(s)x_1^{a_1} + B(s)x_1^{b_1}x_2^{b_2}]^\lambda} = \frac{1}{2\pi i\ \Gamma(\lambda)} \int\limits_{-i\infty}^{i\infty} dz [A(s)x_1^{a_1}]^z [B(s)x_1^{b_1}x_2^{b_2}]^{\lambda+z}\ \Gamma(\lambda+z)\Gamma(-z)$$

The integration path has to separate the chains of poles of $\Gamma(\lambda+z)$ and $\Gamma(-z)$:



$$\begin{aligned}
\mathrm{ResF[z]\Gamma(A+z)}|_{\mathrm{z=-n}} &= \frac{(-1)^{n-A}}{(n-A)!}F[-n], n=-A,-A-1,\cdots \\
\mathrm{ResF[z]\Gamma(1+z)^2}|_{\mathrm{z=-n}} &= \frac{1}{\Gamma[n]^2}(2F[-n]\mathrm{PolyGamma[n]}+\mathrm{F'[-n]}) \\
\mathrm{ResF[z]\Gamma[1+z]PolyGamma[1+z]}|_{\mathrm{z=-n}} &= \frac{(-1)^n}{\Gamma[n]}F'[-n]
\end{aligned}$$

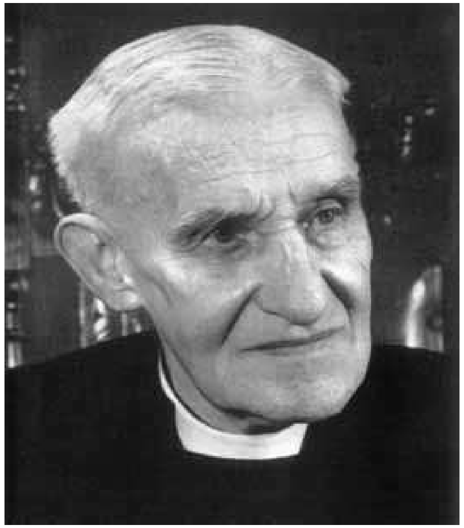
with the definitions

$$S_k[N] = \sum_{i=1}^{N} \frac{1}{i^k}$$

and

$S_1[N]$ **= HarmonicNumber[n-1] - EulerGamma = PolyGamma[n]**

Mellin, Robert Hjalmar, 1854-1933
Barnes, Ernest William, 1874-1953



A little history

- N. Usyukina, 1975: "ON A REPRESENTATION FOR THREE POINT FUNCTION", Teor. Mat. Fiz. 22;
  a finite massless off-shell 3-point 1-loop function represented by 2-dimensional MB-integral
- E. Boos, A. Davydychev, 1990: "A Method of evaluating massive Feynman integrals", Theor. Math. Phys. 89 (1991);
  N-point 1-loop functions represented by n-dimensional MB-integral
- V. Smirnov, 1999: "Analytical result for dimensionally regularized massless on-shell double box", Phys. Lett. B460 (1999);
  treat UV and IR divergencies by analytical continuation: shifting contours and taking residues 'in an appropriate way'
- B. Tausk, 1999: "Non-planar massless two-loop Feynman diagrams with four on-shell legs", Phys. Lett. B469 (1999);
  nice algorithmic approach to that, starting from search for some unphysical space-time dimension d for which the MB-integral is finite and well-defined
- M. Czakon, 2005 (with experience from common work with J. Gluza and TR): "Automatized analytic continuation of Mellin-Barnes integrals", Comput. Phys. Commun. (2006);
  Tausk's approach realized in Mathematica program MB.m, published and available for use

We derive MB-representations with AMBRE, a publicly available Mathematica package

J. Gluza, K. Kajda, T. Riemann, arXiv:0704.2423 [hep-ph], to appear in CPC

AMBRE – Automatic Mellin-Barnes Representations for Feynman diagrams

For the Mathematica package AMBRE, many examples, and the program description, see:

http://prac.us.edu.pl/∼gluza/ambre/
http://www-zeuthen.desy.de/theory/research/CAS.html

See also here:
http://www-zeuthen.desy.de/∼riemann/Talks/capp07/
with additional material presented at the CAPP – School on Computer Algebra in Particle Physics, DESY, Zeuthen, March 2007

A AMBRE functions list

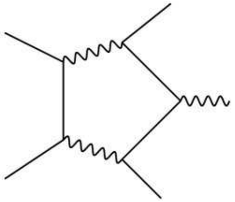
The basic functions of AMBRE are:

- **Fullintegral**[**{numerator},{propagators},{internal momenta}**] – is the basic function for input Feynman integrals
- **invariants** – is a list of invariants, e.g. **invariants = {p1*p1 → s}**
- **IntPart**[**iteration**] – prepares a subintegral for a given internal momentum by collecting the related numerator, propagators, integration momentum
- **Subloop**[**integral**] – determines for the selected subintegral the U and F polynomials and an MB-representation
- **ARint**[**result,i_**] – displays the MB-representation number i for Feynman integrals with numerators
- **Fauto**[**0**] – allows user specified modifications of the F polynomial **fupc**
- **BarnesLemma**[**repr,1,Shifts->True**] – function tries to apply Barnes' first lemma to a given MB-representation; when **Shifts->True** is set, AMBRE will try a simplifying shift of variables
  BarnesLemma[**repr,2,Shifts->True**] – function tries to apply Barnes' second lemma

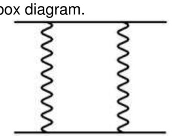
AMBRE - Automatic Mellin-Barnes REpresentation (arXiv:0704.2423)

To download 'right click' and 'save target as'.

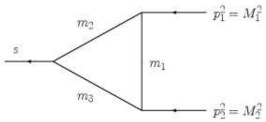
- The package AMBRE.m
- Kinematics generator for 4- 5- and 6- point functions with any external legs KinematicsGen.m
- Tarball with examples given below examples.tar.gz
  - example1.nb, example2.nb - Massive QED pentagon diagram.



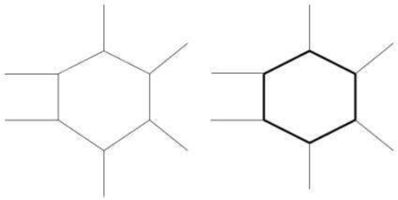
  - example3.nb - Massive QED one-loop box diagram.



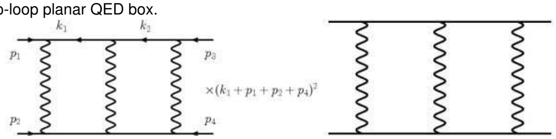
  - example4.nb - General one-loop vertex.



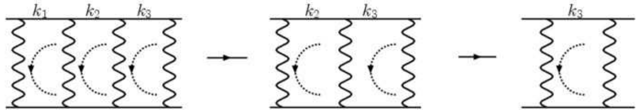
  - example5.nb - Six-point scalar functions;
    left: massless case,
    right: massive case.

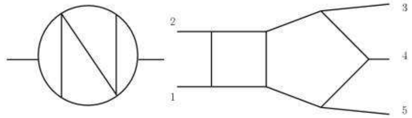


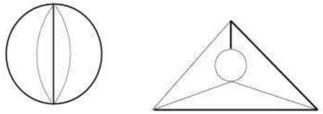
  - example6.nb - **left**, example7.nb - **right**
    Massive two-loop planar QED box.



  - example8.nb - The loop-by-loop iterative procedure.







MB-representation for the scalar massive pentagon

In our example we get a seven-fold MB-representation, reduce to a four-fold representations after three times applying Barnes' lemma in order to eliminate 2 spurious integrations from the mass term. and one from setting $t' = t$ (Born kinematics assumed here).

$$I_5 = \frac{-e^{\epsilon\gamma_E}}{(2\pi i)^4} \prod_{i=1}^{4} \int_{-i\infty+u_i}^{+i\infty+u_i} dz_i (-s)^{z_2} (-t)^{z_4} (-V_2)^{z_3} (-V_4)^{-3-\epsilon-z_1-z_2-z_3-z_4} \frac{\prod_{j=1..12} \Gamma_j}{\Gamma_0 \Gamma_{13} \Gamma_{14}},$$

with a normalization $\Gamma_0 = \Gamma[-1-2\epsilon]$, and the other Γ-functions are:

$$\begin{aligned}
\Gamma_1 &= \Gamma[-z_1], \quad \Gamma_2 = \Gamma[-z_2], \quad \Gamma_3 = \Gamma[-z_3], \quad \Gamma_4 = \Gamma[1+z_3], \\
\Gamma_5 &= \Gamma[1+z_2+z_3], \quad \Gamma_6 = \Gamma[-z_4], \quad \Gamma_7 = \Gamma[1+z_4], \quad \Gamma_8 = \Gamma[-1-\epsilon-z_1-z_2], \\
\Gamma_9 &= \Gamma[-2-\epsilon-z_1-z_2-z_3-z_4], \quad \Gamma_{10} = \Gamma[-2-\epsilon-z_1-z_3-z_4], \\
\Gamma_{11} &= \Gamma[-\epsilon+z_1-z_2+z_4], \; \Gamma_{12} = \Gamma[3+\epsilon+z_1+z_2+z_3+z_4],
\end{aligned}$$

and

$$\Gamma_{13} = \Gamma[-1-\epsilon-z_1-z_2-z_4], \quad \Gamma_{14} = \Gamma[-\epsilon-z_1-z_2+z_4].$$

This is a finite integral if all Γ-functions in the numerator have positive real parts of the arguments.

May be fulfilled with:

$$\epsilon = -3/4$$

The real shifts u_i of the integration strips r_i are:

$$\begin{aligned}
u_1 &= -5/8 \\
u_2 &= -7/8 \\
u_3 &= -1/16 \\
u_4 &= -5/8 \\
u_5 &= -1/32
\end{aligned}$$

Analytical continuation in ϵ and deformation of integration contours

A well-defined MB-integral was found with the finite parameter ϵ and the strips parallel to the imaginary axis.

Now look at the real parts of arguments of Γ-functions (in the numerator only) and find out, which of them change sign (become negative) when $\epsilon \to 0$

Rule:
Moving $\epsilon \to 0$ corresponds to a step-wise analytical continuation of the contour integral ($dimension = n$) and so we have to add or subtract the residues at these values of the integration varables.

The residues have the dimension of integration $n-1, n-2, \cdots$.

This procedure may be automatized "easily" and it is done in the publicly available Mathematica package MB.m (M. Czakon, hep-ph/0511200, CPC)

Analytical continuation, $0 \neq \epsilon << 1$ and ϵ-expansion

After the analytical continuation and the expansion in ϵ, the scalar pentagon function is represented by 11 MB-integrals with different dimensions.
The IR-non-save parts are contained in only few and relatively simple of them:

$$
\begin{aligned}
I_5^{IR} &= I_5^{IR}(V_2) + I_5^{IR}(V_4), \\
I_5^{IR}(V_2) &= \frac{I_{-1}}{\epsilon} + I_0 \\
\frac{I_{-1}}{\epsilon} &= \frac{e^{\epsilon\gamma_E}}{2\pi i}\int_{-i\infty-5/8}^{+i\infty-5/8} dz_1 \frac{(-t)^{-1-z_1}}{2\epsilon s V_2}\frac{\Gamma[-z_1]^3\Gamma[1+z_1]}{\Gamma[-2z_1]}
\end{aligned}
$$

$$
\begin{aligned}
I_0 =& \frac{e^{\epsilon\gamma_E}}{2\pi i}\int_{-i\infty-5/8}^{+i\infty-5/8}\frac{dz_1}{2sV_2}\left[F_1[z_1]\Gamma[1+z_1] + F_2[z_1]\Gamma[1+z_1]\mathrm{PolyGamma}[1+\mathrm{z}_1]\right] \\
&+\frac{e^{\epsilon\gamma_E}}{(2\pi i)^2}\int_{-i\infty-7/8}^{+i\infty-7/8} dz_2 \int dz_1 (-s)^{z_2}(-t)^{-z_1+z_2}(-V_2)^{-2-z_2}(-V_4)^{-1-z_2} \\
&\frac{\Gamma[-z_1]\Gamma[-1-z_2]\Gamma[-1-z_1-z_2]\Gamma[z_1-z_2]\Gamma[-z_2]^2\Gamma[1+z_2]\Gamma[2+z_2]\Gamma[1-z_1+z_2]}{\Gamma[-2z_1]\Gamma[-1-2z_2]}
\end{aligned}
$$

Before taking sums of residua by closing contours to the left (anti-clockwise), look at powers of $(-V_2)$.

Its real part gives $(-V_2)^{-9/8}$, this would be not integrable for small V_2.

Shift the contour z_2 by a unit to the left.
This changes: $(-V_2)^{-9/8} \to (-V_2)^{-1/8}$ and after that, the 2-dim.integral is IR-safe.

One residue is crossed and has to be added to the resulting 2-dim. contour integral.

So take here instead of the original 2-dim. integral only the residue as the contribution of interest:

$$I_0 = \frac{e^{\epsilon\gamma_E}}{2\pi i}\int_{-i\infty-5/8}^{+i\infty-5/8}\frac{dz_1}{2sV_2}\left[(F_2+F_4)\Gamma[1+z_1]+(F_1+F_5)[z_1]\Gamma[1+z_1]\mathrm{PolyGamma}[1+\mathrm{z}_1]+rest\right.$$

$$\begin{aligned}
F_1 &= (-t)^{-1-z_1}\frac{\Gamma[-z_1]^3}{\Gamma[-2z_1]} \\
F_2 &= F_1(\gamma_E - 2\ln[-s] - \ln[-t] + 2\ln[-V_4]) \\
F_4 &= 2F_1(-\gamma_E + \ln[-s] + \ln[-t] - \ln[-V_2] - \ln[-V_4]) \\
F_5 &= -2F_1
\end{aligned} \qquad (9)$$

IR-divergencies as inverse binomial sums

Now take the residues and get:

$$\frac{I_{-1}}{\epsilon} = \frac{1}{2sV_2\epsilon}\frac{1}{2\pi i}\int_{-i\infty+u}^{+i\infty+u} dr(-t)^{-1-r}\ \frac{\Gamma[-r]^3\Gamma[1+r]}{\Gamma[-2r]}.$$

With Mathematica or using Kalmykov et al. or Huber & Maitre:

$$\frac{I_{-1}}{\epsilon} = \frac{1}{2sV_2\epsilon}\sum_{n=0}^{\infty}\frac{(t)^n}{\binom{2n}{n}(2n+1)} = \frac{4\arcsin(\sqrt{t/2})}{V_2\sqrt{4-t}\sqrt{t}} = -\frac{2y\ln(y)}{V_2(1-y^2)},$$

with

$$y \equiv y(t) = \frac{\sqrt{1-4/t}-1}{\sqrt{1-4/t}+1}.$$

and for the constant term in ϵ:

$$I_0 = \frac{1}{2sV_2}\sum_{n=0}^{\infty}\frac{(t)^n}{\binom{2n}{n}(2n+1)}[-2\ln[-V_2]-3S_1[n]+2S_1[2n]] \to \text{Polylogs, HPL's}$$

Rewrite into Polylogs and/or Harmonic PolyLogs

The inverse binomial sums may be summed:

See Davyvdychev,Kalmykov and quite recently also Huber,Maitre.

Here, the following question is of some interest:

→ Why these harmonic numbers?

Look at intermediate 11 MB-integrals, e.g.:

One of the 4 contributing MB-integrals – out of the 11 – is Int07:

$$\begin{aligned}\text{Int07} &= \text{Sum of residues} \\ &= \frac{e^{\epsilon\gamma_E}\epsilon\sqrt{\pi}(-s)^{-1-2\epsilon}(-V_2)^{2\epsilon}}{2^{2\epsilon}V_4} \\ &\quad \frac{\Gamma[3/2+\epsilon]\Gamma[-2\epsilon]\Gamma[2\epsilon]\Gamma[1+2\epsilon]}{\Gamma[3/2+\epsilon]} \\ &\quad \text{HypergeometricPFQ}[[1,1+2\epsilon],[3/2+\epsilon],\text{t}/4]\end{aligned}$$

Without taking the sum:

$$\begin{aligned}
\mathrm{Int07} &= \text{Sum of residues} \\
&= \frac{e^{\epsilon\gamma_E}(-s)^{-1-2\epsilon}(-V_2)^{2\epsilon}}{V_4}\Gamma[-2\epsilon]\Gamma[1+2\epsilon] \\
& \quad \sum_{n=1}^{\infty} t^{-1+n}\frac{\Gamma[\epsilon+n]\Gamma[2\epsilon+n]}{\Gamma[2\epsilon+2n]} \qquad (10)
\end{aligned}$$

The well-known formula (Weinzierl 0402131 eq. 35 and maybe many others)

$$\Gamma[n+1+\epsilon] = \Gamma[1+\epsilon]\Gamma[1+n]e^{-\sum_{k=1}^{\infty}\frac{(-\epsilon)^k}{k}\mathrm{HarmonicNumber[n,k]}}$$

shows why we meet the inverse harmonic sums with the harmonic numbers $S_1[n]$ **and** $S_1[2n]$**.**

Summary

- We present a general algorithm for the evaluation of mixed IR-divergencies from virtual and real emission in terms of inverse binomial sums.
- With AMBRE.m (May 2007) and MB.m (2005) and maybe in more complicated situations also with HypExp 2 on Expanding Hypergeometric Functions about Half-Integer Parameters, arXiv:0708.2443 [hep-ph] this may be automatized.
- The cases of more masses or more legs or more loops or of tensor integrals should not get much more complicated. But: Take sums for e.g. a massive 2-loop case ...
- For relatively simple applications like IR-divergent parts, an analytical treatment with MB-integrals may be quite useful.