Evaluation of Feynman Integrals: Advanced Methods

Getting Ready for Physics at the LHC

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exercises part: Janusz Gluza, Katowice

The slides of lecture

and

the files for the exercises

=>

`http://www-zeuthen.desy.de/~riemann/Talks/RECAPP-2009`

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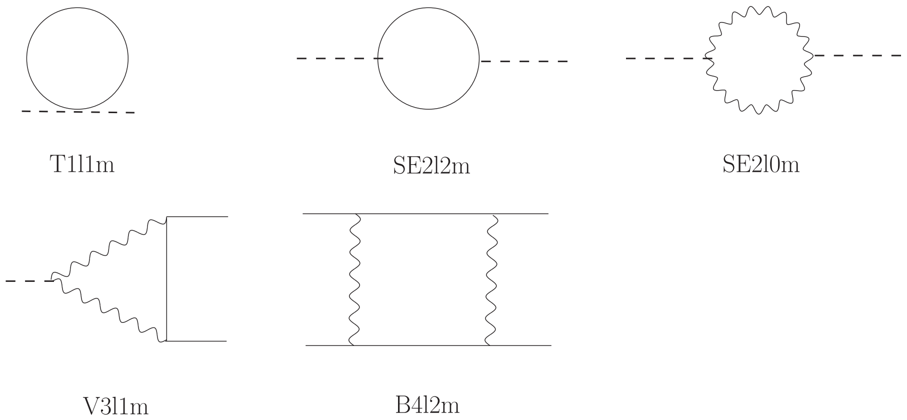
Introductory remarks

For many problems of the past, a relatively simple approach to the evaluation of Feynman integrals was sufficient.

At most 1-loop (massless: 2-loop), typically $2 \longrightarrow 2$ scattering (plus bremsstrahlung)



$$\texttt{T1l1m} = \frac{1}{\epsilon} + 1 + (1 + \frac{\zeta_2}{2})\epsilon + (1 + \frac{\zeta_2}{2} - \frac{\zeta_3}{3})\epsilon^2 +$$

$$\texttt{B4l2m} = [-\frac{1}{\epsilon} + \ln(-s)]\frac{2y\ln(y)}{s(1-y^2)} + c_1\epsilon + \cdots$$

with $d = 4 - 2\epsilon$ and $m = 1$ and

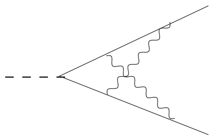
$$y = \frac{\sqrt{1-4/t}-1}{\sqrt{1-4/t}+1}$$

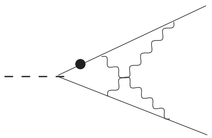
Figure shows so-called master integrals.

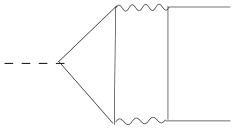
Feynman parameters may be used and by direct integration over them one gets things like: $\frac{23}{57}$, $\ln\frac{t}{s}$, $\ln\frac{t}{s}\ln\frac{m^2}{s}$, $Li_2(\frac{t}{s})$ etc.

With more complexity of the reaction (more legs) and more perturbative accuracy (more loops), this approach appears to be not sufficiently sophisticated.

More loops

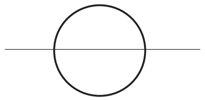

V6l4m1

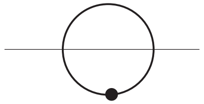

V6l4m1d

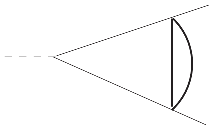

V6l4m2

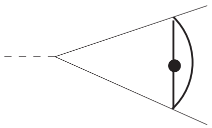
Two-loop vertex integrals with six internal lines

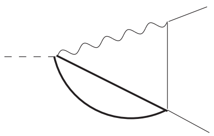
massless case: only fixed numbers and one scale factor

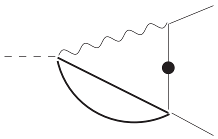

SE3l2M1m

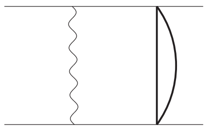

SE3l2M1md

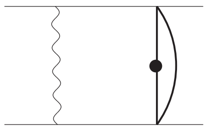

V4l2M2m


V4l2M2md

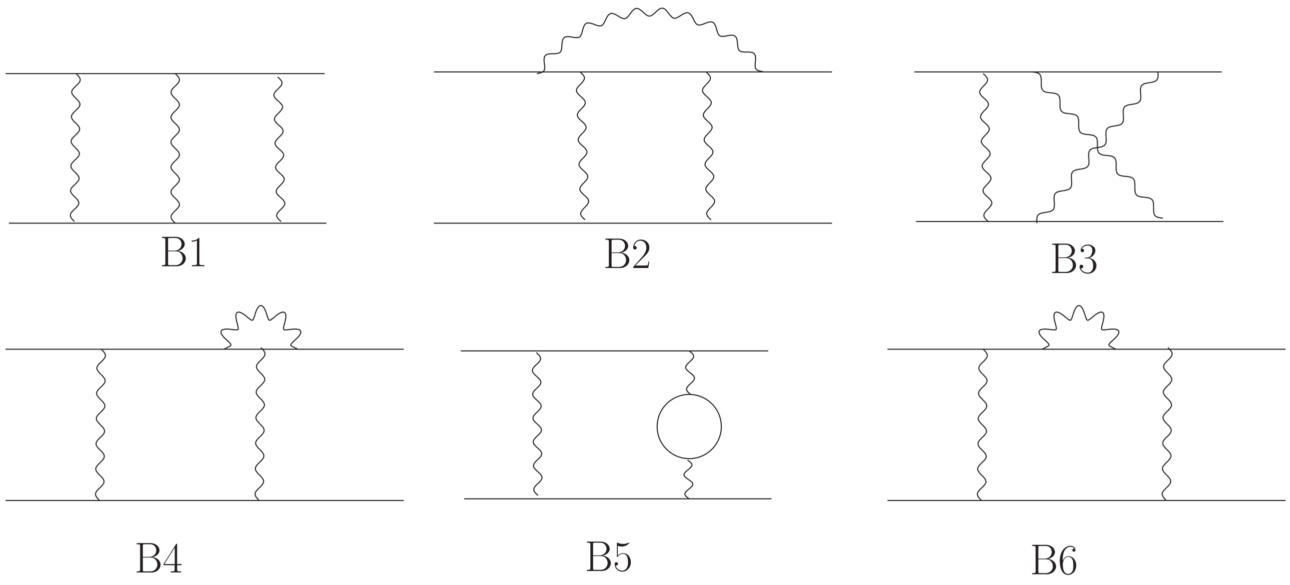

V4l2M1m


V4l2M1md

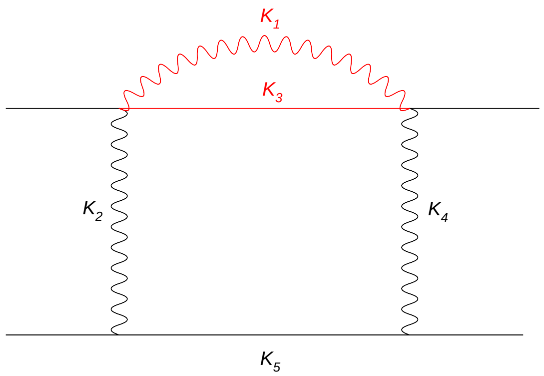

B5l2M2md


B5l2M2m

Integrals with two different mass scales m and M

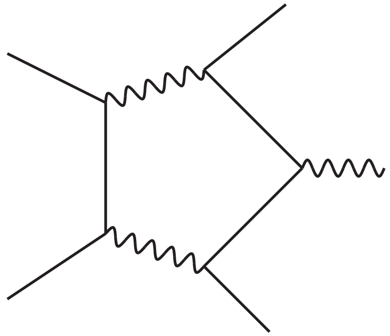


Two-loop box diagrams for massive $2 \to 2$ scattering

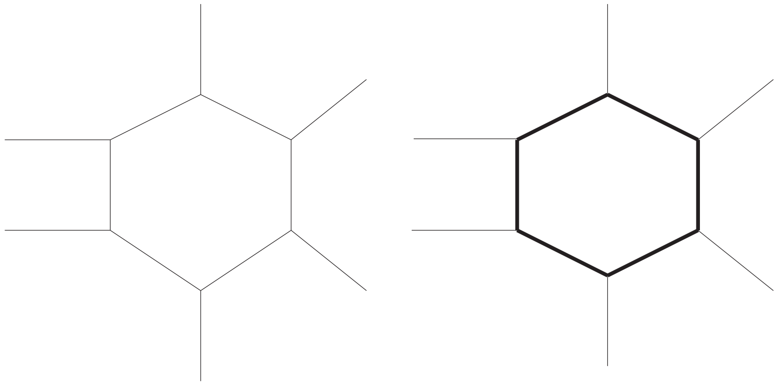


A box master integral B5l2m2, related to B2 = B7l2m2 by shrinking two lines

More legs



Massive pentagon: 5 kinematic variables + several masses



Massless and massive hexagons: 8 kinematic variables + several masses

Variables for $2 \to 2$ **scattering, i.e. box diagrams:** s, t **or** s **and** $\cos\theta$

Variables for $2 \to 3$ **scattering: 5 = 2 + 3** (three additional momenta of a particle)

Variables for $2 \to 4$ **scattering: 8 = 5 + 3** (another three additional)

What are "Advanced Methods?"

- High degree of automatization
- Publicly available – or not . . .
- Go beyond the methods a la Passarino-Veltman and the relatively simple polylogs
- Not yet completely worked out
- Allow the solution of "Advanced Problems"

L-loop n-point Feynman Integrals of tensor rank R with N internal lines

- **Internal loop momenta are** $k_l, \quad l = 1 \cdots L$
- **Propagators have mass** m_i **and momentum** $q_i, \quad i = 1 \cdots N$ **and indices** ν_i **– see** $G(X)$
- **External legs have momentum** $p_e, \quad e = 1 \cdots n$**, with** $p_e^2 = M_e^2$

The N propagators are:

$$D_i \quad = \quad q_i^2 - m_i^2 \quad = \quad [\sum_{l=1}^{L} c_i^l k_l + \sum_{e=1}^{n} d_i^e p_e]^2 - m_i^2$$

Feynman integrals have the following general form:

$$G(X) \quad = \quad \frac{e^{\epsilon \gamma_E L}}{(i\pi^{d/2})^L} \int \frac{d^d k_1 \ldots d^d k_L \quad X(k_{l_1}, \cdots, k_{l_R})}{D_1^{\nu_1} \ldots D_i^{\nu_i} \ldots D_N^{\nu_N}}.$$

The numerator X may contain a tensor structure (see later for more on that):

$$X(k_{l_1}, \cdots, k_{l_R}) \;=\; (k_{l_1} P_{e_1}) \cdots (k_{l_R} P_{e_R}) \;=\; (P_{e_1}^{\alpha_1} \cdots P_{e_R}^{\alpha_R}) \; (k_{l_1}^{\alpha_1} \cdots k_{l_R}^{\alpha_R})$$

Tensor integrals

Tensor integrals appear naturally in Feynman diagrams, due to

- **fermion propagators**
- **non-abelian triple-boson vertices**
- **boson propagators in R_ξ gauges and unitary gauge**

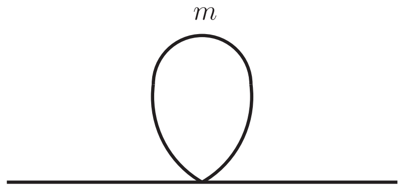
Example: Fermionic vacuum polarization

$$\Pi^{\alpha\beta} \sim \frac{1}{(i\pi^{d/2})} \int d^d k \mathrm{Tr} \left[\frac{[\gamma k + m_1]}{D_1} \gamma^\beta \frac{[\gamma(k+p_1)+m_2]}{D_2} \gamma^\alpha \right]$$

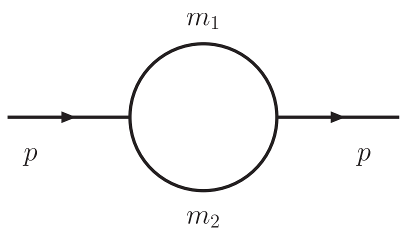
$$\sim \frac{1}{(i\pi^{d/2})} \int \frac{d^d k}{D_1 D_2} \left[(m_1 m_2 - k^2 - k p_1) g^{\alpha\beta} + 2k^\alpha k^\beta + k^\alpha p_1^\beta + p_1^\alpha k^\beta \right]$$

So, one needs also efficient ways to evaluate tensor integrals – see later

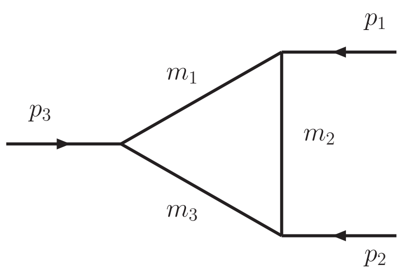
Simple examples of scalar integrals



$$A_0 = \frac{1}{(i\pi^{d/2})} \int \frac{d^d k}{D_1} \quad \rightarrow \mathrm{UV-divergent:} \quad \sim \frac{d^4 k}{k^2}$$



$$B_0 = \frac{1}{(i\pi^{d/2})} \int \frac{d^d k}{D_1 D_2} \quad \rightarrow \mathrm{UV-divergent} \quad \sim \frac{d^4 k}{k^4}$$



$$C_0 = \frac{1}{(i\pi^{d/2})} \int \frac{d^d k}{D_1 D_2 D_3} \quad \rightarrow \mathrm{UV-finite} \quad \sim \frac{d^4 k}{k^6}$$

Dependent on conventions, where k starts to run in the loop, it is:

$$\begin{aligned} D_1 &= k^2 - m_1^2 \\ D_2 &= (k+p_1)^2 - m_2^2 \\ D_3 &= (k+p_1+p_2)^2 - m_3^2 \end{aligned}$$

Evaluate Feynman integrals

There are two strategies to solve a Feynman integral:

- Reduction
  Express the integral with the aid of recurrence relations by other, known integrals.
  These are then the Master Integrals.
- Direct evaluation

Introduce Feynman parameters

$$\frac{1}{D_1^{\nu_1} D_2^{\nu_2} \ldots D_N^{\nu_N}} = \frac{\Gamma(\nu_1 + \ldots + \nu_N)}{\Gamma(\nu_1) \ldots \Gamma(\nu_N)} \int_0^1 dx_1 \ldots \int_0^1 dx_N \frac{x_1^{\nu_1 - 1} \ldots x_N^{\nu_N - 1} \delta(1 - x_1 \ldots - x_N)}{(x_1 D_1 + \ldots + x_N D_N)^{N_\nu}},$$

with $N_\nu = \nu_1 + \ldots \nu_N$.

The denominator of G contains, after introduction of Feynman parameters x_i, the momentum dependent function m^2 with index-exponent N_ν:

$$(m^2)^{-(\nu_1 + \ldots + \nu_N)} = (x_1 D_1 + \ldots + x_N D_N)^{-N_\nu} = (k_i M_{ij} k_j - 2 Q_j k_j + J)^{-N_\nu}$$

Here M is an (LxL)-matrix, $Q = Q(x_i, p_e)$ an L-vector and $J = J(x_i x_j, m_i^2, p_{e_j} p_{e_l})$.
M, Q, J are linear in x_i. The momentum integration is now simple:
Shift the momenta k such that m^2 has no linear term in $\bar{k}$:

$$\begin{aligned} k &= \bar{k} + (M^{-1}) Q, \\ m^2 &= \bar{k} M \bar{k} - Q M^{-1} Q + J. \end{aligned}$$

Remember: $M_{1-\text{loop}} = 1$, in general:

$$M^{-1} = \frac{1}{(\text{det } M)} \tilde{M},$$

where $\tilde{M}$ is the transposed matrix to M. The shift leaves the integral unchanged.

The shift leaves the integral unchanged (rename $\bar{k} \to k$):

$$G(1) = \int \frac{Dk_1 \ldots Dk_L}{(kMk + J - QM^{-1}Q)^{N_\nu}}.$$

Go Euclidean: Rotate now the $k^0 \to iK_E^0$ with $k^2 \to -k_E^2$ (and again rename $k^E \to k$):

$$G(1) \to (i)^L \int \frac{Dk_1^E \ldots Dk_L^E}{(-k^E M k^E + J - QM^{-1}Q)^{N_\nu}} = (-1)^{N_\nu} (i)^L \int \frac{Dk_1 \ldots Dk_L}{[kMk - (J - QM^{-1}Q)]^{N_\nu}}.$$

Call

$$\mu^2(x) = -(J - QM^{-1}Q)$$

and get

$$G(1) = (-1)^{N_\nu} (i)^L \int \frac{Dk_1 \ldots Dk_L}{(kMk + \mu^2)^{N_\nu}}.$$

For 1-loop integrals it is $L = 1, M = 1$ - and we will use nearly only those - we are ready to do the k-integration.

Additional step for L-loop integrals

For L-loops go on and now diagonalize the matrix M by a rotation:

$$\begin{aligned} k \to k'(x) &= V(x)\ k, \\ kMk &= k' M_{diag} k' \\ &\to \sum \alpha_i(x) k_i^2(x), \\ M_{diag}(x) &= (V^{-1})^{+} M V^{-1} \quad = \ (\alpha_1, \ldots, \alpha_L). \end{aligned}$$

This leaves both the integration measure and the integral invariant:

$$G(1) = (-1)^{N_\nu} (i)^L \int \frac{Dk_1 \ldots Dk_L}{\left(\sum_i \alpha_i k_i^2 + \mu^2\right)^{N_\nu}}.$$

Rescale now the k_i,

$$\bar{k}_i = \sqrt{\alpha_i} k_i,$$

with

$$\begin{aligned} d^d k_i &= (\alpha_i)^{-d/2} d^d \bar{k}_i, \\ \prod_{i=1}^{L} \alpha_i &= \text{det } M, \end{aligned}$$

and get the Euclidean integral to be calculated (and rename $\bar{k} \to k$):

$$G(1) = (-1)^{N_\nu}(i)^L(\det M)^{-d/2}\int \frac{Dk_1 \ldots Dk_L}{(k_1^2+\ldots+k_L^2+\mu^2)^{N_\nu}}.$$

Use now (remembering that $Dk = dk/(i\pi^{d/2})$):

$$i^L \int \frac{Dk_1 \ldots Dk_L}{(k_1^2+\ldots+k_L^2+\mu^2)^{N_\nu}} = \frac{\Gamma\left(N_\nu - \frac{d}{2}L\right)}{\Gamma\left(N_\nu\right)} \frac{1}{(\mu^2)^{N_\nu - dL/2}},$$

$$i^L \int \frac{Dk_1 \ldots Dk_L \; k_1^2}{(k_1^2+\ldots+k_L^2+\mu^2)^{N_\nu}} = \frac{d}{2}\,\frac{\Gamma\left(N_\nu - \frac{d}{2}L - 1\right)}{\Gamma\left(N_\nu\right)} \frac{1}{(\mu^2)^{N_\nu - dL/2-1}}.$$

These formulae follow for $L = 1$ immediately from any textbook.

See 'Mathematical Interlude'.

For $L > 1$, get it iteratively, with setting $(k_1^2+k_2^2+m^2)^N = (k_1^2+M^2)^N$, $M^2 = k_2^2+m^2$, etc.

Mathematical interlude: d-dimensional integrals (I)

After the Wick rotation, the integrand of the momentum integration is positive definite.
Further it is independent of the angular variables.
The integral is understood as symmetric limit the infinity boundaries.

$$\int d^dk \; k_\mu \; F(k^2) = 0$$

$$\int d^dk \; F(k+C) = \int d^dk \; F(k).$$

Introduce d-dim. spherical coordinates. The vector k has d components:

$$
\begin{aligned}
k_d &= r\cos\theta_d \equiv \rho_d \cos\theta_d \\
k_{d-1} &= \rho_{d-1}\cos\theta_{d-1} \\
&\dots \\
k_3 &= \rho_3\cos\theta_3 \\
k_2 &= \rho_2\sin\phi \\
k_1 &= \rho_2\cos\phi \\
\rho_{d-1} &= \rho_d\sin\theta_d
\end{aligned}
$$

Mathematical interlude: d-dimensional integrals (II)

The above is the direct generalization of the 3- or 4-dimensional phase space parametrization.

With these variables, the integral over the complete d-dimensional phase space gets the following form:

$$\int_{-\infty}^{\infty} d^d k \; F(k) = \lim_{R\to\infty}\int_0^R dr r^{d-1}\int_0^\pi d\theta_{d-1}\sin^{d-2}\theta_{d-1} \int_0^\pi d\theta_{d-2}\sin^{d-3}\theta_{d-2}\ldots\int_0^{2\pi} d\theta_1 F(k)$$

The integrations met in the loop calculations may be performed using the following two integrals:

$$\int_0^\pi d\theta \sin^m\theta = \sqrt{\pi}\frac{\Gamma\left[\frac{1}{2}(m+1)\right]}{\Gamma\left[\frac{1}{2}(m+2)\right]},$$

$$\int_0^\infty dr \frac{r^\beta}{(r^2+M^2)^\alpha} = \frac{1}{2}\frac{\Gamma\left(\frac{\beta+1}{2}\right)\Gamma\left(\alpha-\frac{\beta+1}{2}\right)}{\Gamma(\alpha)}\frac{1}{(M^2)^{\alpha-(\beta+1)/2}}.$$

In general, the angular integrations are influenced by the integrand too. (Remember phase space integrals of bremsstrahlung!)

Mathematical interlude: d-dimensional integrals (III)

If $F(k) \to F(r),\ r = |k|$, the angular integrations yield the surface of the d-dimensional sphere with radius r:

$$\omega_d(r) = \frac{2\pi^{d/2}}{\Gamma\left[\frac{d}{2}\right]} r^{d-1}.$$

The remaining integration, over r, yields for $F(r) = 1$ the volume of the sphere with radius R:

$$V_d(R) = \frac{\pi^{d/2}}{\Gamma\left[1 + \frac{d}{2}\right]} R^d,$$

$$G(1) = \int d^dk \frac{1}{\left(k^2 + M^2\right)^{N_\nu}} = \int_0^\infty dr \frac{\omega_d(r)}{(r^2 + M^2)^{N_\nu}}$$

and we get immediately, with $M^2 \equiv M^2(x_1, x_2, \ldots)$:

$$G(1) = \left[\frac{i\pi^{d/2}\Gamma\left(N_\nu - d/2\right)}{\Gamma(N_\nu)} \frac{1}{\left(M^2\right)^{N_\nu - d/2}}\right].$$

The Γ-function

The Γ-function may be defined by a difference equation:

$$z\Gamma(z) - \Gamma(z+1) = 0$$

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

Series[Gamma[ep], {ep, 0, 2}] =

$$\Gamma[\epsilon] = \frac{1}{\epsilon} - \gamma_E + \frac{1}{12}(6{\gamma_E}^2 + \pi^2)\epsilon + \frac{1}{12}(-2{\gamma_E}^3 - {\gamma_E}^2\pi + 2\Psi(2,1))\epsilon^2 + \cdots$$

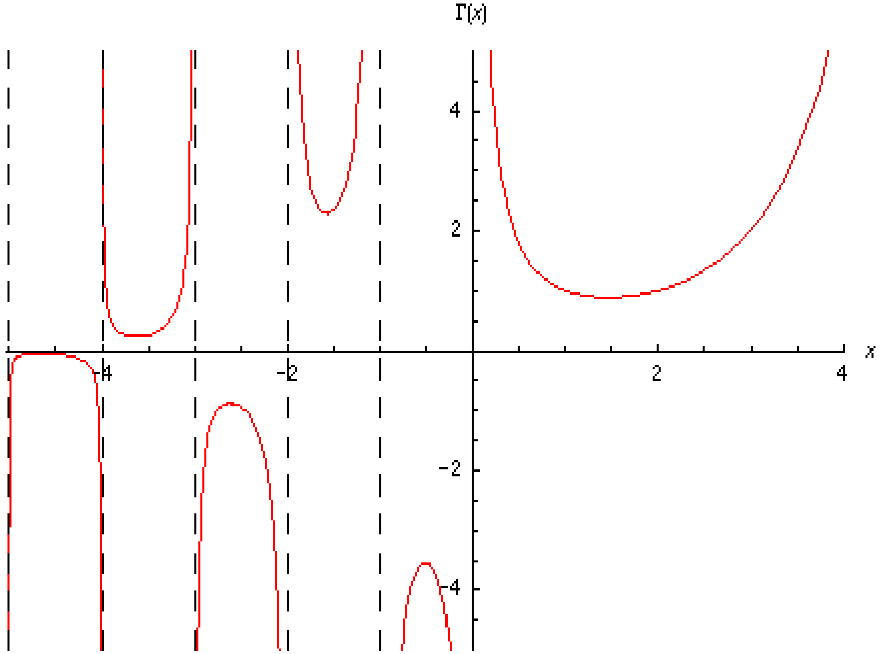
$\Psi(2,1)) = PolyGamma(2,1) = -2\zeta_3$

exp(ep EulerGamma)Series[Gamma[ep], {ep, 0, 2}] =

$$e^{\epsilon\gamma_E}\Gamma[\epsilon] = \frac{1}{\epsilon} + \frac{1}{12}(\pi^2)\epsilon + \frac{1}{6}(\Psi(2,1))\epsilon^2 + \cdots$$

Look at the singularities in the complex plane (figure shows real part of Γ):

When applying the Cauchy-theorem, one may close the integral to the left or to the right ...



Mathematical interlude: d-dimensional integrals (IV)

We will often use:

$$a^{\epsilon} = e^{\epsilon \ln a} = 1 + \epsilon \ln a + \ldots$$

Finally, one gets for **Scalar integrals:**

$$G(1) \quad = \quad (-1)^{N_\nu} \frac{\Gamma\left(N_\nu - \frac{d}{2}L\right)}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)} \int_0^1 \prod_{j=1}^{N} dx_j \; x_j^{\nu_j-1} \delta\left(1 - \sum_{i=1}^{N} x_i\right) \frac{(\text{det } M)^{-d/2}}{(\mu^2)^{N_\nu - dL/2}},$$

or

$$G(1) \quad = \quad (-1)^{N_\nu} \frac{\Gamma\left(N_\nu - \frac{d}{2}L\right)}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)} \int_0^1 \prod_{j=1}^{N} dx_j \; x_j^{\nu_j-1} \delta\left(1 - \sum_{i=1}^{N} x_i\right) \frac{U(x)^{N_\nu - d(L+1)/2}}{F(x)^{N_\nu - dL/2}}$$

with

$$U(x) \quad = \quad (\text{det } M) \; (\rightarrow 1 \text{ for } L = 1)$$

$$F(x) \quad = \quad (\text{det } M) \; \mu^2 \quad = \quad -(\text{det } M) \; J + Q \; \tilde{M} \; Q \; (\rightarrow -J + Q^2 \text{ for } L = 1)$$

Trick for one-loop functions:
$U = \text{det } M = 1 = \sum x_i$ and so U 'disaapears' and the construct $F_1(x)$ is bilinear in $x_i x_j$:

$$F_1(x) = -J(\sum x_i) + Q^2 = \sum A_{ij} x_i x_j.$$

The vector integral differs by some numerator $k_i p_e$ and thus there is a single shift in the integrand

$$k \to \bar{k} + U(x)^{-1}\tilde{M}Q$$

the $\int d^d\bar{k}\ \bar{k}/(\bar{k}^2+\mu^2) \to 0$, and no further changes:

$$G(k_{1\alpha}) = (-1)^{N_\nu} \frac{\Gamma\left(N_\nu - \frac{d}{2}L\right)}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)} \int_0^1 \prod_{j=1}^N dx_j\ x_j^{\nu_j-1} \delta\left(1-\sum_{i=1}^N x_i\right) \frac{U(x)^{N_\nu - d(L+1)/2-1}}{F(x)^{N_\nu - dL/2}} \left[\sum_l \tilde{M}_{1l}Q_l\right]_\alpha ,$$

Here also a tensor integral:

$$\begin{aligned} G(k_{1\alpha}k_{2\beta}) &= (-1)^{N_\nu} \frac{\Gamma\left(N_\nu - \frac{d}{2}L\right)}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)} \int_0^1 \prod_{j=1}^N dx_j\ x_j^{\nu_j-1} \delta\left(1-\sum_{i=1}^N x_i\right) \frac{U(x)^{N_\nu-2-d(L+1)/2}}{F(x)^{N_\nu-dL/2}} \\ &\times \sum_l \left[[\tilde{M}_{1l}Q_l]_\alpha [\tilde{M}_{2l}Q_l]_\beta - \frac{\Gamma\left(N_\nu - \frac{d}{2}L - 1\right)}{\Gamma\left(N_\nu - \frac{d}{2}L\right)} \frac{g_{\alpha\beta}}{2} U(x)F(x) \frac{(V_{1l}^{-1})^+(V_{2l}^{-1})}{\alpha_l} \right]. \end{aligned}$$

The 1-loop case will be used in the following L times for a sequential treatment of an L-loop integral (remember $\sum x_j D_j = k^2 - 2Qk + J$ and $F(x) = Q^2 - J$):

$$G([1, kp_e]) = (-1)^{N_\nu} \frac{\Gamma\left(N_\nu - \frac{d}{2}\right)}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)} \int_0^1 \prod_{j=1}^N dx_j\ x_j^{\nu_j-1} \delta\left(1-\sum_{i=1}^N x_i\right) \frac{[1, Qp_e]}{F(x)^{N_\nu-d/2}}$$

Examples for one-loop F-polynomials

One-loop vertex:

$$F(t,m^2) = m^2(x_1+x_2)^2 + [-t]x_1x_2$$

one-loop box:

$$F(s,t,m^2) = m^2(x_1+x_2)^2 + [-t]x_1x_2 + [-s]x_3x_4$$

one-loop pentagon:

$$F(s,t,t',v_1,v_2,m^2) = m^2(x_1+x_3+x_4)^2 + [-t]x_1x_3 + [-t']x_1x_4 + [-s]x_2x_5 + [-v_1]x_3x_5 + [-v_2]x_2x_4$$

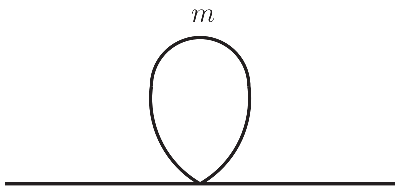
2-loop example: B7l4m2, has a box-type sub-loop with 2 off-shell legs:

$$F^{-(a_{4567}-d/2)} = \Big\{[-t]x_4x_7 + [-s]x_5x_6 + m^2(x_5+x_6)^2 + (m^2-Q_1^2)x_7(x_4+2x_5+x_6) + (m^2-Q_2^2)x_7x_5\Big\}^{-(a_{4567}-d/2)}$$

2-loop: B5l2m2, sub-loop with 2 off-shell legs (diagram see p.4):

$$F_{2lines}(k_1^2,m^2) = m^2(x_3)^2 + [-k_1^2+m^2]x_1x_3$$

The Tadpole $A_0(m)$



$$T1l1m[a] \;=\; A_0 \;=\; \frac{e^{\epsilon\gamma_E}}{(i\pi^{d/2})}\int\frac{d^dk}{(k^2-m^2)^a} \qquad \to \mathrm{UV-divergent}$$

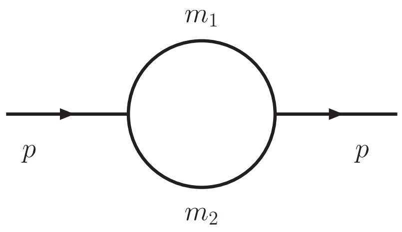
With our general formulae we get, in the 1-dimensional Feynman parameter integral, for the numerator

$$\begin{aligned} N &= (k^2-m^2)x_1 \equiv k^2+J \\ F &= m^2x_1 \equiv m^2x_1^2 \end{aligned}$$

and thus

$$\begin{aligned} T1l1m[a] &= (-1)^a e^{\epsilon\gamma_E}\frac{\Gamma[a-d/2]}{\Gamma[a]}\int_0^1 dx x^{a-1}\delta[1-x]\frac{1}{F^{a-d/2}} \\ &= (-1)^a e^{\epsilon\gamma_E}(m^2)^{2-a-\epsilon}\frac{\Gamma[a-2+\epsilon]}{\Gamma[a]} \\ &\to -e^{\epsilon\gamma_E}\Gamma[-1+\epsilon] \quad \mathrm{for\ a=1,\ m=1} \\ &= \frac{1}{\epsilon}+1+\left(1+\frac{\zeta_2}{2}\right)\epsilon+\left(1+\frac{\zeta_2}{2}-\frac{\zeta_3}{3}\right)\epsilon^2+\cdots \end{aligned}$$

The Self-energy $B_0(s, m_1, m_2)$



$$SE2l = B_0[s, m_1, m_2] = (2\sqrt{\pi}\mu)^{4-d} \frac{e^{\epsilon\gamma_E}}{(i\pi^{d/2})} \int \frac{d^dk}{[k^2 - m^2][(k+p)^2 - m_2^2]}$$

The $SE2l$ is UV-divergent and the corresponding F-function is:

$$F[s, m_1, m_2] = m_1^2 x_1^2 + m_2^2 x_2^2 - [s - m_1^2 - m_2^2] x_1 x_2$$

and for special cases:

$$\begin{aligned} F[s, m_1, 0] &= m_1^2 x_1^2 - [s - m_1^2] x_1 x_2 \\ F[s, m_1, m_1] &= m_1^2 (x_1 + x_2)^2 - [s] x_1 x_2 \\ F[s, 0, 0] &= -[s] x_1 x_2 \end{aligned}$$

The 'conventional' Feynman parameter integral is 1-dimensional because $x_2 \equiv 1 - x_1$:

$$F(x) = -sx(1-x) + m_2^2(1-x) + m_1^2 x \equiv -s(x - x_a)(x - x_b)$$

The result is of logarithmic type for the constant term in ϵ:

$$\begin{aligned} B_0[s,m_1,m_2] &= (4\pi\mu^2)^\epsilon \; e^{\epsilon\gamma_E} \; \frac{\Gamma(1+\epsilon)}{\epsilon} \int_0^1 \frac{dx}{F(x)^\epsilon} \\ &= \frac{1}{\epsilon} - \int_0^1 dx \ln\left(\frac{F(x)}{4\pi\mu^2}\right) \\ &\quad + \epsilon \; \left\{ \frac{\zeta_2}{2} + \frac{1}{2}\int_0^1 dx \ln^2\left(\frac{F(x)}{4\pi\mu^2}\right) \right\} + \mathcal{O}(\epsilon^2). \end{aligned}$$

Here we used the expansion:

$$e^{\epsilon\gamma_E}\Gamma(1+\epsilon) = 1 + \frac{\zeta_2}{2}\epsilon^2 - \frac{\zeta_3}{3}\epsilon^3 \cdots$$

When using `LoopTools`**, the corresponding call returns exactly the constant term of B_0 in ϵ (with use of $e^{\epsilon\gamma_E} = 1 + \epsilon\gamma_E + \cdots \to 1$):**

$$B_0^{(0)}(s, m_1^2, m_2^2) \quad = \quad \mathtt{b0}(\mathtt{s}, \mathtt{am12}, \mathtt{am22})$$

For $4\pi\mu^2 \to 1$ B_0 looks quite compact:

$$B_0(s,m_1,m_2) \quad = \quad \frac{1}{\epsilon} - \int_0^1 dx \;\; \ln[F(x)] + \frac{\epsilon}{2}\left[\zeta_2 + \;\; \int_0^1 dx \ln^2[F(x)]\right] + \cdots$$

Explicitly, one has to integrate

$$\ln[F(x)] = \ln[-s(x-x_a)(x-x_b)]$$
$$\ln^2[F(x)] = \ln^2[-s(x-x_a)(x-x_b)]$$

So we will need the integrals:

$$\int dx_0^1 \{\ln(x-x_a),\ ln(x-x_a)ln(x-x_b)\}$$

which is trivial, together with some complex algebra rules how to handle complex arguments of logarithms with

$$s \to s + i\epsilon$$

wherever needed.

For the case $m_1 = m_2 = 1$, one gets for the first terms in ϵ:

$$
\begin{aligned}
B_0[s,1,1] &= \frac{1}{\epsilon} + 2 + \frac{1+y}{1-y} H(0,y), \\
H(0,y) &= \ln(y).
\end{aligned}
$$

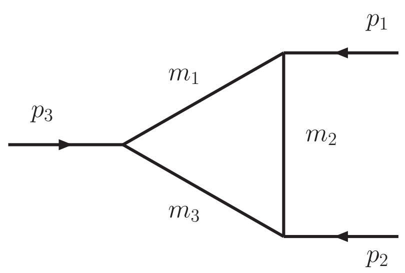
The $H(0,y)$ is a harmonic polylogarithmic function, and

$$
\begin{aligned}
y &= \frac{\sqrt{-s+4} - \sqrt{-s}}{\sqrt{-s+4} + \sqrt{-s}} \\
s &= -\frac{(1-x)^2}{x}
\end{aligned}
$$

The other case treated later again is $m_1 = 0, m_2 = m$:

$$
B_0[s,m^2,0] = \frac{1}{\epsilon} + 2 + \frac{1 - s/m^2}{s/m^2} \ \ln(1 - s/m^2)
$$

The massive one-loop vertex $C_0(s, m_1, m_2)$



$$C_0 = \frac{e^{\epsilon\gamma_E}}{(i\pi^{d/2})} \int \frac{d^d k}{[(k+p_1)^2 - m^2][k^2][(k-p_2)^2 - m^2]} \sim |_{k\to\infty} \frac{d^4 k}{k^6} \to \mathrm{UV-fin.}$$

The massive vertex (all $m_1, m_2, m_3 \neq 0$) is a finite quantity.

We assume immediately $m_1 = m_3 = 0$.

A problem now is IR-divergence.

Appears when a massive internal line is between two external on-shell lines.

Incoming $p_1^2 = m^2$ and $p_2^2 = m^2$, look at $k \to 0$:

$$\begin{aligned} & d^4k \frac{1}{(k-p_2)^2 - m^2} \frac{1}{(k)^2} \frac{1}{(k+p_1)^2 - m^2} \\ = \; & d^4k \frac{1}{k^2 - 2kp_2} \frac{1}{(k)^2} \frac{1}{k^2 + 2kp_1} \\ \to \; & \frac{d^4k}{k^{1+2+1}} \sim \frac{k^3 dk}{k^4} \sim \frac{dk}{k}|_{k\to 0} \longrightarrow div \end{aligned}$$

An IR-regularization is needed, must take $d > 4$.

Both UV-div (with $d < 4$) and IR-div together: must allow for a complex $d = 4 - 2\epsilon$, and take limit at the end.

First we have a look, for later use, at the F-function:

$$\begin{aligned} N &= D_1x + D_2y + D_3z \\ &= k^2x + (k^2 + 2kp_1)y + (k^2 - 2kp_2)z \\ &= k^2(x+y+z) + 2k(p_1y - p_2z) \\ &= (k+Q)^2 - Q^2 \end{aligned}$$

We used $1 = x + y + z$ here. And the F-function is $F = Q^2 - J = Q^2$ (there is no constant term in N here), as was shown before:

$$F = m^2(y+z)^2 + [-s]yz$$

This F-function does not factorize in y and z. But now back to the direct Feynman parameter integration.

Start with change $y \to y' = (1-x)y$**, then** $y' \to y$**:**

$$\frac{1}{D_1 D_2 D_3} = \int_0^1 dx dy dz \frac{\delta(1-x-y-z)}{(D_2 x + D_1 y + D_3 z)^3}$$
$$= \int_0^1 dx \int_0^{1-x} \frac{dy}{(D_2 x + D_1 y + D_3 z)^3}$$
$$= \int_0^1 dx \int_0^1 \frac{x dy}{(D_2 x + D_1 y + D_3 z)^3}$$

After this change of variables, the integrand factorizes in x **and** y**:**

$$N = (k + x p_y)^2 - x^2 p_y^2$$
$$= (k+Q)^2 - Q^2$$

resulting into

$$F = Q^2 = x^2 p_y^2$$
$$p_y^2 = -sy(1-y) + m^2$$

For C_0 we obtain (with $N_\nu = 3$ and $N_\nu - d/2 = 1 + \epsilon$):

$$C_0[s,m,m,0] = (-1)e^{\epsilon\gamma_E}\Gamma[1+\epsilon]\int_0^1 \frac{dx}{x^{1+2\epsilon}}\int_0^1 \frac{dy}{(p_y^2)^{1-\epsilon}}$$

This is integrable for $\epsilon < 0$, or $d > 4$, or more general: $d \not< 4$.

The x-integral made simple here:

$$\int_0^1 \frac{dx}{x^{1+2\epsilon}} = \frac{x^{-2\epsilon}|_0^1}{-2\epsilon} = -\frac{1^{-2\epsilon} - 0^{-2\epsilon}}{2\epsilon} = -\frac{1}{2\epsilon}$$

We see that the IR-singularity is an end-point-singularity in Feynman parameter space.

Further:

$$-\frac{1}{2\epsilon}\frac{dy}{(p_y^2)^{1-\epsilon}} = -\frac{1}{2\epsilon}\frac{dy}{(p_y^2)}\ (y^2)^\epsilon = -\frac{1}{2\epsilon}\frac{dy}{(p_y^2)}\ e^{\epsilon\ln(y^2)} = -\frac{1}{2\epsilon}\frac{dy}{(p_y^2)}\left[1+\epsilon\ln(y^2)+\epsilon^2\ln^2(y^2)+\cdots\right]$$

Here I stop this study.

We see that the further integrations proceed quite similar as for the 2-point function, in fact the $p_y^2 = -sy(1-y) + m^2$ is the same building block.
The integrals to be solved now are more general, they include also denominators $1/p_y^2$:

Some integrals

$$\begin{aligned}
\int dy \ln(y-y_0) &= (y-y_0)\ln(y-y_0) - y + C \\
\int dy \frac{1}{y-y_0} &= \ln(y-y_0) + C \\
\int dy \frac{\ln(y-y_0)}{y-y_0} &= \frac{1}{2}\ln^2(y-y_0) + C
\end{aligned}$$

Here, often y is real and y_0 is complex. Then no special care about phases is necessary.

$$\int_0^1 \frac{dx}{x-x_0}\left[\ln(x-x_A) - \ln(x_0-x_A)\right] = Li_2\left(\frac{x_0}{x_0-x_A}\right) - Li_2\left(\frac{x_0-1}{x_0-x_A}\right).$$

This formula is valid if x_0 is real.

C_0 with a small photon mass λ

In

[Berends:1976zp,'tHooft:1979xw]

, the C_0-integral is treated with a finite photon mass:

$$\int \frac{d^4k}{(k^2-\lambda^2)(k^2+2kp_1)(k^2-2kp_2)}$$

$$= -i\pi^2 \int_0^1 dydx \frac{y}{x^2p_y^2+(1-x)\lambda^2}$$

$$= i\pi^2 \int_0^1 dy \left[\frac{1}{2p_y^2}\ln\frac{\lambda^2}{p_y^2} + \mathcal{O}\left(\lambda/\sqrt{p_y^2}\right)\right],$$

It is easy to see from the term $1/(2p_y^2)\ \ln(\lambda^2)$ the correspondence of $(d-4)$ and λ^2, which is a universal relation in all 1-loop cases.

16.02.2009: End of Lecture 1.

Now using Mellin-Barnes Representations

Perform the x-integrations

Find an as-general-as-possible general formula

Make it ready for algorithmic analytical and/or numerical evaluation

Computercodes:

- Ambre.m - Derive Mellin-Barnes representations for Feynman integrals *[Gluza:2007rt]*
- MB.m - Find an ϵ-expansion and evaluate numerically in Euclidean region *[Czakon:2005rk]*

Integrating the Feynman parameters – get MB-Integrals

We derived:

$$SE2l1m = B_0(s,m,0) \quad = \quad e^{\epsilon\gamma_E}\ \Gamma(\epsilon) \int_0^1 dx_1 dx_2 \delta(1-x_1-x_2) \frac{\delta(1-x_1-x_2)}{F(x)^{\epsilon}}$$

$$V3l2m = C_0(s,m,m,0) \quad = \quad e^{\epsilon\gamma_E}\ \Gamma(1+\epsilon) \int_0^1 dx_1 dx_2 dx_3 \frac{\delta(1-x_1-x_2-x_3)}{F(x)^{1+\epsilon}}$$

and

$$F_{SE2l1m} \quad = \quad m^2 x_1^2 - (s-m^2) x_1 x_2$$

$$F_{V3l2m} \quad = \quad m^2 (x_1+x_2)^2 - (s) x_1 x_2$$

We want to apply now:

$$\int_0^1 \prod_{j=1}^N dx_j\ x_j^{\alpha_j-1}\ \delta\left(1-\sum x_i\right) \quad = \quad \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\cdots\Gamma(\alpha_7 N)}{\Gamma\left(\alpha_1+\alpha_2+\cdots+\alpha_N\right)}$$

with coefficients α_i dependent on ν_i and on the structure of the F

See in a minute:
For this, we have to apply one or several MB-integrals here.

$$\int_0^1 \prod_{j=1}^{N} dx_j \ x_j^{\alpha_j-1} \ \delta\left(1-\sum_{i=1}^{N} x_i\right) \quad = \quad \frac{\prod_{i=1}^{N}\Gamma(\alpha_i)}{\Gamma\left(\sum_{i=1}^{N}\alpha_i\right)}$$

Simplest cases:

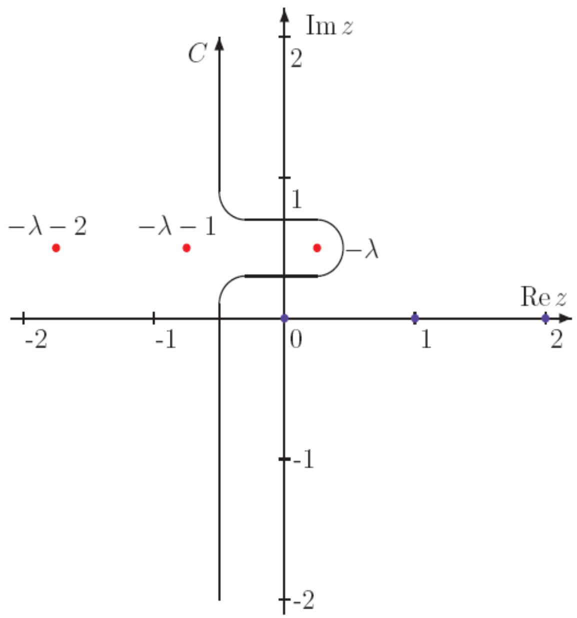
$$\int_0^1 dx_1 \ x_1^{\alpha_1-1} \ \delta\left(1-x_1\right) \quad = \quad 1$$

$$\int_0^1 \prod_{j=1}^{2} dx_j \ x_j^{\alpha_j-1} \ \delta\left(1-\sum_{i=1}^{N} x_i\right) = \int_0^1 dx_1 x_1^{\alpha_1-1}(1-x_1)^{\alpha_2-1} \quad = \quad B(\alpha_1,\alpha_2)$$

$$= \quad \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma\left(\alpha_1+\alpha_2\right)}$$

Here we want to go:

$$\frac{1}{(A+B)^{\lambda}} = \frac{1}{\Gamma(\lambda)}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\Gamma(\lambda+z)\Gamma(-z)\frac{B^z}{A^{\lambda+z}}$$

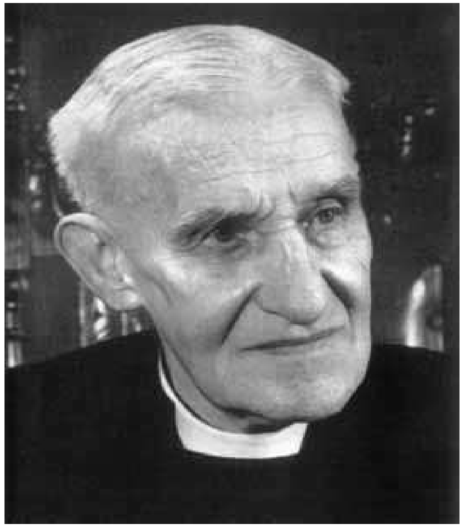


The formula looks a bit unusual to loop people, but for persons with a mathematical background it is common.

One might well assume that these two gentlemen did not dream of so heavy use of their results in basic research $\cdots$

Mellin, Robert, Hjalmar, 1854-1933

Barnes, Ernest, William, 1874-1953



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Barnes' contour integrals for the hypergeometric function

Exact proof and further reading: Whittaker & Watson (CUP 1965) 14.5 - 14.52, pp. 286-290

Consider

$$F(z) \quad = \quad \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} d\sigma(-z)^{\sigma}\frac{\Gamma(a+\sigma)\Gamma(b+\sigma)\Gamma(-\sigma)}{\Gamma(c+\sigma)}$$

where $|\mathrm{arg(-z)}| < \pi$ **(i.e.** $(-z)$ **is not on the neg. real axis) and the path is such that it separates the poles of** $\Gamma(a+\sigma)\Gamma(b+\sigma)$ **from the poles of** $\Gamma(-\sigma)$**.**
$1/\Gamma(c+\sigma)$ **has no pole.**
Assume $a \neq -n$ **and** $b \neq -n, n = 0, 1, 2, \cdots$ **so that the contour can be drawn.**

The poles of $\Gamma(\sigma)$ **are at** $\sigma = -n, n = 1, 2, \cdots$**, and it is:**

```
Residue[ F[s] Gamma[-s] , {s,n} ]  =  (-1)^n/n!  F(n)
```

Closing the path to the right gives then, by Cauchy's theorem, for $|z| < 1$ **the**

hypergeometric function $_2F_1(a,b,c,z)$ **(for proof see textbook):**

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} d\sigma(-z)^{\sigma}\frac{\Gamma(a+\sigma)\Gamma(b+\sigma)\Gamma(-\sigma)}{\Gamma(c+\sigma)} = \sum_{n=0}^{N\to\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\frac{z^n}{n!}$$

$$= \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} \quad {}_2F_1(a,b,c,z)$$

The continuation of the hypergeometric series for $|z|>1$ is made using the intermediate formula

$$F(z) = \sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(1-c+a+n)\sin[(c-a-n)\pi]}{\Gamma(1+n)\Gamma(1-a+b+n)\cos(n\pi)\sin[(b-a-n)\pi]}(-z)^{-a-n}$$

$$+\sum_{n=0}^{\infty}\frac{\Gamma(b+n)\Gamma(1-c+b+n)\sin[(c-b-n)\pi]}{\Gamma(1+n)\Gamma(1-a+b+n)\cos(n\pi)\sin[(a-b-n)\pi]}(-z)^{-b-n}$$

and yields

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} \quad {}_2F_1(a,b,c,z) = \frac{\Gamma(a)\Gamma(a-b)}{\Gamma(a-c)}(-z)^{-a} \quad {}_2F_1(a,1-c+a,1-b+ac,z^{-1})$$

$$+\frac{\Gamma(b)\Gamma(b-a)}{\Gamma(b-c)}(-z)^{-b} \quad {}_2F_1(b,1-c+b,1-a+b,z^{-1})$$

Corollary I

Putting $b = c$, we see that

$$
{}_2F_1(a,b,b,z) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(a)} \frac{z^n}{n!} = \frac{1}{(1-z)^a} = \frac{1}{2\pi i\ \Gamma(a)} \int_{-i\infty}^{+i\infty} d\sigma\ (-z)^\sigma\ \Gamma(a+\sigma)\Gamma(-\sigma)
$$

This allows to replace sum by product:

$$
\frac{1}{(A+B)^a} = \frac{1}{B^a[1-(-A/B)]^a} = \frac{1}{2\pi i \Gamma(a)} \int_{-i\infty}^{i\infty} d\sigma A^\sigma\ B^{-\sigma-a}\ \Gamma(a+\sigma)\Gamma(-\sigma)
$$

Barnes' lemma

If the path of integration is curved so that the poles of $\Gamma(c-\sigma)\Gamma(d-\sigma)$ lie on the right of the path and the poles of $\Gamma(a+\sigma)\Gamma(b+\sigma)$ lie on the left, then

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} d\sigma\Gamma(a+\sigma)\Gamma(b+\sigma)\Gamma(c-\sigma)\Gamma(d-\sigma) \quad = \quad \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}$$

It is supposed that a, b, c, d are such that no pole of the first set coincides with any pole of the second set.

Scetch of proof: Close contour by semicircle C to the right of imaginary axis. The integral exists and $\int_C$ vanishes when $\Re(a+b+c+d-1)<0$. Take sum of residues of the integrand at poles of $\Gamma(c-\sigma)\Gamma(d-\sigma)$. The double sum leads to two hypergeometric functions, expressible by ratios of Γ-functions, this in turn by combinations of sin, may be simplifies finally to the r.h.s.

Analytical continuation: The relation is proved when $\Re(a+b+c+d-1)<0$. Both sides are analytical functions of e.g. a. So the relation remains true for all values of a, b, c, d for which none of the poles of $\Gamma(a+\sigma)\Gamma(b+\sigma)$, as a function of σ, coincide with any of the poles of $\Gamma(c-\sigma)\Gamma(d-\sigma)$.

Corollary II Any real shift k: $\sigma+k, a-k, b-k, c+k, d+k$ together with $\int_{-k-i\infty}^{-k+i\infty}$ leaves the result true.

How can the Mellin-Barnes formula be made useful in the context of Feynman integrals?

- Apply corollary I to propagators and get:

$$\frac{1}{(p^2 - m^2)^a} = \frac{1}{2\pi i\ \Gamma(a)} \int_{-i\infty}^{i\infty} d\sigma \frac{(-m^2)^\sigma}{(p^2)^{a+\sigma}}\ \Gamma(a+\sigma)\Gamma(-\sigma)$$

  which transforms a massive propagator to a massless one (with index a of the line changed to $(a+\sigma)$).

- Apply corollary I after introduction of Feynman parameters and after the momentum integration to the resulting F- and U-forms, in order to get a single monomial in the x_i, which allows the integration over the x_i:

$$\frac{1}{[A(s)x_1^{a_1} + B(s)x_1^{b_1}x_2^{b_2}]^a} = \frac{1}{2\pi i\ \Gamma(a)} \int_{-i\infty}^{i\infty} d\sigma [A(s)x_1^{a_1}]^\sigma [B(s)x_1^{b_1}x_2^{b_2}]^{a+\sigma}\ \Gamma(a+\sigma)\Gamma(-\sigma)$$

Both methods leave Mellin-Barnes (MB-) integrals to be performed afterwards.

A short remark on history

- N. Usyukina, 1975: "ON A REPRESENTATION FOR THREE POINT FUNCTION", Teor. Mat. Fiz. 22;
  a finite massless off-shell 3-point 1-loop function represented by 2-dimensional MB-integral
- E. Boos, A. Davydychev, 1990: "A Method of evaluating massive Feynman integrals", Theor. Math. Phys. 89 (1991);
  N-point 1-loop functions represented by n-dimensional MB-integral
- V. Smirnov, 1999: "Analytical result for dimensionally regularized massless on-shell double box", Phys. Lett. B460 (1999);
  treat UV and IR divergencies by analytical continuation: shifting contours and taking residues 'in an appropriate way'
- B. Tausk, 1999: "Non-planar massless two-loop Feynman diagrams with four on-shell legs", Phys. Lett. B469 (1999);
  nice algorithmic approach to that, starting from search for some unphysical space-time dimension d for which the MB-integral is finite and well-defined
- M. Czakon, 2005 (with experience from common work with J. Gluza and TR): "Automatized analytic continuation of Mellin-Barnes integrals", Comput. Phys. Commun. (2006);
  Tausk's approach realized in Mathematica program MB.m, published and available for use

The Γ-function

The Γ-function may be defined by a difference equation:

$$z\Gamma(z) - \Gamma(z+1) = 0$$

$$\Gamma(z) \quad = \quad \int_0^\infty t^{z-1} e^{-t} dt$$

Series[Gamma[ep], {ep, 0, 2}] =

$$\Gamma[\epsilon] = \frac{1}{\epsilon} - \gamma_E + \frac{1}{12}(6{\gamma_E}^2 + \pi^2)\epsilon + \frac{1}{12}(-2{\gamma_E}^3 - {\gamma_E}^2\pi + 2\Psi(2,1))\epsilon^2 + \cdots$$

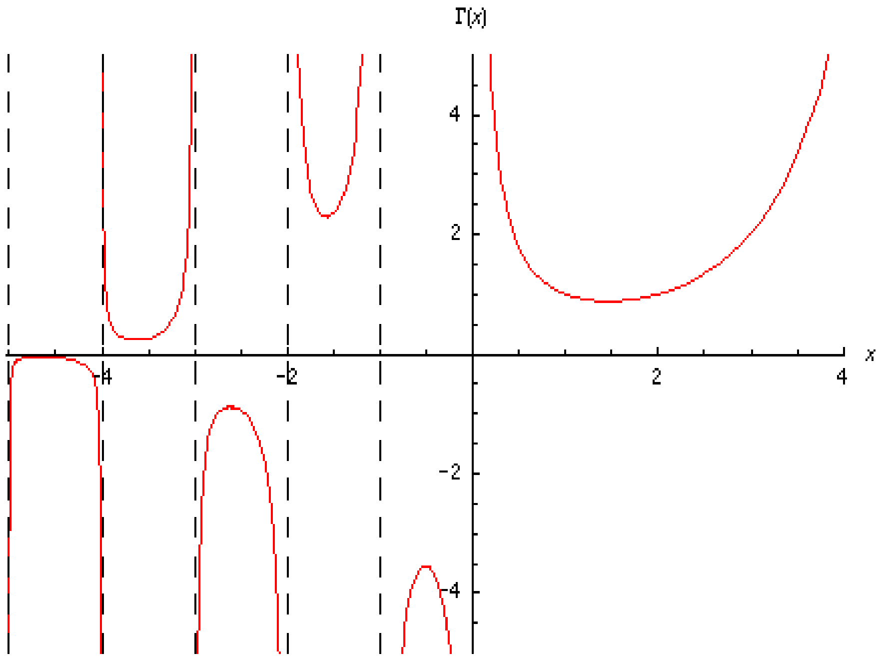
$\Psi(2,1)) = PolyGamma(2,1) = -2\zeta_3$

exp(ep EulerGamma)Series[Gamma[ep], {ep, 0, 2}] =

$$e^{\epsilon\gamma_E}\Gamma[\epsilon] = \frac{1}{\epsilon} + \frac{1}{12}(\pi^2)\epsilon + \frac{1}{6}(\Psi(2,1))\epsilon^2 + \cdots$$

Look at the singularities in the complex plane:

When applying the Cauchy-theorem, one may close the integral to the left or to the right . . .



Some facts on residua

The function

$$F(z) \quad = \sum_{i=-N}^{\infty} \frac{a_i}{(z-z_0)^i}$$

has the residue

$$\text{Res } F(z)|_{z=z_0} = a_{-1}$$

An integral over an anti-clockwize directed closed path C around z_0 then is

$$\frac{1}{2\pi i} \int_C dz F(z) = 2\pi i a_{-1}$$

If $G(z)$ has a Taylor expansion around z_0 and $F(z)$ has a Laurent expansion beginning with $a_{-N}/(z-z_0)^N + \ldots$, then their product has the residue:

$$\text{Res}[G(z) \ F(z)]|_{z=z_0} = \sum_{k=1}^{N} \frac{a_{-k} G(z_0)^{(k)}}{k!}$$

Some residua with $\Gamma(z)$ and $\Psi(z)$

$\Psi(z) = PolyGamma[z] = PolyGamma[0, z]$ → See also next slide

Residue[F[z]Gamma[z], {z, -n}] $= \frac{(-1)^n}{n!} F[-n]$

Residue[F[z]Gamma[z]^2, {z, -n}] $= \frac{2PolyGamma[n+1]F[-n]+F'[-n]}{(n!)^2}$

Residue[F[z]Gamma[z - 1]^2, {z, -3}] $= \frac{25F[-3]-12\gamma_E F[-3]+6F'[-3]}{3456}$

F[z] $= \frac{F[-3]}{576(z+3)^2} + \frac{25F[-3]-12\gamma_E F[-3]+6F'[-3]}{3456(z+3)}$

$$+ a_0 + a_1(z+3) + \cdots$$

Series[Gamma[z + a]Gamma[z - 1]^2, {z, -3, -1 }] $= \frac{\Gamma[-3+a]}{576(z+3)^2}$

$$+ \frac{(25\Gamma[-3+a]-12\gamma_E\Gamma[-3+a]+6(\Gamma[-3+a]PolyGamma[0,-3+a])}{3456} + a_0 + a_1(z+3) + \cdots$$

Where

$$\begin{aligned} \mathsf{P}olygamma[n+1] &\equiv \mathsf{P}olygamma[0,n+1] \\ &= \Psi(n+1) = \frac{\Gamma'(n+1)}{\Gamma(n+1)} = S_1(n) - \gamma_E = \sum_{k=1}^{n} \frac{1}{k} - \gamma_E \end{aligned}$$

The following properties hold:

$$\begin{aligned} \Psi(z+1) &= \Psi(z) + 1/z \\ \Psi(1+\epsilon) &= -\gamma_E + \zeta_2\ \epsilon + \ldots \\ \Psi(1) &= -\gamma_E \\ \Psi(2) &= 1 - \gamma_E \\ \Psi(3) &= 3/2 - \gamma_E \end{aligned}$$

$$PolyGamma[n,z] = \frac{\partial^n}{\partial z^n} \Psi(z)$$

Some sums `Mathematica` can do

Sum[s^(n) Gamma[n + 1]^3/(n!Gamma[2 + 2n]), n, 0, Infinity] =

. (4*ArcSin[Sqrt[s]/2])/(Sqrt[4 - s]*Sqrt[s])

Sum[s^(n) PolyGamma[0, n + 1], n, 0, Infinity] =

. (EulerGamma + Log[1 - s])/(-1 + s)

The above were done with Mathematica 5.2.
Mathematica 6 is more powerful.

A self-energy: SE2l1m

This is a nice example, being simple but showing [nearly] all essentials in a nutshell.

We get for this $F(x) = m^2 x_1^2 - (s - m^2) x_1 x_2$ the following representation:

$$SE2l1m \quad = \quad \frac{e^{\epsilon\gamma_E}}{2\pi i} \frac{(m^2)^{-\epsilon}}{\Gamma[2-2\epsilon]} \int_{\Re z = -1/8} dz \left[\frac{-s+m^2}{m^2}\right]^{-\epsilon-z} \Gamma_1[1-\epsilon-z]\Gamma_2[-z]\Gamma_3[1-\epsilon+z]\Gamma_4[\epsilon+z]$$

Tausk approach:

Seek a configuration where all arguments of Γ-functions have positive real part. Then the $SE2l1m$ is well-defined and finite.

For small ϵ this is - here - evidently impossible; set $\epsilon \to 0$ and look at $\Gamma_2[-z]\Gamma_4[+z]$:

$$\Gamma_1[1-z]\Gamma_2[-z]\Gamma_3[1-+z]\Gamma_4[+z]$$

What to do ????

Tausk: Set ϵ such that it happens, e.g.:

$$\epsilon = 3/8$$

To make physics we have now to deform the integrand or the path such that $\epsilon \to 0$; when crossing a residue, take it and add it up.

Varying $\epsilon \to 0$ from 3/8 makes crossing in $\Gamma_4[\epsilon + z]$ a pole at $\epsilon = -z = +1/8$; there is $\epsilon + z = 0$:

$$Residue[SE2l1m, \{z, -\epsilon\}] = e^{\epsilon\gamma_E} \frac{(m^2)^{-\epsilon}}{\Gamma[2-2\epsilon]} \Gamma_1[1-2\epsilon]\Gamma_2[\epsilon]$$

Here we 'loose' one integration (easier term!) and catch the IR-singularity in $\Gamma_2[\epsilon] \sim 1/\epsilon$!
The function becomes now, for small ϵ:

$$\begin{aligned} SE2l1m &= \frac{e^{\epsilon\gamma_E}}{2\pi i} \frac{(m^2)^{-\epsilon}}{\Gamma[2-2\epsilon]} \int_{\Re z=-1/8} dz \left[\frac{-s+m^2}{m^2}\right]^{-\epsilon-z} \Gamma_1[1-\epsilon-z]\Gamma_2[-z]\Gamma_3[1-\epsilon+z]\Gamma_4[\epsilon+z] \\ &+ e^{\epsilon\gamma_E} \frac{(m^2)^{-\epsilon}}{\Gamma[2-2\epsilon]} \Gamma_1[1-2\epsilon]\Gamma_2[\epsilon] \end{aligned}$$

Now we may take the limit of small ϵ because the integral will stay finite and well-defined:

$$SE2l1m = \frac{e^{\epsilon\gamma_E}}{2\pi i} \int_{\Re z=-1/8} dz \left[\frac{-s+m^2}{m^2}\right]^{-z} \Gamma_3[1-z]\Gamma_4[-z]\Gamma[z]\Gamma[1+z] + e^{\epsilon\gamma_E}\left(2 + \frac{1}{\epsilon} - \ln[m^2]\right) +$$

Now we close the integration path to the left, catch all residues from $\Gamma_3[1-z]$ for $\Re z < -1/8$, i.e. at $z = -n, \; n = 1, 2, \ldots$:

$$Residue\left\{\left[\frac{-s+m^2}{m^2}\right]^{-z} \Gamma_1[1-\epsilon-z]\Gamma_2[-z]\Gamma_3[1-\epsilon+z]\Gamma_4[\epsilon+z], \{z, -n\}\right\} = (-s+m^2)^n \ln(-s+m$$

The sum to be done is trivial (in this trivial case!!):

$$\sum_{n=1}^{\infty}\left[\frac{-s+m^2}{m^2}\right]^n = \frac{1}{1-\frac{-s+m^2}{m^2}} - 1$$

and we end up with:

$$SE2l1m \quad = \quad \frac{1}{\epsilon} + 2 + \left[\frac{1-s/m^2}{s/m^2}\ln(1-s/m^2)\right]$$

This is what we had also from the direct Feynman parameter integration above

A vertex: V3l2m

The Feynman integral V3l2m is the QED one-loop vertex function, which is no master.
It is infrared-divergent (see this by counting of powers of loop integration momentum k or know it from: massless line between two external on-shell lines)

$$F = m^2(x_1+x_2)^2 + [-s]x_1x_2$$

We will also use the variable

$$y = \frac{\sqrt{-s+4}-\sqrt{-s}}{\sqrt{-s+4}+\sqrt{-s}}$$

$$\begin{aligned}
\texttt{V3l2m} &= \frac{e^{\epsilon\gamma_E}\Gamma(-2\epsilon)}{2\pi i}\int dz(-s)^{-\epsilon-1-z}\frac{\Gamma^2(-\epsilon-z)\Gamma(-z)\Gamma(1+\epsilon+z)}{\Gamma(1-2\epsilon)\Gamma(-2\epsilon-2z)} \\
&= \frac{\texttt{V3l2m}[-1]}{\epsilon} + \texttt{V3l2m}[0] + \epsilon\ \texttt{V3l2m}[1] + \cdots .
\end{aligned}$$

One may slightly shift the contour by $(-\epsilon)$ and then close the path to the left and get residua from (and only from) $\Gamma(1+z)$:

[Gluza:2007bd]

:

$$V(s) = \frac{1}{2s\epsilon}\frac{e^{\epsilon\gamma_E}}{2\pi i}\int_{-i\infty-1/2}^{-i\infty-1/2} dz \quad (-s)^{-z}\frac{\Gamma^2(-z)\Gamma(-z+\epsilon)\Gamma(1+z)}{\Gamma(-2z)}$$

$$= -\frac{e^{\epsilon\gamma_E}}{2\epsilon}\sum_{n=0}^{\infty}\frac{s^n}{\binom{2n}{n}(2n+1)}\frac{\Gamma(n+1+\epsilon)}{\Gamma(n+1)}.$$

This series may be summed directly with Mathematica![a], and the vertex becomes:

$$V(s) = -\frac{e^{\epsilon\gamma_E}}{2\epsilon}\Gamma(1+\epsilon) \;\; {}_2F_1\left[1, 1+\epsilon; 3/2; s/4\right].$$

Alternatively, one may derive the ϵ-expansion by exploiting the well-known relation with harmonic numbers $S_k(n) = \sum_{i=1}^{n} 1/i^k$:

$$\frac{\Gamma(n+a\epsilon)}{\Gamma(n)} = \Gamma(1+a\epsilon)\exp\left[-\sum_{k=1}^{\infty}\frac{(-a\epsilon)^k}{k}S_k(n-1)\right].$$

The product $\exp(\epsilon\gamma_E)\Gamma(1+\epsilon) = 1 + \frac{1}{2}\zeta[2]\epsilon^2 + O(\epsilon^3)$ yields expressions with zeta numbers

[a] The expression for $V(s)$ was also derived in

[Huber:2007dx]

; see additionally

[Davydychev:2000na]

.

$\zeta[n]$, and, taking all terms together, one gets a collection of inverse binomial sums[b]; the first of them is the IR divergent part:

$$
\begin{aligned}
V(s) &= \frac{V_{-1}(s)}{\epsilon} + V_0(s) + \cdots \\
V_{-1}(s) &= \frac{1}{2}\sum_{n=0}^{\infty} \frac{s^n}{\binom{2n}{n}(2n+1)} = \frac{1}{2}\frac{4\arcsin(\sqrt{s}/2)}{\sqrt{4-s}\sqrt{s}} = \frac{y}{y^2-1}\ln(y).
\end{aligned}
$$

[b]For the first four terms of the ϵ-expansion in terms of inverse binomial sums or of polylogarithmic functions, see

[Gluza:2007bd]

.

The constant term:

$$\texttt{V312m}[0] = \frac{1}{2\pi i}\int_{-i\infty+u}^{+i\infty+u} dr(-s)^{-1-r}\ \frac{\Gamma^3[-r]\Gamma[1+r]}{\Gamma[-2r]}$$
$$\frac{1}{2}\left[\gamma_E - \ln(-s) + 2\Psi[-2r] - 2\Psi[-r] + \Psi[1+r]\right]$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{s^n}{\binom{2n}{n}(2n+1)}S_1(n),$$

There is also the opportunity to evaluate the MB-integrals numerically by following with e.g. a Fortran routine the straight contour.
This applies after the ϵ-expansion.

$\int_{-5i+\Re z}^{+5i+\Re z}$ is usually sufficient.

But: This works fast and stable for Euclidean kinematics where $-s > 0$.

and the ϵ-term:

$$\begin{aligned}
\texttt{V312m}[1] &= \frac{1/4}{2\pi i}\int_{-i\infty+u}^{+i\infty+u} dr(-s)^{-1-r}\ \frac{\Gamma^3[-r]\Gamma[1+r]}{\Gamma[-2r)]} \\
&\Big[\gamma_E^2 + Log[-s]^2 + Log[-s](-2\gamma_E - 4\Psi[-2z] + 4\Psi[-z] - 2\Psi[1+z]) \\
&+\gamma_E(4\Psi[-2z] - 4\Psi[-z] + 2\Psi[1+z]) \\
&-4\Psi[1,-2z] + 2\Psi[1,-z] + \Psi[1,1+z] \\
&+4(\Psi[-2z]^2 - 2\Psi[-2z]\Psi[-z] + \Psi[-z]^2 + \Psi[-2z]\Psi[1+z] \\
&-\Psi[-z]\Psi[1+z]) + \Psi[1+z]^2\Big] \\
&= [const = 1?] \times \frac{1}{4}\sum_{n=0}^{\infty} \frac{(t)^n}{\begin{pmatrix}2n\\n\end{pmatrix}(2n+1)}\left[S_1(n)^2 + \zeta_2 - S_2(n)\right].
\end{aligned}$$

Here, $\Psi[r] = ...$ and $\Psi[1,r] = ...$, and the harmonic numbers $S_k(n)$ are

$$S_k(n) = \sum_{i=1}^{n} \frac{1}{i^k},$$

The sums appearing above may be obtained from sums listed in Table 1 of Appendix D in [Gluza:2007bd,Davydychev:2003mv]

:

$$\sum_{n=0}^{\infty} \frac{s^n}{\binom{2n}{n}(2n+1)} = \frac{y}{y^2-1} 2\ln(y),$$

$$\sum_{n=0}^{\infty} \frac{s^n}{\binom{2n}{n}(2n+1)} S_1(n) = \frac{y}{y^2-1}\left[-4\mathrm{Li}_2(-y) - 4\ln(y)\ln(1+y) + \ln^2(y) - 2\zeta_2\right],$$

$$\sum_{n=0}^{\infty} \frac{s^n}{\binom{2n}{n}(2n+1)} S_1(n)^2 = \frac{y}{y^2-1}\Bigg[16S_{1,2}(-y) - 8\mathrm{Li}_3(-y) + 16\mathrm{Li}_2(-y)\ln(1+y)$$

$$+8\ln^2(1+y)\ln(y) - 4\ln(1+y)\ln^2(y) + \frac{1}{3}\ln^3(y) + 8\zeta_2\ln(1+y)$$

$$-4\zeta_2\ln(y) - 8\zeta_3\Bigg],$$

$$\sum_{n=0}^{\infty} \frac{s^n}{\binom{2n}{n}(2n+1)} S_2(n) = -\frac{y}{3(y^2-1)}\ln^3(y),$$

Expansion in a small parameter: vertex V3l2m for m^2/s

Use as an example for determining the small mass expansion:

$$\begin{aligned} V3coefm1 &= Coefficient[V3l2m[[1,1]], \epsilon, -1] \\ &= -\frac{1}{2s}\frac{1}{2\pi i}\int_{-i\infty-1/2}^{+i\infty-1/2} dz \left(-\frac{m^2}{s}\right)^z \frac{\Gamma_1[-z]^3\Gamma_2[1+z]}{\Gamma_3[-2z]} \end{aligned}$$

If $|m^2/s| << 1$, then the smallest power of it gives the biggest contribution, i.e. its exponent has to be positive and small.
So, close the contour to the right (positive $\Re z$), and leading terms come from the residua expansion of $\Gamma_1[-z]^3/\Gamma_3[-2z]$ at $z = -1, -2, \cdots$. The residues are terms of a binomial sum:

$$Residue = -\frac{1}{s}\left(\frac{m^2}{s}\right)^n \frac{(2n)!}{(n!)^2}\left[2HarmonicNumber[n] - 2HarmonicNumber[2n] - \ln\left(-\frac{m^2}{s}\right)\right]$$

with first terms equal to (-1)*Residua:

$$V3l2m = \frac{1}{s}\ln\left(-\frac{m^2}{s}\right) + \frac{m^2}{s^2}\left[\ln\left(2+2\frac{m^2}{s}\right)\right] + \frac{m^4}{s^3}\left[\ln\left(7+6\frac{m^2}{s}\right)\right] + O(m^6/s^3)$$

.

Another nice box with numerator, B5l3m($p_e.k_1$)

We used it for the determination if the small mass expansion.

$$\mathtt{B5l3m}(p_e \cdot k_1) = \frac{m^{4\epsilon}(-1)^{a_{12345}}\ e^{2\epsilon\gamma_E}}{\prod_{j=1}^{5}\Gamma[a_i]\Gamma[5-2\epsilon-a_{123}](2\pi i)^4}\int_{-i\infty}^{+i\infty}d\alpha\int_{-i\infty}^{+i\infty}d\beta\int_{-i\infty}^{+i\infty}d\gamma\int_{-i\infty}^{+i\infty}d\delta$$

$$(-s)^{(4-2\ \epsilon)-a_{12345}-\alpha-\beta-\delta}\ (-t)^{\delta}$$

$$\frac{\Gamma[-4+2\ \epsilon+a_{12345}+\alpha+\beta+\delta]}{\Gamma[6-3\ \epsilon-a_{12345}-\alpha]}\frac{\Gamma[-\alpha]\ \Gamma[-\beta]}{\Gamma[7-3\ \epsilon-a_{12345}-\alpha]\ \Gamma[5-2\ \epsilon-a_{123}]}\frac{\Gamma[-\delta]}{\Gamma[4-2\ \epsilon-a_{1123}-2\ \alpha-\gamma]\ \Gamma[5-2\ \epsilon-a_{1123}-2\ \alpha \ldots}$$

$$\frac{\Gamma[2-\epsilon-a_{13}-\alpha-\gamma]}{\Gamma[8-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \beta-2\ \delta-\gamma]}\frac{\Gamma[4-2\ \epsilon-a_{12345}-\alpha-\beta-\delta-\gamma]}{\Gamma[9-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \beta-2\ \delta-\gamma]}\Big\{(p_e\cdot p_3)\ \Gamma[1+a_4+\delta]\ \Gamma[6-3\ \epsilon-\ldots$$

$$\Gamma[4-2\ \epsilon-a_{1234}-\alpha-\beta-\delta]\ \Gamma[3-\epsilon-a_{12}-\alpha]\ \Gamma[8-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \delta-\gamma]\Gamma[9-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \beta-2\ldots$$

$$\Gamma[5-2\ \epsilon-a_{1123}-\gamma]\ \Gamma[4-2\ \epsilon-a_{1123}-2\ \alpha-\gamma]\ \Gamma[a_1+\gamma]\ \Gamma[-2+\epsilon+a_{123}+\alpha+\delta+\gamma]+\Gamma[a_4+\delta]\Big[-(p_e\cdot p_1)\ \Gamma[7-3\ \epsilon-\ldots$$

$$\Gamma[4-2\ \epsilon-a_{1234}-\alpha-\beta-\delta]\Gamma[8-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \delta-\gamma]\Gamma[9-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \beta-2\ \delta-\gamma]$$

$$\Big[\Gamma[3-\epsilon-a_{12}-\alpha]\ \Gamma[5-2\ \epsilon-a_{1123}-\gamma]\Gamma[4-2\ \epsilon-a_{1123}-2\ \alpha-\gamma]\ \Gamma[a_1+\gamma]+\Gamma[2-\epsilon-a_{12}-\alpha]\ \Gamma[4-2\ \epsilon-a_{1123}-\gamma]$$

$$\Gamma[5-2\ \epsilon-a_{1123}-2\ \alpha-\gamma]\ \Gamma[1+a_1+\gamma]\Big]\ \Gamma[-2+\epsilon+a_{123}+\alpha+\delta+\gamma]+\Gamma[6-3\ \epsilon-a_{12345}-\alpha]\ \Gamma[3-\epsilon-a_{12}+\alpha]$$

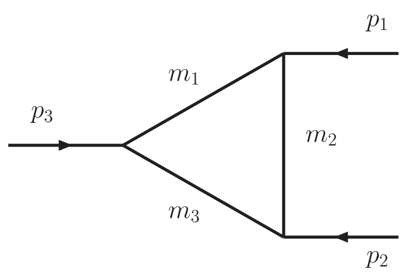
$$\Gamma[5-2\ \epsilon-a_{1123}-\gamma]\ \Gamma[4-2\ \epsilon-a_{1123}-2\ \alpha-\gamma]\Gamma[a_1+\gamma]\Big[((p_e\cdot(p_1+p_2))\ \Gamma[5-2\ \epsilon-a_{1234}-\alpha-\beta-\delta]\Gamma[9-4\ \epsilon-a_{11223}\ldots$$

$$\Gamma[8-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \beta-2\ \delta-\gamma]\Gamma[-2+\epsilon+a_{123}+\alpha+\delta+\gamma]+(p_e\cdot p_1)\ \Gamma[4-2\ \epsilon-a_{1234}-\alpha-\beta-\delta]$$

$$\Gamma[8-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \delta-\gamma]\Gamma[9-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \beta-2\ \delta-\gamma]\Gamma[-1+\epsilon+a_{123}+\alpha+\delta+\gamma]\Big]\Big\}$$

Differential equations: Example V3l2m

Example V3l2m, the massive QED 1-loop vertex



$$
\begin{aligned}
V3l2m &= \int \frac{d^d k}{D_1 D_2 D_3} \\
D_1 &= (k+p_1)^2 - m_1^2 \\
D_2 &= (k-p_2)^2 - m_2^2 \\
D_3 &= (k)^2 - m_0^2
\end{aligned}
$$

$$
s = (p_1+p_2)^2 \ = \ 2p_1p_2 + p_1^2 + p_2^2
$$

Dimensional arguments prove:

$$
V3l2m \sim \frac{1}{s} \ \text{ at } \ s \to \infty
$$

We derive now a differential eqation for the vertex, $V = V(s, p_1^2, p_2^2))$

$$s = (p_1 + p_2)^2 = 2p_1p_2 + p_1^2 + p_2^2 \rightarrow \; 2p_1p_2 + 2m^2 \; or \; \rightarrow \; 2p_1p_2 + m^2 \;\; etc.$$

First derive a differential operator, which is completely independent of the internal details the vertex.

$$\begin{aligned} p_1 \frac{\partial}{\partial p_2} &= p_1 \left[\frac{\partial}{\partial p_2^2} \frac{\partial p_2^2}{\partial p_2} + \frac{\partial}{\partial s} \frac{\partial s}{\partial p_2} \right] \\ &= p_1 \left[2p_2 \frac{\partial}{\partial p_2^2} + 2(p_1 + p_2) \frac{\partial}{\partial s} \right] \\ p_2 \frac{\partial}{\partial p_2} &= p_2 \left[2p_2 \frac{\partial}{\partial p_2^2} + 2(p_1 + p_2) \frac{\partial}{\partial s} \right] \end{aligned}$$

Not Using $p_1^2 = p_2^2 = m^2$ and $p_1p_2 = (s - 2m^2)/2$, we get:

$$\begin{aligned} p_1 \frac{\partial}{\partial p_2} &= (s - p_1^2 - p_2^2) \frac{\partial}{\partial p_2^2} + s \frac{\partial}{\partial s} \\ p_2 \frac{\partial}{\partial p_2} &= (p_1^2 + p_2^2) \frac{\partial}{\partial p_2^2} + s \frac{\partial}{\partial s} \end{aligned}$$

Eliminate $\partial / \partial p_2^2$:

$$[s - 2(p_1^2 + p_2^2)] s \frac{\partial}{\partial s} = \left[[s - (p_1^2 + p_2^2)] \; p_2 - (p_1^2 + p_2^2) \; p_1 \right] \frac{\partial}{\partial p_2} = [R_2 \; p_2 + R_1 \; p_1] \frac{\partial}{\partial p_2}$$

Now apply this universal differential vertex-operator to a specific vertex, e.g. V=V3l2m:
Use on-mass-shell condition $p_1^2 = m_1^2, p_2^2 = m_2^2$.

$$\frac{\partial V}{\partial p_2} = \int \frac{d^d k}{D_3 D_1} \frac{\partial}{\partial p_2} \frac{1}{D_2}$$

With

$$\frac{\partial}{\partial p_2} \frac{1}{D_2} = -\frac{1}{D_2^2} \frac{\partial D_2}{\partial p_2} = \frac{2(k-p_2)}{D_2^2}$$

Compensate numerator against propagators:

$$\begin{aligned} 2p_1 k &= -D_3 + D_1 - m_0^2 - p_1^2 + m_1^2 \\ 2p_2 k &= D_3 - D_2 + m_0^2 + p_2^2 - m_2^2 \\ p_i^2 &= m_i^2 \\ p_1 p_2 &= (s - m_1^2 - m_2^2)/2 \end{aligned}$$

we get, for this vertex, the equation:

$$(s - 2(m_1^2 + m_2^2))\, s \frac{\partial V}{\partial s} = s \int \frac{d^d k}{D_1 D_2^2} - (m_1^2 + m_2^2) \int \frac{d^d k}{D_3 D_2^2} - (s - (m_1^2 + m_2^2))\, V$$

There might also arise a dotted Vertex at the right hand side, but here it drops out from the result; in general:

$$\left[s - 2(p_1^2 + p_2^2)\right] s\frac{\partial}{\partial s} = \int dk \Bigg[\frac{-R_2}{D_1 D_2 D_3} + \frac{R_2 - R_1}{D_1 D_2^2} + \frac{R_1}{D_2^2 D_3} + \frac{R_2(m_0^2 - m_2^2 - p_2^2) + R_1(-m_0^2 + p_2^2 + m_1^2 - s)}{D_1 D_2^2 D_3}\Bigg]$$

This is a simple Euler differential equation!

The inhomogeneity depends on two dotted 2-point functions SE2l2md and SE2l1md, so it is simpler and assumed to be known already.

In principle, one might get also a term with $\int d^dk/D_1D_2^2D_3$, i.e. the dotted vertex V3l2md, but it drops out here.

Then one would have a system of coupled equations, and this often happens.

We see that dotted objects may appear naturally,.

All the above has nothing to do with dimension d – if it is well-defined . . .

In

[Czakon:2004wm]

we use the operator for deriving and solving massive 2-loop vertex functions and give also a corresponding operator for box diagrams, $m^2 \to 1$:

$$s\frac{\partial}{\partial s} = \frac{1}{4-s}\left[2p_2^\mu + (2-s)p_1^\mu\right]\frac{\partial}{\partial p_1^\mu} \qquad \text{for vertex}$$

$$s\frac{\partial}{\partial s} = \frac{1}{2}\left[(p_1^\mu + p_2^\mu) + \frac{s\,(p_2^\mu - p_3^\mu)}{s+t-4}\right]\frac{\partial}{\partial p_2^\mu} \qquad \text{for box}$$

Scetch of: Solve Euler Diff. Equations

Basics developed in

[Kotikov:1991hm,Kotikov:1991kg,Laporta:1996mq]

. The choice of variables depends on the problem.
For the QED vertex it will prove useful to take

$$s/m^2 = -(1-x)^2/x$$

For our other beloved example $B \equiv B_0(s,m,0)$, see nice lecture

[Aglietti:2004vs]

$$s/m^2 = -x$$

For details, I prefer to jump to this case.
Differential operator:

$$s\frac{\partial}{\partial s} = \frac{1}{2}p^\mu \frac{\partial}{\partial p^\mu}$$

The equation becomes here:

$$\frac{\partial}{\partial s}B = \left[-\frac{1}{x}+\frac{1}{1+x}\right]B+\epsilon\left[\frac{1}{x}-\frac{2}{1+x}\right]B+(1-\epsilon)\left[\frac{1}{x}-\frac{1}{1+x}\right]T1l1m$$

$$T1l1m = \frac{1}{\epsilon}+1+(1+\zeta_2/2)\epsilon+\dots$$

We need also a boundary condition; look at explicit expression above:

$$B_0(0, m, 0) = T1l1m$$

Make an ansatz as series in ϵ:

$$B = \frac{1}{\epsilon} B_1 + B_0 + \epsilon B_1 + \ldots$$

and get a series of equations:

$$\begin{aligned}
\frac{\partial}{\partial s} B_{-1} &= A_0 B_{-1} + \Omega_{-1} \\
\frac{\partial}{\partial s} B_0 &= A_0 B_0 + A_1 B_{-1} + \Omega_0 \\
\frac{\partial}{\partial s} B_1 &= A_0 B_1 + A_1 B_0 + \Omega_1 \\
etc
\end{aligned}$$

It is:

$$
\begin{aligned}
A_0 &= -\frac{1}{x} + \frac{1}{1+x} \\
A_1 &= \frac{1}{x} - \frac{2}{1+x} \\
\Omega_{-1} &= \frac{1}{x} - \frac{1}{1+x} \\
\Omega_0 &= -\frac{1}{x} + \frac{1}{1+x} \\
\Omega_{i>0} &= 0
\end{aligned}
$$

You may realize a simple, iterative structure.

The coefficients in the equation are of the form

$$
\frac{A_1}{x - B_1} + \frac{A_2}{x - B_2} + ..
$$

General solution for the homogeneous equations:

$$\frac{B_n'}{B_n} = -\frac{1}{x} + \frac{1}{1+x}$$

Solve it by

$$\ln(B_n^{hom}) \equiv B^{hom} = const + \int \left[-\frac{1}{x} + \frac{1}{1+x} \right]$$

gets

$$B^{hom} = 1 + \frac{1}{x}$$

Solution of the inhomogenious equations is obtained by the 'variation of constants':

$$B_k(x) = B^{hom}(x) \left\{ const + \int dx' \frac{1}{B^{hom}(x')} \left[A_1(x') B_{k-1}(x') + \Omega_k(x') \right] \right\}$$

This may be sequentially evaluated.

Result:

nested integrals over 'simple' iterated integrands

The method leads to the HPLs $H(\{a\},x)$ and similar functions.

Harmonic Polylogarithms $H(x)$

$$\begin{aligned} H[-1,1,x] &= \int_0^x \frac{dx''}{(1+x'')} \int_0^{x''} \frac{dx'}{(1-x')} \\ &= Li_2\left(\frac{1+x}{2}\right) + \ldots \end{aligned}$$

but it works only if the polynomial structure is simple enough for a solution with this class of functions

Method is absolutely 'super' if it works.

But:

one needs complete chains of masters of lower complexity.

Harmonic Polylogs

[Remiddi:1999ew]

(I borrowed some formulas from there)

$$
\begin{aligned}
\mathsf{H}(0;x) &= \ln x \ , \\
\mathsf{H}(1;x) &= \int_0^x \frac{dx'}{1-x'} = -\ln(1-x) \ , \\
\mathsf{H}(-1;x) &= \int_0^x \frac{dx'}{1+x'} = \ln(1+x) \ .
\end{aligned}
$$

Their derivatives are:

$$
\frac{d}{dx}\mathsf{H}(a;x) = f(a;x) \ , \tag{1}
$$

where the index a can take $0, +1, -1$, and

$$
\begin{aligned}
f(0;x) &= \frac{1}{x} \ , \\
f(1;x) &= \frac{1}{1-x} \ , \\
f(-1;x) &= \frac{1}{1+x} \ .
\end{aligned}
$$

The harmonic polylogarithms of weight w are then defined as follows:

$$\mathsf{H}(\vec{0}_w;x) = \frac{1}{w!}\ln^w x \ ,$$

while, if $\vec{m}_w \neq \vec{0}_w$

$$\mathsf{H}(\vec{m}_w;x) = \int_0^x dx' \ f(a;x') \ \mathsf{H}(\vec{m}_{w-1};x') \ .$$

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$$\mathsf{H}(\vec{m}_w;x) = \int_0^x dx' \ f(a;x') \ \mathsf{H}(\vec{m}_{w-1};x') \ .$$

Examples:

$$
\begin{aligned}
\mathsf{H}(0,1;x) &= Li_2(x) \ , \\
\mathsf{H}(0,-1;x) &= -Li_2(-x) \ , \\
\mathsf{H}(1,0;x) &= -\ln x \ln(1-x) + Li_2(x) \ , \\
\mathsf{H}(1,1;x) &= \frac{1}{2!}\ln^2(1-x) \ , \\
\mathsf{H}(1,-1;x) &= Li_2\left(\frac{1-x}{2}\right) - \ln 2 \ln(1-x) - Li_2\left(\frac{1}{2}\right) \ , \\
\mathsf{H}(-1,0;x) &= \ln x \ln(1+x) + Li_2(-x) \ , \\
\mathsf{H}(-1,1;x) &= Li_2\left(\frac{1+x}{2}\right) - \ln 2 \ln(1+x) - Li_2\left(\frac{1}{2}\right) \ , \\
\mathsf{H}(-1,-1;x) &= \frac{1}{2!}\ln^2(1+x) \ .
\end{aligned}
$$

In general, they are more general than Nielsen's Polylogarithms, see e.g.:

$$\mathsf{H}(-1,0,0,1;x) = \int_0^x \frac{dx'}{1+x'} Li_3(x')$$

cannot be expressed in terms of Nielsen's polylogarithms

Example beyond Harmonic Polylogs: QED Box B4l2m

[Fleischer:2006ht]

```
F[x]      = (x5+x6)^2 + (-s)x5x6 + (-t)x4x7
```

`B4l2m`, the 1-loop QED box, with two photons in the s-channel; the Mellin-Barnes representation reads for finite ϵ:

$$\texttt{B4l2m} = \texttt{Box}(t,s) = \frac{e^{\epsilon\gamma_E}}{\Gamma[-2\epsilon](-t)^{(2+\epsilon)}} \frac{1}{(2\pi i)^2} \int_{-i\infty}^{+i\infty} dz_1 \int_{-i\infty}^{+i\infty} dz_2 \frac{(-s)^{z_1}(m^2)^{z_2}}{(-t)^{z_1+z_2}} \Gamma[2+\epsilon+z_1+z_2]\Gamma^2[1+z_1]\Gamma[-z_1]\Gamma[-z_2] \Gamma^2[-1-\epsilon-z_1-z_2] \frac{\Gamma[-2-2\epsilon-2z_1]}{\Gamma[-2-2\epsilon-2z_1-2z_2]} \quad (2)$$

`Mathematica` package `MB` used for analytical expansion $\epsilon \to 0$:

[Czakon:2005rk]

$$\texttt{B412m} = -\frac{1}{\epsilon}\texttt{J}_1 + \ln(-s)\texttt{J}_1 + \epsilon\left(\frac{1}{2}\left[\zeta(2) - \ln^2(-s)\right]\texttt{J}_1 - 2\texttt{J}_2\right). \tag{3}$$

with J_1 being also the divergent part of the vertex function $C_0(t; m, 0, m)/s = \texttt{V312m}/s$ (as is well-known):

$$\texttt{J}_1 = \frac{e^{\epsilon\gamma_E}}{st}\frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dz_1 \left(\frac{m^2}{-t}\right)^{z_1} \frac{\Gamma^3[-z_1]\Gamma[1+z_1]}{\Gamma[-2z_1]} = \frac{1}{m^2 s}\frac{2y}{1-y^2}\ln(y) \tag{4}$$

with

$$y = \frac{\sqrt{1-4m^2/t} - 1)}{(\sqrt{1-4m^2/t}+1)}$$

The J_2 is more complicated:

$$\begin{aligned}\texttt{J}_2 = \frac{e^{\epsilon\gamma_E}}{t^2}\frac{1}{(2\pi i)^2} & \int_{-\frac{3}{4}-i\infty}^{-\frac{3}{4}+i\infty} dz_1 \left(\frac{s}{t}\right)^{z_1} \Gamma[-z_1]\Gamma[-2(1+z_1)]\Gamma^2[1+z_1] \\ & \times\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dz_2 \left(-\frac{m^2}{t}\right)^{z_2}\Gamma[-z_2]\frac{\Gamma^2[-1-z_1-z_2]}{\Gamma[-2(1+z_1+z_2)]}\Gamma[2+z_1+z_2].\end{aligned} \tag{5}$$

The expansion of `B4l2m` at small m^2 and fixed value of t

With

$$m_t = \frac{-m^2}{t}, \tag{6}$$

$$r = \frac{s}{t}, \tag{7}$$

Look, under the integral, at $(-m^2/t)^{z_2}$,
and close the path to the right.
Seek the residua from the poles of Γ-functions with the smallest powers in m^2 and sum the resulting series.
we have obtained a compact answer for J_2 with the additional aid of XSUMMER

[Moch:2005uc]

The box contribution of order ϵ in this limit becomes:

$$\begin{aligned} \texttt{B4l2m}[t,s,m^2;+1] = & \frac{1}{st}\Big\{4\zeta_3 - 9\zeta_2\ln(m_t) + \frac{2}{3}\ln^3(m_t) + 6\zeta_2\ln(r) - \ln^2(m_t)\ln(r) \\ & +\frac{1}{3}\ln^3(r) - 6\zeta_2\ln(1+r) + 2\ln(-r)\ln(r)\ln(1+r) - \ln^2(r)\ln(1+r) \\ & +2\ln(r)\mathrm{Li}_2(1+r) + 2\mathrm{Li}_3(-r)\Big\} + \mathcal{O}(m_t). \end{aligned} \tag{8}$$

Remark:
The exact Box function is NOT expressible by Harmonic Polylogs, one may introduce a

generalization of them: Generalized HPLs.
Automatized tools for this might be developed.

A scetch of the small mass expansion may be made as follows.

First the 1-dim. integral J_1.
The leading term comes from the first residue:

```
J1 =  Residue[m_t^z1 Gamma[-z1]^3 Gamma[1 + z1]/Gamma[-2 z1], {z1, 0}]

   = 2 Log[m_t]
```

WE get a logarithmic mass dependence.
The second integral: Start with z_2, first residue is:

```
I2 = Residue[ m_t^z2 Gamma[-z2] Gamma[-1 - z1 - z2]^2
                     Gamma[2 + z1 + z2]/Gamma[-2 - 2 z1 - 2 z2], {z2, 0}]
```

$$= -\frac{\Gamma[-1-z_1]^2\Gamma[2+z_1]}{\Gamma[-2-2z_1]}$$

The residue is independent of m^2/t.

It has to be integrated over z_1 yet, together with the terms which were independent of z_2:

$$I_2 \sim \int dz_1 r^{z_1+1}\Gamma[-z_1]\Gamma[-2-2z_1]\Gamma[1+z_1]^2 \frac{\Gamma[-1-z_1]^2\Gamma[2+z_1]}{\Gamma[-2-2z_1]}$$

Sum over residues, close path to the left:

$$Residue[z_1=-n] \quad = \quad \frac{(-1)^n r^{1-n}}{2(-1+n)^3}\left[2+(-1+n)^2\pi^2+(-1+n)\ln[r](2+(-1+n)\ln[r])\right]$$

$$Residue[z_1=-1] \quad = \quad \frac{1}{6}(3\pi^2\ln[r]+\ln[r]^3)$$

and finally:

$$I_2 \sim Residue[z_1=-1]+\sum_{n=2}^{\infty} Residue[z_1=-n]$$

The sum can be done also withoput using XSUMMER (here at least), e.g.

$$\ln[r]\sum_{n=2}^{Infty}\frac{(-1)^n(2+\pi^2-2n\pi^2+n^2\pi^2)r^{1-n}}{2(-1+n)^3} \quad = \quad \frac{1}{2}\left[\pi^2\ln(1+1/r)-2Li_3\left(-1/r\right)\right]$$

etc

Integration-by-parts identities

[Tkachov:1981wb,Chetyrkin:1981qh]

The Integration-by-parts identities relate different scalar Feynman integrals in d dimensions algebraically.

They may be used to determine a list of basic so-called master integrals.

These masters have to be solved, the others are then - more or less easily - derived from them by algebra.

First systematic algorithm realized in a computer code (unpublished) by *[Laporta:1996mq,Laporta:2000dc,Laporta:2001dd]*

, the masters then were evaluated by difference equations.

Public codes with Laporta algorithm: Maple package AIR *[Anastasiou:2005cb]*

and Fire *[Smirnov:2008iw]*

A nice lecture, where also integration by parts, differential equations, HPLs and all that are introduced, is *[Aglietti:2004vs]*

Lagrange/1762–Gauss/1813–Green/1825–Ostrogadski/1831 integral theorem (or divergence theorem)

$$\int_V d^d k \; \frac{\partial}{\partial k_\mu} [F_\mu] \;\; = \;\; \oint_{S_\infty} [F_\mu] \, dS^\mu \to 0$$

where for a tadpole and self-energy e.g.:

$$F_{T,\mu} \;\; = \;\; \frac{k_\mu}{D_1^{n_1}}$$

$$F_{SE,\mu} \;\; = \;\; \frac{a k_\mu + b p_\mu}{D_1^{n_1} D_2^{n_2}}$$

etc., and $D_1 = k^2 - m^2$ and $D_2 = (k-p)^2 - m^2$. For the tadpole one gets due to:

$$\partial_{k_\mu} \frac{k_\mu}{D_1^{n_1}} \;\; = \;\; \partial_{k_\mu} k_\mu \frac{1}{D_1^{n_1}} + k_\mu \frac{-2k_\mu}{D_1^{n_1+1}}$$

with $\partial_{k_\mu} k_\mu = d$ the relation for tadpoles with different indices n_1 and $n_1 + 1$:

$$T(n_1 + 1) \;\; = \;\; \frac{(d - 2n_1)}{2n_1} \, \frac{T(n_1)}{m^2}$$

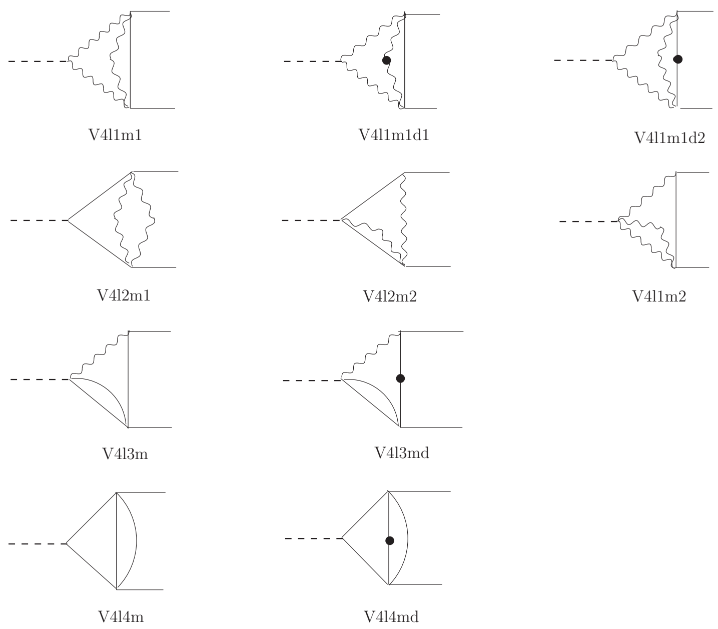
For more complicated Fyenman integrals one gets several equations and one may then hope to formulate a simple basis.

Regularisation by subtraction: V4l1m2

Simplest case was factorization as in V3l2m.
Systematic isolation of divergences in regions of phase space, and in each of them the divergence is in only ONE variable: sector decomposition

Intermediate case:

It is a singularity in one variable only, but not sufficiently isolated.
Then one may try to perform a subtraction.



The Feynman integral V4l1m2 has a massless UV-divergent subloop

$$\text{V4l1m2} = -\frac{e^{2\epsilon\gamma_E}}{\pi^D}\int\frac{d^Dk_1d^Dk_2}{[k_2^2][(k_1+k_2-p_1)^2][k_1^2-1][(k_1+p_2)^2]}.$$

We will use a subtraction procedure in order to isolate the remaining UV singularity. The two momentum integrations may be performed subsequently:

$$\int\frac{d^Dk_2}{[k_2^2][(k_2+k_1-p_1)^2]} = i\pi^{D/2}\frac{\Gamma(1-\epsilon)^2\Gamma(\epsilon)}{\Gamma(2-2\epsilon)}\frac{1}{[(k_1-p_1)^2]^\epsilon},$$

$$\int\frac{d^Dk_1}{[(k_1-p_1)^2]^\epsilon[k_1^2-1][(k_1+p_2)^2]} = i\pi^{D/2}\frac{\epsilon(1+\epsilon)\Gamma(2\epsilon)}{\Gamma(2+\epsilon)}\,I_{\rm div},$$

$$I_{\rm div} = \int_0^1\frac{dxdy\;x^{-1+\epsilon}(1-x)^{1-2\epsilon}}{([(1-x)(1-y)^2-xys]^2)^{2\epsilon}}.$$

The Feynman parameter integral $I_{\rm div}$ has a singularity at $x=0$ - like the QED vertex or SE - and may be regulated by a subtraction:

$$I_{\rm div} = \int_0^1 dx x^{-1+\epsilon}(1-x)^{1-2\epsilon}\int_0^1 dy\left\{\left[f(x,y)^{-2\epsilon}-f(0,y)^{-2\epsilon}\right]+f(0,y)^{-2\epsilon}\right\}$$

$$= \frac{\Gamma(\epsilon)\Gamma(2-2\epsilon)}{(1-4\epsilon)\Gamma(2-\epsilon)}+I_{reg},$$

$$I_{reg} = \int_0^1 dx\;(1-x)\left[x(1-x)^2\right]^\epsilon\int_0^1 dy\frac{f(x,y)^{-2\epsilon}-f(0,y)^{-2\epsilon}}{x},$$

with

$$f(x,y) = (1-x)(1-y)^2 - xys.$$

The remaining integrations in I_{reg} are regular and can be performed analytically or numerically after the ϵ-expansion:

$$I_{reg} = \int_0^1 dx(1-x)e^{\epsilon \ln[x/(1-x)^2]} \int_0^1 \frac{dy}{x} \ln\left(\frac{f(x,y)}{f(0,y)}\right) \sum_{n=1}^{\infty} \frac{(-2\epsilon)^n}{n!} \left[\sum_{k=0}^{n} \ln^{n-k-1} f(x,y) \ln^k f(0,y)\right].$$

The first terms of the series expansion in ϵ for `V4l1m2` are (see (1)):

$$\texttt{V4l1m2} = \frac{1}{2\epsilon^2} + \frac{5}{2\epsilon} + \frac{19}{2} - \frac{3-13x}{2(1+x)}\zeta_2 - \frac{1-x}{2(1+x)}\left[\ln^2(x) + 4\mathrm{Li}_2(x)\right] + \mathcal{O}(\epsilon).$$

In general, the situation is more involved.

Sector decomposition

For Euclidean kinematics, the integrand for the multi-dimensional x-integrations is positive semi-definite.

In numerical integrations, one has to separate the poles in $(d-4)$, and in doing so one has to avoid overlapping singularities.

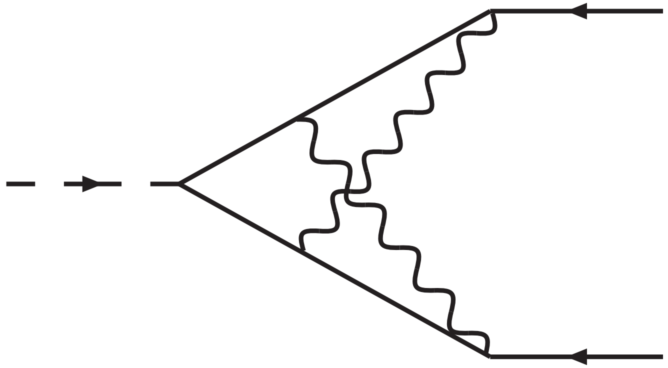
A method for that is sector decomposition.

There are quite a few recent papers on that, and also nice reviews are given [Binoth:2000ps,Denner:2004iz,Bogner:2007cr,Heinrich:2008si,Smirnov:2008aw]

The intention is to separate singular regions in different variables from each other, as is nicely demonstrated by an example borrowed from [Heinrich:2008si]

:

$$
\begin{aligned}
I &= \int_0^1 dx \int_0^1 dy \frac{1}{x^{1+a\epsilon} y^{b\epsilon}[x+(1-x)y]} \\
&= \int_0^1 \frac{dx}{x^{1+(a+b)\epsilon}} \int_0^1 \frac{dt}{t^{b\epsilon}[1+(1-x)t]} + \int_0^1 \frac{dy}{y^{1+(a+b)\epsilon}} \int_0^1 \frac{dt}{t^{1+a\epsilon}[1+(1-y)t]}. \qquad (9)
\end{aligned}
$$



The master integral V6l4m1

At several occasions, we used for cross checks the package sector_decomposition [Bogner:2007cr]

built on the C++ library GINAC [Bauer:2000cp]

For that reason, the interface CSectors was written; it will be made publicly available soon.
The syntax is similar to that of AMBRE.
Example:
The program input for the evaluation of the integral V6l4m1 is simple; we choose $m = 1, s = -11$, and the topology may be read from the arguments of propagator functions PR:

```
<< CSectors.m

Options[DoSectors]
SetOptions[DoSectors, TempFileDelete -> False, SetStrategy -> C]

n1 = n2 = n3 = n4 = n5 = n6 = n7 = 1;
m = 1; s = -11;
invariants = {p1^2 -> m^2, p2^2 -> m^2, p1 p2 -> (s - 2 m^2)/2};

DoSectors[{1},
  {PR[k1,0,n1]       PR[k2,0,n2]       PR[k1+p1,m,n3]
   PR[k1+k2+p1,m,n5] PR[k1+k2-p2,m,n6] PR[k2-p2,m,n7]},
   {k2, k1}, invariants][-4, 2]
```

Here, the numerator is 1 (see the first argument $\{1\}$ of `DoSectors`), and the output contains the functions U_2 and F_2:

```
Using strategy C
U = x3 x4+x3 x5+x4 x5+x3 x6+x5 x6+x2 (x3+x4+x6)+x1 (x2+x4+x5+x6)

F = x1 x4^2+13 x1 x4 x5+x4^2 x5+x1 x5^2+x4 x5^2+13 x1 x4 x6
+2 x1 x5 x6+13 x4 x5 x6+x5^2 x6+x1 x6^2+x5 x6^2+x3^2 (x4+x5+x6)
```

```
+x2(x3^2+x4^2+13 x4 x6+x6^2+x3 (2 x4+13 x6))+x3 (x4^2+(x5+x6)^2
+x4 (2x5+13 x6))
```

Notice the presence of a U-function and the complexity of the F-function (compared to $U = 1$ and `f1` and `f2` in the loop-by-loop MB-approach) due to the non-sequential, direct performance of both momentum integrals at once. Both U and F are evidently positive semi-definite. The numerical result for the Feynman integral is:

$$\texttt{V6l4m1}(-\texttt{s})^{2\epsilon} = -0.052210\ \frac{1}{\epsilon} - 0.17004 + 0.24634\ \epsilon + 4.8773\ \epsilon^2 + \mathcal{O}(\epsilon^3). \qquad (10)$$

The numbers may be compared to (13). We obtained a third numerical result, also by sector decomposition, with the Mathematica package FIESTA

[Smirnov:2008py]

$$\texttt{V6l4m1}(-\texttt{s})^{2\epsilon} = -0.052208\ \frac{1}{\epsilon} - 0.17002 + 0.24622\ \epsilon + 4.8746\ \epsilon^2 + \mathcal{O}(\epsilon^3). \qquad (11)$$

Most accurate result: obtained with an analytical representation based on harmonic polylogarithmic functions obtained by solving a system of differential equations

[Gluza, TR, unpubl.;Remiddi:1999ew,Maitre:2005uu]

$$\texttt{V6l4m1}(-\texttt{s})^{2\epsilon} = -0.0522082\ \frac{1}{\epsilon} - 0.170013 + 0.246253\ \epsilon + 4.87500\ \epsilon^2 + \mathcal{O}(\epsilon^3). \qquad (12)$$

All displayed digits are accurate here.

V6l4m: Compare to MB-integrals

In a loop-by-loop approach, after the first momentum integration one gets here $U = 1$ and a first F-function (**??**), which depends yet on one internal momentum k_1:

```
f1 = m^2 [X[2]+X[3]+X[4]]^2 - s X[2]X[4] - PR[k1+p1,m] X[1]X[2]
               - PR[k1+p1+p2,0] X[2]X[3] - PR[k1-p2,m] X[1]X[4]
                     - PR[k1,0] X[3]X[4] ,
```

leading to a 7-dimensional MB-representation; after the second momentum integration, one has:

```
f2 = m^2 [X[2]+X[3]]^2 - s X[2]X[3] - s X[1]X[4] - 2s X[3]X[4],
```

leading to another 4-dimensional integral.

After several applications of Barnes' first lemma, an 8-dimensional integral has to be treated.

We made no attempt here to simplify the situation by any of the numerous tricks and reformulations etc. known to experts.

The package `AMBRE.m` is designed for a semi-automatic derivation of Mellin-Barnes (MB) representations for Feynman diagrams; for details and examples of use see the webpage `http://prac.us.edu.pl/~gluza/ambre/`. The package is also available from `http://projects.hepforge.org/mbtools/`.

Version 1.0 is described in

[Gluza:2007rt]

, the last released version is 1.2. We are releasing now version 2.0, which allows to construct MB-representations for two-loop *tensor* integrals.
The package is yet restricted to the so-called loop-by-loop approach, which yields compact representations, but is known to potentially fail for non-planar topologies with several scales. An instructive example has been discussed in

[Czakon:2007wk]

.

For one-scale problems, one may safely apply `AMBRE.m` to non-planar diagrams. For our example `V6l4m1`, one gets e.g. with the 8-dimensional MB-representation scetched above the following numerical output after running also `MB.m`

[Czakon:2005rk]

(see also the webpage `http://projects.hepforge.org/mbtools/`),

at $s = -11$:

$$\texttt{V6l4m}\ (-\texttt{s})^{2\epsilon} = -0.0522082\ \frac{1}{\epsilon} - 0.17002 + 0.25606\ \epsilon + 4.67\ \epsilon^2 + \mathcal{O}(\epsilon^3). \tag{13}$$

Compare this to an MB-integral for V6l0m

Was solved first in

[Gonsalves:1983nq]

```
fupc1 = m^2 FX[X[3] + X[4]]^2 - PR[k1 + p1, 0] X[1] X[2] -
 PR[k1 + p1 + p2, m] X[2] X[3] - PR[k1 - p2, m] X[1] X[4] -
 S X[2] X[4] - PR[k1, 0] X[3] X[4]

fupc2 = -s X[2] X[3]-s X[1] X[4]-2 s X[3] X[4]
```

The function is a scale factor times a pure number

$$V6l0m = const(-s)^{-2-2\epsilon}\left[\frac{A_4}{\epsilon^4}+\frac{A_3}{\epsilon^3}+\frac{A_2}{\epsilon^2}+\frac{A_1}{\epsilon}+A_0+\dots\right]$$

```
V6l0m = (-s)^{-2 - 2 eps}
 *
Gamma[-z1] Gamma[-1 - eps - z1 - z2] Gamma[
    1 + z1 + z3] Gamma[-1 - eps - z1 - z3 - z4] Gamma[-z4] Gamma[
    1 + z1 + z2 + z4] Gamma[
    2 + eps + z1 + z2 + z3 + z4] Gamma[-2 - 2 eps - z2 - z4 -
     z5] Gamma[-z5] Gamma[
    1 - z1 + z5] Gamma[-2 - 2 eps - z3 - z4 - z6] Gamma[-z6] Gamma[
```

```
    3 + eps + z1 + z2 + z3 + z4 + z6] Gamma[
    2 + 2 eps + z4 + z5 + z6])/(Gamma[-2 eps] Gamma[
    1 - z1] Gamma[-3 eps - z4] Gamma[3 + eps + z1 + z2 + z3 + z4])


integrals = MBcontinue[(-s)^(2 + 2 eps) fin, eps -> 0, rules];

26 integral(s) found
```

From $1/\epsilon^4$ until ϵ^0: at most 4-dimensional

AMBRE output:

```
{-47.8911 + 1/eps^4 - 9.8696/eps^2 - 33.2569/eps}

1

0

-9.869604401660688      -> -Pi^2

-33.25689147976733      -> -(83/3*Zeta[3])

-47.89108304541082     -> (-59 Pi^4/120)
```

Try PSQL for a systematic study

Tensor integrals

At the LHC, we need also massive 5-point and 6-point functions at least, for NLO corrections.

In general, the treatment of tensor integrals is a non-trivial task.

- One might think that all tensor numerators may be reduced to (even simpler) scalar integrands. Important notice: For $L > 1$, this is not true, we have *irreducible numerators*, see next slide.
- For many problems, it i preferrable to evaluate the tensors without knowing the scalar products. The reasons are different.

Simple tensor integrals

$$B^{\mu} \equiv p_1^{\mu} B_1 = \frac{1}{(i\pi^{d/2})} \int \frac{d^d k \quad k^{\mu}}{D_1 D_2}$$

$$B^{\mu\nu} \equiv p_1^{\mu} p_1^{\nu} B_{22} + g^{\mu\nu} B_{20} = \frac{1}{(i\pi^{d/2})} \int \frac{d^d k \quad k^{\mu} k^{\nu}}{D_1 D_2}$$

and B_1 **and** B_{22}, B_{20} **have to be determined.**

Reducible numerators

Some numerators are reducible – one may divide them out against the denominators:

$$\frac{2kp_1}{D_1\ [(k+p_1)^2-m_2^2]\ \ldots D_N} \equiv \frac{[(k+p_1)^2-m_2^2]-[k^2-m_1^2]+(-m_1^2+m_2^2-p_1^2)]}{D_1[(k+p_1)^2-m_2^2]\ldots D_N}$$

$$= \frac{1}{D_1D_3\ldots D_N}-\frac{1}{D_2D_3\ldots D_N}+\frac{-m_1^2+m_2^2-p_1^2}{D_1D_2D_3\ldots D_N}$$

This way one derives:

$$p_1^\mu B^\mu = p^2 B_1 = \frac{1}{(i\pi^{d/2})}\int d^dk\frac{p_1k}{D_1D_2}$$

$$= \frac{1}{(i\pi^{d/2})}\frac{1}{2}\int d^dk\left[\frac{1}{D_1}-\frac{1}{D_2}+\frac{-m_1^2+m_2^2-p_1^2}{D_1D_2}\right]$$

and finally:

$$B_1 = \frac{1}{2p_1^2}\left[A_0(m_1)-A_0(m_2)+(-m_1^2+m_2^2-p_1^2)B_0(m_1,m_2,p^2)\right]$$

Known to everybody: The Passarino-Veltman reduction scheme for 1-loop tensors worked out in

[Passarino:1979jh]

until 4-point functions.

Irreducible numerators

For a two-loop QED box diagram, it is e.g. $L = 2$, $E = 4$, and we have as potential simplest numerators:

$$k_1^2, k_2^2, k_1 k_2 \text{ and } 2(E-1) \text{ products } k_1 p_e, k_2 p_e$$

compared to N internal lines, $N = 5, 6, 7$. This gives

$$I = L + L(L-1)/2 + L(E-1) - N \text{ irreducible numerators}$$

of this type. Here:

$$I(N) = 9 - N = 4, 3, 2$$

This observation is of practical importance:

Imagine you search for potential masters. Then you may take into the list of masters at most (here e.g.) $I(5) = 4$, or $I(6) = 3$, or $I(7) = 2$ such integrals.

Which momenta combinations are irreducible is dependent on the choice of momenta flows.

Message:

When evaluating all Feynman integrals by Mellin-Barnes-integrals, one should also learn to handle numerator integrals

. . . and it is - in some cases - not too complicated compared to scalar ones

The one-loop case: $L = 1, E = N$, so

$$I(N) = 1 + (E-1) - N = 0$$

irreducible numerators

The vector integral differs by some numerator $k_i p_e$ and thus there is a single shift in the integrand

$$k \to \bar{k} + U(x)^{-1}\tilde{M}Q$$

the $\int d^d\bar{k}\ \bar{k}/(\bar{k}^2+\mu^2) \to 0$, and no further changes:

$$G(k_{1\alpha}) = (-1)^{N_\nu} \frac{\Gamma\left(N_\nu - \frac{d}{2}L\right)}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)} \int_0^1 \prod_{j=1}^{N} dx_j\ x_j^{\nu_j-1} \delta\left(1-\sum_{i=1}^{N} x_i\right) \frac{U(x)^{N_\nu - d(L+1)/2-1}}{F(x)^{N_\nu - dL/2}} \left[\sum_l \tilde{M}_{1l} Q_l\right]_\alpha,$$

Here also a tensor integral:

$$\begin{aligned} G(k_{1\alpha}k_{2\beta}) &= (-1)^{N_\nu} \frac{\Gamma\left(N_\nu - \frac{d}{2}L\right)}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)} \int_0^1 \prod_{j=1}^{N} dx_j\ x_j^{\nu_j-1} \delta\left(1-\sum_{i=1}^{N} x_i\right) \frac{U(x)^{N_\nu-2-d(L+1)/2}}{F(x)^{N_\nu - dL/2}} \\ &\times \sum_l \left[[\tilde{M}_{1l}Q_l]_\alpha [\tilde{M}_{2l}Q_l]_\beta - \frac{\Gamma\left(N_\nu - \frac{d}{2}L - 1\right)}{\Gamma\left(N_\nu - \frac{d}{2}L\right)} \frac{g_{\alpha\beta}}{2} U(x)F(x) \frac{(V_{1l}^{-1})^+(V_{2l}^{-1})}{\alpha_l} \right]. \end{aligned}$$

The 1-loop case will be used in the following L times for a sequential treatment of an L-loop integral (remember $\sum x_j D_j = k^2 - 2Qk + J$ and $F(x) = Q^2 - J$):

$$G([1, kp_e]) = (-1)^{N_\nu} \frac{\Gamma\left(N_\nu - \frac{d}{2}\right)}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)} \int_0^1 \prod_{j=1}^{N} dx_j\ x_j^{\nu_j-1} \delta\left(1-\sum_{i=1}^{N} x_i\right) \frac{[1, Qp_e]}{F(x)^{N_\nu - d/2}}$$

.

Another nice box with numerator, B5l3m($p_e.k_1$)

We used it for the determination if the small mass expansion.

$$\mathtt{B5l3m}(p_e\cdot k_1) = \frac{m^{4\epsilon}(-1)^{a_{12345}}\; e^{2\epsilon\gamma_E}}{\prod_{j=1}^{5}\Gamma[a_i]\Gamma[5-2\epsilon-a_{123}](2\pi i)^4}\int_{-i\infty}^{+i\infty}d\alpha\int_{-i\infty}^{+i\infty}d\beta\int_{-i\infty}^{+i\infty}d\gamma\int_{-i\infty}^{+i\infty}d\delta$$

$$(-s)^{(4-2\,\epsilon)-a_{12345}-\alpha-\beta-\delta}\;(-t)^{\delta}$$

$$\frac{\Gamma[-4+2\,\epsilon+a_{12345}+\alpha+\beta+\delta]}{\Gamma[6-3\,\epsilon-a_{12345}-\alpha]}\frac{\Gamma[-\alpha]\,\Gamma[-\beta]}{\Gamma[7-3\,\epsilon-a_{12345}-\alpha]\,\Gamma[5-2\,\epsilon-a_{123}]}\frac{\Gamma[-\delta]}{\Gamma[4-2\,\epsilon-a_{1123}-2\,\alpha-\gamma]\,\Gamma[5-2\,\epsilon-a_{1123}-2\,\alpha\ldots}$$

$$\frac{\Gamma[2-\epsilon-a_{13}-\alpha-\gamma]}{\Gamma[8-4\,\epsilon-a_{112233445}-2\,\alpha-2\,\beta-2\,\delta-\gamma]}\frac{\Gamma[4-2\,\epsilon-a_{12345}-\alpha-\beta-\delta-\gamma]}{\Gamma[9-4\,\epsilon-a_{112233445}-2\,\alpha-2\,\beta-2\,\delta-\gamma]}\Big\{(p_e\cdot p_3)\;\Gamma[1+a_4+\delta]\;\Gamma[6-3\,\epsilon-\ldots$$

$$\Gamma[4-2\,\epsilon-a_{1234}-\alpha-\beta-\delta]\;\Gamma[3-\epsilon-a_{12}-\alpha]\;\Gamma[8-4\,\epsilon-a_{112233445}-2\,\alpha-2\,\delta-\gamma]\Gamma[9-4\,\epsilon-a_{112233445}-2\,\alpha-2\,\beta-\ldots$$

$$\Gamma[5-2\,\epsilon-a_{1123}-\gamma]\;\Gamma[4-2\,\epsilon-a_{1123}-2\,\alpha-\gamma]\;\Gamma[a_1+\gamma]\;\Gamma[-2+\epsilon+a_{123}+\alpha+\delta+\gamma]+\Gamma[a_4+\delta]\Big[-(p_e\cdot p_1)\;\Gamma[7-3\,\epsilon-\ldots$$

$$\Gamma[4-2\,\epsilon-a_{1234}-\alpha-\beta-\delta]\Gamma[8-4\,\epsilon-a_{112233445}-2\,\alpha-2\,\delta-\gamma]\Gamma[9-4\,\epsilon-a_{112233445}-2\,\alpha-2\,\beta-2\,\delta-\gamma]$$

$$\Big[\Gamma[3-\epsilon-a_{12}-\alpha]\;\Gamma[5-2\,\epsilon-a_{1123}-\gamma]\Gamma[4-2\,\epsilon-a_{1123}-2\,\alpha-\gamma]\;\Gamma[a_1+\gamma]+\Gamma[2-\epsilon-a_{12}-\alpha]\;\Gamma[4-2\,\epsilon-a_{1123}-\gamma]$$

$$\Gamma[5-2\,\epsilon-a_{1123}-2\,\alpha-\gamma]\;\Gamma[1+a_1+\gamma]\Big]\;\Gamma[-2+\epsilon+a_{123}+\alpha+\delta+\gamma]+\Gamma[6-3\,\epsilon-a_{12345}-\alpha]\;\Gamma[3-\epsilon-a_{12}+\alpha]$$

$$\Gamma[5-2\,\epsilon-a_{1123}-\gamma]\;\Gamma[4-2\,\epsilon-a_{1123}-2\,\alpha-\gamma]\Gamma[a_1+\gamma]\Big[((p_e\cdot(p_1+p_2))\;\Gamma[5-2\,\epsilon-a_{1234}-\alpha-\beta-\delta]\Gamma[9-4\,\epsilon-a_{11223\ldots}$$

$$\Gamma[8-4\,\epsilon-a_{112233445}-2\,\alpha-2\,\beta-2\,\delta-\gamma]\Gamma[-2+\epsilon+a_{123}+\alpha+\delta+\gamma]+(p_e\cdot p_1)\;\Gamma[4-2\,\epsilon-a_{1234}-\alpha-\beta-\delta]$$

$$\Gamma[8-4\,\epsilon-a_{112233445}-2\,\alpha-2\,\delta-\gamma]\Gamma[9-4\,\epsilon-a_{112233445}-2\,\alpha-2\,\beta-2\,\delta-\gamma]\Gamma[-1+\epsilon+a_{123}+\alpha+\delta+\gamma]\Big]\Big\}$$

Interlude: Tensor integrals, a bit background

Introducing α parameter representations for the propagators,

$$\frac{1}{(k^2-m^2+i\epsilon)^\nu} = \frac{i^{-\nu}}{\Gamma(\nu)}\int_0^\infty d\alpha\alpha^{\nu-1}\exp i\alpha(k^2-m^2+i\epsilon)$$

one may write e.g. for the L-loop integral with I internal lines and E external lines

$$G^{(d)} \equiv \int \frac{d^dk_1\dots d^dk_L}{c_1^{\nu_1}\dots c_I^{\nu_I}}\exp i(a_1k_1+\dots+a_Lk_L)$$

$$= i^2\left(\frac{\pi}{i}\right)^{dL/2}\prod_{i=1}^{I}\frac{i^{-\nu_i}}{\Gamma(\nu)}\int_0^\infty\frac{d\alpha_i\alpha_i^{\nu_i-1}}{[D(\alpha)]^{d/2}}\exp i\left[\frac{Q(\alpha,a_1,\dots a_L)}{D(\alpha)}-\sum_{i=1}^{I}\alpha_i(m_i^2-i\epsilon)\right],$$

with the following notations:

$$c_i = q_i^2-m_i^2+i\epsilon,$$

$$q_i^\mu = \sum_{l=1}^{L}\omega_{il}k_l^\mu+\sum_{e=1}^{E}\eta_{ie}p_e^\mu = \sum_{l=1}^{L}\omega_{il}k_l^\mu+e_i^\mu, \qquad i=1,\dots I,$$

and the ω_{il} and η_{ie} are equal to $+1,-1,0$, depending on the topology and on the distribution of internal momenta. They are incidence matrices of a diagram.

This formula represents a scalar integral for $a_l=0$, and for $a_i\neq 0$ it is an auxiliary representation for the treatment of tensor integrals.

The Q and D forms may be determined to be as follows for the 1-loop case:

$$\begin{aligned} D &= A, \\ Q(\alpha, a) &= DF - \left(\frac{a}{2} + E\right)^2, \end{aligned}$$

with

$$\begin{aligned} A &= \sum_{i=1}^{I} \omega_i^2 \ \alpha_i, \\ E^\mu &= \sum_{i=1}^{I} \omega_i e_i^\mu \ \alpha_i, \\ F &= \sum_{i=1}^{I} (e_i^\mu)^2 \ \alpha_i. \end{aligned}$$

Explicit expressions are given for D and the a-dependent part of Q for the convention used in Figure ??. We may add the a-independent part $Q(a=0)$ of the Q form here:

$$\begin{aligned} D &= \sum \alpha_j, \\ Q &= a_\mu \sum \alpha_j p_j^\mu - \frac{1}{4} a^2 + Q(a=0), \\ Q(a=0) &= DF - E^2 \ = \ \left(\sum \alpha_j\right)\left(\sum \alpha_k p_k^2\right) - \left(\sum \alpha_k p_k\right)^2. \end{aligned}$$

The Q and D forms may be determined to be as follows for the 2-loop case

$$
\begin{aligned}
D &= A_1 A_2 - M^2, \\
Q(\alpha, a_1, a_2) &= DF - A_2 E_1^2 - A_1 E_2^2 + 2 M E_1 E_2 + Q(a_1, a_2),
\end{aligned}
$$

and

$$
\begin{aligned}
A_1 &= \sum_{i=1}^{I} \omega_{i1}^2 \ \alpha_i, \\
A_2 &= \sum_{i=1}^{I} \omega_{i2}^2 \ \alpha_i, \\
M &= \sum_{i=1}^{I} \omega_{i1} \omega_{i2} \ \alpha_i, \\
E_l^\mu &= \sum_{i=1}^{I} \omega_{il} e_i^\mu \ \alpha_i, \\
F &= \sum_{i=1}^{I} \left(e_i^\mu\right)^2 \ \alpha_i.
\end{aligned}
$$

The $Q(a_1, a_2)$ is given in Section ??.

General tensor reduction to scalars, $L = 1$

The idea for 1-loop integrals and generalization to n-loops; I follow Tausk et al.
[Davydychev:1991va, Tarasov:1996br, Tarasov:1997kx]

$$
\begin{aligned}
J^{(N)}_{\mu_1\ldots\mu_R}(d;\nu_1,\ldots,\nu_N) &= \int \frac{d^dk}{i\pi^{d/2}} \frac{k_{\mu_1}\ldots k_{\mu_R}}{D_1^{\nu_1}\ldots D_N^{\nu_N}} \\
&= (-1)^R \sum_{\lambda,\kappa_1,\ldots,\kappa_N} \left(-\frac{1}{2}\right)^{\lambda} \left\{[g]^{\lambda}[q_1]^{\kappa_1}\ldots[q_N]^{\kappa_N}\right\}_{\mu_1\ldots\mu_R} (\nu_1)_{\kappa_1}\ldots(\nu_N)_{\kappa_N} \\
&\quad J^{(N)}(d+2(R-\lambda);\nu_1+\kappa_1,\ldots,\nu_N+\kappa_N)
\end{aligned}
$$

where $(\nu)_\kappa = \frac{\Gamma(\nu+\kappa)}{\Gamma(\nu)}$ are Pochhammer symbols and the sum runs over non-negative integers such that $2\lambda+\kappa_1+\ldots+\kappa_N = R$. The next step is to use recurrence relations to reduce the scalar coefficients $J^{(N)}$ appearing in the decomposition to a set of master integrals.

It is useful to introduce a notation for certain determinants that occur in the recurrence relations and their solutions. First, the determinant of an $(N+1)\times(N+1)$ matrix, known as the modified Cayley determinant

[Melrose:1965kb]

,

$$()_N \equiv \begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & Y_{11} & Y_{12} & \dots & Y_{1N} \\ 1 & Y_{12} & Y_{22} & \dots & Y_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & Y_{1N} & Y_{2N} & \dots & Y_{NN} \end{vmatrix}, \tag{14}$$

with coefficients

$$Y_{ij} = -(q_i - q_j)^2 + m_i^2 + m_j^2 , \quad (i, j = 1 \dots N) . \tag{15}$$

Although the masses of the propagators appear in the coefficients Y_{ij}, the determinant $()_N$ does not depend on them and it is actually proportional to the Gram determinant of the external momenta of the N-point function in eq. (14). All other determinants we need are signed minors of $()_N$, constructed by deleting m rows and m columns from $()_N$, and multiplying with a sign factor. They will be denoted by

$$\begin{pmatrix} j_1 & j_2 & \cdots & j_m \\ k_1 & k_2 & \cdots & k_m \end{pmatrix}_N \equiv (-1)^{\sum_l (j_l + k_l)}$$

$$\mathbf{sgn}_{\{j\}}\, \mathbf{sgn}_{\{k\}} \begin{vmatrix} \textbf{rows } j_1 \cdots j_m \textbf{ deleted} \\ \textbf{columns } k_1 \cdots k_m \textbf{ deleted} \end{vmatrix}, \tag{16}$$

where $\mathrm{sgn}_{\{j\}}$ and $\mathrm{sgn}_{\{k\}}$ are the signs of permutations that sort the deleted rows $j_1 \cdots j_m$ and columns $k_1 \cdots k_m$ into ascending order.

Combining integration by parts identities with relations connecting integrals in different space-time dimensions

[Tarasov:1996br]

, one obtains the following basic recurrence relations

[Fleischer:1999hq]

:

$$
\begin{aligned}
()_N \, \nu_j \mathbf{j}^+ J^{(N)}(d+2) &= \left[-\binom{j}{0}_N + \sum_{k=1}^{n} \binom{j}{k}_N \mathbf{k}^- \right] J^{(N)}(d) \,, \\
(d - \sum_{i=1}^{n} \nu_i + 1) \, ()_N \, J^{(N)}(d+2) &= \left[\binom{0}{0}_N - \sum_{k=1}^{n} \binom{0}{k}_N \mathbf{k}^- \right] J^{(N)}(d) \,, \\
\binom{0}{0}_N \nu_j \mathbf{j}^+ J^{(N)}(d) &= \sum_{k=1}^{n} \binom{0j}{0k}_N \left[d - \sum_{i=1}^{n} \nu_i (\mathbf{k}^- \mathbf{i}^+ + 1) \right] J^{(N)}(d) \,.
\end{aligned}
$$

where the operator $\mathbf{j}^{\pm}$ acts by shifting the index ν_j by ± 1.

Recurrence relations: An application

$\frac{1}{d-4} = -\frac{1}{2\epsilon}$

$$C_0(m,0,m;m^2,m^2,s) = \frac{1}{s-4m^2}\left[\frac{d-2}{d-4}\,\frac{A_0(m^2)}{m^2} + \frac{2d-3}{d-4}\,B_0(m,m;s)\right]$$

$$V3l2m = \frac{-1}{s-4}\left[\frac{1-\epsilon}{\epsilon}\,T1l1m + \frac{5-4\epsilon}{2\epsilon}\,SE2l2m\right] \qquad (17)$$

This relates UV-divergent quantities with one which is IR-divergent!

Derivation:

I do not know a simple one.

With use of recurrence relations published in

[Tarasov:1996br,Fleischer:1999hq]

it may be done. See

[Fleischer:2006ht]

.

Because the am determinant vanishes, $\binom{0}{0}_3 = 0$**, one of the recurrence relations**

expresses the vertex by a self-energy with 2 massive lines in $d-2$ dimensions:

$$(d-4)\,()_3\, I_3^{(d)} \quad = \quad \left[\binom{0}{0}_3 - \sum_{k=1}^{3}\binom{0}{k}_3 \mathbf{k}^-\right] I_3^{(d-2)} = I_2^{(d-2)}(m,m)$$

The dimension of the 2-point function may be raised by another recurrence relation, but now one line will be dotted:

$$I_2^{(d-2)}(m,m) = -2\mathbf{1}^+ I_2^d(m,m), \tag{18}$$

The dot may be removed, in general form, by expressing a dotted function by itself, but undotted, plus extra terms with one line less, but a dotted one:

$$\binom{0}{0}_2 \mathbf{1}^+ I_2^{(d)} \qquad = \left[(3-d)\binom{0}{1}_2\right] I_2^{(d)} - \sum_{i,k\; i\neq k}^{2} \binom{01}{0k}_2 \mathbf{k}^- \mathbf{i}^+ I_2^{(d)}, \tag{19}$$

Here the latter is a dotted tadpole, and that is proportional to the tadpole itself, and we get the relation needed. (In general, iterations do the job here.)

Direct evaluation shows:

$$A_0(m)^{dot} = \frac{1}{2}(d-2)\frac{A_0(m)}{m^2}$$

- **hexagon package in Mathematica and (yet) unpublished Fortran code**

The packages evaluate

- **6-point functions up to tensor rank 4**
- **5-point functions up to tensor rank 3**

6-point functions up to tensor rank 4

[Fleischer:1999hq,Diakonidis:0901.4455,Diakonidis:2008ij]

One reduction: 5-point tensor, rank three

$$I_5^{\mu\,\nu\,\lambda} = \sum_{i,j,k=1}^{4} q_i^{\mu}\, q_j^{\nu}\, q_k^{\lambda} E_{ijk} + \sum_{k=1}^{4} g^{[\mu\nu} q_k^{\lambda]} E_{00k}\,,$$

where

$$g^{[\mu\nu} q_k^{\lambda]} = g^{\mu\nu}\, q_k^{\lambda} \,+\, g^{\mu\lambda}\, q_k^{\nu} \,+\, g^{\nu\lambda}\, q_k^{\mu}\,, \tag{20}$$

with scalar coefficients defined by

$$E_{ijk} = \sum_{s=1}^{5} S_{ijk}^{4,s} I_4^{s} + \sum_{s,t=1}^{5} S_{ijk}^{3,st} I_3^{st} + \sum_{s,t,u=1}^{5} S_{ijk}^{2,stu} I_2^{stu}\,, \tag{21}$$

with

$$
\begin{aligned}
S_{ijk}^{4,s} = & \frac{1}{3\binom{0}{0}_5\binom{s}{s}_5^2}\times \\
& \left\{-\binom{0s}{0k}_5\left[\binom{0s}{is}_5\binom{0s}{js}_5+\binom{is}{js}_5\binom{0s}{0s}_5\right]\right. \\
& \left.+\binom{0s}{0s}_5\left[\binom{0i}{sk}_5\binom{0s}{js}_5+\binom{0j}{sk}_5\binom{0s}{is}_5\right]\right\} \\
& +(i\leftrightarrow k)+(j\leftrightarrow k)\,,
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
S_{ijk}^{3,st} = & \frac{1}{3\binom{0}{0}_5\binom{s}{s}_5^2}\left\{\binom{0s}{0k}_5\left[\binom{ts}{is}_5\binom{0s}{js}_5\right.\right. \\
& \left.+\binom{is}{js}_5\binom{ts}{0s}_5+\frac{\binom{s}{s}_5\binom{0st}{ist}_5}{\binom{st}{st}_5}\binom{ts}{js}_5\right] \\
& -\left[\binom{0i}{sk}_5\binom{0s}{js}_5+\binom{0j}{sk}_5\binom{0s}{is}_5\right]\binom{ts}{0s}_5 \\
& -\left[\binom{0i}{sk}_5\binom{ts}{js}_5+\binom{0j}{sk}_5\binom{ts}{is}_5\right] \\
& \left.\times\frac{\binom{s}{s}_5\binom{0st}{0st}_5}{2\binom{st}{st}_5}\right\}+(i\leftrightarrow k)+(j\leftrightarrow k)\,,
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
S^{2,stu}_{ijk} = & -\frac{1}{3\binom{0}{0}_5\binom{s}{s}_5\binom{st}{st}_5}\times \\
& \left\{ \binom{0s}{0k}_5 \binom{ts}{js}_5 \binom{ust}{ist}_5 - \frac{1}{2}\left[\binom{0j}{sk}_5 \binom{ust}{ist}_5 \right.\right. \\
& \left.\left. + \binom{0i}{sk}_5 \binom{ust}{jst}_5 \right] \binom{ts}{0s}_5 \right\} \\
& + (i \leftrightarrow k) + (j \leftrightarrow k)\,, \qquad (24)
\end{aligned}
$$

and

$$E_{00j} = \frac{1}{6\binom{0}{0}_5} \left\{ -\sum_{s=1}^{5} \frac{1}{\binom{s}{s}_5^2} \right.$$
$$\times \left[3\binom{s}{0}_5 \binom{0s}{js}_5 - \binom{s}{j}_5 \binom{0s}{0s}_5 \right] \binom{0s}{0s}_5 I_4^s$$
$$+ \sum_{s,t=1}^{5} \frac{1}{\binom{s}{s}_5^2}$$
$$\times \left[3\binom{s}{0}_5 \binom{0s}{js}_5 - \binom{s}{j}_5 \frac{\binom{ts}{0s}_5^2}{\binom{st}{st}_5} \right] \binom{ts}{0s}_5 I_3^{st}$$
$$\left. - \sum_{s,t,u=1}^{5} \binom{s}{j}_5 \frac{\binom{ust}{0st}_5}{\binom{s}{s}_5 \binom{st}{st}_5} \binom{ts}{0s}_5 I_2^{stu} \right\} . \qquad (25)$$

The decomposition in eq. (20) is equivalent to the one found in ref.

[Denner:2002ii]

, where the coefficients E_{ijk} and E_{00j} are expressed in terms of tensor 4-point functions. Here, instead, they are completely reduced to a basis of scalar master integrals consisting of boxes I_4^s, vertices I_3^{st}, and 2-point functions I_2^{stu} obtained by removing lines s, s and t, or s, t and u from the pentagon.

Summary

- We have introduced to the representation of L-loop N-point Feynman integrals of general type
- The determination of the ϵ-poles is generally solved
- The remaining problem is the evaluation of the multi-dimensional, finite MB-Integrals
- This is unsolved in the general case, . . . so you have something to do if you like to . . .

Problem: Determine the small mass limit of B5l2m2 or of any other of the 2-loop boxes for Bhabha scattering.
Prof. Gluza may check your solution.
He leaves soon.

On-shell example: B4l2m, the 1-loop on-shell box

```
den = (x4 d4 + x5 d5 + x6 d6 + x7 d7 // Expand) /. kinBha /. m^2 -> 1 //  Expand

Q = -Coefficient[den, k]/2 // Simplify
    =  p3 x4 + p2 x5 - p1 (x4 + x6)

M = Coefficient[den, k^2] // Simplify
    = x4 + x5 + x6 + x7 -> 1

J = den /. k -> 0 // Simplify
    = t x4

F[x] = (Q^2 - J  M // Expand) /. kinBha /. m^2 -> 1 /. u -> -s - t + 4 // Expand
    = (x5+x6)^2 + (-s)x5x6 + (-t)x4x7

B4l2ma = mb[(x5+x6)^2, -tx7x4 - sx5x6, nu, ga]

B4l2mb = B4l2ma /. (-sx5x6 - tx4x7)^(-ga - nu) ->
            mb[(-s)x5x6, (-t)x7x4, nu+ga, de]
            /.((-s)x5x6)^de_ -> (-s)^de x5^de x6^de
            /.((x56^2)^ga    -> (x5 + x6)^(2ga)
```

```
=
   (inv2piI^2(-s)^de x5^de x6^de ((x5 + x6)^(2ga)((-t)x4x7)^(-de-ga-nu)
        Gamma[-de] Gamma[-ga] Gamma[de + ga + nu] /Gamma[nu]

B4l2mc = B4l2mb /. (x5 + x6)^(2ga) ->
            mb[x5, x6, -2ga, si]
            /. ((-t)x4x7)^si_ -> (-t)^si x4^si x7^si // ExpandAll
=
   1/(Gamma[-2ga] Gamma[nu])
inv2piI^3  (-s)^de (-t)^(-de - ga - nu)
            x4^(-de - ga - nu) x5^(de + si) x6^(de + 2 ga - si) x7^(-de - ga - nu)
            Gamma[-de] Gamma[-ga] Gamma[ de + ga + nu] Gamma[-si] Gamma[-2ga + si]

B4l2md =  xfactor4[a4, x4, a5, x5, a6, x6, a7, x7] B4l2mc
=
.... (-s)^de (-t)^(-de-ga-nu)
x4^(-1+a4-de-ga-nu) x5^(-1+a5+de+si) x6^(-1+a6+de+2ga-si) x7^(-1+a7- de-ga-nu)

B4l2me =
  B4l2md /.
      x4^B4_  x5^B5_  x6^B6_  x7^B7_ -> xint4[x4^B4 x5^B5 x6^B6 x7^B7]
= ...
```

```
B4l2mf =  B4l2me /.
      Gamma[a6 + de + 2 ga - si]Gamma[-si]Gamma[ a5 + de + si] Gamma[-2 ga + si]
    ->  barne1[si, a5 + de, -2 ga, a6 + de + 2 ga, 0]
```

This finishes the evaluation of the MB-representation for `B4l2m`.

Package: AMBRE.m (K. Kajda, with support by J. Gluza and TR)

Example beyond Harmonic Polylogs: QED Box B4l2m

[Fleischer:2006ht]

```
F[x]       = (x5+x6)^2 + (-s)x5x6 + (-t)x4x7
```

`B4l2m`, the 1-loop QED box, with two photons in the s-channel; the Mellin-Barnes representation reads for finite ϵ:

$$\mathtt{B4l2m} = \mathtt{Box}(t,s) = \frac{e^{\epsilon\gamma_E}}{\Gamma[-2\epsilon](-t)^{(2+\epsilon)}} \frac{1}{(2\pi i)^2} \int_{-i\infty}^{+i\infty} dz_1 \int_{-i\infty}^{+i\infty} dz_2$$
$$\frac{(-s)^{z_1}(m^2)^{z_2}}{(-t)^{z_1+z_2}} \Gamma[2+\epsilon+z_1+z_2]\Gamma^2[1+z_1]\Gamma[-z_1]\Gamma[-z_2]$$
$$\Gamma^2[-1-\epsilon-z_1-z_2] \frac{\Gamma[-2-2\epsilon-2z_1]}{\Gamma[-2-2\epsilon-2z_1-2z_2]} \quad (26)$$

`Mathematica` package `MB` used for analytical expansion $\epsilon \to 0$:

[Czakon:2005rk]

$$\mathtt{B412m} = -\frac{1}{\epsilon}\mathtt{J}_1 + \ln(-s)\mathtt{J}_1 + \epsilon\left(\frac{1}{2}\left[\zeta(2) - \ln^2(-s)\right]\mathtt{J}_1 - 2\mathtt{J}_2\right). \tag{27}$$

with J_1 being also the divergent part of the vertex function $C_0(t;m,0,m)/s = \mathtt{V312m}/s$ (as is well-known):

$$\mathtt{J}_1 = \frac{e^{\epsilon\gamma_E}}{st}\frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dz_1 \left(\frac{m^2}{-t}\right)^{z_1} \frac{\Gamma^3[-z_1]\Gamma[1+z_1]}{\Gamma[-2z_1]} = \frac{1}{m^2 s}\frac{2y}{1-y^2}\ln(y) \tag{28}$$

with

$$y = \frac{\sqrt{1-4m^2/t} - 1)}{(\sqrt{1-4m^2/t} + 1)}$$

The J_2 is more complicated:

$$\mathtt{J}_2 = \frac{e^{\epsilon\gamma_E}}{t^2}\frac{1}{(2\pi i)^2} \int_{-\frac{3}{4}-i\infty}^{-\frac{3}{4}+i\infty} dz_1 \left(\frac{s}{t}\right)^{z_1} \Gamma[-z_1]\Gamma[-2(1+z_1)]\Gamma^2[1+z_1] \times \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dz_2 \left(-\frac{m^2}{t}\right)^{z_2}\Gamma[-z_2]\frac{\Gamma^2[-1-z_1-z_2]}{\Gamma[-2(1+z_1+z_2)]}\Gamma[2+z_1+z_2]. \tag{29}$$

The expansion of B4l2m at small m^2 and fixed value of t

With

$$m_t = \frac{-m^2}{t}, \quad (30)$$

$$r = \frac{s}{t}, \quad (31)$$

Look, under the integral, at $(-m^2/t)^{z_2}$,

and close the path to the right.

Seek the residua from the poles of Γ-functions with the smallest powers in m^2 and sum the resulting series.

we have obtained a compact answer for J_2 with the additional aid of XSUMMER

[Moch:2005uc]

The box contribution of order ϵ in this limit becomes:

$$\begin{aligned} \texttt{B4l2m}[t,s,m^2;+1] &= \frac{1}{st}\Big\{4\zeta_3 - 9\zeta_2\ln(m_t) + \frac{2}{3}\ln^3(m_t) + 6\zeta_2\ln(r) - \ln^2(m_t)\ln(r) \\ &+\frac{1}{3}\ln^3(r) - 6\zeta_2\ln(1+r) + 2\ln(-r)\ln(r)\ln(1+r) - \ln^2(r)\ln(1+r) \\ &+2\ln(r)\mathrm{Li}_2(1+r) + 2\mathrm{Li}_3(-r)\Big\} + \mathcal{O}(m_t). \end{aligned} \quad (32)$$

Remark:

The exact Box function is NOT expressible by Harmonic Polylogs, one may introduce a

generalization of them: Generalized HPLs.
Automatized tools for this might be developed.

A scetch of the small mass expansion may be made as follows.

First the 1-dim. integral J_1.
The leading term comes from the first residue:

```
J1 =  Residue[m_t^z1 Gamma[-z1]^3 Gamma[1 + z1]/Gamma[-2 z1], {z1, 0}]

   = 2 Log[m_t]
```

WE get a logarithmic mass dependence.
The second integral: Start with z_2, first residue is:

```
I2 = Residue[ m_t^z2 Gamma[-z2] Gamma[-1 - z1 - z2]^2
                     Gamma[2 + z1 + z2]/Gamma[-2 - 2 z1 - 2 z2], {z2, 0}]
```

$$= -\frac{\Gamma[-1-z_1]^2\Gamma[2+z_1]}{\Gamma[-2-2z_1]}$$

The residue is independent of m^2/t.

It has to be integrated over z_1 yet, together with the terms which were independent of z_2:

$$I_2 \sim \int dz_1 r^{z_1+1}\Gamma[-z_1]\Gamma[-2-2z_1]\Gamma[1+z_1]^2 \frac{\Gamma[-1-z_1]^2\Gamma[2+z_1]}{\Gamma[-2-2z_1]}$$

Sum over residues, close path to the left:

$$\begin{aligned} Residue[z_1=-n] &= \frac{(-1)^n r^{1-n}}{2(-1+n)^3}\left[2+(-1+n)^2\pi^2+(-1+n)\ln[r](2+(-1+n)\ln[r])\right] \\ Residue[z_1=-1] &= \frac{1}{6}(3\pi^2\ln[r]+\ln[r]^3) \end{aligned}$$

and finally:

$$I_2 \sim Residue[z_1=-1] + \sum_{n=2}^{\infty} Residue[z_1=-n]$$

The sum can be done also withoput using XSUMMER (here at least), e.g.

$$\ln[r]\sum_{n=2}^{Infty} \frac{(-1)^n(2+\pi^2-2n\pi^2+n^2\pi^2)r^{1-n}}{2(-1+n)^3} = \frac{1}{2}\left[\pi^2\ln(1+1/r) - 2Li_3\left(-1/r\right)\right]$$

etc

Some routines in mathematica which may be used:

```
(* Barnes' first lemma: \int d(si) Gamma(si1p+si)Gamma(si2p+si)Gamma(si1m-si)Gamma(si2m-si)
                        with 1/inv2piI = 2 Pi I *)

barne1[si_, si1p_, si2p_, si1m_, si2m_] :=
  1/inv2piI  Gamma[si1p + si1m] Gamma[si1p + si2m] Gamma[
      si2p + si1m] Gamma[si2p + si2m] /Gamma[si1p + si2p + si1m + si2m]

 (* Mellin-Barnes integral: (A+B)^(-nu) = 1/(2 Pi I) \int d(si) a^si b^(-nu - si)
                            Gamma[-si]Gamma[nu+si]/Gamma[nu]  *)

mb[a_, b_, nu_, si_]:=inv2piI a^si b^(-nu-si)Gamma[-si]Gamma[nu+si]/Gamma[nu]

 (* After the k-integration, the integrand for \int\prod(dxi xi^(ai -  1))\delta(1-\sum xi)
              will be (L=1 loop) : xfactorn F^(-nu) Q(xi).pe with  nu = a1 + .. + an - d/2 *)

xfactor3[a1_, x1_, a2_, x2_, a3_, x3_] :=
  I Pi^(d/2) (-1)^(a1 + a2 + a3) x1^(a1 - 1) x2^(a2 - 1)x3^(a3 - 1)Gamma[
        a1 + a2 + a3 - d/2]/(Gamma[a1]Gamma[a2]Gamma[a3])

 (* xinti - the i-dimensional x - integration over Feynman parameters /16 06 2005 *)

xint3[x1_^(a1_) x2_^(a2_) x3_^(a3_)  ] :=
  Gamma[a1 + 1] Gamma[a2 + 1] Gamma[a3 + 1]  / Gamma[a1 + a2 + a3 + 3]
```

quote also:

[Argeri:2007up]
[Smirnov:2008py]
[Ellis:2007qk]
[Binoth:2008uq]
[Diakonidis:2008ij]
[Bogner:2007cr]
[Heinrich:2008si]
[Fleischer:2007ph]
[Diakonidis:2008dt]

References

Feynman integrals: scalar and tensor integrals, shrinked and dotted ones

Just to mention what kind of integrals may appear:

- **diagrams with numerators:**
  tensors arise from internal fermion lines: $\int d^dk_i \cdots \frac{\Gamma_\nu(k_i^\nu - p_n^\nu) - m_n}{(k_i - p_n)^2 - m_n^2} \cdots$
- **diagrams with shrinked and/or with dotted lines: a sample relation see next page**
- **relations to simpler diagrams may shift the complexity:** $\frac{1}{d-4} = -\frac{1}{2\epsilon}$

$$C_0(m,0,m;m^2,m^2,s) = \frac{1}{s-4m^2}\left[\frac{d-2}{d-4}\,\frac{A_0(m^2)}{m^2} + \frac{2d-3}{d-4}\,B_0(m,m;s)\right]$$

$$V3l2m = \frac{-1}{s-4}\left[\frac{1-\epsilon}{\epsilon}\,T1l1m + \frac{5-4\epsilon}{2\epsilon}\,SE2l2m\right] \qquad (33)$$

$$\texttt{SE3l2m}(a,b,c,d) \quad = \quad -\frac{e^{2\epsilon\Gamma_E}}{\pi^D}\int \frac{d^D k_1 d^D k_2 \quad (k_1 k_2)^{-d}}{[(k_1+k_2-p)^2-m^2]^b \; [k_1^2]^a \; [k_2^2-m^2]^c}.$$

$$\begin{aligned} \texttt{SE3l2m} &= \texttt{SE3l2m}(1,1,1,0) \\ \texttt{SE3l2md} &= \texttt{SE3l2m}(1,1,2,0) \\ \texttt{SE3l2mN} &= \texttt{SE3l2m}(1,1,1,-1) \end{aligned}$$

$$\begin{aligned} \texttt{SE3l2md} &= \frac{-(1+s)+\epsilon(2+s)}{s-4}\,\texttt{SE3l2m} + \frac{2(1-\epsilon)}{s-4}\left(\texttt{T1l1m}^2 + 3\,\texttt{SE3l2mN}\right), \\ &= \frac{1}{2\epsilon^2} + \frac{1}{2\epsilon} - \left(\frac{1-\zeta_2}{2} + \frac{1+x}{1-x}\ln(x) + \frac{1+x^2}{(1-x)^2}\frac{1}{2}\ln^2(x)\right) + \mathcal{O}(\epsilon) \end{aligned}$$

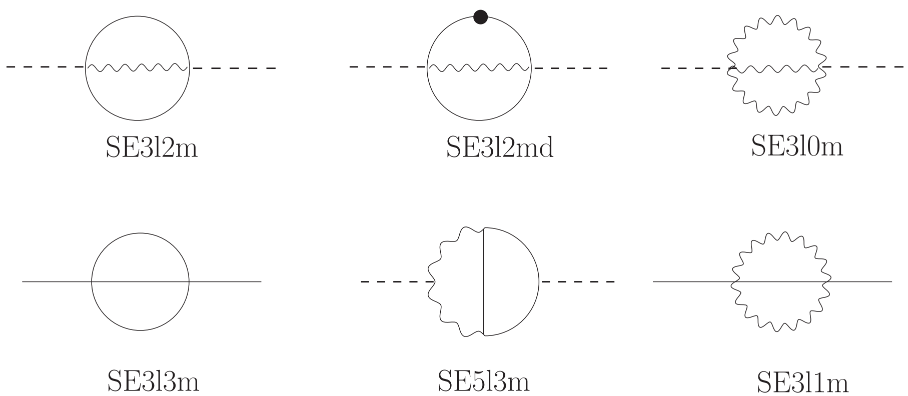


Figure 1: The six two-loop 2-point masters of Bhabha scattering.

More legs, more loops

Seek methods for an (approximated) analytical or numerical evaluation of more involved diagrams

Remember: UltraViolet (UV) und InfraRed (IR) divergencies appear

Might try:

- **same as for simple one-loop: Feynman parameters, direct integration**
- **use algebraic relations between integrals and find a (minimal) basis of master integrals – is a preparation of the final evaluation**
- **derive and solve (system of) differential equations**
- **derive and solve (system of) difference equations**
- **do something else for a direct evaluation of single integrals – of all of them or of the masters only**

Into the last category falls what we present here:
Use Feynman parameters and transform the problem