A recursive reduction of tensor Feynman integrals

Tord Riemann

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based on work with

Th.Diakonidis, J.Fleischer, J.B.Tausk [arXiv:0907.2115]

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- **Introduction**
- **Algebraic Davydychev-Tarasov approach** **[AD,OT,FJT + Diak.et.al.PRD 2009]**
- **Recursions** **[arXiv:0907.2115]** **+ Cancellations** **new**
- **Numbers**
- **Summary**

11-2009: 2 typos on p.5 corr. and p.7 after p.8; date of print: 2009-11-26 12:01

Introduction

Why not this talk in the NLO session of the workshop?

Reduction of tensor integrals = expressing them by a (very) small set of scalar integrals

– is needed for practically any Feynman diagram calculation.

Feynman diagrams with loops contain tensor integrals due to:

- fermion propagators
- three-gauge boson couplings
- e.g. unitary gauge propagators

Examples:

- LO (Lowest order) of $Z \to e + \mu$ is one-loop; processes forbidden in tree diagrams of Standard Model
- NLO: one-loop corrections to Born diagrams; prominent here at Radcor: $2 \to 3, 4, \cdots$
- NNLO: radiative loop corrections to Born diagrams: $e^+e^- \to e^+e^-\gamma$ with a loop (including 5-point functions)

My first tensor reduction [Proc. Ahrenshoop Symposium, 1981] was for:

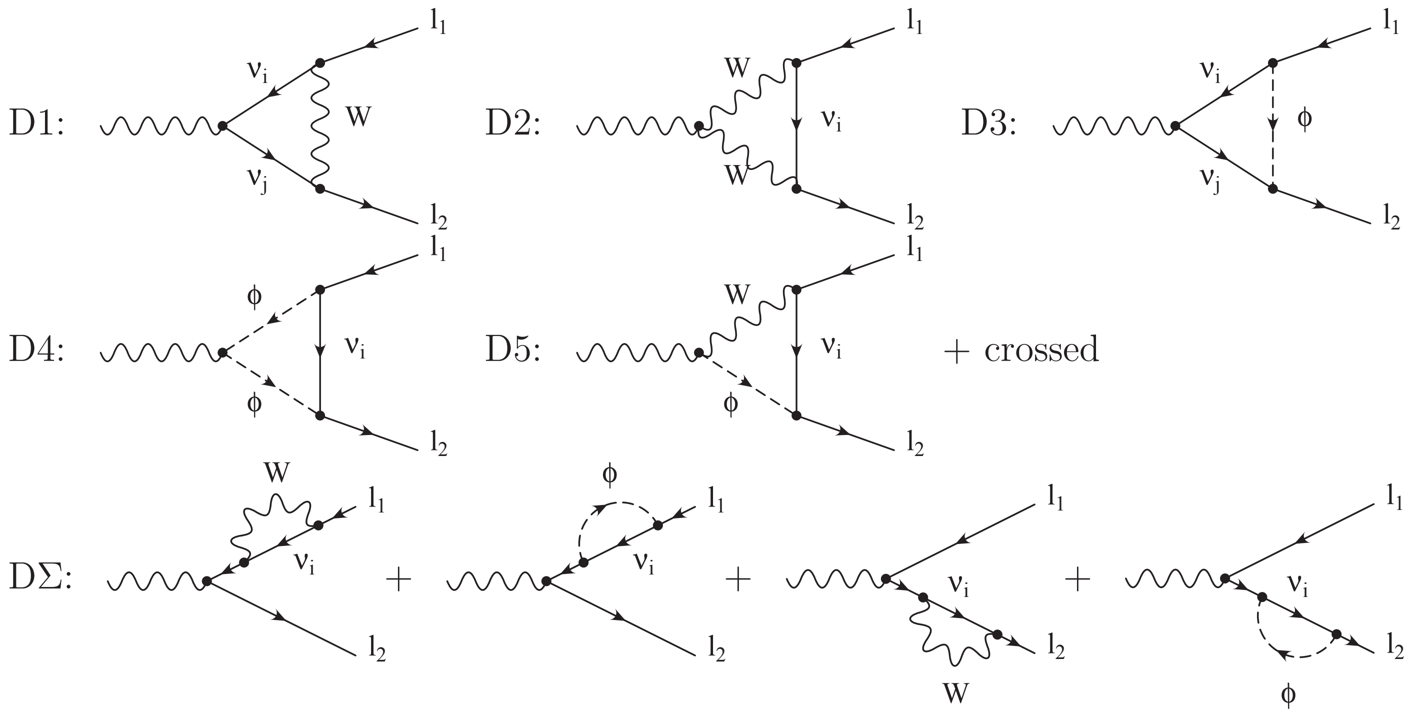
$$Z \to \mu + e, \quad b + s, \quad \text{etc.}$$

Use of the 'conventional' Passarino-Veltman reduction (Nucl.Phys.B160, 1979), invented not so much earlier.

This was done in 1980.

Use of the ’conventional’ Passarino-Veltman reduction (Nucl.Phys.B160, 1979), invented not so much earlier.

The process is, in the Standard Model, possible due to one-loop vertex diagrams with internal flavor-changing W bosons and mixing heavy fermions:



J. Illana,TR, PRD 2000 study repeated for ILC applications for SM and models with heavy neutrinos

J. Illana studied also supersymmetric cases.

In this talk, we discuss the n-point tensor integrals of rank R, in short: (n,R)-integrals:

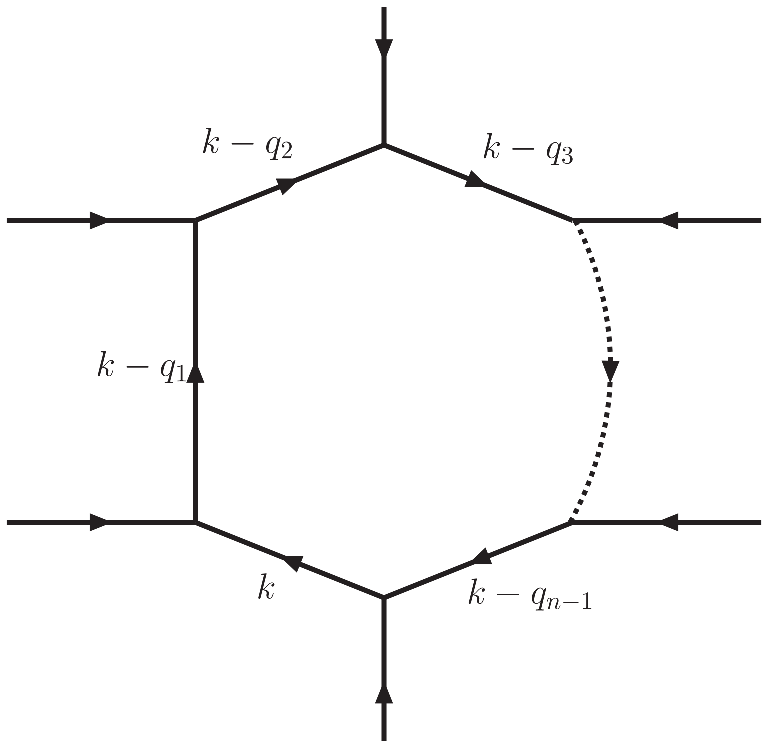
$$I_n^{\mu_1\cdots\mu_R} = \int \frac{d^dk}{i\pi^{d/2}} \frac{\prod_{r=1}^{R} k^{\mu_r}}{\prod_{j=1}^{n} c_j^{\nu_j}}, \tag{1}$$

where the denominators c_j have *indices* ν_j and *chords* q_j:

$$c_j = (k-q_j)^2 - m_j^2 + i\varepsilon \tag{2}$$

We use

$$d = 4 - 2\epsilon$$



The problems

It is well-known how to evaluate n-point tensor integrals for $n = 1, 2, 3, 4$.

This here is not a review, so I just mention as publicly available:

- **Passarino,Veltman NPB (1979) for $n = 1, 2, 3, 4$**
- **FF by van Oldenborgh (1990), a realization of that scheme**
- **LoopTools/FF by Hahn (1998,2006) – covers also 5-point functions, rank $R \leq 4$**

  We observed problems in certain configurations with light-like external particles
- **Golem by Binoth al. (2008) for $n \leq 6$, but only massless propagators**
- **many interesting details may be found in: Review talk by Denner (DESY TH workshop, 2009)**

So, if you want to evaluate something less trivial, you have to create your own tensor reduction library.

Algebraic Davydychev-Tarasov approach (I): Davydychev's higher dimensional integrals

Davydychev PLB (1991) has shown that one may write any tensor integral as a composition of scalar integrals

but the price is two-fold:

- **higher dimensions** $[d^+]^l = 4 - 2\epsilon + 2l$
- **higher indices – the powers ν_i of the denominators of the propagators**

In fact, we will need the scalar integrals with shifted indices and shifted dimensions:

$$I_{p,\,i\,j\,k\cdots}^{[d+]^l,stu\cdots} = \int \frac{d^{4-2\epsilon+2l}k}{\pi^{d/2}} \prod_{r=1}^{n} \frac{1}{c_r^{1+\delta_{ri}+\delta_{rj}+\delta_{rk}+\cdots-\delta_{rs}-\delta_{rt}-\delta_{ru}-\cdots}} \tag{3}$$

The $I_{p,\,i\,j\,k\cdots}^{[d+]^l,stu\cdots}$ is a p-point function, $p \leq n$, and some propagators may have higher power. Simplest cases:

$$I_n^{\mu} = \int \frac{d^{4-2\epsilon}k}{\pi^{d/2}} k^{\mu} \prod_{r=1}^{n} c_r^{-1} = -\sum_{i=1}^{n-1} q_i^{\mu}\, I_{n,i}^{[d+]}$$

$$I_n^{\mu\,\nu} = \int \frac{d^{4-2\epsilon}k}{\pi^{d/2}} k^{\mu}\, k^{\nu} \prod_{r=1}^{n} c_r^{-1} = \sum_{i,j=1}^{n-1} q_i^{\mu}\, q_j^{\nu}\, (1+\delta_{ij})\, I_{n,ij}^{[d+]^2} - \frac{1}{2} g^{\mu\nu}\, I_n^{[d+]}$$

Algebraic Davydychev-Tarasov approach (II): Recursions with change of dimensions

Tarasov NPB (1996) and Fleischer-Jegerlehner-Tarasov (1999) derive and apply recurrence relations, some of them relating scalar integrals of different dimensionality in order to get rid of the dimensionality $D = d - 2\epsilon + 2l$:

$$\begin{aligned}
\left(\right)_N \nu_j \mathbf{j}^+ J^{(N)}(d+2) &= \left[-\binom{j}{0}_N + \sum_{k=1}^{n} \binom{j}{k}_N \mathbf{k}^- \right] J^{(N)}(d)\,, \\
(d - \sum_{i=1}^{n} \nu_i + 1) \left(\right)_N J^{(N)}(d+2) &= \left[\binom{0}{0}_N - \sum_{k=1}^{n} \binom{0}{k}_N \mathbf{k}^- \right] J^{(N)}(d)\,, \\
\binom{0}{0}_N \nu_j \mathbf{j}^+ J^{(N)}(d) &= \sum_{k=1}^{n} \binom{0j}{0k}_N \left[d - \sum_{i=1}^{n} \nu_i (\mathbf{k}^- \mathbf{i}^+ + 1) \right] J^{(N)}(d)\,.
\end{aligned}$$

where the operator $\mathbf{j}^\pm$ acts by shifting the index ν_j by ± 1.

An important problem for applications is the appearence of the Gram determinant $\left(\right)_N$ at the left-hand sides of recursions with dimensional shifts.

Their inverse powers – as overall factors – cause numerical instabilities.

This might have been a reason, why the project FIRCLA by Tarasov-Jegerlehner Nucl.Phys.Proc.Suppl.116:83-87,2003 for automated treatment of $2 \to 3$ processes like $e^+e^- \to H\nu\bar{\nu}$ was not completed.

Special cases for later use here (eqn. references go to FJT (1999)):

Reduction of 4-point integrals

$$(1+\delta_{ij})\, I_{4,ij}^{[d+]^2,s} = -\frac{\binom{0s}{js}_5}{\binom{s}{s}_5} I_{4,i}^{[d+],s} + \sum_{t=1,t\neq s,i}^{5} \frac{\binom{ts}{js}_5}{\binom{s}{s}_5} I_{3,i}^{[d+],st} + \frac{\binom{is}{js}_5}{\binom{s}{s}_5} I_4^{[d+],s} \tag{4}$$

$$I_{4,i}^{[d+],s} = -\frac{\binom{0s}{is}_5}{\binom{s}{s}_5} I_4^s + \sum_{t=1,t\neq s}^{5} \frac{\binom{ts}{is}_5}{\binom{s}{s}_5} I_3^{st}, \tag{5}$$

$$I_4^{[d+],s} = \left[\frac{\binom{0s}{0s}_5}{\binom{s}{s}_5} I_4^s - \sum_{t=1,t\neq s}^{5} \frac{\binom{ts}{0s}_5}{\binom{s}{s}_5} I_3^{st} \right] \frac{1}{d-3} \tag{6}$$

Reduction of 3-point integrals

$$I_{3,i}^{[d+],st} = -\frac{\binom{0st}{ist}_5}{\binom{st}{st}_5} I_3^{st} + \sum_{u=1,u\neq s,t}^{5} \frac{\binom{ust}{ist}_5}{\binom{st}{st}_5} I_2^{stu} \tag{7}$$

In applications we can put $d=4$ since $I_4^{[d+]}$ is UV and IR finite.

Alternatively, instead of the reductions, $I_4^{[d+]}$ can as well be used directly as “master integral” (see e.g. Binoth 2005 ff)

Some notations

The determinant of an $(N+1)\times(N+1)$ matrix, known as the

modified Cayley determinant [Melrose 1965]

$$\left(\right)_N \equiv \left| \begin{array}{ccccc} 0 & 1 & 1 & \ldots & 1 \\ 1 & Y_{11} & Y_{12} & \ldots & Y_{1N} \\ 1 & Y_{12} & Y_{22} & \ldots & Y_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & Y_{1N} & Y_{2N} & \ldots & Y_{NN} \end{array} \right| , \tag{8}$$

with coefficients

$$Y_{ij} = -(q_i - q_j)^2 + m_i^2 + m_j^2 , \quad (i,j = 1 \ldots N) . \tag{9}$$

Although the masses of the propagators appear in the coefficients Y_{ij}, the determinant $\left(\right)_N$ does not depend on the masses and it is actually proportional to the Gram determinant of the external momenta of the N-point function.

All other determinants we need are signed minors of $()_N$, constructed by deleting m rows and m columns from $()_N$, and multiplying with a sign factor:

$$\left(\begin{array}{cccc} j_1 & j_2 & \cdots & j_m \\ k_1 & k_2 & \cdots & k_m \end{array}\right)_N \equiv (-1)^{\sum_l (j_l+k_l)}$$

$$\textbf{sgn}_{\{j\}}\ \textbf{sgn}_{\{k\}} \left| \begin{array}{c} \textbf{rows } j_1 \cdots j_m \textbf{ deleted} \\ \textbf{columns } k_1 \cdots k_m \textbf{ deleted} \end{array} \right| , \qquad (10)$$

where sgn$_{\{j\}}$ and sgn$_{\{k\}}$ are the signs of permutations that sort the deleted rows $j_1 \cdots j_m$ and columns $k_1 \cdots k_m$ into ascending order.

Try: Perform the reductions for pentagons (and hexagons) as given in FJT 1999 and search for the elimination of inverse Gram determinants in the Davydychev-Tarasov approach.

Inverse Gram determinants of pentagons may be avoided, as demonstrated e.g. in

- **Campbell-Glover-Miller NPB, 1997 [they discuss FJT 1999]**
- **Denner-Dittmaier NPB, 2003-2006**
- **Binoth et al. JHEP,NPB, 2000-2005**

In fact it was demonstrated in Diakonidis-Fleischer-Gluza-Kajda-Riemann-Tausk PRD 2009, that in the Davydychev-Tarasov approach the cancellation of inverse Gram determinants can also be achieved. Explicitly for $n = 5$ and for rank $R = 2, 3$.

The derivations are trickier with higher rank, and by studying them we found a much better approach.

Recursions

Several recursive tensor reductions have been worked out [see next slides].

- Denner-Dittmaier NPB 2005 – the so-called Passarino-Veltman recursion and an alternative Passarino-Veltman recursion
- van Hameren 2009 describes another recursive tensor reduction
- Diakonidis-Fleischer-Riemann-Tausk arXiv:0907.2115 (2009): Our recursive approach

See arXiv:0907.2115 for the proof:

$$(n, R) = (n, R-1) \times Q_0^\mu + \sum_{s=1}^{n} (n-1, R-1)^s \times Q_s^\mu$$

with auxiliary vectors

$$Q_s^\mu = \sum_{i=1}^{n} q_i^\mu \frac{\binom{s}{i}_n}{()_n}, \quad s = 0, \ldots, n.$$

In the Q_s^μ we create inverse Gram determinants, but the recursions define a convenient calculational scheme.

See the recursion triangle.

Proof

relies on a simple relation for the products:

$$
\begin{aligned}
q_i Q_0 &= -\frac{1}{2}\left(Y_{in} - Y_{nn}\right), \; i = 1, \cdots, n-1, \\
q_i Q_s &= \frac{1}{2}\left(\delta_{is} - \delta_{ns}\right), \; i = 1, \ldots, n-1, \; s = 1, \cdots, n,
\end{aligned}
$$

with

$$
Y_{jk} = -(q_j - q_k)^2 + m_j^2 + m_k^2.
$$

Show that the relations hold:

$$
q_j \times (n, R) = q_j \times (n, R-1) \times Q_0^\mu + q_j \times \sum_{s=1}^{n} (n-1, R-1)^s \times Q_s^\mu
$$

for any chord q_j.

The chords form a basis.

q.e.d.

Fleischer's recursion triangle [arXiv:0907.2115]:

$$
\begin{tikzcd}
 & & & & I_5^{\mu\nu\lambda\rho\sigma} \arrow[dl] \arrow[dr] & & & & \\
 & & & I_5^{\mu\nu\lambda\rho} \arrow[dl] \arrow[dr] & & I_4^{\mu\nu\lambda\rho,s} \arrow[dl] \arrow[dr] & & & \\
 & & I_5^{\mu\nu\lambda} \arrow[dl] \arrow[dr] & & I_4^{\mu\nu\lambda,s} \arrow[dl] \arrow[dr] & & I_3^{\mu\nu\lambda,st} \arrow[dl] \arrow[dr] & & \\
 & I_5^{\mu\nu} \arrow[dl] \arrow[dr] & & I_4^{\mu\nu,s} \arrow[dl] \arrow[dr] & & I_3^{\mu\nu,st} \arrow[dl] \arrow[dr] & & I_2^{\mu\nu,stu} \arrow[dl] \arrow[dr] & \\
I_5^{\mu} \arrow[d] \arrow[drr] & & I_4^{\mu,s} \arrow[d] \arrow[drr] & & I_3^{\mu,st} \arrow[d] \arrow[drr] & & I_2^{\mu,stu} \arrow[d] \arrow[drr] & & I_1^{\mu,stuv} \arrow[d] \\
E_0 & & D_0 & & C_0 & & B_0 & & A_0
\end{tikzcd}
$$

The ’conventional’ Passarino-Veltman recursion triangle

[Denner/Dittmaier hep-ph/0509141]:

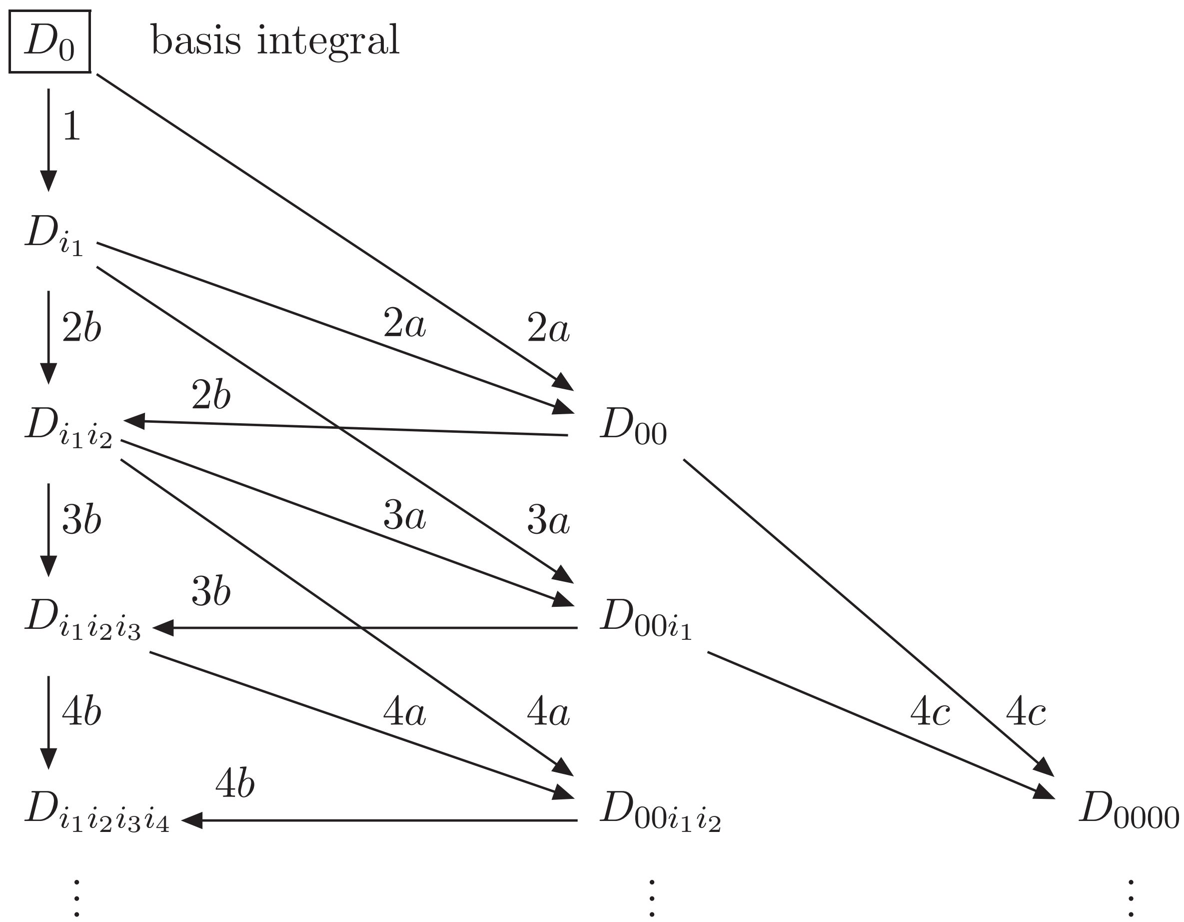


Figure 1: Schematic illustration of conventional Passarino–Veltman reduction.

The 'alternative' Passarino-Veltman triangle

[Denner/Dittmaier hep-ph/0509141]:

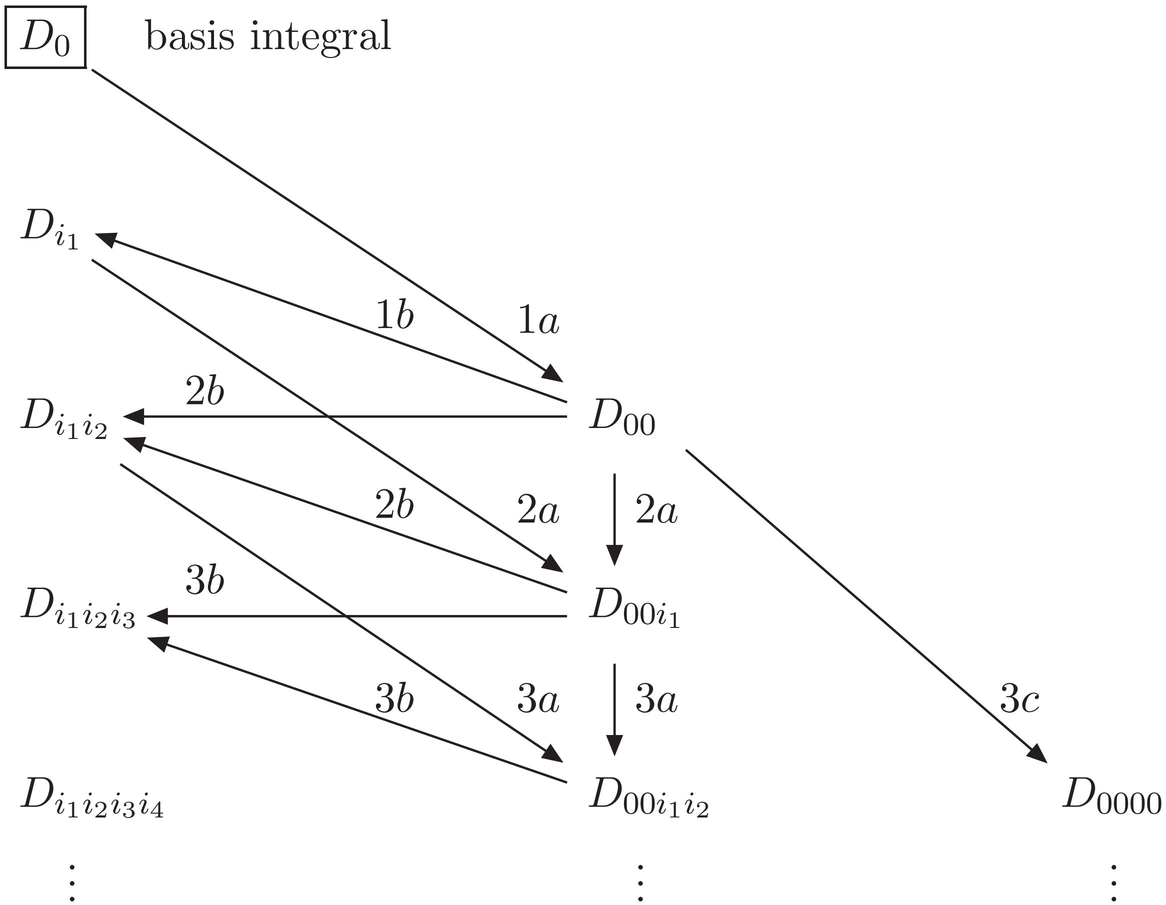


Figure 2: Schematic illustration of alternative Passarino–Veltman reduction.

Compared to

$$(n, R) = (n, R-1) \times Q_0^\mu + \sum_{s=1}^{n} (n-1, R-1)^s \times Q_s^\mu$$

for $n < 5$ things are a bit more involved.

There the chords are not a complete basis for the 4-vectors, and one has to introduce another type of auxiliary quantities [van Oldenborgh-Vermaseren 1990]:

$$\begin{aligned} v^\mu &= \epsilon^{\mu\lambda\rho\sigma}(q_1 - q_4)_\lambda (q_2 - q_4)_\rho (q_3 - q_4)_\sigma, \\ v^{\mu\lambda} &= \epsilon^{\mu\lambda\rho\sigma}(q_1 - q_3)_\rho (q_2 - q_3)_\sigma \end{aligned}$$

They may be used to represent the metric tensor by chords (plus additional terms):

$$\begin{aligned} g^{\mu\nu} &= 2 \sum_{i,j=1}^{6} q_i^\mu q_j^\nu \frac{\binom{0i}{0j}_6}{\binom{0}{0}_6}, \\ g^{\mu\nu} &= 2 \sum_{i,j=1}^{5} q_i^\mu q_j^\nu \frac{\binom{i}{j}_5}{()_5}, \\ g^{\mu\nu} &= 2 \sum_{i,j=1}^{4} q_i^\mu q_j^\nu \frac{\binom{i}{j}_4}{()_4} + \frac{8 v^\mu v^\nu}{()_4}, \\ g^{\mu\nu} &= 2 \sum_{i,j=1}^{3} q_i^\mu q_j^\nu \frac{\binom{i}{j}_3}{()_3} + \frac{4 v^{\mu\lambda} v^\nu_{\ \lambda}}{()_3}. \end{aligned}$$

Then, one may find

recursions for two-, three-, four-point functions

e.g.:

$$I_4^{\mu\nu\lambda} = I_4^{\mu\nu} Q_0^{\lambda} - \sum_{t=1}^{4} I_3^{\mu\nu,t} Q_t^{\lambda} - G^{\mu\lambda} I_4^{\nu,[d+]} - G^{\nu\lambda} I_4^{\mu,[d+]},$$

with

$$G^{\mu\lambda} = \frac{4 v^{\mu} v^{\nu}}{()_4},$$

and the higher-dimensional integrals $I_4^{\nu,[d+]}$ **are given above in terms of the usual scalar** A**-,** B**-,** C**-,** D**-functions.**

Similarly:

$$I_3^{\mu\nu,t} = I_3^{\mu,t} Q_0^{t,\nu} - \sum_{u=1}^{4} I_2^{\mu,tu} Q_u^{t,\nu} - I_3^{[d+],t} \frac{2 v^{t,\mu\lambda} v_{\lambda}^{t,\nu}}{\binom{t}{t}_4}. \qquad (11)$$

The $Q_s^{t,\nu}$ **are defined analogously to the** Q_s^{ν}**.**

All this is nice, but we re-introduce the inverse Gram determinants.

In the next part, we indicate how one may get rid of them in a systematic, relatively simple way, if they are not wanted.

Cancellations (I) - inverse Gram determinants, repeated

The recursions introduce at each step inverse Gram determinants, hidden in the auxiliary vectors Q_s^ν.

It is relatively easy to show that, for pentagons, they may be eliminated by eliminating these auxilary vectors and applying some algebra.

Of course, in that case, the recursion breaks.

Using relations like:

$$
\begin{aligned}
\begin{pmatrix}0\\0\end{pmatrix}_5 \begin{pmatrix}s\\i\end{pmatrix}_5 &= \begin{pmatrix}0s\\0i\end{pmatrix}_5 ()_5 + \begin{pmatrix}0\\i\end{pmatrix}_5 \begin{pmatrix}s\\0\end{pmatrix}_5 \\
\begin{pmatrix}s\\i\end{pmatrix}_5 \frac{\binom{0}{j}_5}{()_5} &= -\begin{pmatrix}0i\\sj\end{pmatrix}_5 + \begin{pmatrix}s\\0\end{pmatrix}_5 \frac{\binom{i}{j}_5}{()_5} \\
\begin{pmatrix}s\\0\end{pmatrix}_5 \begin{pmatrix}0s\\is\end{pmatrix}_5 &= \begin{pmatrix}s\\i\end{pmatrix}_5 \begin{pmatrix}0s\\0s\end{pmatrix}_5 - \begin{pmatrix}s\\s\end{pmatrix}_5 \begin{pmatrix}0s\\0i\end{pmatrix}_5
\end{aligned}
$$

and one (or the other) of the already quoted representations of the metric tensor:

$$
g^{\mu\,\nu} = 2 \sum_{i,j=1}^{4} \frac{\binom{i}{j}_5}{()_5} \, q_i^\mu \, q_j^\nu
$$

An example: $I_5^{\mu\nu}$

$$
\begin{aligned}
\binom{0}{0}_5 I_5^{\mu\nu} &= \binom{0}{0}_5 \left[I_5^{\mu} Q_0^{\nu} - \sum_{s=1}^{5} I_4^{\mu,s} Q_s^{\nu} \right] \\
&= \left[\binom{0}{0}_5 I_5^{\mu} - \sum_{s=1}^{5} \binom{s}{0}_5 I_4^{\mu,s} \right] Q_0^{\nu} - \sum_{s=1}^{5} I_4^{\mu,s} \overline{Q}_s^{0,\nu} \\
&= - \left[\sum_{i=1}^{5} q_i^{\mu} \binom{s}{i}_5 I_4^{[d+],s} \right] Q_0^{\nu} - \sum_{s=1}^{5} I_4^{\mu,s} \overline{Q}_s^{0,\nu} \\
&= g^{\mu\nu} \left[-\frac{1}{2} \sum_{s=1}^{5} \binom{s}{0}_5 I_4^{[d+],s} \right] + \sum_{i,j=1}^{4} q_i^{\mu}\, q_j^{\nu} \left[\sum_{s=1}^{5} \left[\binom{0i}{sj}_5 I_4^{[d+],s} + \binom{0s}{0j}_5 I_{4,i}^{[d+],s} \right] \right] \\
&= g^{\mu\nu} \binom{0}{0}_5 E_{00} + \sum_{i,j=1}^{4} q_i^{\mu}\, q_j^{\nu} \binom{0}{0}_5 E_{ij}
\end{aligned}
\tag{12}
$$

This presentation is evidently free of an inverse Gram determinant.

$$
\overline{Q}_s^{0,\mu} = \sum_{i=1}^{n} q_i^{\mu} \binom{0s}{0i}_n , \quad s = 1, \ldots, n.
$$

In a next step one has to treat the dimensionally shifted scalar integrals.

Cancellations (II) - isolation of inverse sub-Grams

When reducing the higher-dimensional self-energy, vertex, box integrals by the dimensional reductions introduced, we also, as in the case of the 5-point integrals, meet inverse Gram determinants, but these cannot be cancelled completely.

Nevertheless, one can do something to get a much improved representation.

In fact, it is another alternative Passarino-Veltman recursion to those described in Denner-Dittmaier [2005].

An example demonstrating the method

We know [from FJT 2005] the limit

$$\lim_{()_n \to 0} I_n^d = \sum_{k=1}^{n} \frac{\binom{k}{0}_n}{\binom{0}{0}_n} I_{n-1}^{d,k}$$

Introduce:

$$\lim_{\binom{s}{s}_5 \to 0} I_4^{d,s} = \sum_{t=1}^{5} \frac{\binom{ts}{0s}_5}{\binom{0s}{0s}_5} I_3^{d,st} \equiv Z_4^{d,s}$$

For $I_4^{[d+],s}$, we get with this notations ($Z_4^{d\to 4,s} \equiv Z_4^s$):

$$I_4^{[d+],s} = \binom{0s}{0s}_5 \frac{[I_4^s - Z_4^s]}{\binom{s}{s}_5}$$

The relation makes evident that $I_4^{[d+],s}$ stays finite at $\binom{s}{s}_5 \to 0$.

A less trivial example appearing in (12) is $I_{4,i}^{[d+],s}$.

With the aid of

$$\binom{0s}{is}_5 \binom{ts}{0s}_5 - \binom{0s}{0s}_5 \binom{ts}{is}_5 = -\binom{s}{s}_5 \binom{0st}{0si}_5,$$

one may cancel out the factor $\binom{s}{s}_5$ from the combination

$$\frac{\binom{0s}{is}_5}{\binom{s}{s}_5} Z_4^s - \sum_{t=1}^{5} \frac{\binom{ts}{is}_5}{\binom{s}{s}_5} I_3^{st} = \cdots = -\frac{1}{\binom{0s}{0s}_5} \sum_{t=1}^{5} \binom{0st}{0si}_5 I_3^{st}$$

This allows to derive finally

$$I_{4,i}^{[d+],s} = -\binom{0s}{is}_5 \frac{[I_4^s - Z_4^s]}{\binom{s}{s}_5} + \frac{1}{\binom{0s}{0s}_5} \sum_{t=1}^{5} \binom{0st}{0si}_5 I_3^{st}$$

This is a nice expression, because it contains an explicit difference quotient for the case that $\binom{s}{s}_5$ becomes zero or if it becomes small but finite.

A proper use will stabilize numerics [considerably?].

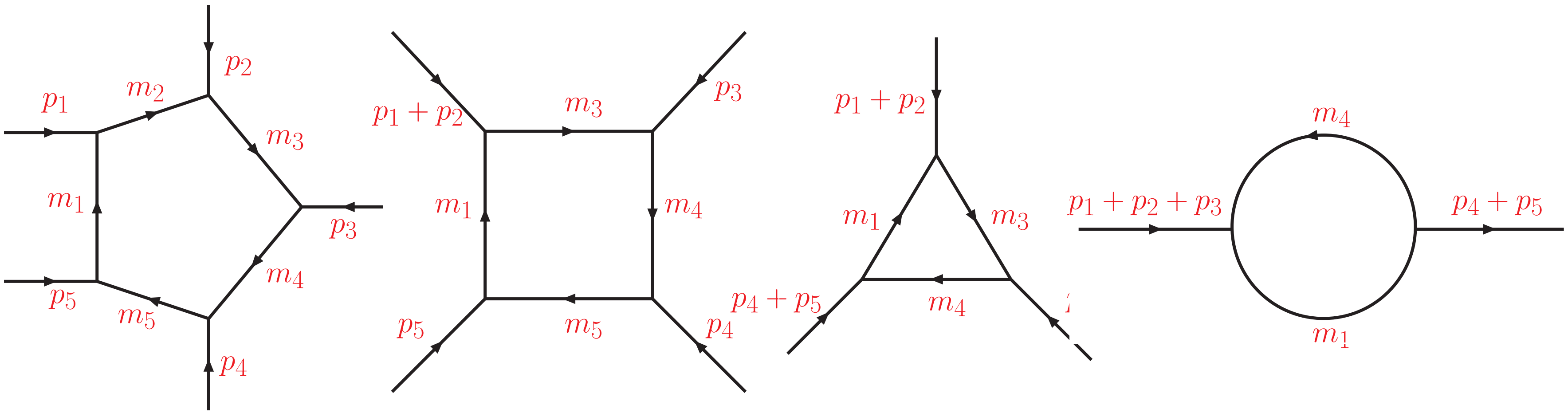
The practical numerics is not yet worked out.

Numbers (I) – Pentagons

Randomly chosen phase space point with massive and massless internal particles

p_1	5.00000000000 E+00	0.0000000000 E+00	0.00000000000 E+00	4.00000000000 E+00
p_2	5.00000000000 E+00	0.0000000000 E+00	0.00000000000 E+00	-4.00000000000 E+00
p_3	– 0.30770034895 E+01	0.5359484673 E+00	–0.37447035150 E+00	– 0.20120057390 E+00
p_4	– 0.34048537280 E+01	0.2184763540 E-01	– 0.10479394969 E+01	0.12224460727 E+01
p_5	– 0.35181427825 E+01	– 0.5577961027 E+00	0.14224098484 E+01	– 0.10212454988 E+01

$$m_1 = 0.0, \quad m_2 = 2.0, \quad m_3 = 3.0, \quad m_4 = 4.0, \quad m_5 = 5.0$$



Selected tensor components

Shown are the constant terms of the tensor components

	$Pentagon.F$
E^2	(2.80450709388539E-05 , -1.08461817406464E-05)
E^{12}	(-5.41333978667301E-06 , 6.26985967678899E-06)
E^{232}	(-1.20374858970726E-04 , 4.07974751672555E-04)
E^{0321}	(-9.11194535703727E-06 , 4.39187998675819E-05)
E^{01230}	(4.37928367160152E-05 , -2.18183151665913E-04)

	$Box.F$	$LoopTools$
D^1	(6.81403420828588E-03 , -5.74298462683219E-03)	(6.8140342082847463E-03 , -5.7429846268324187E-03)
D^{33}	(2.40138809967981E-03 , 1.11591328775015E-02)	(2.4013880996803092E-03,1.1159132877500448E-02)
D^{212}	(-1.69702786278243E-03 , -2.83731121595478E-03)	(-1.6970278627700630E-03,-2.8373112159962330E-03)
D^{0123}	(-1.92190388316994E-04 , -4.04730302413490E-04)	(-1.9219038693301300E-04,-4.0473030187772325E-04)

	$Triangle.F$	$LoopTools$
C^2	(2.44757827793318E-04 , -7.50688449850356E-03)	(2.4475782779342707E-04,-7.5068844985030472E-03)
C^{01}	(-1.28259813172255E-02 , -6.73809718907549E-02)	(-1.2825981317215014E-02,-6.7380971890795340E-02)
C^{133}	(-7.00360822297110E-02 , 7.24628606014397E-02)	(-7.0036082229746830E-02,7.2462860601566081E-02)

	$Bubble.F$	$LoopTools$
B^3	(-0.141525070262337E+00 , 0.138870631815383E+00)	(-0.1415250702623366,0.1388706318153829)
B^{12}	(0.102490343329085E+00 , -6.12154531068256E-02)	(0.1024903433290848,-6.1215453106825706E-02)

Numbers (II) – Hexagons

p_1	0.21774554 E+03	0.0	0.0	0.21774554 E+03
p_2	0.21774554 E+03	0.0	0.0	– 0.21774554 E+03
p_3	– 0.20369415 E+03	– 0.47579512 E+02	0.42126823 E+02	0.84097181 E+02
p_4	– 0.20907237 E+03	0.55215961 E+02	– 0.46692034 E+02	– 0.90010087 E+02
p_5	– 0.68463308 E+01	0.53063195 E+01	0.29698267 E+01	– 0.31456871 E+01
p_6	– 0.15878244 E+02	– 0.12942769 E+02	0.15953850 E+01	0.90585932 E+01
$m_1 = 110.0,\ m_2 = 120.0,\ m_3 = 130.0,\ m_4 = 140.0,\ m_5 = 150.0,\ m_6 = 160.0$				

		F_0
		– 0.223393 E–18 – i 0.396728 E–19
μ		F^{μ}
0		0.192487 E–17 + i 0.972635 E–17
1		– 0.363320 E–17 – i 0.11940 E–17
2		0.365514 E–17 + i 0.106928 E–17
3		0.239793 E–16 + i 0.341928 E–17
μ	ν	$F^{\mu\nu}$
0	0	0.599459 E–14 – i 0.114601 E–14
0	1	0.323869 E–15 + i 0.423754 E–15
0	2	– 0.294252 E–15 – i 0.375481 E–15
0	3	– 0.255450 E–14 – i 0.195640 E–14
1	1	– 0.164562 E–14 – i 0.993796 E–16
1	2	0.920944 E–16 + i 0.706487 E–17
1	3	0.347694 E–15 – i 0.127190 E–16
2	2	– 0.163339 E–14 – i 0.994148 E–16
2	3	– 0.341773 E–15 + i 0.818678 E–17
3	3	– 0.413909 E–14 + i 0.670676 E–15

μ	ν	λ	$F^{\mu\nu\lambda}$
0	0	0	– 0.227754 E–11 – i 0.267244 E–12
0	0	1	0.140271 E–13 – i 0.119448 E–12
0	0	2	– 0.201270 E–13 + i 0.101968 E–12
0	0	3	0.102976 E–12 + i 0.624467 E–12
0	1	1	0.183904 E–12 + i 0.142429 E–12
0	1	2	– 0.131028 E–13 – i 0.610343 E–14
0	1	3	– 0.543316 E–13 – i 0.158809 E–13
0	2	2	0.181352 E–12 + i 0.141686 E–12
0	2	3	0.506408 E–13 + i 0.163568 E–13
0	3	3	0.600542 E–12 + i 0.130733 E–12
1	1	1	– 0.563539 E–13 + i 0.178403 E–13
1	1	2	0.210641 E–13 – i 0.584990 E–14
1	1	3	0.120482 E–12 – i 0.574688 E–13
1	2	2	– 0.201182 E–13 + i 0.620591 E–14
1	2	3	– 0.686164 E–14 + i 0.205457 E–14
1	3	3	– 0.447329 E–13 + i 0.193180 E–13
2	2	2	0.582201 E–13 – i 0.163889 E–13
2	2	3	0.119659 E–12 – i 0.570084 E–13
2	3	3	0.457464 E–13 – i 0.181141 E–13
3	3	3	0.557081 E–12 – i 0.374359 E–12

Tensor components for a massive rank $R = 3$ six-point function

μ	ν	λ	ρ	$F^{\mu\nu\lambda\rho}$
0	0	0	0	0.666615 E–09 + i 0.247562 E–09
0	0	0	1	– 0.200049 E–10 + i 0.294036 E–10
0	0	0	2	0.200975 E–10 – i 0.237333 E–10
0	0	0	3	0.645477 E–10 – i 0.162236 E–09
0	0	1	1	– 0.116956 E–10 – i 0.516760 E–10
0	0	1	2	0.160357 E–11 + i 0.222284 E–11
0	0	1	3	0.792692 E–11 + i 0.729502 E–11
0	0	2	2	– 0.111838 E–10 – i 0.513133 E–10
0	0	2	3	– 0.681086 E–11 – i 0.708933 E–11
0	0	3	3	– 0.804454 E–10 – i 0.801909 E–10
0	1	1	1	0.100498 E–10 – i 0.151735 E–13
0	1	1	2	– 0.348984 E–11 – i 0.195436 E–12
0	1	1	3	– 0.211111 E–10 + i 0.295212 E–11
0	1	2	2	0.357455 E–11 + i 0.662809 E–14
0	1	2	3	0.121595 E–11 – i 0.807388 E–13
0	1	3	3	0.825803 E–11 – i 0.142086 E–11
0	2	2	2	– 0.958961 E–11 – i 0.585948 E–12
0	2	2	3	– 0.209232 E–10 + i 0.289031 E–11
0	2	3	3	– 0.802359 E–11 + i 0.994701 E–12
0	3	3	3	– 0.102576 E–09 + i 0.378476 E–10
1	1	1	1	– 0.246426 E–10 + i 0.276326 E–10
1	1	1	2	0.915670 E–12 – i 0.660629 E–12
1	1	1	3	0.303529 E–11 – i 0.287480 E–11
1	1	2	2	– 0.822697 E–11 + i 0.919635 E–11
1	1	2	3	– 0.116294 E–11 + i 0.100024 E–11
1	1	3	3	– 0.146918 E–10 + i 0.183799 E–10
1	2	2	2	0.908296 E–12 – i 0.654735 E–12
1	2	2	3	0.109510 E–11 – i 0.100875 E–11
1	2	3	3	0.717342 E–12 – i 0.557293 E–12
1	3	3	3	0.450661 E–11 – i 0.485065 E–11
2	2	2	2	– 0.245154 E–10 + i 0.274313 E–10
2	2	2	3	– 0.318500 E–11 + i 0.279750 E–11
2	2	3	3	– 0.146317 E–10 + i 0.182912 E–10
2	3	3	3	– 0.477335 E–11 + i 0.477368 E–11
3	3	3	3	– 0.730168 E–10 + i 0.112865 E–09

For the phase space point given by:

p_1 =(0.5, 0, 0,0.5)

p_2 =(0.5, 0, 0,–0.5)

p_3 =(–0.19178191,–0.12741180,–0.08262477,–0.11713105)

p_4 =(–0.33662712, 0.06648281, 0.31893785, 0.08471424)

p_5 =(-0.21604814, 0.20363139,-0.04415762,-0.05710657)

$p_6 = -(p_1 + p_2 + p_3 + p_4 + p_5)$

$M_1 = 0, M_2 = 0, M_3 = 0, M_4 = 0, M_5 = 0$

Golem95: Binoth, Guillet, Heinrich, Pilon, Reiter [arXiv:hep-ph/0810.0992]

scalar masters QCDloop, agreement of 8 digits

Shown are only the constant terms of the tensor components.

	$Hexagon.F$	$Golem95$
F^{03121}	(0.158428986740235E+00 , 0.416706979843194E-01)	(0.158428980552600E+00 , 0.416706995132716E-01)
F^{11020}	(-0.143913859903552E+01 , -0.164647048275408E+00)	(-0.143913852754709E+01 , -0.164647075385477E+00)
F^{20200}	(0.242928799509288E+02 , 0.555041844207877E+02)	(0.242928775936564E+02 , 0.555041824180155E+02)
F^{22130}	(0.225563941055782E+00 , 0.231928571404353E+00)	(0.225563949300093E+00 , 0.231928509918651E+00)
F^{33333}	(0.244568134868438E+00 , 0.740146041525474E+00)	(0.244568138432017E+00 , 0.740146095196997E+00)

Summary

- **We derived a recursive treatment of tensor integrals of the type (n,R) in terms of those with lower rank, (n,R-1) and (n-1,R-1)**
- **This is realized in a Fortran code and checked against other codes until (5,4) and (6,5) for massive and massless diagrams**
- **We derived, in a systematic way, expressions which are explicitely free of inverse Gram determinants** $()_5$
- **We derived expressions with properly isolated terms with inverse Gram determinants of subdiagrams of the type** $\binom{s}{s}_n$**, which cannot be completely avoided.**
  We assume that this will stabilize numerics in sensible phase space points considerably.