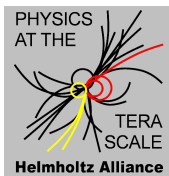


Feynman Integrals and Mellin-Barnes representations



Computer Algebra and Particle Physics

The DESY CAPP School

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Intro: Tord Riemann, DESY, Zeuthen

Exercises: Janusz Gluza, U. Katowice



Contents

- Introduction + Motivation
- Mathematical Reminder on Γ -function, Residues, Cauchy-theorem
- The Feynman parameter representation
- Few simple Feynman integrals, made conventionally
- Mellin-Barnes representations and their evaluation
- Expansions in a small parameter, e.g. $m^2/s \ll 1$
- Sector decomposition

For longer versions of my lectures and of the exercises worked out by Janusz Gluza, with much more material included, see:

<http://www-zeuthen.desy.de/~riemann/>

<http://prac.us.edu.pl/~gluza/ambre/>

and, of course, <http://www.us.edu.pl/~gluza/capp2011/>

We profited much from collaborations with [Krzysztof Kajda](#) and [Valery Yundin](#).

See also for this and related software:

<http://projects.hepforge.org/mbtools/>

Introductory remarks

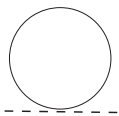
For many problems of the past, a relatively simple approach to the evaluation of Feynman integrals was sufficient:

- ★ Tensor reduction a la Passarino/Veltmann
- ★ Evaluate Feynman parameter integrals by direct integration

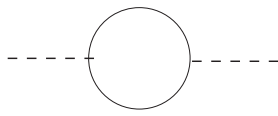
Typically 1-loop (massless: 2-loop), typically $2 \rightarrow 2$ scattering (plus bremsstrahlung)

Feynman parameters may be used and by direct integration over them one gets objects like: $\frac{23}{57}$, $\zeta(3)$, $\ln(\frac{t}{s})$, $\ln(\frac{t}{s}) \cdot \ln(\frac{s}{m^2})$, $\text{Li}_2(\frac{t}{s+i\epsilon})$ etc.

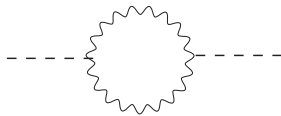
With more complexity of the reaction (more legs) and more perturbative accuracy (more loops), this approach appears to be not sufficiently sophisticated.



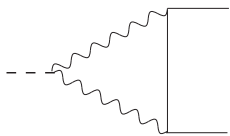
T111m



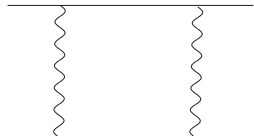
SE2l2m



SE2l0m



V3l1m



B4l2m

$$T111m = \frac{1}{\epsilon} + 1 + (1 + \frac{\zeta_2}{2})\epsilon + (1 + \frac{\zeta_2}{2} - \frac{\zeta_3}{3})\epsilon^2 + \dots$$

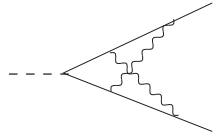
$$B4l2m = [-\frac{1}{\epsilon} + \ln(-s)] \frac{2y \ln(y)}{s(1-y^2)} + c_1\epsilon + \dots$$

with $d = 4 - 2\epsilon$ and $m = 1$ and

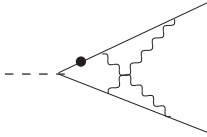
$$y = \frac{\sqrt{1-4/t}-1}{\sqrt{1-4/t}+1}$$

Figure shows so-called **master integrals**.

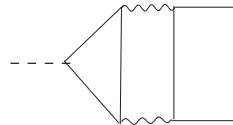
More loops



V6l4m1

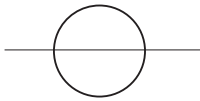


V6l4m1d

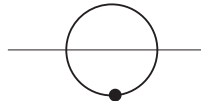


V6l4m2

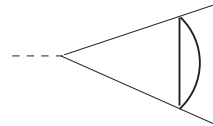
Two-loop vertex integrals with six internal lines
massless case: only fixed numbers and one scale factor



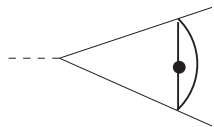
SE3l2M1m



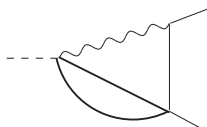
SE3l2M1md



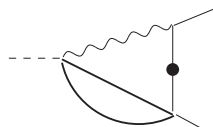
V4l2M2m



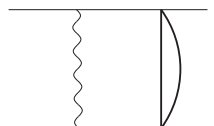
V4l2M2md



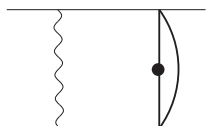
V4l2M1m



V4l2M1md



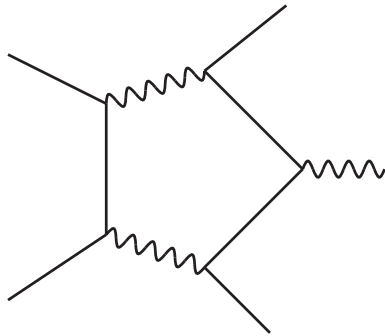
B5l2M2md



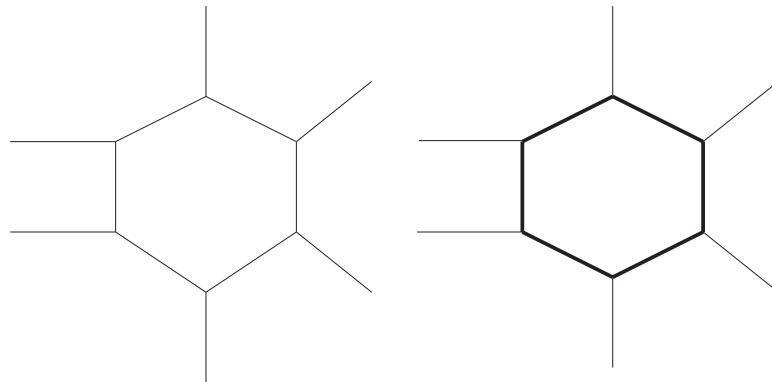
B5l2M2m

Integrals with two different mass scales m and M

More legs



Massive pentagon: 5 kinematic variables + several masses



Massless and massive hexagons: 8 kinematic variables + several masses

Variables for $2 \rightarrow 2$ scattering: 2 – i.e. **box diagrams: s, t or s and $\cos \theta$**

Variables for $2 \rightarrow 3$ scattering: $5 = 2 + 3$ (three additional momenta of a particle)

Variables for $2 \rightarrow 4$ scattering: $8 = 5 + 3$ (another three additional momenta)

What we will do here (i)

Want to evaluate some Feynman integral in momentum space:

$$I = \frac{e^{\epsilon\gamma_E L}}{(i\pi^{d/2})^L} \int \frac{d^d k_1 \dots d^d k_L \quad k_1^\mu \dots k_R^\nu}{D_1^{\nu_1} \dots D_i^{\nu_i} \dots D_N^{\nu_N}}.$$

L ... number of loops

n ... external lines with momenta p_e

N ... internal lines with momenta q_i

$$D_i = q_i^2 - m_i^2 = \left[\sum_{l=1}^L c_i^l k_l + \sum_{e=1}^n d_i^e p_e \right]^2$$

ν_i ... index of propagator/line

μ, ν ... tensor degrees of integral

Instrumentarium

- Just tackle the integrals needed directly by e.g. Feynman parameter integration
- Perform tensor reduction, get scalar master integrals, solve the latter
- Use integration-by-parts and related methods to determine master integrals, solve the latter
- Combine all this with solving of (a system of) differential equation(s) for the integrals
- Rewrite the (master) integrals by using Mellin-Barnes representations and try to solve the latter

May be combined with other methods

Advantage: one has to deal with single objects, not with systems of them

There are few masters of using Mellin-Barnes representations, among them:

- V. Smirnov – massless planar double box and many other
- B. Tausk – massless non-planar double box and others
- M. Czakon – some massive non-planar double box and many others

Using tools lets you come close to the masters.

Some of the tools are:

- **MB** – M. Czakon
- **Ambre** – J. Gluza, K. Kajda, T.R., V. Yundin
- **MBresolve** – 2 Smirnovs
- maybe others

What we will do here (ii)

Solve the momentum integral, get a Feynman parameter integral
(a la textbook)

$$I = \frac{e^{\epsilon\gamma_E L}}{(-1)^{N_\nu}} \frac{\Gamma(N_\nu - \frac{d}{2}L)}{\Gamma(\nu_1) \dots \Gamma(\nu_N)} \int_0^1 \prod_{j=1}^N dx_j x_j^{\nu_j-1} \delta\left(1 - \sum_{i=1}^N x_i\right) \frac{U(x)^{N_\nu - d(L+1)/2}}{F(x)^{N_\nu - dL/2}}$$

with

N_ν = sum of indices

$U(x)$ = $(\det M)$ ($\rightarrow 1$ for $L = 1$)

$F(x)$ = $(\det M) \mu^2 = -(\det M) J + Q \tilde{M} Q$ ($\rightarrow -J + Q^2$ for $L = 1$)

M is a matrix in terms of the x_i and J, Q_μ depend on external momenta, masses and x_i

What we will do here (iii)

Mellin-Barnes formula transforms sums into products:

$$\frac{1}{[A(s)x_1^{a_1} + B(s)x_1^{b_1}x_2^{b_2}]^a} = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} d\sigma [A(s)x_1^{a_1}]^\sigma [B(s)x_1^{b_1}x_2^{b_2}]^{a+\sigma} \frac{\Gamma(a+\sigma)\Gamma(-\sigma)}{\Gamma(a)}$$

Depending on complexity, get multi-dimensional complex path integral, and the x -dependence can be integrated out – generalization of the Beta-function:

$$\int_0^1 \prod_{j=1}^N dx_j x_j^{\alpha_j-1} \delta\left(1 - \sum x_i\right) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\cdots\Gamma(\alpha_N)}{\Gamma(\alpha_1 + \alpha_2 + \cdots + \alpha_N)}$$

with coefficients α_i dependent on ν_i and on the structure of the F

Some Mathematical Preparations

We will often use, for $d = 4 - 2\epsilon$:

$$a^\epsilon = e^{\epsilon \ln(a)} = 1 + \ln(a) \epsilon + \frac{1}{2} \ln^2(a) \epsilon^2 + \dots$$

The Γ -function

The Γ -function may be defined by a difference equation:

$$z\Gamma(z) - \Gamma(z + 1) = 0$$

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt$$

$$\Gamma(0) = \infty$$

$$\Gamma(1) = 1$$

$$\Gamma(n) = (n-1)!, \quad n = 2, 3, \dots$$

You remember that $\Gamma(z)$ has poles at $z = -n, n = 0, 1, 2, 3, \dots$, and it is

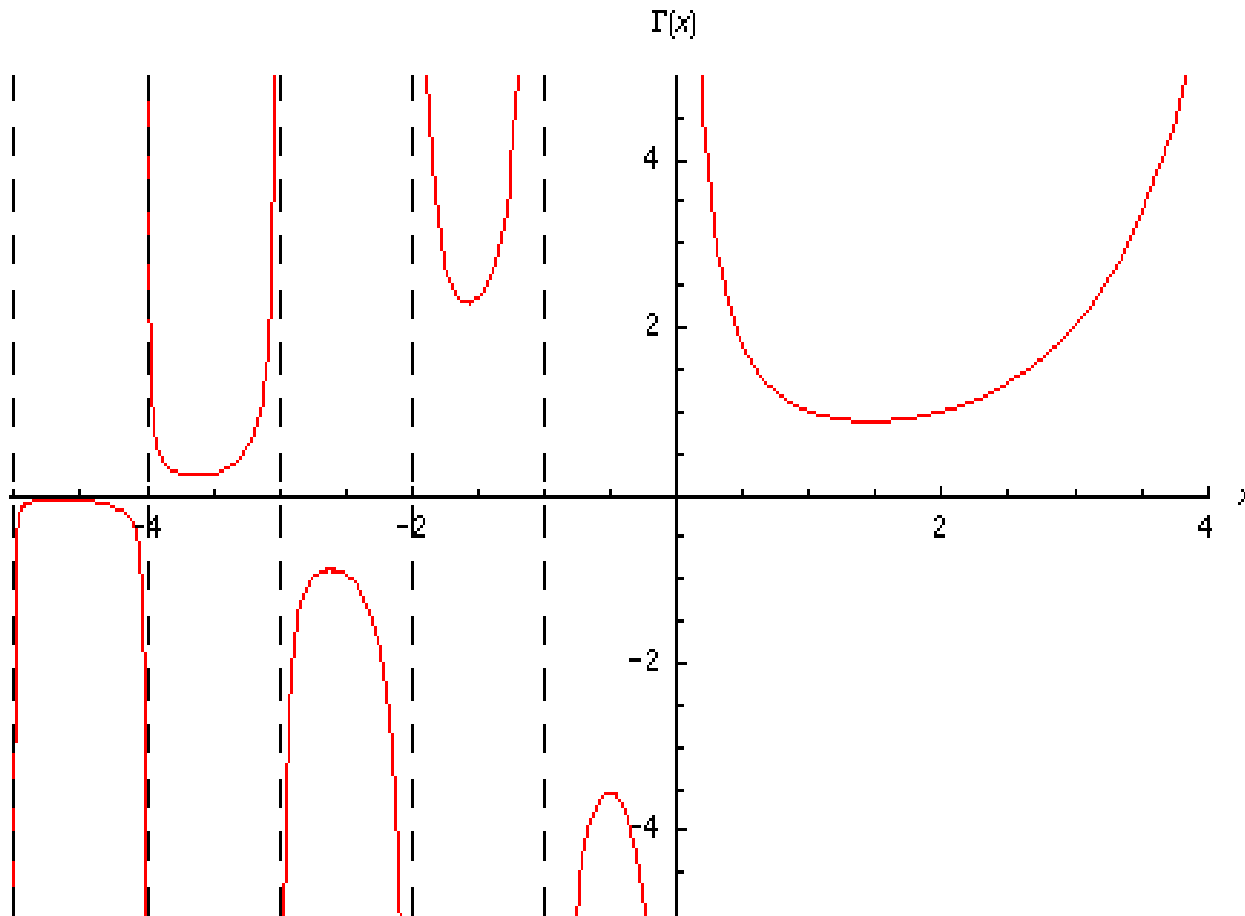
$$\Gamma[\epsilon] = \frac{1}{\epsilon} - \gamma_E + \frac{1}{2} [\gamma_E^2 + \zeta(2)] \epsilon + \frac{1}{6} [-\gamma_E^3 - 3\gamma_E^2 \zeta(2) - 2\zeta(3)] \epsilon^2 + \dots$$

$$e^{\epsilon \gamma_E} \Gamma[\epsilon] = \frac{1}{\epsilon} + \frac{1}{2} \zeta(2) \epsilon - \frac{1}{3} \zeta(3) \epsilon^2 + \dots$$

For definitions of **Riemann's zeta-numbers** $\zeta(N)$ and the **Euler constant** γ_E see next slides.

Look at the singularities in the complex plane.

Figure shows the real part of Γ :



$$\text{Gamma}[-1 \pm 10i] = -4.9974 \cdot 10^{-9} \pm 1.07847 \cdot 10^{-8}i$$

$$\text{Gamma}[-1 \pm 100i] = 1.51438 \cdot 10^{-71} \pm 1.27644 \cdot 10^{-73}i \tag{1}$$

$$\text{Gamma}[\pm 100.1] \approx \pm 10^{\pm 157} \tag{2}$$

Just to remind you:

$$\text{Riemann zeta numbers } \zeta(a) = \sum_{k=1}^{\infty} \frac{1}{k^a} \quad \zeta(2) = \pi^2/6, \quad \zeta(3) = 1.20206, \quad \zeta(4) = \pi^4/90$$

$$\text{HarmonicNumber}[N, a] = S_a(N) = \sum_{k=1}^N \frac{1}{k^a} = H_{N,a} \quad (3)$$

$$\zeta(1) \rightarrow \gamma_E = \lim_{N \rightarrow \infty} \left[\sum_{k=1}^N \frac{1}{k^1} - \ln(N) \right] = 0.57721 \dots \quad (4)$$

$$\text{HarmonicNumber}[N] = S_1(N) = \sum_{k=1}^N \frac{1}{k^1} = H_N \quad (5)$$

We will also need derivatives of $\Gamma(z)$:

$$\text{PolyGamma}[z] = \Psi(z) \equiv \text{PolyGamma}[0, z] = \frac{1}{\Gamma(z)} \frac{d}{dz} \Gamma(z)$$

At integer values:

$$\Psi(N+1) = \sum_{k=1}^N \frac{1}{k} - \gamma_E = S_1(N) - \gamma_E$$

Cauchy Theorem and Residues

An integral over an anti-clockwise directed closed path C is:

$$\oint F(z)dz = 2\pi i \sum_{z=z_i} \text{Res}[F(z)]$$

where the residues $\text{Res}[F(z)]|_{z=z_i}$ are coefficients a_{-1}^i of the **Laurent series of $F(z)$** around z_i :

$$F(z) = \sum_{n=-N}^{\infty} a_n^i (z - z_i)^n = \frac{a_{-N}^i}{(z - z_i)^N} + \dots + \frac{a_{-1}^i}{(z - z_i)} + a_0^i + \dots \quad (6)$$

$$\text{Res}[F(z)]|_{z=z_i} = a_{-1}^i$$

If $G(z)$ has a **Taylor expansion** around z_0 , then it is:

$$\text{Res}[G(z) F(z)]|_{z=z_i} = \sum_{n=1}^N \frac{a_{-n}^i}{k!} \frac{d^n}{dz^n} G(z)|_{z=z_i}$$

Due to this property, we need for applications not only $\Gamma(z)$, but also its derivatives.

Some residues with $\Gamma(z)$

$$\begin{aligned} \Psi(z) &= \text{PolyGamma}[z] = \text{PolyGamma}[0, z] \\ \text{Residue}[F[z]\Gamma[z], \{z, -n\}] &= \frac{(-1)^n}{n!} F[-n] \end{aligned} \quad (7)$$

$$\text{Residue}[F[z]\Gamma[z]^2, \{z, -n\}] = \frac{2\text{PolyGamma}[n+1]F[-n] + F'[-n]}{(n!)^2} \quad (8)$$

Integrals + Some sums Mathematica can do

$$\oint_{-1/3-9i}^{-1/3+9i} dz \Gamma[z] = (-i) 3.97173$$

close path to the left : $2\pi i \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} = (2\pi i) \frac{1-e}{e} = (-i) 3.97173$ (9)

while

close path to the right : $(-1) * 2\pi i \sum_{n=0}^0 \frac{(-1)^n}{n!} = (2\pi i) \neq (-i) 3.97173$

$\text{Sum}[s^{(n)} \text{Gamma}[n + 1]^3 / (n! \text{Gamma}[2 + 2n]), n, 0, \text{Infinity}] =$
 $(4 * \text{ArcSin}[\text{Sqrt}[s]/2]) / (\text{Sqrt}[4 - s] * \text{Sqrt}[s])$

$\text{Sum}[s^{(n)} \text{PolyGamma}[0, n + 1], n, 0, \text{Infinity}] =$
 $(\text{EulerGamma} + \text{Log}[1 - s]) / (-1 + s)$

The above sums were done with Mathematica 5.2.

Later Mathematica versions are much more powerful.

Now come back to

L -loop n -point Feynman Integrals of tensor rank R with N internal lines

- Internal loop momenta are k_l , $l = 1 \dots L$
- Propagators have mass m_i and momentum q_i , $i = 1 \dots N$ and indices ν_i – see $G(X)$
- External legs have momentum p_e , $e = 1 \dots n$, with $p_e^2 = M_e^2$

Feynman integrals have the following general form:

$$G(X) = \frac{e^{\epsilon\gamma_E L}}{(i\pi^{d/2})^L} \int \frac{d^d k_1 \dots d^d k_L X(k_{l_1}, \dots, k_{l_R})}{D_1^{\nu_1} \dots D_i^{\nu_i} \dots D_N^{\nu_N}}.$$

The N propagators are:

$$D_i = q_i^2 - m_i^2 = \left[\sum_{l=1}^L c_i^l k_l + \sum_{e=1}^n d_i^e p_e \right]^2 - m_i^2$$

The numerator X may contain a tensor structure (see later for more on that):

$$X(k_{l_1}, \dots, k_{l_R}) = (k_{l_1} P_{e_1}) \dots (k_{l_R} P_{e_R}) = (P_{e_1}^{\alpha_1} \dots P_{e_R}^{\alpha_R}) (k_{l_1}^{\alpha_1} \dots k_{l_R}^{\alpha_R})$$

Tensor integrals

Tensor integrals appear naturally in Feynman diagrams, due to

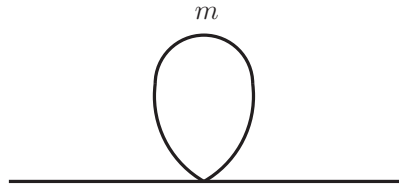
- fermion propagators
- non-abelian triple-boson vertices
- boson propagators in R_ξ gauges and unitary gauge

Example: Fermionic vacuum polarization

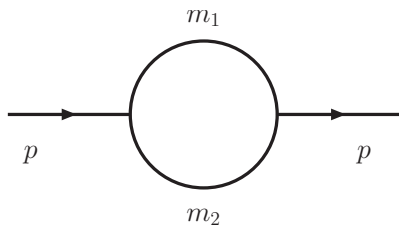
$$\begin{aligned}\Pi^{\alpha\beta} &\sim \frac{1}{(i\pi^{d/2})} \int d^d k \text{Tr} \left[\frac{[\gamma k + m_1]}{D_1} \gamma^\beta \frac{[\gamma(k + p_1) + m_2]}{D_2} \gamma^\alpha \right] \\ &\sim \frac{1}{(i\pi^{d/2})} \int \frac{d^d k}{D_1 D_2} \left[(m_1 m_2 - k^2 - k p_1) g^{\alpha\beta} + 2k^\alpha k^\beta + k^\alpha p_1^\beta + p_1^\alpha k^\beta \right]\end{aligned}$$

So, one needs also efficient ways to evaluate tensor integrals – see later

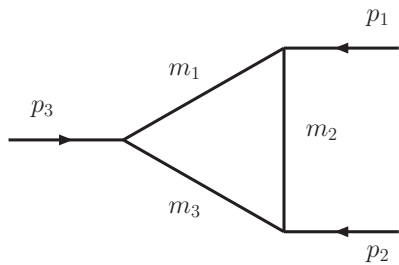
Simple examples of scalar integrals



$$A_0 = \frac{1}{(i\pi^{d/2})} \int \frac{d^d k}{D_1} \rightarrow \text{UV - divergent} : \sim \frac{d^4 k}{k^2}$$



$$B_0 = \frac{1}{(i\pi^{d/2})} \int \frac{d^d k}{D_1 D_2} \rightarrow \text{UV - divergent} \sim \frac{d^4 k}{k^4}$$



$$C_0 = \frac{1}{(i\pi^{d/2})} \int \frac{d^d k}{D_1 D_2 D_3} \rightarrow \text{UV - finite} \sim \frac{d^4 k}{k^6}$$

Dependent on conventions, where k starts to run in the loop, it is:

$$\begin{aligned} D_1 &= k^2 - m_1^2 \\ D_2 &= (k + p_1)^2 - m_2^2 \\ D_3 &= (k + p_1 + p_2)^2 - m_3^2 \end{aligned}$$

Evaluate Feynman integrals

There are two strategies to solve a Feynman integral:

- **Reduction**
Express the integral with the aid of **recurrence relations** by other, known integrals.
These are then the **Master Integrals**.
- **Direct evaluation**

Introduce Feynman parameters

$$\frac{1}{D_1^{\nu_1} D_2^{\nu_2} \dots D_N^{\nu_N}} = \frac{\Gamma(\nu_1 + \dots + \nu_N)}{\Gamma(\nu_1) \dots \Gamma(\nu_N)} \int_0^1 dx_1 \dots \int_0^1 dx_N \frac{x_1^{\nu_1-1} \dots x_N^{\nu_N-1} \delta(1 - x_1 \dots - x_N)}{(x_1 D_1 + \dots + x_N D_N)^{N_\nu}},$$

with $N_\nu = \nu_1 + \dots + \nu_N$.

The denominator of G contains, after introduction of Feynman parameters x_i , the momentum dependent function m^2 with index-exponent N_ν :

$$(m^2)^{-(\nu_1 + \dots + \nu_N)} = (x_1 D_1 + \dots + x_N D_N)^{-N_\nu} = (k_i M_{ij} k_j - 2Q_j k_j + J)^{-N_\nu}$$

Here M is an $(L \times L)$ -matrix, $Q = Q(x_i, p_e)$ an L -vector and $J = J(x_i, m_i^2, p_{e_j} p_{e_l})$.

M, Q, J are linear in x_i . The momentum integration is now simple:

Shift the momenta k such that m^2 has no linear term in \bar{k} :

$$\begin{aligned} k &= \bar{k} + (M^{-1})Q, \\ m^2 &= \bar{k} M \bar{k} - Q M^{-1} Q + J. \end{aligned}$$

Remember: $M_{1\text{-loop}} = 1$, in general:

$$M^{-1} = \frac{1}{(\det M)} \tilde{M},$$

where \tilde{M} is the transposed matrix to M . The shift leaves the integral unchanged.

The shift leaves the integral unchanged (rename $\bar{k} \rightarrow k$):

$$G(1) = \int \frac{Dk_1 \dots Dk_L}{(kMk + J - QM^{-1}Q)^{N_\nu}}.$$

Go Euclidean: Rotate now the $k^0 \rightarrow iK_E^0$ with $k^2 \rightarrow -k_E^2$ (and again rename $k^E \rightarrow k$):

$$G(1) \rightarrow (i)^L \int \frac{Dk_1^E \dots Dk_L^E}{(-k^E M k^E + J - QM^{-1}Q)^{N_\nu}} = (-1)^{N_\nu} (i)^L \int \frac{Dk_1 \dots Dk_L}{[kMk - (J - QM^{-1}Q)]^{N_\nu}}.$$

Call

$$\mu^2(x) = -(J - QM^{-1}Q)$$

and get

$$G(1) = (-1)^{N_\nu} (i)^L \int \frac{Dk_1 \dots Dk_L}{(kMk + \mu^2)^{N_\nu}}.$$

For 1-loop integrals it is $L = 1, M = 1$ - and we will use nearly only those - we are ready to do the k -integration.

Additional step for L -loop integrals

For L -loops go on and now **diagonalize the matrix M** by a rotation:

$$\begin{aligned}
 k \rightarrow k'(x) &= V(x) k, \\
 k M k &= k' M_{diag} k' \\
 &\rightarrow \sum \alpha_i(x) k_i^2(x), \\
 M_{diag}(x) &= (V^{-1})^+ M V^{-1} = (\alpha_1, \dots, \alpha_L).
 \end{aligned}$$

This leaves both the integration measure and the integral invariant:

$$G(1) = (-1)^{N_\nu} (i)^L \int \frac{Dk_1 \dots Dk_L}{(\sum_i \alpha_i k_i^2 + \mu^2)^{N_\nu}}.$$

Rescale now the k_i ,

$$\bar{k}_i = \sqrt{\alpha_i} k_i,$$

with

$$\begin{aligned}
 d^d k_i &= (\alpha_i)^{-d/2} d^d \bar{k}_i, \\
 \prod_{i=1}^L \alpha_i &= \det M,
 \end{aligned} \tag{10}$$

and get the Euclidean integral to be calculated (and rename $\bar{k} \rightarrow k$):

$$G(1) = (-1)^{N_\nu} (i)^L (\det M)^{-d/2} \int \frac{Dk_1 \dots Dk_L}{(k_1^2 + \dots + k_L^2 + \mu^2)^{N_\nu}}.$$

Use now (remembering that $Dk = dk/(i\pi^{d/2})$):

$$\begin{aligned} i^L \int \frac{Dk_1 \dots Dk_L}{(k_1^2 + \dots + k_L^2 + \mu^2)^{N_\nu}} &= \frac{\Gamma(N_\nu - \frac{d}{2}L)}{\Gamma(N_\nu)} \frac{1}{(\mu^2)^{N_\nu - dL/2}}, \\ i^L \int \frac{Dk_1 \dots Dk_L k_1^2}{(k_1^2 + \dots + k_L^2 + \mu^2)^{N_\nu}} &= \frac{d}{2} \frac{\Gamma(N_\nu - \frac{d}{2}L - 1)}{\Gamma(N_\nu)} \frac{1}{(\mu^2)^{N_\nu - dL/2 - 1}}. \end{aligned} \quad (11)$$

These formulae follow for $L = 1$ immediately from any textbook.

See 'Mathematical Interlude'.

For $L > 1$, get it iteratively, with setting $(k_1^2 + k_2^2 + m^2)^N = (k_1^2 + M^2)^N$, $M^2 = k_2^2 + m^2$, etc.

Finally, one gets for **Scalar integrals**:

$$G(1) = (-1)^{N_\nu} \frac{\Gamma(N_\nu - \frac{d}{2}L)}{\Gamma(\nu_1) \dots \Gamma(\nu_N)} \int_0^1 \prod_{j=1}^N dx_j x_j^{\nu_j-1} \delta\left(1 - \sum_{i=1}^N x_i\right) \frac{(\det M)^{-d/2}}{(\mu^2)^{N_\nu - dL/2}},$$

or

$$G(1) = (-1)^{N_\nu} \frac{\Gamma(N_\nu - \frac{d}{2}L)}{\Gamma(\nu_1) \dots \Gamma(\nu_N)} \int_0^1 \prod_{j=1}^N dx_j x_j^{\nu_j-1} \delta\left(1 - \sum_{i=1}^N x_i\right) \frac{U(x)^{N_\nu - d(L+1)/2}}{F(x)^{N_\nu - dL/2}}$$

with

$$U(x) = (\det M) \quad (\rightarrow 1 \text{ for } L = 1)$$

$$F(x) = (\det M) \mu^2 = -(\det M) J + Q \tilde{M} Q \quad (\rightarrow -J + Q^2 \text{ for } L = 1)$$

Trick for one-loop functions:

$U = \det M = 1 = \sum x_i$ and so U 'disappears' and the construct $F_1(x)$ is bilinear in $x_i x_j$:

$$F_1(x) = -J(\sum x_i) + Q^2 = \sum A_{ij} x_i x_j.$$

The vector integrals differ by some numerator $k_i p_e$ and thus there is a single shift in the integrand

$$k \rightarrow \bar{k} + U(x)^{-1} \tilde{M} Q$$

the $\int d^d \bar{k} \bar{k} / (\bar{k}^2 + \mu^2) \rightarrow 0$, and no further changes:

$$G(k_{1\alpha}) = (-1)^{N_\nu} \frac{\Gamma(N_\nu - \frac{d}{2}L)}{\Gamma(\nu_1) \dots \Gamma(\nu_N)} \int_0^1 \prod_{j=1}^N dx_j x_j^{\nu_j-1} \delta\left(1 - \sum_{i=1}^N x_i\right) \frac{U(x)^{N_\nu - d(L+1)/2 - 1}}{F(x)^{N_\nu - dL/2}} \left[\sum_l \tilde{M}_{1l} Q_l \right]_\alpha,$$

Tensor integrals also follow this scheme: The x -integrals have the same structure like the scalar ones.

Often, the 1-loop presentation may be used for a **sequential treatment of an L -loop integral** by iterating the basic representation L times.

Beware: This does NOT work for all cases. Keyword: **Non-planar diagrams**

Examples for one-loop F -polynomials

One-loop vertex:

$$F(t, m^2) = m^2(x_1 + x_2)^2 + [-t]x_1x_2$$

one-loop box:

$$F(s, t, m^2) = m^2(x_1 + x_2)^2 + [-t]x_1x_2 + [-s]x_3x_4$$

one-loop pentagon:

$$F(s, t, t', v_1, v_2, m^2) = m^2(x_1 + x_3 + x_4)^2 + [-t]x_1x_3 + [-t']x_1x_4 + [-s]x_2x_5 + [-v_1]x_3x_5 + [-v_2]x_2x_4$$

Euclidean kinematics:

s, t, t', v_1, v_2 are negative, so that F is positive semi-definite.

Otherwise, analytic continuations have to be carefully performed if needed, having in mind e.g. that

$$s \rightarrow s + i\epsilon$$

The Tadpole $A_0(m)$



$$T1l1m[a] = A_0 = \frac{e^{\epsilon\gamma_E}}{(i\pi^{d/2})} \int \frac{d^d k}{(k^2 - m^2)^a} \rightarrow \text{UV - divergent}$$

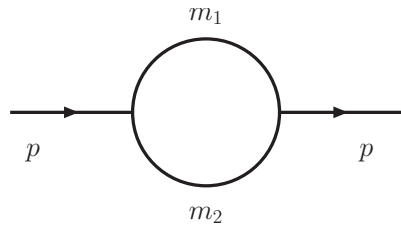
With our general formulae we get, in the 1-dimensional Feynman parameter integral, for the numerator

$$\begin{aligned} N &= (k^2 - m^2)x_1 \equiv k^2 + J \\ F &= m^2 x_1 \equiv m^2 x_1^2 \end{aligned} \tag{12}$$

and thus

$$\begin{aligned} T1l1m[a] &= (-1)^a e^{\epsilon\gamma_E} \frac{\Gamma[a - d/2]}{\Gamma[a]} \int_0^1 dx x^{a-1} \delta[1 - x] \frac{1}{F^{a-d/2}} \\ &= (-1)^a e^{\epsilon\gamma_E} (m^2)^{2-a-\epsilon} \frac{\Gamma[a - 2 + \epsilon]}{\Gamma[a]} \\ &\rightarrow -e^{\epsilon\gamma_E} \Gamma[-1 + \epsilon] \text{ for } a = 1, m = 1 \\ &= \frac{1}{\epsilon} + 1 + \left(1 + \frac{\zeta_2}{2}\right) \epsilon + \left(1 + \frac{\zeta_2}{2} - \frac{\zeta_3}{3}\right) \epsilon^2 + \dots \end{aligned} \tag{13}$$

The Self-energy $B_0(s, m_1, m_2)$



$$SE2l = B_0[s, m_1, m_2] = (2\sqrt{\pi}\mu)^{4-d} \frac{e^{\epsilon\gamma_E}}{(i\pi^{d/2})} \int \frac{d^d k}{[k^2 - m^2][(k+p)^2 - m_2^2]}$$

The $SE2l$ is UV-divergent and the corresponding F -function is:

$$F[s, m_1, m_2] = m_1^2 x_1^2 + m_2^2 x_2^2 + [-s + m_1^2 + m_2^2] x_1 x_2$$

and for special cases:

$$\begin{aligned} F[s, m_1, 0] &= m_1^2 x_1^2 + [-s + m_1^2] x_1 x_2 \\ F[s, m_1, m_1] &= m_1^2 (x_1 + x_2)^2 + [-s] x_1 x_2 \\ F[s, 0, 0] &= [-s] x_1 x_2 \end{aligned} \tag{14}$$

The 'conventional' Feynman parameter integral is 1-dimensional because $x_2 \equiv 1 - x_1$:

$$F(x) = -sx(1-x) + m_2^2(1-x) + m_1^2 x \equiv -s(x - x_a)(x - x_b)$$

The result is of logarithmic type for the constant term in ϵ :

$$\begin{aligned}
 B_0[s, m_1, m_2] &= (4\pi\mu^2)^\epsilon e^{\epsilon\gamma_E} \frac{\Gamma(1+\epsilon)}{\epsilon} \int_0^1 \frac{dx}{F(x)^\epsilon} \\
 &= \frac{1}{\epsilon} - \int_0^1 dx \ln\left(\frac{F(x)}{4\pi\mu^2}\right) \\
 &\quad + \epsilon \left\{ \frac{\zeta_2}{2} + \frac{1}{2} \int_0^1 dx \ln^2\left(\frac{F(x)}{4\pi\mu^2}\right) \right\} + \mathcal{O}(\epsilon^2). \tag{15}
 \end{aligned}$$

Here we used the expansion:

$$e^{\epsilon\gamma_E} \Gamma(1+\epsilon) = 1 + \frac{\zeta_2}{2} \epsilon^2 - \frac{\zeta_3}{3} \epsilon^3 \dots$$

When using `LoopTools`, the corresponding call returns exactly the constant term of B_0 in ϵ (with use of $e^{\epsilon\gamma_E} = 1 + \epsilon\gamma_E + \dots \rightarrow 1$):

$$B_0^{(0)}(s, m_1^2, m_2^2) = \text{b0}(s, \text{am12}, \text{am22})$$

For $4\pi\mu^2 \rightarrow 1$ B_0 looks quite compact:

$$B_0(s, m_1, m_2) = \frac{1}{\epsilon} - \int_0^1 dx \ln[F(x)] + \frac{\epsilon}{2} \left[\zeta_2 + \int_0^1 dx \ln^2[F(x)] \right] + \dots$$

Explicitly, one has to integrate

$$\begin{aligned}\ln[F(x)] &= \ln[-s(x - x_a)(x - x_b)] \\ \ln^2[F(x)] &= \ln^2[-s(x - x_a)(x - x_b)]\end{aligned}\tag{16}$$

So we will need the integrals:

$$\int dx_0^1 \{ \ln(x - x_a), \ln(x - x_a)\ln(x - x_b) \}$$

which is trivial, together with some complex algebra rules how to handle complex arguments of logarithms with

$$s \rightarrow s + i\epsilon$$

wherever needed.

For the case $m_1 = m_2 = 1$, one gets for the first terms in ϵ :

$$\begin{aligned} B_0[s, 1, 1] &= \frac{1}{\epsilon} + 2 + \frac{1+y}{1-y} H(0, y), \\ H(0, y) &= \ln(y). \end{aligned} \tag{17}$$

The $H(0, y)$ is a harmonic polylogarithmic function, and

$$\begin{aligned} y &= \frac{\sqrt{-s+4} - \sqrt{-s}}{\sqrt{-s+4} + \sqrt{-s}} \\ s &= -\frac{(1-x)^2}{x} \end{aligned} \tag{18}$$

The other case treated later again is $m_1 = 0, m_2 = m$:

$$B_0[s, m^2, 0] = \frac{1}{\epsilon} + 2 + \frac{1-s/m^2}{s/m^2} \ln(1-s/m^2)$$

Now using Mellin-Barnes Representations

Perform the x -integrations

Find an as-general-as-possible general formula

Make it ready for algorithmic analytical and/or numerical evaluation

Computer codes:

- **Ambre.m** – Derive Mellin-Barnes representations for Feynman integrals
- **MB.m** – Find an ϵ -expansion and evaluate numerically in Euclidean region

[Gluza:2007rt]

[Czakon:2005rk]

Integrating the Feynman parameters – get MB-Integrals

We derived:

$$\begin{aligned}
 SE2l1m = B_0(s, m, 0) &= e^{\epsilon\gamma_E} \Gamma(\epsilon) \int_0^1 dx_1 dx_2 \frac{\delta(1 - x_1 - x_2)}{F(x)^\epsilon} \\
 V3l2m = C_0(s, m, m, 0) &= e^{\epsilon\gamma_E} \Gamma(1 + \epsilon) \int_0^1 dx_1 dx_2 dx_3 \frac{\delta(1 - x_1 - x_2 - x_3)}{F(x)^{1+\epsilon}} \quad (19)
 \end{aligned}$$

and

$$\begin{aligned}
 F_{SE2l1m} &= m^2 x_1^2 + (-s + m^2) x_1 x_2 \\
 F_{V3l2m} &= m^2 (x_1 + x_2)^2 + (-s) x_1 x_2 \quad (20)
 \end{aligned}$$

We want to apply now:

$$\int_0^1 \prod_{j=1}^N dx_j x_j^{\alpha_j - 1} \delta\left(1 - \sum x_i\right) = \frac{\Gamma(\alpha_1) \Gamma(\alpha_2) \cdots \Gamma(\alpha_N)}{\Gamma(\alpha_1 + \alpha_2 + \cdots + \alpha_N)}$$

with coefficients α_i dependent on ν_i and on the structure of the F

See in a minute:

For this, we have to apply one or several MB-integrals here.

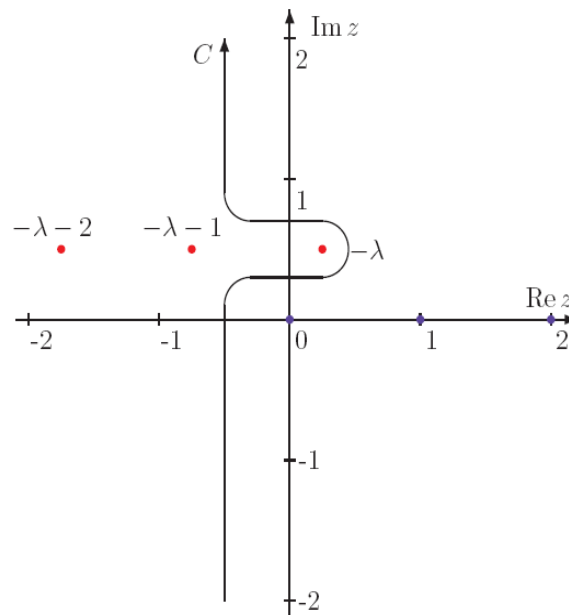
$$\int_0^1 \prod_{j=1}^N dx_j x_j^{\alpha_j-1} \delta\left(1 - \sum_{i=1}^N x_i\right) = \frac{\prod_{i=1}^N \Gamma(\alpha_i)}{\Gamma\left(\sum_{i=1}^N \alpha_i\right)}$$

Simplest cases:

$$\begin{aligned} \int_0^1 dx_1 x_1^{\alpha_1-1} \delta(1-x_1) &= 1 \\ \int_0^1 \prod_{j=1}^2 dx_j x_j^{\alpha_j-1} \delta\left(1 - \sum_{i=1}^N x_i\right) &= \int_0^1 dx_1 x_1^{\alpha_1-1} (1-x_1)^{\alpha_2-1} = B(\alpha_1, \alpha_2) \\ &= \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1 + \alpha_2)} \end{aligned}$$

Here we want to go:

$$\frac{1}{(A+B)^\lambda} = \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz \Gamma(\lambda+z) \Gamma(-z) \frac{B^z}{A^{\lambda+z}}$$



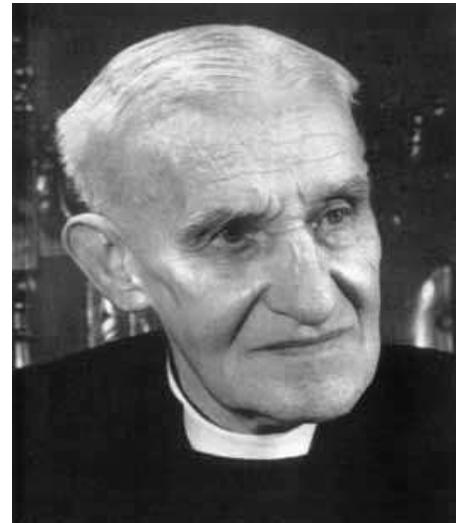
The integration path **separates poles of $\Gamma[\lambda+z]$ and $\Gamma[-z]$.**

The formula looks a bit unusual to loop people, but for persons with a mathematical background it is common knowledge.

One might well assume that these two gentlemen did not dream of so heavy use of their results in basic research ...

Mellin, Robert, Hjalmar, 1854-1933

Barnes, Ernest, William, 1874-1953



Barnes' contour integrals for the hypergeometric function

Exact proof and further reading: Whittaker & Watson (CUP 1965) 14.5 - 14.52, pp. 286-290

Consider

$$F(z) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} d\sigma (-z)^\sigma \frac{\Gamma(a + \sigma)\Gamma(b + \sigma)\Gamma(-\sigma)}{\Gamma(c + \sigma)}$$

where $|\arg(-z)| < \pi$ (i.e. $(-z)$ is not on the neg. real axis) and the path is such that it separates the poles of $\Gamma(a + \sigma)\Gamma(b + \sigma)$ from the poles of $\Gamma(-\sigma)$.

$1/\Gamma(c + \sigma)$ has no pole.

Assume $a \neq -n$ and $b \neq -n, n = 0, 1, 2, \dots$ so that the contour can be drawn.

The poles of $\Gamma(\sigma)$ are at $\sigma = -n, n = 1, 2, \dots$, and it is:

$$\text{Residue}[F[s] \Gamma(-s) , \{s, n\}] = (-1)^n / n! F(n)$$

Closing the path to the right gives then, by Cauchy's theorem, for $|z| < 1$ the

hypergeometric function ${}_2F_1(a, b, c, z)$ (for proof see textbook):

$$\begin{aligned} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} d\sigma (-z)^\sigma \frac{\Gamma(a+\sigma)\Gamma(b+\sigma)\Gamma(-\sigma)}{\Gamma(c+\sigma)} &= \sum_{n=0}^{N \rightarrow \infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!} \\ &= \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} {}_2F_1(a, b, c, z) \end{aligned}$$

The **continuation** of the hypergeometric series for $|z| > 1$ is made using the intermediate formula

$$\begin{aligned} F(z) &= \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(1-c+a+n) \sin[(c-a-n)\pi]}{\Gamma(1+n)\Gamma(1-a+b+n) \cos(n\pi) \sin[(b-a-n)\pi]} (-z)^{-a-n} \\ &\quad + \sum_{n=0}^{\infty} \frac{\Gamma(b+n)\Gamma(1-c+b+n) \sin[(c-b-n)\pi]}{\Gamma(1+n)\Gamma(1-a+b+n) \cos(n\pi) \sin[(a-b-n)\pi]} (-z)^{-b-n} \end{aligned}$$

and yields

$$\begin{aligned} \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} {}_2F_1(a, b, c, z) &= \frac{\Gamma(a)\Gamma(a-b)}{\Gamma(a-c)} (-z)^{-a} {}_2F_1(a, 1-c+a, 1-b+ac, z^{-1}) \\ &\quad + \frac{\Gamma(b)\Gamma(b-a)}{\Gamma(b-c)} (-z)^{-b} {}_2F_1(b, 1-c+b, 1-a+b, z^{-1}) \end{aligned}$$

Corollary I

Putting $b = c$, we see that

$$\begin{aligned} {}_2F_1(a, b, b, z) &= \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(a)} \frac{z^n}{n!} \\ &= \frac{1}{(1-z)^a} = \frac{1}{2\pi i \Gamma(a)} \int_{-i\infty}^{+i\infty} d\sigma (-z)^\sigma \Gamma(a+\sigma) \Gamma(-\sigma) \end{aligned}$$

This allows to **replace sum by product**:

$$\frac{1}{(A+B)^a} = \frac{1}{B^a [1 - (-A/B)]^a} = \frac{1}{2\pi i \Gamma(a)} \int_{-i\infty}^{i\infty} d\sigma A^\sigma B^{-\sigma-a} \Gamma(a+\sigma) \Gamma(-\sigma)$$

Barnes' lemma

If the path of integration is curved so that the poles of $\Gamma(c - \sigma)\Gamma(d - \sigma)$ lie on the right of the path and the poles of $\Gamma(a + \sigma)\Gamma(b + \sigma)$ lie on the left, then

$$\frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} d\sigma \Gamma(a + \sigma)\Gamma(b + \sigma)\Gamma(c - \sigma)\Gamma(d - \sigma) = \frac{\Gamma(a + c)\Gamma(a + d)\Gamma(b + c)\Gamma(b + d)}{\Gamma(a + b + c + d)}$$

It is supposed that a, b, c, d are such that no pole of the first set coincides with any pole of the second set.

Scetch of proof: Close contour by semicircle C to the right of imaginary axis. The integral exists and \int_C vanishes when $\Re(a + b + c + d - 1) < 0$. Take sum of residues of the integrand at poles of $\Gamma(c - \sigma)\Gamma(d - \sigma)$. The double sum leads to two hypergeometric functions, expressible by ratios of Γ -functions, this in turn by combinations of *sin*, may be simplifies finally to the r.h.s.

Analytical continuation: The relation is proved when $\Re(a + b + c + d - 1) < 0$.

Both sides are analytical functions of e.g. a . So the relation remains true for all values of a, b, c, d for which none of the poles of $\Gamma(a + \sigma)\Gamma(b + \sigma)$, as a function of σ , coincide with any of the poles of $\Gamma(c - \sigma)\Gamma(d - \sigma)$.

Corollary II Any real shift k : $\sigma + k, a - k, b - k, c + k, d + k$ together with $\int_{-k-i\infty}^{-k+i\infty}$ leaves the result true.

How can the Mellin-Barnes formula be made useful in the context of Feynman integrals?

- Apply corollary I to propagators and get:

$$\frac{1}{(p^2 - m^2)^a} = \frac{1}{2\pi i \Gamma(a)} \int_{-i\infty}^{i\infty} d\sigma \frac{(-m^2)^\sigma}{(p^2)^{a+\sigma}} \Gamma(a + \sigma) \Gamma(-\sigma)$$

which transforms a massive propagator to a massless one (with index a of the line changed to $(a + \sigma)$).

- Apply corollary I after introduction of Feynman parameters and after the momentum integration to the resulting F - and U -forms, in order to get a single monomial in the x_i , which allows the integration over the x_i :

$$\frac{1}{[A(s)x_1^{a_1} + B(s)x_1^{b_1}x_2^{b_2}]^a} = \frac{1}{2\pi i \Gamma(a)} \int_{-i\infty}^{i\infty} d\sigma [A(s)x_1^{a_1}]^\sigma [B(s)x_1^{b_1}x_2^{b_2}]^{a+\sigma} \Gamma(a + \sigma) \Gamma(-\sigma)$$

Both methods leave Mellin-Barnes (MB-) integrals to be performed afterwards.

A short remark on history

- [N. Usyukina, 1975](#): "ON A REPRESENTATION FOR THREE POINT FUNCTION", Teor. Mat. Fiz. 22;
a finite massless off-shell 3-point 1-loop function represented by 2-dimensional MB-integral
- [E. Boos, A. Davydychev, 1990](#): "A Method of evaluating massive Feynman integrals", Theor. Math. Phys. 89 (1991);
N-point 1-loop functions represented by n-dimensional MB-integral
- [V. Smirnov, 1999](#): "Analytical result for dimensionally regularized massless on-shell double box", Phys. Lett. B460 (1999);
treat UV and IR divergencies by analytical continuation: shifting contours and taking residues 'in an appropriate way'
- [B. Tausk, 1999](#): "Non-planar massless two-loop Feynman diagrams with four on-shell legs", Phys. Lett. B469 (1999);
nice algorithmic approach to that, starting from search for some unphysical space-time dimension d for which the MB-integral is finite and well-defined
- [M. Czakon, 2005](#) (with experience from common work with [J. Gluza](#) and [TR](#)): "Automated analytic continuation of Mellin-Barnes integrals", Comput. Phys. Commun. (2006);
Tausk's approach realized in Mathematica program [MB.m](#), published and available for use

A self-energy: SE2l1m

This is a nice example, being simple but showing [nearly] all essentials in a nutshell.

We get for this $F(x) = m^2 x_1^2 + [-s + m^2] x_1 x_2$ the following representation:

$$SE2l1m = \frac{e^{\epsilon\gamma_E} (m^2)^{-\epsilon}}{2\pi i \Gamma[2 - 2\epsilon]} \int_{\Re z = -1/8} dz \left[\frac{-s + m^2}{m^2} \right]^{-\epsilon - z} \Gamma_1[1 - \epsilon - z] \Gamma_2[-z] \Gamma_3[1 - \epsilon + z] \Gamma_4[\epsilon + z]$$

Tausk approach:

Seek a configuration (i.e. values of z here) where all arguments of Γ -functions have **positive real part**. Then the $SE2l1m$ is well-defined **and finite**.

For small ϵ this is - here - evidently impossible; set $\epsilon \rightarrow 0$ and look at $\Gamma_2[-z] \Gamma_4[+z]$:

$$\Gamma_1[1 - z] \Gamma_2[-z] \Gamma_3[1 + z] \Gamma_4[+z]$$

What to do ????

Tausk: Set ϵ such that **all arguments of Γ -functions get positive real parts**, e.g. with the choice:

$$\Re(z) = -\frac{1}{8} \quad \text{and also} \quad \epsilon = 3/8$$

To make physics we have now to deform the integrand or the path such that $\epsilon \rightarrow 0$; when crossing a residue, take it and add it up.

Varying $\epsilon \rightarrow 0$ from $3/8$ makes crossing in $\Gamma_4[\epsilon + z]$ a pole at $\epsilon = -z = +1/8$; there is $\epsilon + z = 0$:

$$\text{Residue}[\text{SE2I1m}, \{z, -\epsilon\}] = e^{\epsilon\gamma_E} \frac{(m^2)^{-\epsilon}}{\Gamma[2 - 2\epsilon]} \Gamma_3[1 - 2\epsilon] \Gamma_2[\epsilon]$$

Here we 'loose' one integration (easier term!) and catch the IR-singularity in $\Gamma_2[\epsilon] \sim 1/\epsilon!$

The function becomes now, for small ϵ :

$$\begin{aligned} \text{SE2I1m} &= \frac{e^{\epsilon\gamma_E}}{2\pi i} \frac{(m^2)^{-\epsilon}}{\Gamma[2 - 2\epsilon]} \int_{\Re z = -1/8} dz \left[\frac{-s + m^2}{m^2} \right]^{-\epsilon - z} \Gamma_1[1 - \epsilon - z] \Gamma_2[-z] \Gamma_3[1 - \epsilon + z] \Gamma_4[\epsilon + z] \\ &+ e^{\epsilon\gamma_E} \frac{(m^2)^{-\epsilon}}{\Gamma[2 - 2\epsilon]} \Gamma_1[1 - 2\epsilon] \Gamma_2[\epsilon] \end{aligned} \quad (21)$$

Now we may take the limit of small ϵ because the integral will stay finite and well-defined:

$$\text{SE2I1m} = \frac{e^{\epsilon\gamma_E}}{2\pi i} \int_{\Re z = -\frac{1}{8}} dz \left[\frac{-s + m^2}{m^2} \right]^{-z} \Gamma[1 - z] \Gamma[-z] \Gamma[z] \Gamma[1 + z] + e^{\epsilon\gamma_E} \left(2 + \frac{1}{\epsilon} - \ln[m^2] \right) + O(\epsilon)$$

Now we close the integration path to the left, catch all residues from $\Gamma[z] \Gamma[1 + z]$ for

$\Re z < -1/8$, i.e. at $z = -n$, $n = 1, 2, \dots$:

$$\text{Res} \left\{ \left[\frac{-s + m^2}{m^2} \right]^{-z} \Gamma_1[1 - \epsilon - z] \Gamma_2[-z] \Gamma_3[1 - \epsilon + z] \Gamma_4[\epsilon + z], \{z, -n\} \right\} = (-s + m^2)^n \ln(-s + m^2)$$

The sum to be done is trivial (in this trivial case!!):

$$\sum_{n=1}^{\infty} \left[\frac{-s + m^2}{m^2} \right]^n = \frac{1}{1 - \frac{-s+m^2}{m^2}} - 1$$

and we end up with:

$$\mathbf{SE2I1m} = \frac{1}{\epsilon} + 2 + \left[\frac{1 - s/m^2}{s/m^2} \ln(1 - s/m^2) \right]$$

This is what we had also from the direct Feynman parameter integration above

Expansion in a small parameter: vertex V3l2m for m^2/s

Use as an example for determining the small mass expansion:

$$\begin{aligned} V3coefm1 &= \text{Coefficient}[V3l2m[[1, 1]], \epsilon, -1] \\ &= -\frac{1}{2s} \frac{1}{2\pi i} \int_{-i\infty-1/2}^{+i\infty-1/2} dz \left(-\frac{m^2}{s}\right)^z \frac{\Gamma_1[-z]^3 \Gamma_2[1+z]}{\Gamma_3[-2z]} \end{aligned} \quad (22)$$

If $|m^2/s| \ll 1$, then the smallest [positive] power of it gives the biggest contribution: its exponent has to be positive and small.

So, close the contour to the right (positive $\Re z$), and leading terms come from the residues expansion of $\Gamma_1[-z]^3/\Gamma_3[-2z]$ at $-z = -1, -2, \dots$. The residues are terms of a binomial sum:

$$\text{Residue} = -\frac{1}{s} \left(\frac{m^2}{s}\right)^n \frac{(2n)!}{(n!)^2} \left[2\text{HarmonicNumber}[n] - 2\text{HarmonicNumber}[2n] - \ln\left(-\frac{m^2}{s}\right) \right]$$

with first terms equal to $(-1)^n \text{Residua}$:

$$V3l2m = \frac{1}{s} \ln\left(-\frac{m^2}{s}\right) + \frac{m^2}{s^2} \left[\ln\left(2 + 2\frac{m^2}{s}\right) \right] + \frac{m^4}{s^3} \left[\ln\left(7 + 6\frac{m^2}{s}\right) \right] + O(m^6/s^3)$$

Sector decomposition

For Euclidean kinematics, the integrand for the multi-dimensional x -integrations is **positive semi-definite**. Proof: See examples

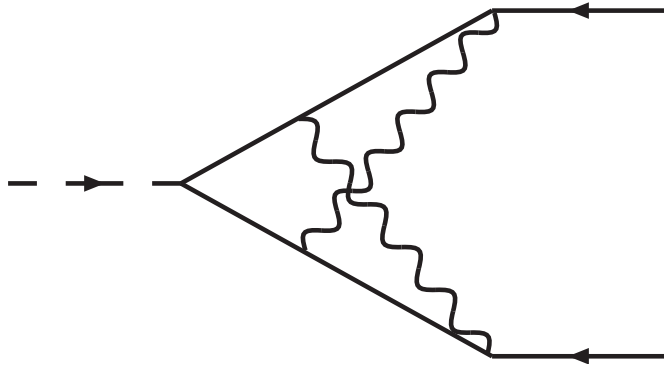
In numerical integrations, one has to separate the poles in $(d - 4)$, and in doing so one has to avoid **overlapping singularities**.

A method for that is **sector decomposition**.

There are quite a few recent papers on that, and also nice reviews are given

[Binoth:2000ps](#), [Denner:2004iz](#), [Bogner:2007cr](#), [Heinrich:2008si](#), [Smirnov:2008aw](#) The intention is to **separate singular regions in different variables from each other**, as is nicely demonstrated by an example borrowed from [Heinrich:2008si](#):

$$\begin{aligned}
 I &= \int_0^1 dx \int_0^1 dy \frac{1}{x^{1+a\epsilon} y^{b\epsilon} [x + (1-x)y]} \\
 &= \int_0^1 \frac{dx}{x^{1+(a+b)\epsilon}} \int_0^1 \frac{dt}{t^{b\epsilon} [1 + (1-x)t]} + \int_0^1 \frac{dy}{y^{1+(a+b)\epsilon}} \int_0^1 \frac{dt}{t^{1+a\epsilon} [1 + (1-y)t]}. \quad (23)
 \end{aligned}$$



The master integral $V614m1$

At several occasions, we used for cross checks the package `sector_decomposition`
[Bogner:2007cr]

built on the C++ library GINAC

[Bauer:2000cp]

For that reason, the interface `CSectors` was written; Gluza, Kajda, Yundin, T.R., Eur.Phys.J.
C71 (2011) 1516

The syntax is similar to that of `AMBRE`.

Example:

The program input for the evaluation of the integral $V614m1$ is simple; we choose
 $m = 1, s = -11$, and the topology may be read from the arguments of propagator functions PR:

```
<< CSectors.m
```

```
Options[DoSectors]
```

```
SetOptions[DoSectors, TempFileDelete -> False, SetStrategy -> C]
```

```
n1 = n2 = n3 = n4 = n5 = n6 = n7 = 1;
```

```
m = 1; s = -11;
```

```
invariants = {p1^2 -> m^2, p2^2 -> m^2, p1 p2 -> (s - 2 m^2)/2};
```

```
DoSectors[{1},
```

```
  {PR[k1,0,n1]          PR[k2,0,n2]          PR[k1+p1,m,n3]
   PR[k1+k2+p1,m,n5] PR[k1+k2-p2,m,n6] PR[k2-p2,m,n7]},
  {k2, k1}, invariants][-4, 2]
```

Here, the numerator is 1 (see the first argument `{1}` of `DoSectors`), and the output contains the functions U_2 and F_2 :

Using strategy C

$$U = x_3 x_4 + x_3 x_5 + x_4 x_5 + x_3 x_6 + x_5 x_6 + x_2 (x_3 + x_4 + x_6) + x_1 (x_2 + x_4 + x_5 + x_6)$$

$$F = x_1 x_4^2 + 13 x_1 x_4 x_5 + x_4^2 x_5 + x_1 x_5^2 + x_4 x_5^2 + 13 x_1 x_4 x_6 + 2 x_1 x_5 x_6 + 13 x_4 x_5 x_6 + x_5^2 x_6 + x_1 x_6^2 + x_5 x_6^2 + x_3^2 x_6 (x_4 + x_5 + x_6)$$

$$+x^2(x^3^2+x^4^2+13x^4x^6+x^6^2+x^3(2x^4+13x^6))+x^3(x^4^2+(x^5+x^6)^2+x^4(2x^5+13x^6))$$

Notice the presence of a U -function and the complexity of the F -function (compared to $U = 1$ and $\mathbb{F}1$ and $\mathbb{F}2$ in the loop-by-loop MB-approach) due to the **non-sequential, direct performance of both momentum integrals at once**. Both U and F are evidently positive semi-definite. The numerical result for the Feynman integral is:

$$V_{614m1}(-s)^{2\epsilon} = -0.052210 \frac{1}{\epsilon} - 0.17004 + 0.24634 \epsilon + 4.8773 \epsilon^2 + \mathcal{O}(\epsilon^3). \quad (24)$$

The numbers may be compared to (27). We obtained a third numerical result, also by sector decomposition, with the Mathematica package **FIESTA**

[Smirnov:2008py]

$$V_{614m1}(-s)^{2\epsilon} = -0.052208 \frac{1}{\epsilon} - 0.17002 + 0.24622 \epsilon + 4.8746 \epsilon^2 + \mathcal{O}(\epsilon^3). \quad (25)$$

Most accurate result: obtained with an analytical representation based on harmonic polylogarithmic functions obtained by solving a system of differential equations

[Gluza, TR, unpubl.; Remiddi:1999ew, Maitre:2005uu]

$$V_{614m1}(-s)^{2\epsilon} = -0.0522082 \frac{1}{\epsilon} - 0.170013 + 0.246253 \epsilon + 4.87500 \epsilon^2 + \mathcal{O}(\epsilon^3). \quad (26)$$

All displayed digits are accurate here.

MB-integrals – Try V614m1 with the loop-by-loop approach

In a **loop-by-loop approach**, after the first momentum integration one gets here $U = 1$ and a first F -function, which depends yet on one internal momentum k_1 :

$$\begin{aligned} f_1 = & m^2 [X[2]+X[3]+X[4]]^2 - s X[2]X[4] - \text{PR}[k_1+p_1, m] X[1]X[2] \\ & - \text{PR}[k_1+p_1+p_2, 0] X[2]X[3] - \text{PR}[k_1-p_2, m] X[1]X[4] \\ & - \text{PR}[k_1, 0] X[3]X[4] , \end{aligned}$$

leading to a **7-dimensional** MB-representation; after the second momentum integration, one has:

$$f_2 = m^2 [X[2]+X[3]]^2 - s X[2]X[3] - s X[1]X[4] - 2s X[3]X[4] ,$$

leading to another **4-dimensional** integral.

After several applications of Barnes' first lemma, an **8-dimensional integral** has to be treated.

We made no attempt here to simplify the situation by any of the numerous tricks and reformulations etc. known to experts.

at $s = -11$:

$$V614m (-s)^{2\epsilon} = -0.0522082 \frac{1}{\epsilon} - 0.17002 + 0.25606 \epsilon + 4.67 \epsilon^2 + \mathcal{O}(\epsilon^3). \quad (27)$$

A nice massive 2-loop box with numerator is e.g. $B5I3m(p_e \cdot k_1)$

We determined its expansion in a small mass **Czakon, Gluza, T.R., NPB 751 (2006).**

$$\begin{aligned}
 B5I3m(p_e \cdot k_1) &= \frac{m^{4\epsilon} (-1)^{a_{12345}} e^{2\epsilon\gamma_E}}{\prod_{j=1}^5 \Gamma[a_j] \Gamma[5 - 2\epsilon - a_{123}] (2\pi i)^4} \int_{-i\infty}^{+i\infty} d\alpha \int_{-i\infty}^{+i\infty} d\beta \int_{-i\infty}^{+i\infty} d\gamma \int_{-i\infty}^{+i\infty} d\delta \\
 & \frac{(-s)^{(4-2\epsilon) - a_{12345} - \alpha - \beta - \delta} (-t)^\delta}{\Gamma[-4 + 2\epsilon + a_{12345} + \alpha + \beta + \delta]} \frac{\Gamma[-\alpha] \Gamma[-\beta]}{\Gamma[6 - 3\epsilon - a_{12345} - \alpha] \Gamma[7 - 3\epsilon - a_{12345} - \alpha] \Gamma[5 - 2\epsilon - a_{123}] \Gamma[4 - 2\epsilon - a_{1123} - 2\alpha - \gamma] \Gamma[5 - 2\epsilon - a_{1123} - 2\alpha - \gamma]} \frac{\Gamma[-\delta]}{\Gamma[5 - 2\epsilon - a_{1123} - 2\alpha - \gamma]} \\
 & \frac{\Gamma[2 - \epsilon - a_{13} - \alpha - \gamma]}{\Gamma[8 - 4\epsilon - a_{112233445} - 2\alpha - 2\beta - 2\delta - \gamma]} \frac{\Gamma[4 - 2\epsilon - a_{12345} - \alpha - \beta - \delta - \gamma]}{\Gamma[9 - 4\epsilon - a_{112233445} - 2\alpha - 2\beta - 2\delta - \gamma]} \left\{ (p_e \cdot p_3) \Gamma[1 + a_4 + \delta] \Gamma[6 - 3\epsilon - a_{12345} - \alpha] \right. \\
 & \Gamma[4 - 2\epsilon - a_{1234} - \alpha - \beta - \delta] \Gamma[3 - \epsilon - a_{12} - \alpha] \Gamma[8 - 4\epsilon - a_{112233445} - 2\alpha - 2\delta - \gamma] \Gamma[9 - 4\epsilon - a_{112233445} - 2\alpha - 2\beta - 2\delta - \gamma] \\
 & \Gamma[5 - 2\epsilon - a_{1123} - \gamma] \Gamma[4 - 2\epsilon - a_{1123} - 2\alpha - \gamma] \Gamma[a_1 + \gamma] \Gamma[-2 + \epsilon + a_{123} + \alpha + \delta + \gamma] + \Gamma[a_4 + \delta] \left[-(p_e \cdot p_1) \Gamma[7 - 3\epsilon - a_{12345} - \alpha] \right. \\
 & \Gamma[4 - 2\epsilon - a_{1234} - \alpha - \beta - \delta] \Gamma[8 - 4\epsilon - a_{112233445} - 2\alpha - 2\delta - \gamma] \Gamma[9 - 4\epsilon - a_{112233445} - 2\alpha - 2\beta - 2\delta - \gamma] \\
 & \left. \left[\Gamma[3 - \epsilon - a_{12} - \alpha] \Gamma[5 - 2\epsilon - a_{1123} - \gamma] \Gamma[4 - 2\epsilon - a_{1123} - 2\alpha - \gamma] \Gamma[a_1 + \gamma] + \Gamma[2 - \epsilon - a_{12} - \alpha] \Gamma[4 - 2\epsilon - a_{1123} - \gamma] \right. \right. \\
 & \left. \left. \Gamma[5 - 2\epsilon - a_{1123} - 2\alpha - \gamma] \Gamma[1 + a_1 + \gamma] \right] \Gamma[-2 + \epsilon + a_{123} + \alpha + \delta + \gamma] + \Gamma[6 - 3\epsilon - a_{12345} - \alpha] \Gamma[3 - \epsilon - a_{12} - \alpha] \right. \\
 & \Gamma[5 - 2\epsilon - a_{1123} - \gamma] \Gamma[4 - 2\epsilon - a_{1123} - 2\alpha - \gamma] \Gamma[a_1 + \gamma] \left[((p_e \cdot (p_1 + p_2)) \Gamma[5 - 2\epsilon - a_{1234} - \alpha - \beta - \delta] \Gamma[9 - 4\epsilon - a_{112233445} - 2\alpha - 2\beta - 2\delta - \gamma] \right. \\
 & \left. \Gamma[8 - 4\epsilon - a_{112233445} - 2\alpha - 2\beta - 2\delta - \gamma] \Gamma[-2 + \epsilon + a_{123} + \alpha + \delta + \gamma] + (p_e \cdot p_1) \Gamma[4 - 2\epsilon - a_{1234} - \alpha - \beta - \delta] \right. \\
 & \left. \left. \Gamma[8 - 4\epsilon - a_{112233445} - 2\alpha - 2\delta - \gamma] \Gamma[9 - 4\epsilon - a_{112233445} - 2\alpha - 2\beta - 2\delta - \gamma] \Gamma[-1 + \epsilon + a_{123} + \alpha + \delta + \gamma] \right] \right\}
 \end{aligned}$$

Summary

- We have introduced the **MB-representations of L -loop N -point Feynman integrals** of general type
- The **determination of the ϵ -poles** is generally solved
- The remaining problem is the **evaluation of the multi-dimensional, finite MB-Integrals**
- This is unsolved in the general case, ... **so you have something to do if you like to ...**

Not discussed, although there is some recent activity also in that field of research:

Phase space integrals

- **Angular integrals in d dimensions**
G. Somogyi (DESY, Zeuthen) arXiv:1101.3557
- A subtraction scheme for ... QCD ...cross sections at NNLO: **integrating the iterated singly-unresolved subtraction terms**
P. Bolzoni, G. Somogyi, Z. Trocsanyi JHEP 1101 (2011), arXiv:1011.1909