

Introduction to Mellin-Barnes Representations

for Feynman Integrals



Tord Riemann, DESY, Zeuthen

CALC, July 2006, Dubna

- Introduction: 2-loop QED contributions to Bhabha scattering
- Barnes' contour integrals for the hypergeometric function
- Loop momentum integrations with Feynman parameters for L -loop n -point functions
- Representation by Mellin-Barnes integrals
- Treatment of divergencies in $d = 4 - 2\epsilon$ (MB package)
- Numerical evaluations, nested infinite series, approximations, and all that
- Summary

Introduction: 2-loop QED contributions to Bhabha scattering

We are interested in a calculation of the virtual second order corrections to

$$\frac{d\sigma}{d\cos\vartheta}(e^+e^- \rightarrow e^+e^-)$$

We are using a scheme with

- (1) $m_e \neq 0$ (**good** with the MC's)
- (2) $m_\gamma = 0$ (**bad** with the MC's; $\rightarrow$ **Mastrolia, Remiddi 2003**)
- (3) **dim.reg.** for UV and **IR** divergences

Also:

Argeri, Bonciani, Ferroglia, Mastrolia, Remiddi, v.d.Bij: all but many **2-boxes**
G. Heinrich, V. Smirnov: Calculation of selected complicated Feynman integrals

There are few topologies only:

- 1-loop: 5
- 2-loop self energies: 5 (3 for external legs)
- 2-loop vertices: 5
- 2-loop boxes: 6 → next slides

The many Feynman integrals may be reduced to 'few' master integrals by sophisticated methods (Remiddi-Laporta algorithm, 1996/2000 → IdSolver (Czakon 2003)).

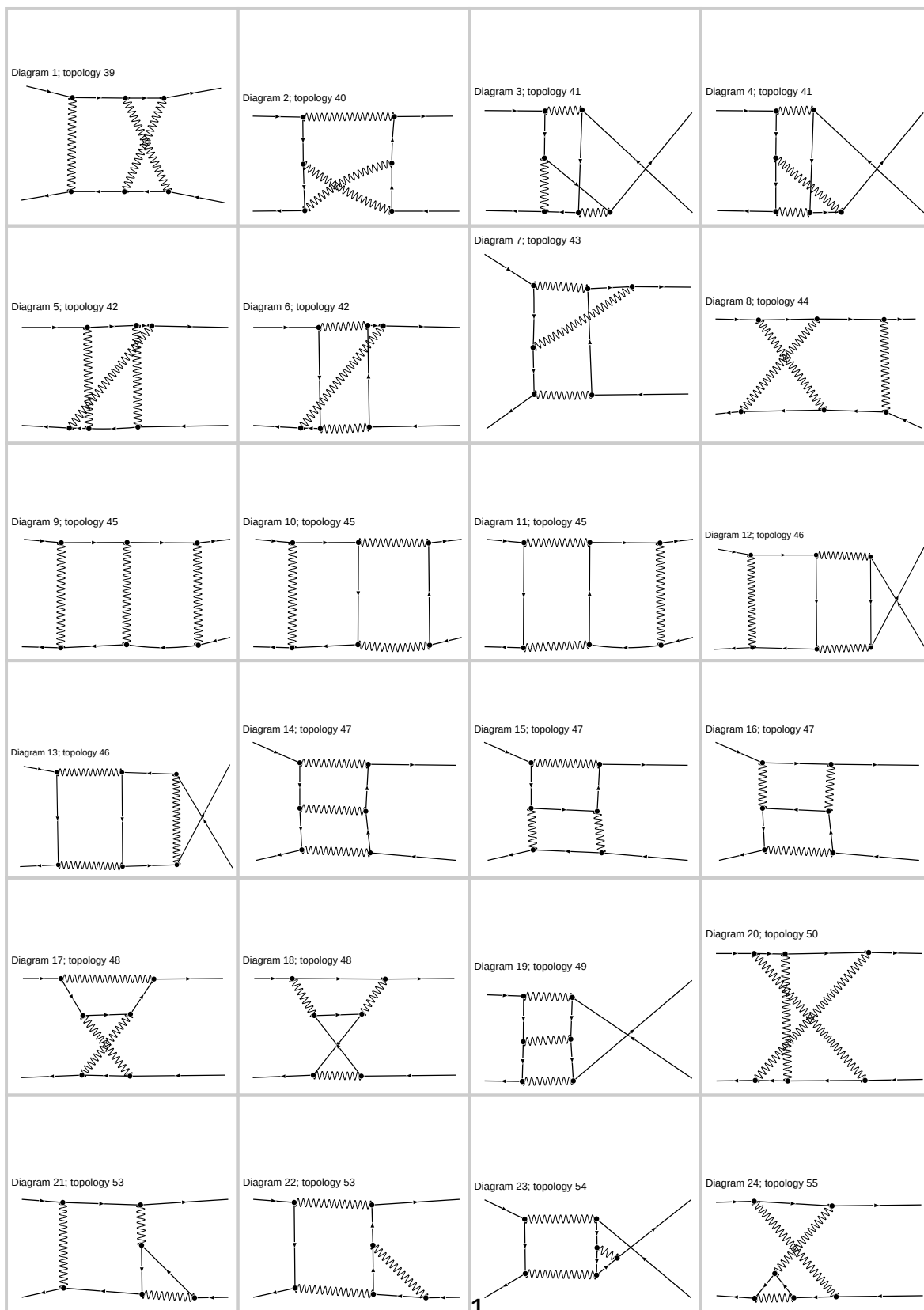
The massive diagrams (See also <http://www-zeuthen.desy.de/theory/research/bhabha>)

Assume 3 leptonic flavors, do not look at loops in external legs.

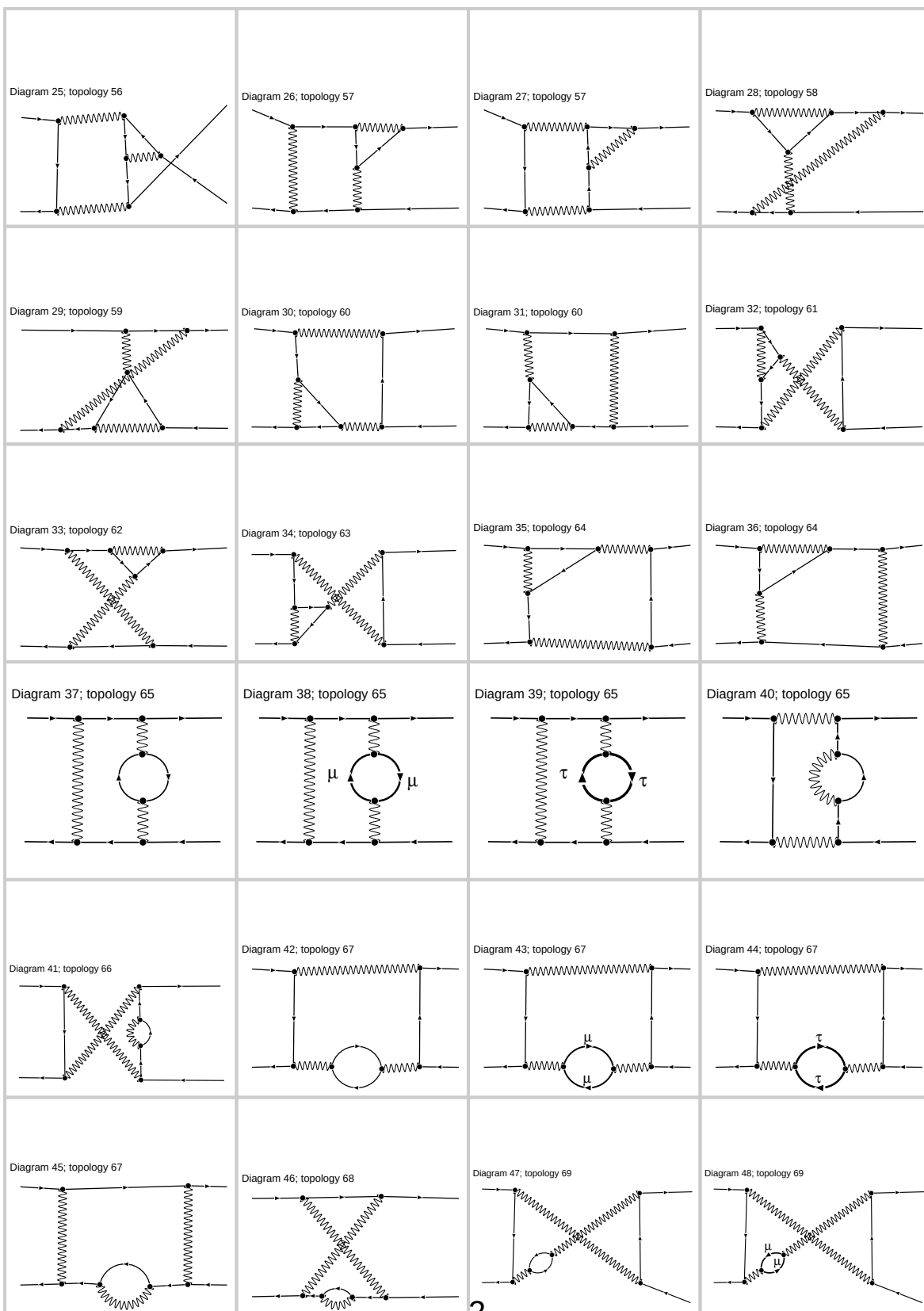
Not too many QED diagrams:

- Born diagrams: 2
- 1-loop diagrams: 14
- 2-loop diagrams: 154 (with 68 double-boxes) interfere with Born

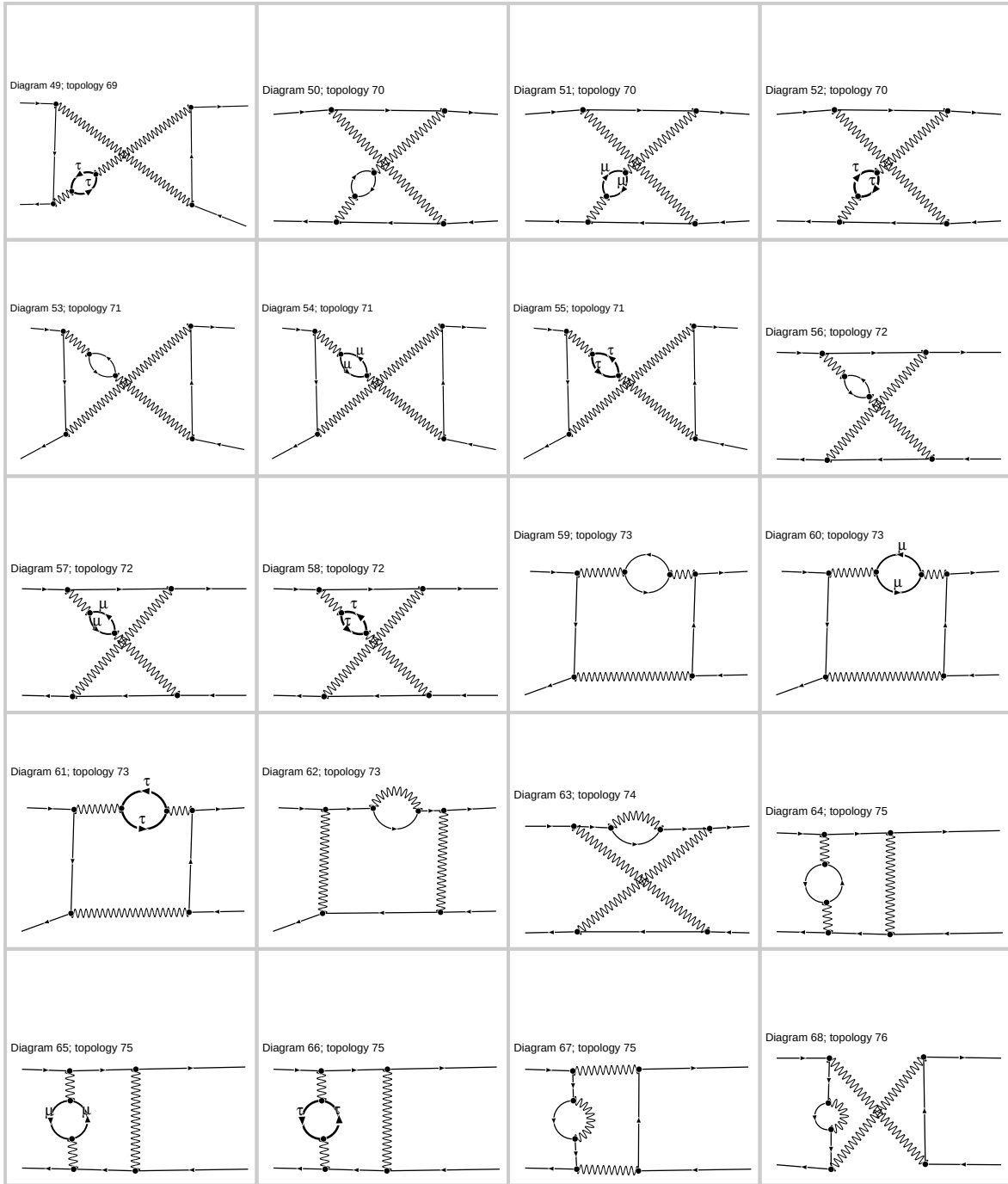
Irreducible two-loop diagrams: 1/3



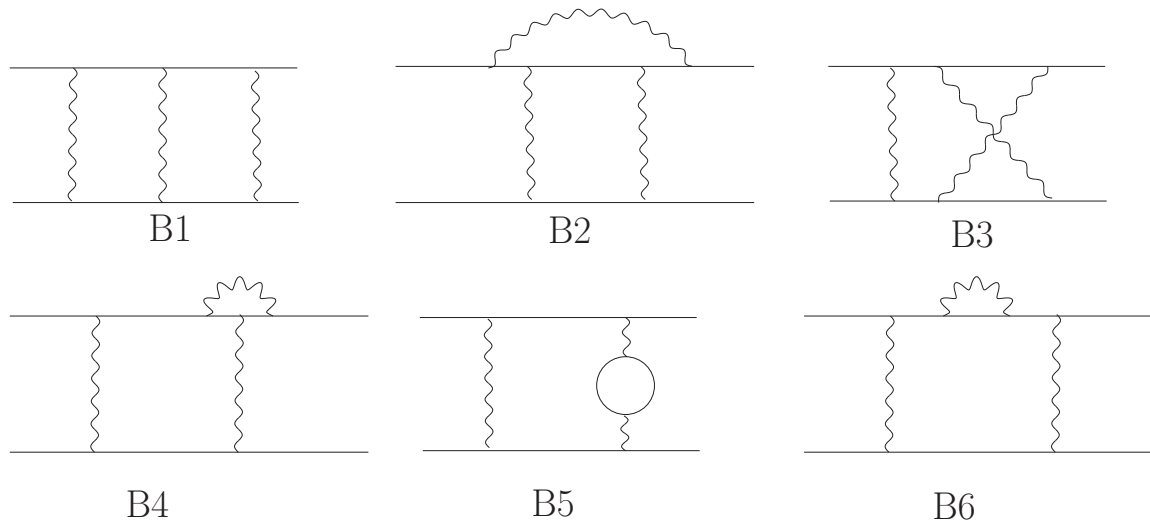
Irreducible two-loop diagrams: 2/3



Irreducible two-loop diagrams: 3/3

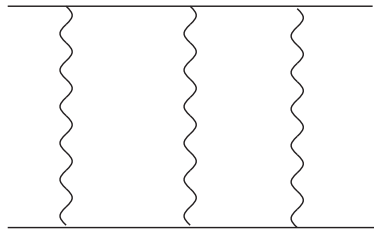


The two-loop box diagrams for massive Bhabha scattering

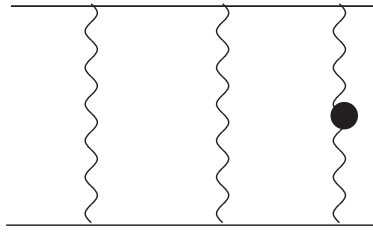


- **B5**: → 5-line masters + simpler, completely known (2004)
 Bonciani, Ferroglia, Mastrolia, Remiddi, van der Bij: hep-ph/0405275, hep-ph/0411321
 Czakon, Gluza, Riemann: <http://www-zeuthen.desy.de/.../MastersBhabha.m> (unpubl.)
- **B1–B3**: → 7-line masters + simpler, few masters known (Smirnov, Heinrich 2002,2004;
 for all planar masters the small mass limit: Czakon et al. 2006)
- **B4, B6**: → planar 6,5-line masters + simpler small mass limit known (Czakon et al. 2006)

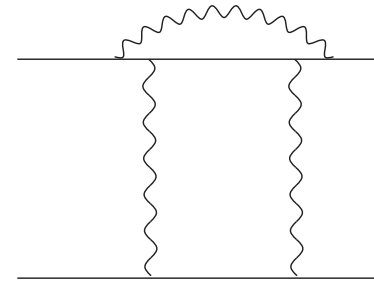
The basic planar 2-box master of **B1**, **B7l4m**, was a breakthrough



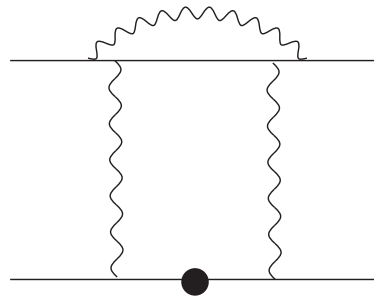
B7l4m1



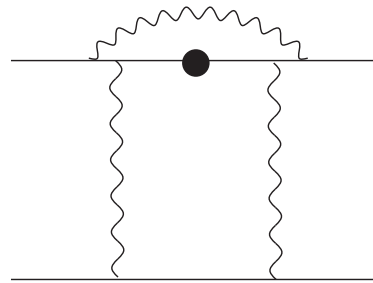
B7l4m1d



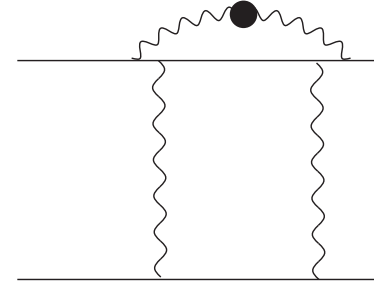
B7l4m2



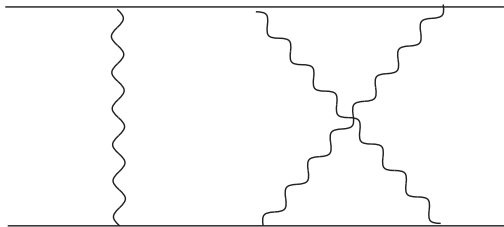
B7l4m2d1



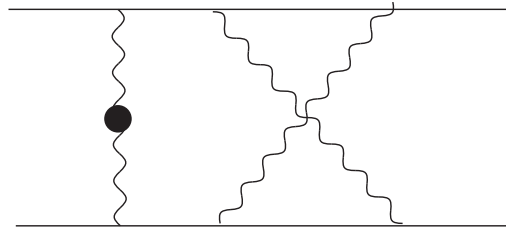
B7l4m2d2



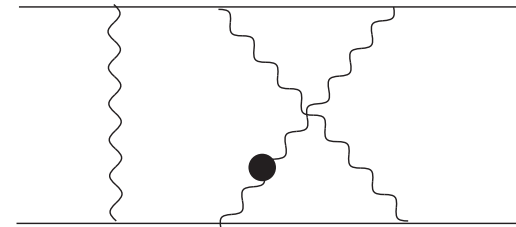
B7l4m2d3



B7l4m3

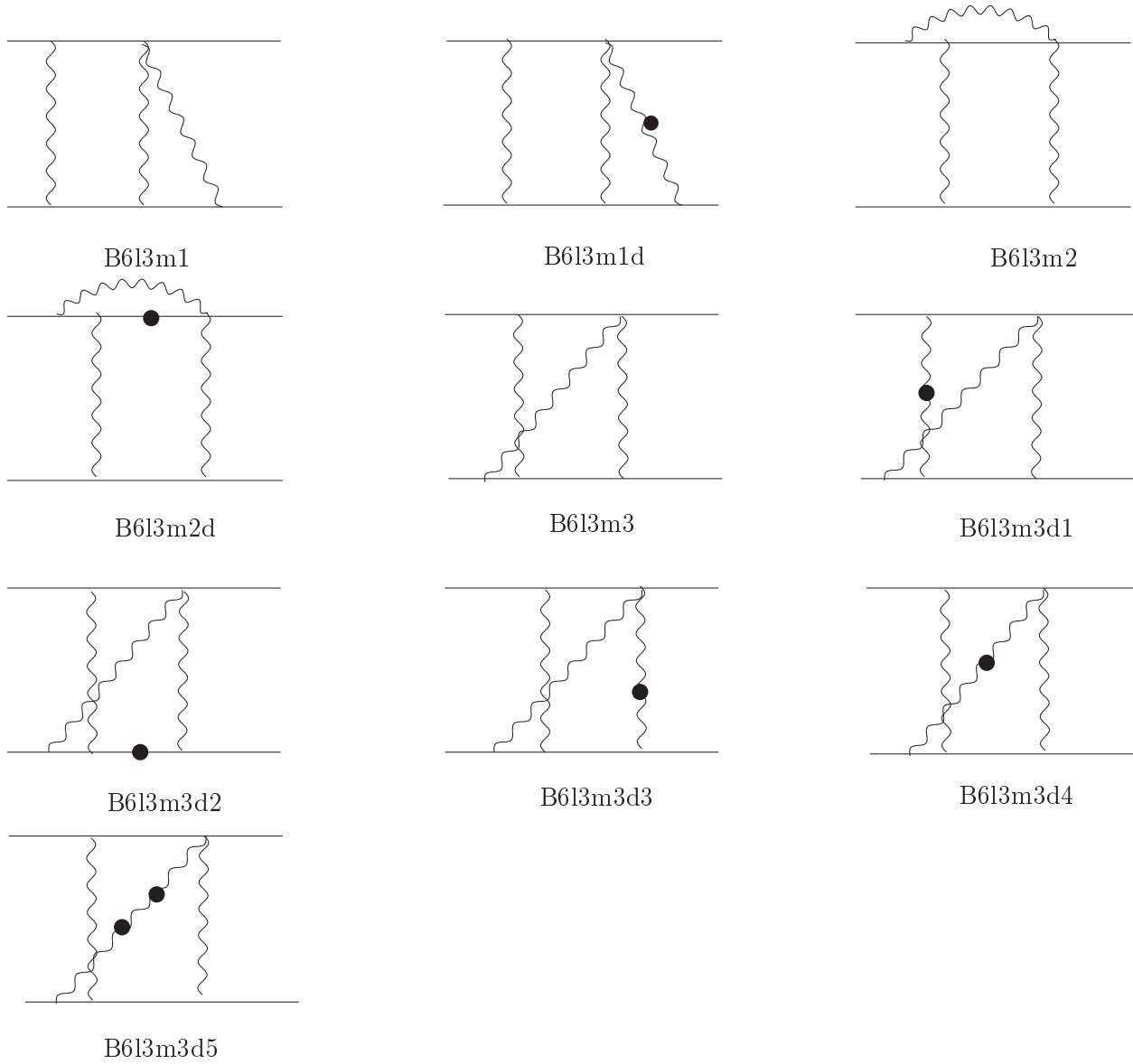


B7l4m3d1

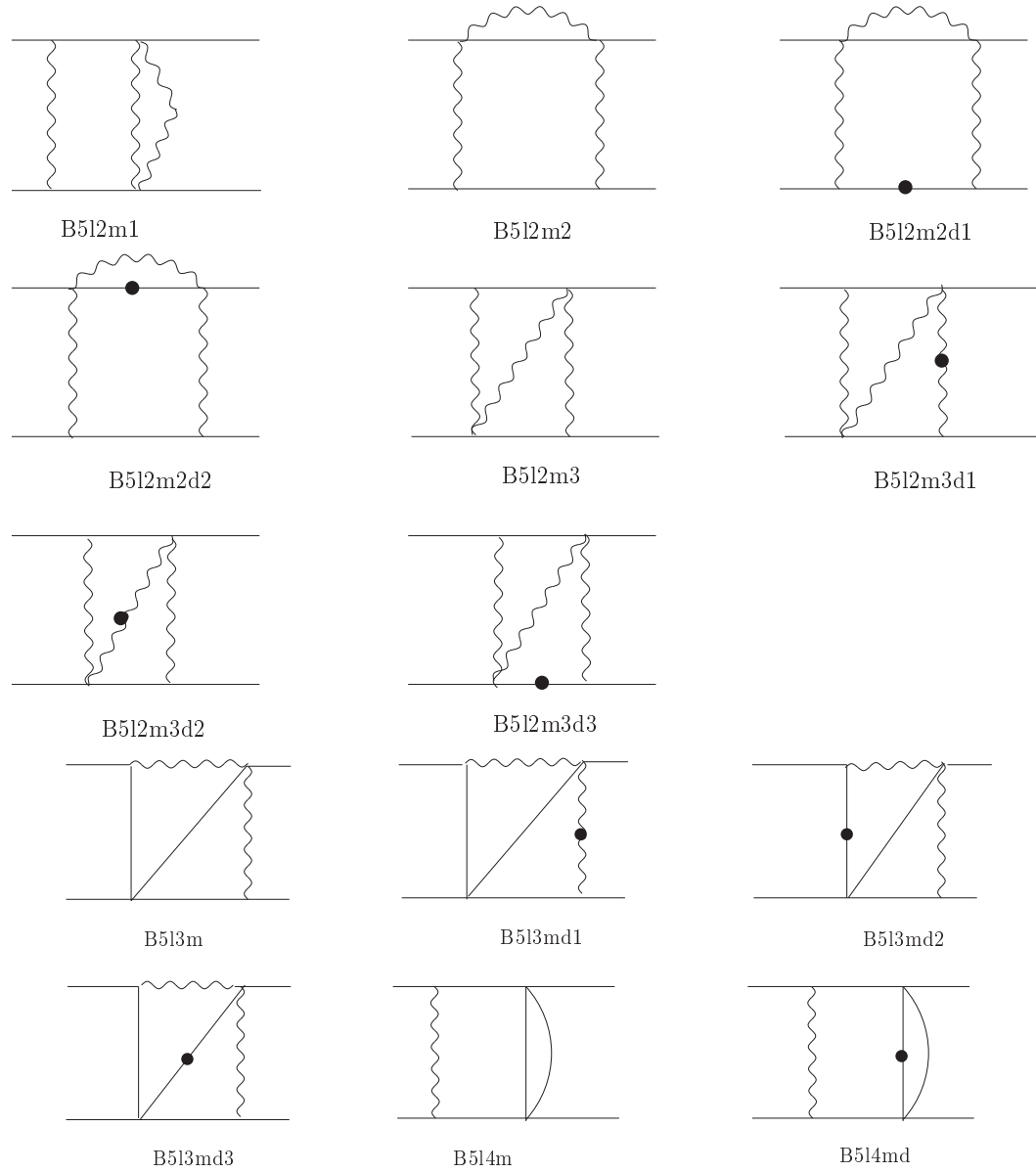


B7l4m3d2

The nine two-loop box MIs with seven internal lines.



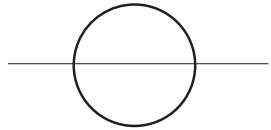
The ten two-loop box MIs with six internal lines.



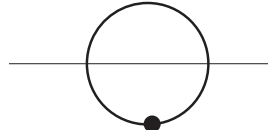
The 15 two-loop box MIs with five internal lines.

The $N_f > 1$ contributions

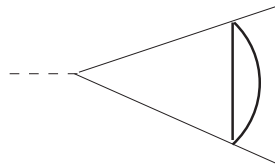
Actis, Czakon, Gluza, TR, under study



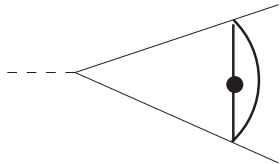
SE3l2M1m



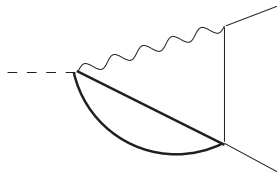
SE3l2M1md



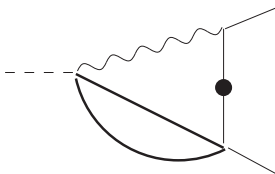
V4l2M2m



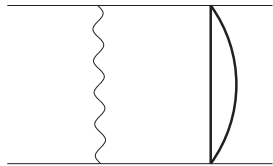
V4l2M2md



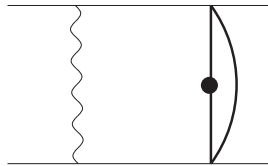
V4l2M1m



V4l2M1md



B5l2M2md



B5l2M2m

The eight additional master integrals with two different mass scales.

These 2-box-diagrams represent a three-scale problem: $s/m^2, t/m^2, M^2/m^2$

Barnes' contour integrals for the hypergeometric function

Exact proof and further reading: Whittaker & Watson (CUP 1965) 14.5 - 14.52, pp. 286-290

Consider

$$F(z) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} d\sigma (-z)^\sigma \frac{\Gamma(a+\sigma)\Gamma(b+\sigma)\Gamma(-\sigma)}{\Gamma(c+\sigma)}$$

where $|\arg(-z)| < \pi$ (i.e. $(-z)$ is not on the neg. real axis) and the path is such that it **separates** the poles of $\Gamma(a+\sigma)\Gamma(b+\sigma)$ from the poles of $\Gamma(-\sigma)$.

$1/\Gamma(c+\sigma)$ has no pole.

Assume $a \neq -n$ and $b \neq -n, n = 0, 1, 2, \dots$ so that the contour can be drawn.

The poles of $\Gamma(\sigma)$ are at $\sigma = -n, n = 1, 2, \dots$, and it is:

$$\text{Residue}[F[s] \Gamma(-s), \{s, n\}] = (-1)^n / n! F(n)$$

Closing the path to the right gives then, by Cauchy's theorem, for $|z| < 1$ the

hypergeometric function ${}_2F_1(a, b, c, z)$ (for proof see textbook):

$$\begin{aligned} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} d\sigma (-z)^\sigma \frac{\Gamma(a+\sigma)\Gamma(b+\sigma)\Gamma(-\sigma)}{\Gamma(c+\sigma)} &= \sum_{n=0}^{N \rightarrow \infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!} \\ &= \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} {}_2F_1(a, b, c, z) \end{aligned}$$

The **continuation** of the hypergeometric series for $|z| > 1$ is made using the intermediate formula

$$\begin{aligned} F(z) &= \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(1-c+a+n) \sin[(c-a-n)\pi]}{\Gamma(1+n)\Gamma(1-a+b+n) \cos(n\pi) \sin[(b-a-n)\pi]} (-z)^{-a-n} \\ &+ \sum_{n=0}^{\infty} \frac{\Gamma(b+n)\Gamma(1-c+b+n) \sin[(c-b-n)\pi]}{\Gamma(1+n)\Gamma(1-a+b+n) \cos(n\pi) \sin[(a-b-n)\pi]} (-z)^{-b-n} \end{aligned}$$

and yields

$$\begin{aligned} \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} {}_2F_1(a, b, c, z) &= \frac{\Gamma(a)\Gamma(a-b)}{\Gamma(a-c)} (-z)^{-a} {}_2F_1(a, 1-c+a, 1-b+ac, z^{-1}) \\ &+ \frac{\Gamma(b)\Gamma(b-a)}{\Gamma(b-c)} (-z)^{-b} {}_2F_1(b, 1-c+b, 1-a+b, z^{-1}) \end{aligned}$$

Corollary I

Putting $b = c$, we see that

$$\begin{aligned} {}_2F_1(a, b, b, z) &= \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(a)} \frac{z^n}{n!} \\ &= \frac{1}{(1-z)^a} = \frac{1}{2\pi i \Gamma(a)} \int_{-i\infty}^{+i\infty} d\sigma (-z)^\sigma \Gamma(a+\sigma) \Gamma(-\sigma) \end{aligned}$$

This allows to **replace sum by product**:

2009-11-25 10:09: 2 typos corrected in next line

$$\frac{1}{(A+B)^a} = \frac{B^{-a}}{(1 - (-A/B))^{-a}} = \frac{B^{-a}}{2\pi i \Gamma(a)} \int_{-i\infty}^{i\infty} d\sigma A^\sigma B^{-\sigma} \Gamma(a+\sigma) \Gamma(-\sigma)$$

Barnes' lemma

If the path of integration is curved so that the poles of $\Gamma(c - \sigma)\Gamma(d - \sigma)$ lie on the right of the path and the poles of $\Gamma(a + \sigma)\Gamma(b + \sigma)$ lie on the left, then

$$\frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} d\sigma \Gamma(a + \sigma)\Gamma(b + \sigma)\Gamma(c - \sigma)\Gamma(d - \sigma) = \frac{\Gamma(a + c)\Gamma(a + d)\Gamma(b + c)\Gamma(b + d)}{\Gamma(a + b + c + d)}$$

It is supposed that a, b, c, d are such that no pole of the first set coincides with any pole of the second set.

Scetch of proof: Close contour by semicircle C to the right of imaginary axis. The integral exists and $\int_C$ vanishes when $\Re(a + b + c + d - 1) < 0$. Take sum of residues of the integrand at poles of $\Gamma(c - \sigma)\Gamma(d - \sigma)$. The double sum leads to two hypergeometric functions, expressible by ratios of Γ -functions, this in turn by combinations of *sin*, may be simplifies finally to the r.h.s.

Analytical continuation: The relation is proved when $\Re(a + b + c + d - 1) < 0$.

Both sides are analytical functions of e.g. a . So the relation remains true for all values of a, b, c, d for which none of the poles of $\Gamma(a + \sigma)\Gamma(b + \sigma)$, as a function of σ , coincide with any of the poles of $\Gamma(c - \sigma)\Gamma(d - \sigma)$.

Corollary II Any real shift k : $\sigma + k, a - k, b - k, c + k, d + k$ together with $\int_{-k-i\infty}^{-k+i\infty}$ leaves the result true.

How can this be made useful in the context of Feynman integrals?

- Apply corollary I to propagators and get:

$$\frac{1}{(\textcolor{red}{p}^2 - m^2)^a} = \frac{1}{2\pi i \Gamma(a)} \int_{-i\infty}^{i\infty} d\sigma \frac{(-m^2)^\sigma}{(\textcolor{red}{p}^2)^{a+\sigma}} \Gamma(a + \sigma) \Gamma(-\sigma)$$

which may allow to perform the (massless) momentum integral (with index a of the line changed to $(a + \sigma)$).

- Apply corollary I after introduction of Feynman parameters and after the momentum integration to the resulting F - and U -forms, in order to get a single monomial in the x_i , which allows the integration over the x_i :

$$\frac{1}{[A(s)\textcolor{red}{x}_1^{a_1} + B(s)\textcolor{red}{x}_1^{b_1}\textcolor{red}{x}_2^{b_2}]^a} = \frac{1}{2\pi i \Gamma(a)} \int_{-i\infty}^{i\infty} d\sigma [A(s)\textcolor{red}{x}_1^{a_1}]^\sigma [B(s)\textcolor{red}{x}_1^{b_1}\textcolor{red}{x}_2^{b_2}]^{a+\sigma} \Gamma(a + \sigma) \Gamma(-\sigma)$$

Both methods leave Mellin-Barnes (MB-) integrals to be performed afterwards.

A short remark on history

- [N. Usyukina, 1975](#): "ON A REPRESENTATION FOR THREE POINT FUNCTION", Teor. Mat. Fiz. 22;
a finite massless off-shell 3-point 1-loop function represented by 2-dimensional MB-integral
- [E. Boos, A. Davydychev, 1990](#): "A Method of evaluating massive Feynman integrals", Theor. Math. Phys. 89 (1991);
N-point 1-loop functions represented by n-dimensional MB-integral
- [V. Smirnov, 1999](#): "Analytical result for dimensionally regularized massless on-shell double box", Phys. Lett. B460 (1999);
treat UV and IR divergencies by analytical continuation: shifting contours and taking residues 'in an appropriate way'
- [B. Tausk, 1999](#): "Non-planar massless two-loop Feynman diagrams with four on-shell legs", Phys. Lett. B469 (1999);
nice algorithmic approach to that, starting from search for some unphysical space-time dimension d for which the MB-integral is finite and well-defined
- [M. Czakon, 2005](#) (with experience from common work with [J. Gluza](#) and [TR](#)): "Automatized analytic continuation of Mellin-Barnes integrals", Comput. Phys. Commun. (2006);
Tausk's approach realized in Mathematica program [MB.m](#), published and available for use

Loop momentum integrations with Feynman parameters for L -loop n -point functions

Consider an arbitrary L -loop integral $G(X)$ with loop momenta k_l , with E external legs with momenta p_e , and with N internal lines with masses m_i and propagators $1/D_i$,

$$G(X) = \frac{1}{(i\pi^{d/2})^L} \int \frac{d^D k_1 \dots d^D k_L}{D_1^{\nu_1} \dots D_i^{\nu_i} \dots D_N^{\nu_N}} X(k_1, \dots, k_L).$$

$$D_i = q_i^2 - m_i^2 = \left[\sum_{l=1}^L c_i^l k_l + \sum_{e=1}^E d_i^e p_e \right]^2 - m_i^2$$

The numerator may contain a tensor structure

$$X = (k_1^{\alpha_1} \dots k_L^{\beta_L}) (p_{e_1}^{\alpha_1} \dots p_{e_L}^{\beta_L})$$

Some numerators are reducible, i.e. one may divide them out against the numerators a la:

$$\begin{aligned} \frac{2kp_e}{D_1[(k+p_e)^2 - m^2] \dots D_N} &\equiv \frac{[(k+p_e)^2 - m^2] - [k^2 - m_1^2] + (m^2 - m_e^2)}{D_1[(k+p_e)^2 - m^2] \dots D_N} \\ &= \frac{1}{D_1 \dots D_N} - \frac{1}{[(k+p_e)^2 - m^2] \dots D_N} + \frac{m^2 - m_e^2}{D_1[(k+p_e)^2 - m^2] \dots D_N} \end{aligned}$$

For a two-loop QED box diagram, it is e.g. $L = 2$, $E = 4$, and we have as potential simplest numerators: $k_1^2, k_2^2, k_1 k_2$ and $2E$ products $k_1 p_e, k_2 p_e$ compared to N internal lines, $N = 5, 6, 7$. This gives $I = L + L(L - 1)/2 + 2E - N$ irreducible numerators of this type: $I(N) = 9 - N = 4, 3, 2$ here.

This observation is of practical importance: imagine you search for potential masters. Then you may take into the list of masters at most (here e.g.) $I(5) = 4$, or $I(6) = 3$, or $I(7) = 2$ such integrals.

Which momenta combinations are irreducible is partly dependent on the choice of momenta conventions (and fixed by that) and partly dependent on choice.

Message: When evaluating all F.I. by MB-integrals, one should consider numerator integrals, and it is not too complicated compared to scalar ones.

Now **introduce Feynman parameters**:

$$\frac{1}{D_1^{n_1} D_2^{n_2} \dots D_N^{n_N}} = \frac{\Gamma(n_1 + \dots + n_N)}{\Gamma(n_1) \dots \Gamma(n_N)} \int_0^1 dx_1 \dots \int_0^1 dx_N \frac{x_1^{n_1-1} \dots x_N^{n_N-1} \delta(1 - x_1 - \dots - x_N)}{(x_1 D_1 + \dots + x_N D_N)^{N_\nu}},$$

with $N_\nu = n_1 + \dots + n_N$.

The denominator of G contains, after introduction of Feynman parameters x_i , the momentum dependent function m^2 with index-exponent N_ν :

$$(m^2)^{-(n_1 + \dots + n_N)} = (x_1 D_1 + \dots + x_N D_N)^{-N_\nu} = (k_i M_{ij} k_j - 2Q_j k_j + J)^{-N_\nu}$$

Here M is an $(L \times L)$ -matrix, $Q = Q(x_i, p_e)$ an L -vector and $J = J(x_i x_j, m_i^2, p_{e_j} p_{e_l})$.

The momentum integration is now simple:

Shift the momenta k such that m^2 has no linear term in $\bar{k}$:

$$\begin{aligned} k &= \bar{k} + (M^{-1})Q, \\ m^2 &= \bar{k}M\bar{k} - QM^{-1}Q + J. \end{aligned}$$

Remember:

$$M^{-1} = \frac{1}{(\det M)} \tilde{M},$$

where $\tilde{M}$ is the transposed matrix to M . The shift leaves the integral unchanged (rename $\bar{k} \rightarrow k$):

$$I_k(1) = \int \frac{Dk_1 \dots Dk_L}{(kMk + J - QM^{-1}Q)^{N_\nu}}.$$

Rotate now the $k^0 \rightarrow iK_E^0$ with $k^2 \rightarrow -k_E^2$ (and again rename $k^E \rightarrow k$):

$$I_k(1) \rightarrow (i)^L \int \frac{Dk_1^E \dots Dk_L^E}{(-k^E M k^E + J - QM^{-1}Q)^{N_\nu}} = (-1)^{N_\nu} (i)^L \int \frac{Dk_1 \dots Dk_L}{[kMk - (J - QM^{-1}Q)]^{N_\nu}}.$$

Call

$$\mu^2(x) = -(J - QM^{-1}Q)$$

and get

$$I_k(1) = (-1)^{N_\nu} (i)^L \int \frac{Dk_1 \dots Dk_L}{(kMk + \mu^2)^{N_\nu}}.$$

For 1-loop integrals - will use only those - we are ready. For L-loops go on and now **diagonalize the matrix M** by a rotation:

$$k \rightarrow k'(x) = V(x) k, \quad (1)$$

$$kMk = k' M_{diag} k' \quad (2)$$

$$\rightarrow \sum \alpha_i(x) k_i^2(x), \quad (3)$$

$$M_{diag}(x) = (V^{-1})^+ M V^{-1} = (\alpha_1, \dots, \alpha_L).$$

This leaves both the integration measure and the integral invariant:

$$I_k(1) = (-1)^{N_\nu} (i)^L \int \frac{Dk_1 \dots Dk_L}{(\sum_i \alpha_i k_i^2 + \mu^2)^{N_\nu}}.$$

Rescale now the k_i ,

$$\bar{k}_i = \sqrt{\alpha_i} k_i,$$

with

$$d^D k_i = (\alpha_i)^{-D/2} d^D \bar{k}_i, \quad (4)$$

$$\prod_{i=1}^L \alpha_i = \det M,$$

and get the Euclidean integral to be calculated (and rename $\bar{k} \rightarrow k$):

$$I_k(1) = (-1)^{N_\nu} (i)^L (\det M)^{-D/2} \int \frac{Dk_1 \dots Dk_L}{(k_1^2 + \dots + k_L^2 + \mu^2)^{N_\nu}}.$$

Use now (remembering that $Dk = dk/(i\pi^{d/2})$):

$$\begin{aligned} i^L \int \frac{Dk_1 \dots Dk_L}{(k_1^2 + \dots + k_L^2 + \mu^2)^{N_\nu}} &= \frac{\Gamma(N_\nu - \frac{D}{2}L)}{\Gamma(N_\nu)} \frac{1}{(\mu^2)^{N_\nu - DL/2}}, \\ i^L \int \frac{Dk_1 \dots Dk_L k_1^2}{(k_1^2 + \dots + k_L^2 + \mu^2)^{N_\nu}} &= \frac{d}{2} \frac{\Gamma(N_\nu - \frac{D}{2}L - 1)}{\Gamma(N_\nu)} \frac{1}{(\mu^2)^{N_\nu - DL/2 - 1}}. \end{aligned} \quad (5)$$

These formulae follow for $L = 1$ immediately from any textbook.

For $L > 1$, get it iteratively, with setting $(k_1^2 + k_2^2 + m^2)^N = (k_1^2 + M^2)^N$, $M^2 = k_2^2 + m^2$, etc.

Finally, one gets:

$$\begin{aligned} G(1) &= (-1)^{N_\nu} \frac{\Gamma(N_\nu - \frac{D}{2}L)}{\Gamma(\nu_1) \dots \Gamma(\nu_N)} \int_0^1 \prod_{j=1}^N dx_j x_j^{\nu_j-1} \delta\left(1 - \sum_{i=1}^N x_i\right) \frac{(\det M)^{-D/2}}{(\mu^2)^{N_\nu - DL/2}}, \\ &= (-1)^{N_\nu} \frac{\Gamma(N_\nu - \frac{D}{2}L)}{\Gamma(\nu_1) \dots \Gamma(\nu_N)} \int_0^1 \prod_{j=1}^N dx_j x_j^{\nu_j-1} \delta\left(1 - \sum_{i=1}^N x_i\right) \frac{U(x)^{N_\nu - D(L+1)/2}}{F(x)^{N_\nu - DL/2}} \end{aligned} \quad (6)$$

with

$$U(x) = (\det M) \text{ (} \rightarrow 1 \text{ for } L = 1 \text{)}$$

$$F(x) = (\det M) \mu^2 = -(\det M) J + Q \tilde{M} Q \text{ (} \rightarrow -J + Q^2 \text{ for } L = 1 \text{)} = \sum A_{ij} x_i x_j.$$

The vector integral differs by some numerator $k_i p_e$ and thus there is a single shift in the integrand, $k \rightarrow \bar{k} + U(x)\tilde{M}Q$, the $\int d^d \bar{k} \bar{k}/(\bar{k}^2 + \mu^2) \rightarrow 0$, and no further changes:

$$G(k_{1\alpha}) = (-1)^{N_\nu} \frac{\Gamma(N_\nu - \frac{D}{2}L)}{\Gamma(\nu_1) \dots \Gamma(\nu_N)} \int_0^1 \prod_{j=1}^N dx_j x_j^{\nu_j-1} \delta\left(1 - \sum_{i=1}^N x_i\right) \frac{U(x)^{N_\nu-1-D(L+1)/2}}{\textcolor{red}{F}(x)^{N_\nu-DL/2}} \left[\sum_l \tilde{M}_{1l} Q_l \right]_\alpha,$$

Here also a tensor integral:

$$\begin{aligned} G(k_{1\alpha} k_{2\beta}) &= (-1)^{N_\nu} \frac{\Gamma(N_\nu - \frac{D}{2}L)}{\Gamma(\nu_1) \dots \Gamma(\nu_N)} \int_0^1 \prod_{j=1}^N dx_j x_j^{\nu_j-1} \delta\left(1 - \sum_{i=1}^N x_i\right) \frac{U(x)^{N_\nu-2-D(L+1)/2}}{F(x)^{N_\nu-DL/2}} \\ &\quad \times \sum_l \left[[\tilde{M}_{1l} Q_l]_\alpha [\tilde{M}_{2l} Q_l]_\beta - \frac{\Gamma(N_\nu - \frac{D}{2}L - 1)}{\Gamma(N_\nu - \frac{D}{2}L)} \frac{g_{\alpha\beta}}{2} U(x) F(x) \frac{(V_{1l}^{-1})^+ (V_{2l}^{-1})}{\alpha_l} \right]. \end{aligned}$$

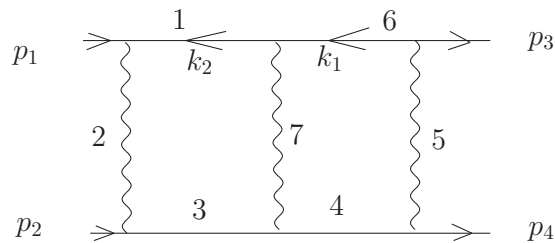
The 1-loop case will be used in the following L times for a sequential treatment of an L -loop integral (remember $\sum x_j D_j = k^2 - 2Qk + J$ and $\textcolor{red}{F}(x) = Q^2 - J$):

$$G([\textcolor{red}{1}, k p_e]) = (-1)^{N_\nu} \frac{\Gamma(N_\nu - \frac{D}{2})}{\Gamma(\nu_1) \dots \Gamma(\nu_N)} \int_0^1 \prod_{j=1}^N dx_j x_j^{\nu_j-1} \delta\left(1 - \sum_{i=1}^N x_i\right) \frac{[\textcolor{red}{1}, Q p_e]}{F(x)^{N_\nu-D/2}}$$

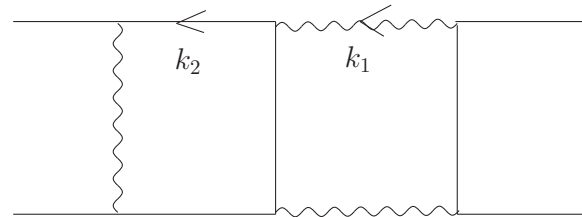
Come back to the evaluation of Bhabha boxes ...

... and look at B7l4m2, the second planar double box:

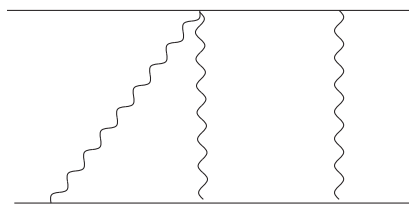
Perform the momentum and Feynman parameter integrations first for the subloop with k_1 , then for the second.



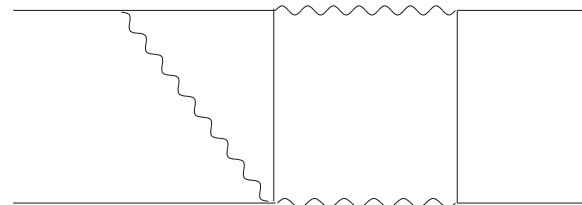
B7l4m1



B7l4m2



B6l3m1



B6l3m2

Figure 1: The planar 6- and 7-line topologies.

Integrating the Feynman parameters – get MB-Integrals

In 2-loops, consider **two subsequent sub-loops** (the first: **off-shell 1-loop**, second **on-shell 1-loop**) and get e.g. for **B7l4m2**, the planar 2nd type 2-box (for momenta see next page):

$$K_{1\text{-loop Box,off}} = \frac{(-1)^{a_{4567}} \Gamma(a_{4567} - d/2)}{\Gamma(a_4) \Gamma(a_5) \Gamma(a_6) \Gamma(a_7)} \int_0^\infty \prod_{j=4}^7 dx_j x_j^{a_j-1} \frac{\delta(1 - x_4 - x_5 - x_6 - x_7)}{F^{a_{4567}-d/2}}$$

where $a_{4567} = a_4 + a_5 + a_6 + a_7$ and the function F is characteristic of the diagram; here for the off-shell 1-box (2nd type):

$$F^{-(a_{4567}-d/2)} = \left\{ [-t]x_4x_7 + [-s]x_5x_6 + m^2(x_5 + x_6)^2 \right. \\ \left. + (m^2 - Q_1^2)x_7(x_4 + 2x_5 + x_6) + (m^2 - Q_2^2)x_7x_5 \right\}^{-(a_{4567}-d/2)}$$

We want to apply now:

$$\int_0^1 \prod_{j=4}^7 dx_j x_j^{\alpha_j-1} \delta(1 - x_4 - x_5 - x_6 - x_7) = \frac{\Gamma(\alpha_4) \Gamma(\alpha_5) \Gamma(\alpha_6) \Gamma(\alpha_7)}{\Gamma(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7)}$$

with coefficients α_i dependent on a_i and on F

For this, we have to apply several MB-integrals here.

And repeat the procedure for the 2nd subloop.

On-shell example: B412m, the 1-loop on-shell box but here use of another sequence of MB-integrals than in Smirnov's book

```
den = (x4 d4 + x5 d5 + x6 d6 + x7 d7 // Expand) /. kinBha /. m^2 -> 1 // Expand
```

```
Q = -Coefficient[den, k]/2 // Simplify
    = p3 x4 + p2 x5 - p1 (x4 + x6)
```

```
M = Coefficient[den, k^2] // Simplify
    = x4 + x5 + x6 + x7 -> 1
```

```
J = den /. k -> 0 // Simplify
    = t x4
```

```
F[x] = (Q^2 - J M // Expand) /. kinBha /. m^2 -> 1 /. u -> -s - t + 4 // Expand
    = (x5+x6)^2 + (-s)x5x6 + (-t)x4x7
```

```
B412ma = mb[(x5+x6)^2, -tx7x4 - sx5x6, nu, ga]
```

```
B412mb = B412ma /. (-sx5x6 - tx4x7)^(-ga - nu) ->
    mb[(-s)x5x6, (-t)x7x4, nu+ga, de]
    /. ((-s)x5x6)^de_ -> (-s)^de x5^de x6^de
    /. ((x5+x6)^2)^ga -> (x5 + x6)^(2ga)
```

```

=
(inv2piI^2(-s)^de x5^de x6^de ((x5 + x6)^(2ga)((-t)x4x7)^(-de-ga-nu)
  Gamma[-de] Gamma[-ga] Gamma[de + ga + nu] /Gamma[nu]

B4l2mc = B4l2mb /. (x5 + x6)^(2ga) ->
  mb[x5, x6, -2ga, si]
  /. ((-t)x4x7)^si_ -> (-t)^si x4^si x7^si // ExpandAll

=
1/(Gamma[-2ga] Gamma[nu])
inv2piI^3 (-s)^de (-t)^(-de - ga - nu)
  x4^(-de - ga - nu) x5^(de + si) x6^(de + 2 ga - si) x7^(-de - ga - nu)
  Gamma[-de] Gamma[-ga] Gamma[de + ga + nu] Gamma[-si] Gamma[-2ga + si]

B4l2md = xfactor4[a4, x4, a5, x5, a6, x6, a7, x7] B4l2mc

=
.... (-s)^de (-t)^(-de-ga-nu)
x4^(-1+a4-de-ga-nu) x5^(-1+a5+de+si) x6^(-1+a6+de+2ga-si) x7^(-1+a7- de-ga-nu)

B4l2me =
  B4l2md /.
    x4^B4_ x5^B5_ x6^B6_ x7^B7_ -> xint4[x4^B4 x5^B5 x6^B6 x7^B7]

= ...

```

B412mf = B412me /.

```
Gamma[a6 + de + 2 ga - si]Gamma[-si]Gamma[ a5 + de + si] Gamma[-2 ga + si]  
-> barne1[si, a5 + de, -2 ga, a6 + de + 2 ga, 0]
```

This finishes the evaluation of the MB-representation for B412m.

Some routines in mathematica which were used:

```
(* Barnes' first lemma: \int d(si) Gamma(si1p+si)Gamma(si2p+si)Gamma(si1m-si)Gamma(si2m-si)
      with 1/inv2piI = 2 Pi I *)

barne1[si_, si1p_, si2p_, si1m_, si2m_] :=
  1/inv2piI Gamma[si1p + si1m] Gamma[si1p + si2m] Gamma[
    si2p + si1m] Gamma[si2p + si2m] /Gamma[si1p + si2p + si1m + si2m]

(* Mellin-Barnes integral: (A+B)^(-nu) = 1/(2 Pi I) \int d(si) a^si b^(-nu - si)
      Gamma[-si]Gamma[nu+si]/Gamma[nu] *)

mb[a_, b_, nu_, si_] := inv2piI a^si b^(-nu-si)Gamma[-si]Gamma[nu+si]/Gamma[nu]

(* After the k-integration, the integrand for \int \prod (dx_i x_i^(a_i - 1)) \delta(1 - \sum x_i)
      will be (L=1 loop) : xfactorn F^(-nu) Q(x_i).pe with nu = a1 + .. + an - d/2 *)

xfactor3[a1_, x1_, a2_, x2_, a3_, x3_] :=
  I Pi^(d/2) (-1)^(a1 + a2 + a3) x1^(a1 - 1) x2^(a2 - 1) x3^(a3 - 1) Gamma[
    a1 + a2 + a3 - d/2] / (Gamma[a1] Gamma[a2] Gamma[a3])

(* xinti - the i-dimensional x - integration over Feynman parameters /16 06 2005 *)

xint3[x1_^(a1_) x2_^(a2_) x3_^(a3_) ] :=
  Gamma[a1 + 1] Gamma[a2 + 1] Gamma[a3 + 1] / Gamma[a1 + a2 + a3 + 3]
```

Two different 2-dim. MB-representations for the massive 1-loop QED box:

```
(* s=(PP1+PP2)^2, t=(PP1-PP3)^2, s+t+u=4, scalar QED box *)
(* PRDfact = E^(ep EulerGamma) 1/(I Pi^(d/2)) *)
(* B4l2m = PRDfact * K4l2m[k-PP1, 0, k, 1, k-PP3, 0, k-PP1 - PP2, 1] *)
(* B4l2mINPUT with (\int dr1 dr3) is the on - shell QED box *)

B4l2mINPUT[s_,t_,m1_,m2_,m3_,m4_] =
  ((-1)^(m1 + m2 + m3 + m4)*E^(ep*EulerGamma)*inv2piI^2*Pi^(2 - d/2 - ep)*
  (-s)^(2 - ep - m1 - m2 - m3 - m4 - r1 - r3)*(-t)^r3*Gamma[-r1]*
  Gamma[4 - 2*ep - 2*m1 - m2 - 2*m3 - m4 - 2*r3]*Gamma[2 - ep - m1 - m2 - m3 - r1 - r3]*
  Gamma[2 - ep - m1 - m3 - m4 - r1 - r3]*Gamma[-r3]*Gamma[m1 + r3]*Gamma[m3 + r3]*
  Gamma[-2 + ep + m1 + m2 + m3 + m4 + r1 + r3])
/(Gamma[m1]*Gamma[m2]*Gamma[m3] * Gamma[4 - 2*ep - m1 - m2 - m3 - m4]*Gamma[m4]*
  Gamma[4 - 2*ep - 2*m1 - m2 - 2*m3 - m4 - 2*r1 - 2*r3])

(* s=(PP1+PP2)^2, t=(PP1-PP3)^2, s+t+u=4, scalar QED box *)
(* PRDfact = E^(ep EulerGamma) 1/(I Pi^(d/2)) *)
(* B4l2m = PRDfact * K4l2m[k-PP1, 0, k, 1, k-PP3, 0, k-PP1 - PP2, 1] *)
(* B4l2mINPUT with (\int dr1 dr3) is the on - shell QED box *)
(* the MB-sequence deviates from e.g. Smirnov book, done for B5l2m3 *)

B4l2mINPUTvar[s_, t_, m1_, m2_, m3_, m4_] = ((-1)^(m1 + m2 + m3 + m4)
  *E^(ep*EulerGamma)*inv2piI^2 * (-s)^r1 * (-t)^(d/2-m1-m2-m3-m4-r1-r3)
  *Gamma[-r1]*Gamma[m2 + r1]*
  Gamma[m4 + r1]*Gamma[d/2 - m1 - m2 - m4 - r1 - r3]*
  Gamma[d/2 - m2 - m3 - m4 - r1 - r3]*Gamma[-r3]*
  Gamma[-d/2 + m1 + m2 + m3 + m4 + r1 + r3]*Gamma[m2 + m4 + 2*r1 + 2*r3])
/ (Gamma[m1]*Gamma[m2]*Gamma[m3]*Gamma[d - m1 - m2 - m3 - m4]*Gamma[m4]*
  Gamma[m2 + m4 + 2*r1])
```

The 2-dim. MB-representation for the 1-loop QED box with numerator $k.pe$ depends on the choice of momentum flow

```
B4l2mnumINPUTvar[pe_, s_, t_, m1_, m2_, m3_, m4_] = -((( -1)^(m1 + m2 + m3 + m4)*E^(ep*EulerGamma)*inv2piI^2
  *pe*(-s)^r1* (-t)^((d - 2*(m1 + m2 + m3 + m4 + r1 + r3))/2)
  *Gamma[-r1]*Gamma[-r3]* Gamma[-d/2 + m1 + m2 + m3 + m4 + r1 + r3]*
  (p1*Gamma[1 + m2 + r1]*Gamma[m4 + r1]*Gamma[m2 + m4 + 2*r1]*
  Gamma[1 + m2 + m4 + 2*r1 + 2*r3]*Gamma[(d - 2*(m1 + m2 + m4 + r1 + r3))/
  2] + Gamma[m2 + r1]*((p1 - p3)*Gamma[m4 + r1]*
  Gamma[1 + m2 + m4 + 2*r1]*Gamma[m2 + m4 + 2*(r1 + r3)]*
  Gamma[(d - 2*(-1 + m1 + m2 + m4 + r1 + r3))/2] -
  p2*Gamma[1 + m4 + r1]*Gamma[m2 + m4 + 2*r1]*
  Gamma[1 + m2 + m4 + 2*r1 + 2*r3]*
  Gamma[(d - 2*(m1 + m2 + m4 + r1 + r3))/2]))*
  Gamma[(d - 2*(m2 + m3 + m4 + r1 + r3))/2])/((Gamma[m1]*Gamma[m2]*Gamma[m3]*
  Gamma[1 + d - m1 - m2 - m3 - m4]*Gamma[m4]*Gamma[m2 + m4 + 2*r1]*
  Gamma[1 + m2 + m4 + 2*r1]))
```

A 3-dim. representation which is not derived by shrinking lines from 7-line box:

```
b5l2m2 = InputForm[((( -1)^(a1 + a2 + a3 + a4 + a5)*E^(2*ep*EulerGamma)*inv2piI^3*
  (-s)^(2 - a2 - a4 - a5 - ep - r1 - r3 + si)*(-t)^r3*Gamma[-r1]*
  Gamma[2 - a2 - a4 - a5 - ep - r1 - r3]*Gamma[-r3]*Gamma[a2 + r3]*
  Gamma[a4 + r3]*Gamma[4 - 2*a1 - a3 - 2*ep - si]*
  Gamma[-2 + a2 + a4 + a5 + ep + r1 + r3 - si]*Gamma[a1 + si]*
  Gamma[-2 + a1 + a3 + ep + si]*Gamma[4 - 2*a2 - 2*a4 - a5 - 2*ep - 2*r3 +
  si]*Gamma[2 - a2 - a4 - ep - r1 - r3 + si])/
  (Gamma[a1]*Gamma[a2]*Gamma[a3]*Gamma[a4]*Gamma[a5]*
  Gamma[4 - a1 - a3 - 2*ep]*Gamma[4 - a2 - a4 - a5 - 2*ep + si]*
  Gamma[4 - 2*a2 - 2*a4 - a5 - 2*ep - 2*r1 - 2*r3 + si])]
```

Another nice box with numerator, $B5l3m(p_e \cdot k_1)$

We used it for the determination if the small mass expansion.

$$\begin{aligned}
 B5l3m(p_e \cdot k_1) &= \frac{m^{4\epsilon} (-1)^{a_{12345}} e^{2\epsilon\gamma_E}}{\prod_{j=1}^5 \Gamma[a_j] \Gamma[5 - 2\epsilon - a_{123}] (2\pi i)^4} \int_{-i\infty}^{+i\infty} d\alpha \int_{-i\infty}^{+i\infty} d\beta \int_{-i\infty}^{+i\infty} d\gamma \int_{-i\infty}^{+i\infty} d\delta \\
 & \quad (-s)^{(4-2\epsilon)-a_{12345}-\alpha-\beta-\delta} (-t)^\delta \\
 & \quad \frac{\Gamma[-4+2\epsilon+a_{12345}+\alpha+\beta+\delta]}{\Gamma[6-3\epsilon-a_{12345}-\alpha]} \frac{\Gamma[-\alpha] \Gamma[-\beta]}{\Gamma[7-3\epsilon-a_{12345}-\alpha] \Gamma[5-2\epsilon-a_{123}]} \frac{\Gamma[-\delta]}{\Gamma[4-2\epsilon-a_{1123}-2\alpha-\gamma] \Gamma[5-2\epsilon-a_{1123}-2\alpha-\gamma]} \\
 & \quad \frac{\Gamma[2-\epsilon-a_{13}-\alpha-\gamma]}{\Gamma[8-4\epsilon-a_{112233445}-2\alpha-2\beta-2\delta-\gamma]} \frac{\Gamma[4-2\epsilon-a_{12345}-\alpha-\beta-\delta-\gamma]}{\Gamma[9-4\epsilon-a_{112233445}-2\alpha-2\beta-2\delta-\gamma]} \left\{ (p_e \cdot p_3) \Gamma[1+a_4+\delta] \Gamma[6-3\epsilon-a_{12345}-\alpha-\beta-\delta-\gamma] \right. \\
 & \quad \Gamma[4-2\epsilon-a_{1234}-\alpha-\beta-\delta] \Gamma[3-\epsilon-a_{12}-\alpha] \Gamma[8-4\epsilon-a_{112233445}-2\alpha-2\delta-\gamma] \Gamma[9-4\epsilon-a_{112233445}-2\alpha-2\beta-2\delta-\gamma] \\
 & \quad \Gamma[5-2\epsilon-a_{1123}-\gamma] \Gamma[4-2\epsilon-a_{1123}-2\alpha-\gamma] \Gamma[a_1+\gamma] \Gamma[-2+\epsilon+a_{123}+\alpha+\delta+\gamma] + \Gamma[a_4+\delta] \left[-(p_e \cdot p_1) \Gamma[7-3\epsilon-a_{12345}-\alpha-\beta-\delta-\gamma] \right. \\
 & \quad \Gamma[4-2\epsilon-a_{1234}-\alpha-\beta-\delta] \Gamma[8-4\epsilon-a_{112233445}-2\alpha-2\delta-\gamma] \Gamma[9-4\epsilon-a_{112233445}-2\alpha-2\beta-2\delta-\gamma] \\
 & \quad \left[\Gamma[3-\epsilon-a_{12}-\alpha] \Gamma[5-2\epsilon-a_{1123}-\gamma] \Gamma[4-2\epsilon-a_{1123}-2\alpha-\gamma] \Gamma[a_1+\gamma] + \Gamma[2-\epsilon-a_{12}-\alpha] \Gamma[4-2\epsilon-a_{1123}-\gamma] \right. \\
 & \quad \left. \Gamma[5-2\epsilon-a_{1123}-2\alpha-\gamma] \Gamma[1+a_1+\gamma] \right] \Gamma[-2+\epsilon+a_{123}+\alpha+\delta+\gamma] + \Gamma[6-3\epsilon-a_{12345}-\alpha] \Gamma[3-\epsilon-a_{12}-\alpha] \\
 & \quad \Gamma[5-2\epsilon-a_{1123}-\gamma] \Gamma[4-2\epsilon-a_{1123}-2\alpha-\gamma] \Gamma[a_1+\gamma] \left[((p_e \cdot (p_1+p_2)) \Gamma[5-2\epsilon-a_{1234}-\alpha-\beta-\delta] \Gamma[9-4\epsilon-a_{112233445}-2\alpha-2\beta-2\delta-\gamma] \right. \\
 & \quad \Gamma[8-4\epsilon-a_{112233445}-2\alpha-2\beta-2\delta-\gamma] \Gamma[-2+\epsilon+a_{123}+\alpha+\delta+\gamma] + (p_e \cdot p_1) \Gamma[4-2\epsilon-a_{1234}-\alpha-\beta-\delta] \\
 & \quad \left. \left. \Gamma[8-4\epsilon-a_{112233445}-2\alpha-2\delta-\gamma] \Gamma[9-4\epsilon-a_{112233445}-2\alpha-2\beta-2\delta-\gamma] \Gamma[-1+\epsilon+a_{123}+\alpha+\delta+\gamma] \right] \right\}
 \end{aligned}$$

B5l2m2

$$\begin{aligned}
\text{B5l2m2} = & \frac{m^{4\epsilon} (-1)^{a_{12345}} e^{2\epsilon\gamma_E}}{\prod_{j=1}^5 \Gamma[a_i] \Gamma[4 - 2\epsilon - a_{13}] (2\pi i)^3} \int_{-i\infty}^{+i\infty} d\alpha \int_{-i\infty}^{+i\infty} d\beta \int_{-i\infty}^{+i\infty} d\gamma (-s)^{2-\epsilon-a_{245}-\gamma-\alpha+\beta} (-t)^\alpha \\
& \Gamma[-2 + \epsilon + a_{13} + \beta] \Gamma[-\gamma] \Gamma[2 - \epsilon - a_{245} - \gamma - \alpha] \Gamma[-\alpha] \\
& \Gamma[a_2 + \alpha] \Gamma[a_4 + \alpha] \Gamma[4 - 2\epsilon - a_{113} - \beta] \Gamma[-2 + \epsilon + a_{245} + \gamma + \alpha - \beta] \Gamma[a_1 + \beta] \\
& \frac{\Gamma[4 - 2\epsilon - a_{2245} - 2\alpha + \beta] \Gamma[2 - \epsilon - a_{24} - \gamma - \alpha + \beta]}{\Gamma[4 - 2\epsilon - a_{245} + \beta] \Gamma[4 - 2\epsilon - a_{22445} - 2\gamma - 2\alpha + \beta]}
\end{aligned}$$

This kind of expression now has to be evaluated:

- Check special cases of indices, set lines to 1 (by setting $a_i \rightarrow 0$ if possible)
- Extract the ϵ -dependence related to UV and IR singularities (see next pages)
- After that: may set $s < 0$, $t < 0$ and evaluate numerically Euclidean case
- Use sector decomposition for a numerical comparison - if you have a program for that
- Try to go Minkowskian in a numerical way (if you like this)
- Go on analytically, e.g. by taking residua $\rightarrow$ get nested infinite sums from the residua
- Try to sum them up

B4l2m2

[Fleischer, Gluza, Lorca, TR 2006] B4l2m, the 1-loop QED box, with two photons in the s -channel; the Mellin-Barnes representation reads for finite ϵ :

$$\begin{aligned}
 \text{B4l2m} = \text{Box}(t, s) = & \frac{e^{\epsilon\gamma_E}}{\Gamma[-2\epsilon](-t)^{(2+\epsilon)}} \frac{1}{(2\pi i)^2} \int_{-i\infty}^{+i\infty} dz_1 \int_{-i\infty}^{+i\infty} dz_2 \\
 & \frac{(-s)^{z_1} (m^2)^{z_2}}{(-t)^{z_1+z_2}} \Gamma[2 + \epsilon + z_1 + z_2] \Gamma^2[1 + z_1] \Gamma[-z_1] \Gamma[-z_2] \\
 & \Gamma^2[-1 - \epsilon - z_1 - z_2] \frac{\Gamma[-2 - 2\epsilon - 2z_1]}{\Gamma[-2 - 2\epsilon - 2z_1 - 2z_2]}
 \end{aligned} \tag{7}$$

Mathematica package MB used for analytical expansion $\epsilon \rightarrow 0$:

[Czakon:2005rk]

$$B_{412m} = -\frac{1}{\epsilon} I_1 + \ln(-s) I_1 + \epsilon \left(\frac{1}{2} [\zeta(2) - \ln^2(-s)] I_1 - 2I_2 \right). \quad (8)$$

with I_1 being also the divergent part of the vertex function $C_0(t; m, 0, m)/s = V_{312m}/s$ (as is well-known):

$$I_1 = \frac{e^{\epsilon\gamma_E}}{st} \frac{1}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dz_1 \left(\frac{m^2}{-t} \right)^{z_1} \frac{\Gamma^3[-z_1] \Gamma[1+z_1]}{\Gamma[-2z_1]} = \frac{1}{m^2 s} \frac{2y}{1-y^2} \ln(y) \quad (9)$$

with $y = (\sqrt{1-4m^2/t} - 1)/(\sqrt{1-4m^2/t} + 1)$: close contour to left, take **residua** at $(1+z_1) = -n$,
sum up with **Mathematica**:

`Residue[F[x]Gamma[1 + x], {x, -n}] // InputForm = -(-1)^n F[-n]/n!`

`Sum[s^(n) Gamma[n + 1]^3/(n!Gamma[2 + 2n]), {n, 0, Infinity}] // InputForm
= (4*ArcSin[Sqrt[s]/2])/(Sqrt[4 - s]*Sqrt[s])`

The I_2 is more complicated:

$$I_2 = \frac{e^{\epsilon\gamma_E}}{t^2} \frac{1}{(2\pi i)^2} \int_{-\frac{3}{4}-i\infty}^{-\frac{3}{4}+i\infty} dz_1 \left(\frac{-s}{-t} \right)^{z_1} \Gamma[-z_1] \Gamma[-2(1+z_1)] \Gamma^2[1+z_1] \quad (10)$$

$$\times \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dz_2 \left(\frac{m^2}{-t} \right)^{z_2} \Gamma[-z_2] \frac{\Gamma^2[-1-z_1-z_2]}{\Gamma[-2(1+z_1+z_2)]} \Gamma[2+z_1+z_2].$$

The expansion of B_{412m} at small m^2 and fixed value of t

With

$$m_t = \frac{-m^2}{t}, \quad (11)$$

$$r = \frac{s}{t}, \quad (12)$$

Look, under the integral, at $(-m^2/t)^{z_2}$,
and close the path to the right.

Seek the residua from the poles of Γ -functions with the smallest powers in m^2 and try to sum the resulting series.

Automatize this, it is not too easy.

we have obtained a compact answer for I2 with the additional aid of XSUMMER

[Moch:2005uc]

. The box contribution of order ϵ in this limit becomes:

$$\begin{aligned} B_{412m}[t, s, m^2; +1] = & \frac{1}{st} \left\{ 4\zeta_3 - 9\zeta_2 \ln(m_t) + \frac{2}{3} \ln^3(m_t) + 6\zeta_2 \ln(r) - \ln^2(m_t) \ln(r) \right. \\ & + \frac{1}{3} \ln^3(r) - 6\zeta_2 \ln(1+r) + 2\ln(-r) \ln(r) \ln(1+r) - \ln^2(r) \ln(1+r) \\ & \left. + 2\ln(r) \text{Li}_2(1+r) + 2\text{Li}_3(-r) \right\} + \mathcal{O}(m_t). \end{aligned} \quad (13)$$

Shrinking of lines; seek the ϵ -expansion

Go on with some study of the 2nd planar 2-box, B7l4m2 (see also Smirnov book 4.73):

$$B_{\text{pl},2} = \frac{\text{const}}{(2\pi i)^6} \int_{-i\infty}^{+i\infty} \left[\frac{m^2}{-s} \right]^{z_5+z_6} \left[\frac{-t}{-s} \right]^{z_1} \prod_{j=1}^6 [dz_j \Gamma(-z_j)] \frac{\prod_{k=7}^{18} \Gamma_k(\{z_i\})}{\prod_{l=19}^{24} \Gamma_l(\{z_i\})}$$

with $a = a_1 + \dots + a_7$ and

$$z_i = \text{const} + i \Im m(z_i)$$

$$d = 4 - 2\epsilon$$

$$\text{const} = \frac{(i\pi^{d/2})^2 (-1)^a (-s)^{d-a}}{\Gamma(a_2) \Gamma(a_4) \Gamma(a_5) \Gamma(a_6) \Gamma(a_7) \Gamma(d - a_{4567})}$$

The integrand includes e.g.:

$$\Gamma_2 = \Gamma(-z_2)$$

$$\Gamma_4 = \Gamma(-z_4)$$

$$\Gamma_7 = \Gamma(a_4 + z_2 + z_4)$$

$$\Gamma_8 = \Gamma(D - a_{445667} - z_2 - z_3 - 2z_4)$$

...

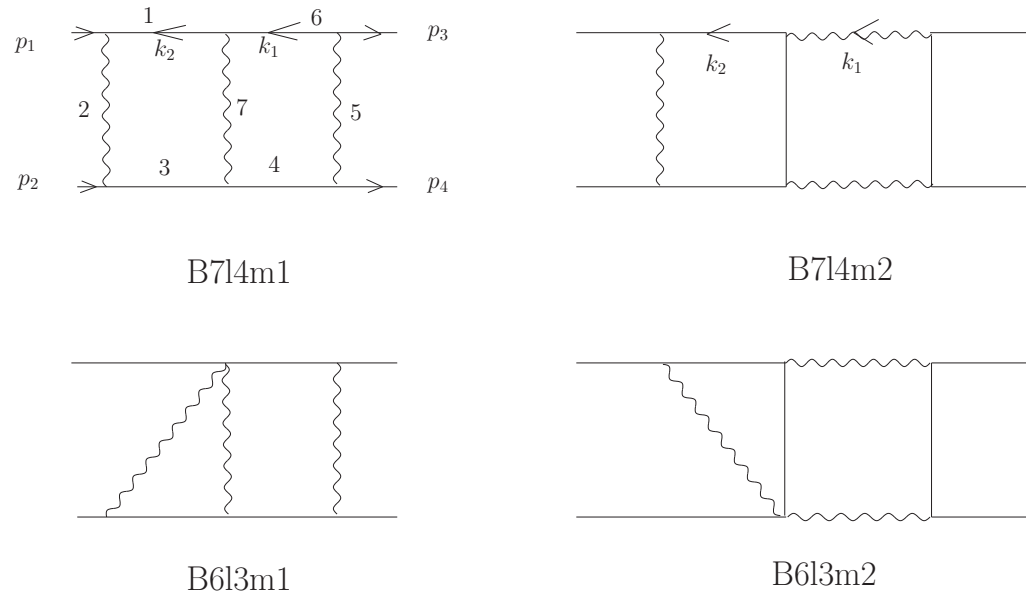


Figure 2: The planar 6- and 7-line topologies.

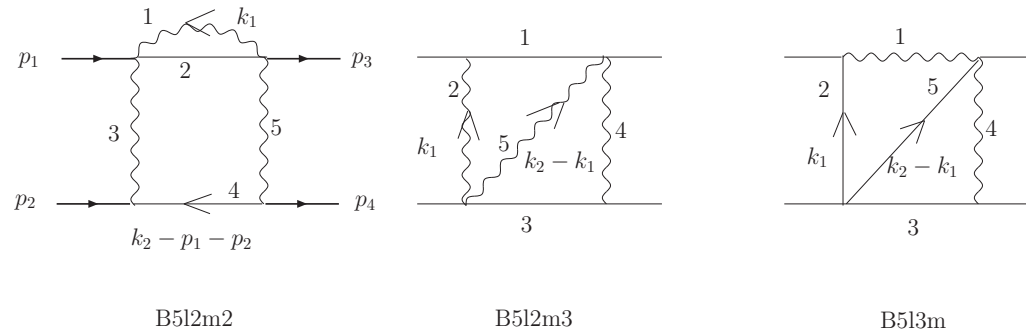


Figure 3: The 5-line topologies. **B7l4m2**: shrink line 1 get **B6l3m2**, then line 4 get **B5l3m**

Example:

derive from B7l4m2 the MB-integral for B5l3m by setting $a_1 = 0$ (trivial, gives B6l3m2) and then setting $a_4 = 0$.

The latter do with care because of

$$\frac{1}{\Gamma(a_4)} \rightarrow \frac{1}{\Gamma(0)} = 0$$

See by inspection that we will get factor $\Gamma(a_4)$ if $z_2, z_4 \rightarrow 0$.

→ Start with the z_2, z_4 integrations by

taking the residues for closing the integration contours to the right:

$$\begin{aligned} I_{2,4} &= \frac{(-1)^2}{(2\pi i)^2} \int dz_2 \Gamma(-z_2) \int dz_4 \frac{\Gamma(a_4 + z_2 + z_4)}{\Gamma(a_4)} \Gamma(-z_4) R(z_i) \\ &= \frac{1}{(2\pi i)} \int dz_2 \Gamma(-z_2) \sum_{n=0,1,\dots} \frac{-(-1)^n}{n!} \frac{\Gamma(a_4 + z_2 + n)}{\Gamma(a_4)} R(z_i) \\ &= \sum_{n,m=0,1,\dots} \frac{(-1)^{n+m}}{n!m!} \frac{\Gamma(a_4 + n + m)}{\Gamma(a_4)} R(z_i) \rightarrow_{a_4=0} 1 \times R(z_i) \end{aligned}$$

So, setting $a_1 = a_4 = 0$ and eliminating $\int dz_2 dz_4$ with setting $z_2 = z_4 = 0$

we got a 4-fold Mellin-Barnes integral for topology B5l3m (by "shrinking of lines")

with $24 - 3 = 21$ z_i -dependent Γ -functions which may yield residues within four-fold sums.

The MB-representation has to be calculated explicitly at **fixed** indices, e.g.

$$B5l3md2 = \frac{B_2}{\epsilon^2} + \frac{B_1}{\epsilon} + B_0$$

General Tasks, first two steps automated by MB.m:

- Find a **region of definiteness** of the n-fold MB-integral

$$\Re(z_1) = -1/80, \Re(z_3) = -33/40, \Re(z_5) = -21/20, \Re(z_6) = -59/160, \Re(\epsilon) = -1/10!$$

- Then go to the physical region where $\epsilon \ll 1$ by distorting the integration path step by step (adding each crossed residuum – **per residue this means one integral less!!!**)
- Take integrals by sums over residua, i.e. introduce infinite sums
- Sum these infinite multiple series into some known functions of a given class, e.g. Nielsen polylogs, Harmonic polylogs or whatever is appropriate.

An important tool is the command **FindInstance** of Mathematica 5:

It allows to solve a system of inequalities.

Here an example for B7l4m3, the non-planar massive double box:

```
sol = FindInstance[
  Cases[B7l4m3 ... Gamma[x_] -> x > 0 /. ep -> -1/10, {z1, z2, z3, z4, z5, z6, z7, z8}]
```

The result is:

```
{z1 -> -1/20, z2 -> -1/40, z3 -> -1/20, z4 -> -29/32,
 z5 -> -67/80, z6 -> -83/160, z7 -> -273/320, z8 -> -5/64}
```

Really, all arguments are positive:

```
G1[11/160] G10[1/320] G11[3/40] G12[3/40] G13[41/80] G14[37/40] G15[1/20] G16[1/40]
G17[1/20] G18[29/32] G19[67/80] G2[7/160] G20[83/160] G21[273/320] G3[7/80] G4[139/160]
G5[143/160] G6[1/320] G7[41/80] G8[1/80] G9[43/80]
```

Now set $\epsilon = 0$:

```
G1[11/160] G1
{z1 -> -1/20, z2 -> -1/40, z3 -> -1/20, z4 -> -29/32,
 z5 -> -67/80, z6 -> -83/160, z7 -> -273/320, z8 -> -5/64, ep -> 0}
```

Determine again the arguments of the Gamma-functions; observe:

2 arguments are negative now: those for G3 and G8

```
G1[11/160] G10[33/320] G11[3/40] G12[7/40] G13[57/80] G14[37/40] G15[1/20] G16[1/40]
G17[1/20] G18[29/32] G19[67/80] G2[7/160] G20[83/160] G21[273/320] G4[123/160]
G5[127/160] G6[1/320] G7[5/16] G9[27/80] G3[-9/80] G8[-31/80]
```

Perform the corresponding shifts of integration curve, add the residua and again perform the test for the arguments of the new, lower-dimensional MB-integrals.

We derived an algorithmic solution for isolating the singularities in $1/\epsilon$

The automatization of that: **MB.m** (M. Czakon)

$$\begin{aligned}
 B5l3md2 &\rightarrow MB(4\text{-dim}, \text{fin}) + MB_3(3\text{-dim}, \text{fin}) \\
 &+ MB_{36}(\text{2-dim}, \epsilon^{-1}, \text{fin}) + MB_{365}(\text{1-dim}, \epsilon^{-2}, \epsilon^{-1}, \text{fin}) \\
 &+ MB_5(3\text{-dim}, \text{fin})
 \end{aligned}$$

After these preparations e.g.:

$$\begin{aligned}
 MB_{365}(\text{1-dim}, \epsilon^{-2}) &\sim \frac{1}{\epsilon^2} \frac{1}{2\pi i} \int dz_6 \frac{(-s)^{(-z_6-1)} \Gamma(-z_6)^3 \Gamma(1+z_6)}{8\Gamma(-2z_6)} \\
 &= \frac{1}{\epsilon^2} \sum_{n=0, \infty} - \frac{(-1)^n (-s)^n \Gamma(1+n)^3}{8n! \Gamma(-2(-1-n))} \\
 &= -\frac{1}{\epsilon^2} \frac{\text{ArcSin}(\sqrt{s}/2)}{2\sqrt{4-s}\sqrt{s}} \\
 &= \frac{1}{\epsilon^2} \frac{-x}{4(1-x^2)} H[0, x]
 \end{aligned}$$

Here residua were taken at $z_6 = -n - 1, n = 0, 1, \dots$, and $H[0, x] = \ln(x)$ and $x = \frac{\sqrt{-s+4} - \sqrt{-s}}{\sqrt{-s+4} + \sqrt{-s}}$.

Summary

- We have introduced to the representation of L -loop N -point Feynman integrals of general type
- The determination of the ϵ -poles is generally solved
- The remaining problem is the evaluation of the multi-dimensional, finite MB-Integrals
- This is unsolved in the general case, ... so you have something to do if you like to ...

Problem: Determine the small mass limit of B5l2m2 or of any other of the 2-loop boxes for Bhabha scattering. Stefano Actis may check your solution. He leaves soon.

Some facts on $\Gamma(z)$ and $\Psi(z)$

Residues may be easily derived from Laurent series expansions obtained with Mathematica, e.g.

$$\text{Series}[\text{Gamma}[a - z]^2, z, a, 1] = \frac{1}{(z - a)^2} + \frac{2\gamma_E}{z - a} + 2\gamma_E^2 + \zeta_2 + \dots \quad (14)$$

$$\text{Series}[\text{Gamma}[z - a]^2, z, a, 1] = \frac{1}{(z - a)^2} - \frac{2\gamma_E}{z - a} + 2\gamma_E^2 + \zeta_2 + \dots \quad (15)$$

$$\text{Series}[\text{Gamma}[a - z], z, a, 1] = -\frac{1}{(z - a)} - \gamma_E + \dots \quad (16)$$

$$\text{Series}[\text{Gamma}[z - a], z, a, 1] = +\frac{1}{(z - a)} - \gamma_E + \dots$$

The following property of Γ is useful:

$$\Gamma(-n + \epsilon) = \frac{(-1)^n}{n!} \left[\frac{1}{\epsilon} + \Psi(n + 1) + \dots \right]$$

The derivation: Express $\Gamma(-n + \epsilon)$ by $\Gamma(\epsilon)$. Thereby factors $1/(-n + \epsilon + k)$ are collected, whose inversion for small ϵ leads to the Ψ term and the factor $(-1)^n/n!$. Potential terms with

γ_E cancel in order ϵ^0 . Have in mind here that:

$$\begin{aligned}
 \text{Polygamma}[n+1] &= \text{Polygamma}[0, n+1] \\
 &= \Psi(n+1) \\
 &= \frac{\Gamma'(n+1)}{\Gamma(n+1)} \\
 &= S_1(n) - \gamma_E \\
 &= \sum_{k=1}^n \frac{1}{k} - \gamma_E
 \end{aligned} \tag{17}$$

The following properties hold:

$$\Psi(z+1) = \Psi(z) + 1/z \tag{18}$$

$$\Psi(1+\epsilon) = -\gamma_E + \zeta_2 \epsilon + \dots \tag{19}$$

$$\Psi(1) = -\gamma_E \tag{20}$$

$$\Psi(2) = 1 - \gamma_E \tag{21}$$

$$\Psi(3) = 3/2 - \gamma_E$$

Some facts on residua

The function

$$F(z) = \sum_{i=-N}^{\infty} \frac{a_i}{(z - z_0)^i}$$

has the residue

$$\text{Res } F(z)|_{z=z_0} = a_{-1}$$

An integral over an anti-clockwise directed closed path C around z_0 then is

$$\frac{1}{2\pi i} \int_C dz F(z) = 2\pi i a_1$$

If $G(z)$ has a Taylor expansion around z_0 and $F(z)$ has a Laurent expansion beginning with $a_{-N}/(z - z_0)^N + \dots$, then their product has the residue:

$$\text{Res}[G(z) F(z)]|_{z=z_0} = \sum_{k=1}^N \frac{a_{-k} G(z_0)^{(k)}}{k!}$$

Some sums Mathematica can do

$$\text{Sum}[s^n \Gamma[n+1]^3/(n! \Gamma[2+2n]), n, 0, \text{Infinity}] =$$

$$\cdot \quad (4 \operatorname{ArcSin}[\sqrt{s}/2]) / (\sqrt{4-s} \sqrt{s})$$

$$\text{Sum}[s^n \operatorname{PolyGamma}[0, n+1], n, 0, \text{Infinity}] =$$

$$\cdot \quad (\operatorname{EulerGamma} + \operatorname{Log}[1-s]) / (-1+s)$$