New results on tensor reduction

Tord Riemann, in collaboration with Jochem Fleischer
and with V. Yundin for the PJFry package

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A simple example

1-loop self-energy:

$$
\begin{aligned}
I_2^{\mu} &= \int \frac{d^d k}{i\pi^{d/2}} \; \frac{k^{\mu}}{[k^2 - M_1^2]\,[(k+p)^2 - M_2^2]} \\
&= p_{\mu} B_1
\end{aligned}
$$

Solve:

$$
\begin{aligned}
p_{\mu} I_2^{\mu} &= p^2 B_1(p, M_1, M_2) \\
&= \int \frac{d^d k}{i\pi^{d/2}} \; \frac{pk}{[k^2 - M_1^2]\,[(k+p)^2 - M_2^2]} = \int \frac{d^d k}{i\pi^{d/2}} \; \frac{pk}{D_1 \; D_2} \\
&= \int \frac{d^d k}{i\pi^{d/2}} \left[\frac{D_2 - (p^2 - M_2^2 - M_1^2) - D_1}{D_1 \; D_2} \right], \\
B_1(p, M_1, M_2) &= \frac{1}{2p^2} \left[A_0(M_1) - A_0(M_2) - (p^2 - M_2^2 - M_1^2) B_0(p, M_1, M_2) \right]
\end{aligned}
$$

A tensor Feynman integral is expressed in terms of scalar Feynman integrals.

This talk: Efficient reduction formulae in the algebraicDavydychev-Fleischer-Tarasov approach

Recent developments in the Fleischer-Davydychev-Tarasov approach:

- Get $n > 4$ tensor reduction with $\cdots$:
- $\cdots$ arbitrary masses
- $\cdots$ killed pentagon Gram determinants
- $\cdots$ treatment of full kinematics, also with small sub-diagram Gram determinants
- $\rightarrow$ c++ code PJFry, $n \leq 5$ by V. Yundin [$\rightarrow$ GOSAM option]
  see talk of Yundin at last SFB meeting
- *NEW* : multiple sums over tensor coefficients made efficient by contracting with external momenta

  arXiv/1104.4067, PLB 701 (2011) 646
- *NEW* : $n \geq 7$ $\rightarrow$ arXiv/1111.5821, PLB 707 (2012) 375

Outline

[7] 1991 Davydychev, . . . *Reducing Feynman diagrams to scalar integrals*
[8] 1996 Tarasov, *Connection [of] Feynman integrals [with] different . . . space-time dimensions*
[9] 1999 Fleischer et al., *Algebraic reduction of one-loop Feynman graph amplitudes*

References:
[10] 2010 Diakonidis et al., PLB 683, . . . *recursive reduction of tensor Feynman integrals*
[6] 2011 Fleischer, T.R., PRD 83, *Complete . . . reduction of . . . tensor Feynman integrals*
[11] 2011 Fleischer, T.R., PLB 701, . . . *contracted tensor Feynman integrals*
Aug. 2011: V. Yundin, PhD thesis [with PJFry code]
[12] 2012 Fleischer, T.R., PLB 707, . . . *tensor Feynman integrals with rank* ≥ 6

Notations: Gram and modified Cayley determinant, signed minors [Melrose:1965]

Gram determinant G_n:

$$G_n = |2q_iq_j|, i,j = 1,\dots n \tag{1}$$

Modified Cayley determinant $()_N$ of a diagram with N internal lines and chords q_j; for a choice $q_n = 0$, both determinants are related:

$$()_N \equiv \begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & Y_{11} & Y_{12} & \dots & Y_{1N} \\ 1 & Y_{12} & Y_{22} & \dots & Y_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & Y_{1N} & Y_{2N} & \dots & Y_{NN} \end{vmatrix} = -G_{N-1}, \tag{2}$$

where $D_i = (k-q_i)^2 - m_i^2$ [with q_i = chord], and the matrix elements

$$Y_{ij} = -(q_i - q_j)^2 + m_i^2 + m_j^2, \quad (i,j = 1\dots N) \tag{3}$$

$\Rightarrow$ The Gram determinant $()_N$ does not depend on the masses.

Notations: signed minors [Melrose:1965]

signed minors of $()_N$ are constructed by deleting m rows and m columns from $()_N$, and multiplying with a sign factor:

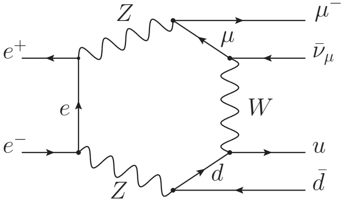
$$\begin{pmatrix} j_1 & j_2 & \cdots & j_m \\ k_1 & k_2 & \cdots & k_m \end{pmatrix}_N \equiv$$
$$\equiv (-1)^{\sum_l (j_l+k_l)} \mathrm{sgn}_{\{j\}} \, \mathrm{sgn}_{\{k\}} \begin{vmatrix} \text{rows } j_1 \cdots j_m \text{ deleted} \\ \text{columns } k_1 \cdots k_m \text{ deleted} \end{vmatrix} \quad (4)$$

where $\mathrm{sgn}_{\{j\}}$ and $\mathrm{sgn}_{\{k\}}$ are the signs of permutations that sort the deleted rows $j_1 \cdots j_m$ and columns $k_1 \cdots k_m$ into ascending order.

Example:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}_N \equiv \begin{vmatrix} Y_{11} & Y_{12} & \dots & Y_{1N} \\ Y_{12} & Y_{22} & \dots & Y_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{1N} & Y_{2N} & \dots & Y_{NN} \end{vmatrix}, \quad (5)$$

Example: Getting a 4-point function from a six-point function I



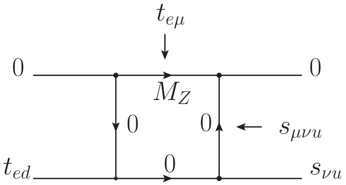


Figure: A six-point topology (a) leading to four-point functions (b) with realistically vanishing Gram determinants.

Example: Getting a 4-point function from a six-point function II

The example is taken from [13].

The corresponding 4-point tensor integrals are, in LoopTools [5, 14] notation:

$$\mathrm{D0i}(\mathrm{id}, 0, 0, s_{\bar{\nu}u}, t_{ed}, t_{\bar{e}\mu}, s_{\mu\bar{\nu}u}, 0, M_Z^2, 0, 0). \tag{6}$$

The Gram determinant is:

$$()_4 \quad = \quad -2t_{\bar{e}\mu}[s^2_{\mu\bar{\nu}u} + s_{\bar{\nu}u}t_{ed} - s_{\mu\bar{\nu}u}(s_{\bar{\nu}u} + t_{ed} - t_{\bar{e}\mu})], \tag{7}$$

It vanishes if:

$$t_{ed} \to t_{ed,\mathrm{crit}} = \frac{s_{\mu\bar{\nu}u}(s_{\mu\bar{\nu}u} - s_{\bar{\nu}u} + t_{\bar{e}\mu})}{s_{\mu\bar{\nu}u} - s_{\bar{\nu}u}}. \tag{8}$$

In terms of a dimensionless scaling parameter x,

$$t_{ed} \quad = \quad (1 + x)t_{ed,\mathrm{crit}}, \tag{9}$$

Example: Getting a 4-point function from a six-point function III

the Gram determinant becomes:

$$()_4 = 2\,x\, s_{\mu\bar{\nu}u} t_{\bar{e}\mu} (s_{\mu\bar{\nu}u} - s_{\bar{\nu}u} + t_{\bar{e}\mu}). \tag{10}$$

We will also need the modified Cayley determinant:

$$\begin{aligned}
\binom{0}{0}_4 &= \begin{pmatrix} 2M_Z^2 & M_Z^2 & M_Z^2 - s_{\mu\bar{\nu}u} & M_Z^2 \\ M_Z^2 & 0 & -s_{\bar{\nu}u} & M_Z^2 \\ M_Z^2 - s_{\mu\bar{\nu}u} & -s_{\bar{\nu}u} & 0 & -t_{ed} \\ M_Z^2 & -t_{\bar{e}\mu} & -t_{ed} & 0 \end{pmatrix} \\
&= s_{\mu\bar{\nu}u}^2 t_{\bar{e}\mu}^2 + 2\,M_Z^2 t_{\bar{e}\mu}[-2s_{\bar{\nu}u} t_{ed} + s_{\mu\bar{\nu}u}(s_{\bar{\nu}u} + t_{ed} - t_{\bar{e}\mu})] \\
&\quad + M_Z^4 (s_{\bar{\nu}u}^2 + (t_{ed} - t_{\bar{e}\mu})^2 - 2s_{\bar{\nu}u}(t_{ed} + t_{\bar{e}\mu})).
\end{aligned}$$

Dimensional shifts and recurrence relations for pentagons (I)

Following [Davydychev:1991 [7]]

Replace tensors by scalar integrals in higher dimensions:

Example $R = 3$:

$$
\begin{aligned}
I_5^{\mu\nu\lambda} &= \int \frac{d^{4-2\epsilon}k}{i\pi^{d/2}} \prod_{r=1}^{5} c_r^{-1} k^\mu k^\nu k^\lambda \qquad (11) \\
&= -\sum_{i,j,k=1}^{4} q_i^\mu q_j^\nu q_k^\lambda n_{ijk} I_{5,ijk}^{[d+]^3} + \frac{1}{2} \sum_{i=1}^{n-1} (g^{\mu\nu} q_i^\lambda + g^{\mu\lambda} q_i^\nu + g^{\nu\lambda} q_i^\mu) I_{5,i}^{[d+]^2},
\end{aligned}
$$

and $n_{ijk} = (1+\delta_{ij})(1+\delta_{ik}+\delta_{jk})$.

$[d+]^l = 4 - 2\epsilon + 2l$

$I_{5,i}^{[d+]^2}$ – scratch the line i from $I_5^{[d+]^2}$.

Dimensional shifts and recurrence relations for pentagons (II)

'Naive', direct approach – just perform dimensional recurrences

Following [Tarasov:1996,Fleischer:1999 [8, 9]]
apply recurrence relations, relating scalar integrals of different dimensions, in order to get rid of the dimensionalities $[d+]^l = 4 - 2\epsilon + 2l$:

$$\nu_j(\mathbf{j}^+ I_5^{[d+]}) = \frac{1}{()_5}\left[-\begin{pmatrix} j \\ 0 \end{pmatrix}_5 + \sum_{k=1}^{5} \begin{pmatrix} j \\ k \end{pmatrix}_5 \mathbf{k}^-\right] I_5 \qquad (12)$$

$$(d - \sum_{i=1}^{5} \nu_i + 1) I_5^{[d+]} = \frac{1}{()_5}\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}_5 - \sum_{k=1}^{5} \begin{pmatrix} 0 \\ k \end{pmatrix}_5 \mathbf{k}^-\right] I_5, \qquad (13)$$

also:

$$\nu_j \mathbf{j}^+ I_5 = \frac{1}{\binom{0}{0}_5} \sum_{k=1}^{5} \begin{pmatrix} 0j \\ 0k \end{pmatrix}_5 \left[d - \sum_{i=1}^{5} \nu_i(\mathbf{k}^- \mathbf{i}^+ + 1)\right] I_5 \qquad (14)$$

where the operators $\mathbf{i}^\pm, \mathbf{j}^\pm, \mathbf{k}^\pm$ act by shifting the indices ν_i, ν_j, ν_k by ± 1.

The result of simplifying manipulations

...and collecting all contributions, our final result for e.g. the tensor of rank $R = 3$ can be written as follows:

$$I_5^{\mu\,\nu\,\lambda} = \sum_{i,j,k=1}^{4} q_i^\mu \, q_j^\nu \, q_k^\lambda E_{ijk} + \sum_{k=1}^{4} g^{[\mu\nu} q_k^{\lambda]} E_{00k}, \tag{15}$$

with:

$$E_{00j} = \sum_{s=1}^{5} \frac{1}{\binom{0}{0}_5} \left[\frac{1}{2} \binom{0s}{0j}_5 I_4^{[d+],s} - \frac{d-1}{3} \binom{s}{j}_5 I_4^{[d+]^2,s} \right], \tag{16}$$

$$E_{ijk} = -\sum_{s=1}^{5} \frac{1}{\binom{0}{0}_5} \left\{ \left[\binom{0j}{sk}_5 I_{4,i}^{[d+]^2,s} + (i \leftrightarrow j) \right] + \binom{0s}{0k}_5 \nu_{ij} I_{4,ij}^{[d+]^2,s} \right\}. \tag{17}$$

✓ no scalar 5-point integrals in higher dimensions
✓ no inverse Gram det. $()_5$

We have yet:

† scalar 4-point integrals in higher dimensions: $I_{4,ij}^{[d+]^2,s}$ etc.

† inverse Gram det. $\binom{0}{0}_5 \equiv ()_4$

Reduce $I_{4,ij\cdots}^{[d+]^l,s}$ to $I_4^{[d+]^L,s}$ plus simpler objects I

By nontrivial manipulations we get e.g.:

$$I_{4,i}^{[d+],s} = \frac{1}{\left(\begin{smallmatrix}0s\\0s\end{smallmatrix}\right)_5}\left[-\begin{pmatrix}0s\\is\end{pmatrix}_5 (d-3) I_4^{[d+],s} + \sum_{t=1}^{5}\begin{pmatrix}0st\\0si\end{pmatrix}_5 I_3^{st}\right] \tag{18}$$

$$\begin{aligned}\nu_{ij} I_{4,ij}^{[d+]^2} = {} & \frac{\left(\begin{smallmatrix}0\\i\end{smallmatrix}\right)_4}{\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)_4}\frac{\left(\begin{smallmatrix}0\\j\end{smallmatrix}\right)_4}{\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)_4}(d-2)(d-1) I_4^{[d+]^2} + \frac{\left(\begin{smallmatrix}0i\\0j\end{smallmatrix}\right)_4}{\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)_4} I_4^{[d+]} \\ & - \frac{\left(\begin{smallmatrix}0\\j\end{smallmatrix}\right)_4}{\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)_4}\frac{d-2}{\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)_4}\sum_{t=1}^{4}\begin{pmatrix}0t\\0i\end{pmatrix}_4 I_3^{[d+],t} + \frac{1}{\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)_4}\sum_{t=1}^{4}\begin{pmatrix}0t\\0j\end{pmatrix}_4 I_{3,i}^{[d+],t}\end{aligned} \tag{19}$$

These equations are free of inverse Gram determinants $()_4$.
But they contain yet the generic 4-point and (partly indexed) 3-point functions in higher dimensions, $I_4^{[d+],s}$, $I_3^{[d+],t}$, etc.

Last step: evaluate the $I_4^{[d+],s}$, $I_3^{[d+],t}$, etc. I

Several strategies are now possible:

- Just evaluate them analytically in $d + 2l - 2\epsilon$ dimensions – if you may do that
- Just evaluate them numerically in $d + 2l - 2\epsilon$ dimensions
- Reduce them further by recurrences – buy the towers of $1/()_4$ $\rightarrow$ apply (13)
- Make a small Gram determinant expansion $\rightarrow$ apply (13) another way round

Last two items are done here.

Reduction of scalars I_4^D to the generic dimension $\rightarrow I_4^d = D_0, I_3^d = C_0$ I

Non-small 4-point Gram determinants:

Direct, iterative use of (13) yields e.g.:

$$I_4^{[d+]^l} = \left[\frac{\binom{0}{0}_4}{()_4} I_4^{[d+]^{l-1}} - \sum_{t=1}^{4} \frac{\binom{t}{0}_4}{()_4} I_3^{[d+]^{l-1},t} \right] \frac{1}{d+2l-5} \tag{20}$$

$$I_3^{[d+]^l,t} = \left[\frac{\binom{0t}{0t}_4}{\binom{t}{t}_4} I_3^{[d+]^{l-1},t} - \sum_{u=1,u\neq t}^{4} \frac{\binom{ut}{0t}_4}{\binom{t}{t}_4} I_2^{[d+]^{l-1},tu} \right] \frac{1}{d+2l-4} \tag{21}$$

And we are done.
This works fine if $()_4$ is not small [and also the $\binom{t}{t}_4$].

Make a small Gram expansion I

Again use (13):

$$()_4 (d - \sum_{i=1}^{4} \nu_i + 1) I_4^{[d+]} = \left[\binom{0}{0}_4 I_4 - \sum_{k=1}^{4} \binom{0}{k}_4 I_3^k \right]$$

If $()_4 = 0$, then it follows ($n = 4$):

$$I_n^D = \sum_k^n \frac{\binom{0}{k}_n}{\binom{0}{0}_n} I_{n-1}^{D,k} \tag{22}$$

If $()_4 << 1$, re-write (13), as follows:

$$I_n^D = \sum_k^n \frac{\binom{0}{k}_n}{\binom{0}{0}_n} I_{n-1}^{D,k} - \frac{()_n}{\binom{0}{0}_n} [(D+1) - \sum_i^n \nu_i] I_n^{D+2}. \tag{23}$$

Effectively we may evaluate I_n^D in terms of simpler functions $I_{n-1}^{D,k}$ with a small correction depending on I_n^{D+2}.

We may go a step further, and insert into (23) for I_n^{D+2} the rhs. of (22), taken now at $D' = D + 2$:

$$I_n^D = \sum_k^n \frac{\binom{0}{k}_n}{\binom{0}{0}_n} I_{n-1}^{D,k} - \frac{()_n}{\binom{0}{0}_n}[(D+1) - \sum_i^n \nu_i] \times \left[\sum_k^n \frac{\binom{0}{k}_n}{\binom{0}{0}_n} I_{n-1}^{D+2,k} - \frac{()_n}{\binom{0}{0}_n}[(D+3) - \sum_i^n \nu_i] I_n^{D+4} \right].$$

The terms proportional to $[()_n/\binom{0}{0}_n]^a, a = 0, 1$ may be evaluated at the correct kinematics. They depend on three-point functions, and their reduction by normal recurrences will not introduce the unwanted powers of $1/()_4$. The last term, suppressed by the factor $[()_n/\binom{0}{0}_n]^2$, depends on I_n^{D+4}. It may either be taken approximately at $()_n = 0$, where it can also be represented by 3-point functions (and their reductions), or it may be evaluated more correctly by another iteration based on (22).

And so on and so on ...

In the numerical example – next section – we worked out up to 10 stable iterations.

Numbers: D_{1111} I

Following Davydychev, [7], one gets

$$I_4^{\mu\nu\lambda} = \int^d \frac{k^\mu k^\nu k^\lambda}{\prod_{r=1}^n c_r} = -\sum_{i,j,k=1}^n q_i^\mu q_j^\nu q_k^\lambda \nu_{ijk} I_{n,ijk}^{[d+]^3} + \frac{1}{2}\sum_{i=1}^n g^{[\mu\nu} q_i^{\lambda]} I_{n,i}^{[d+]^2} \quad (24)$$

We identify the tensor coefficients $D_{11\dots}$ a la LoopTools, e.g.:

$$D_{111} = I_{4,222}^{[d+]^3} \quad (25)$$

Similarly:

$$D_{1111} = I_{4,2222}^{[d+]^4} \quad (26)$$

Rank $R = 4$ tensor D_{1111} – Numerics with dimensional recurrences

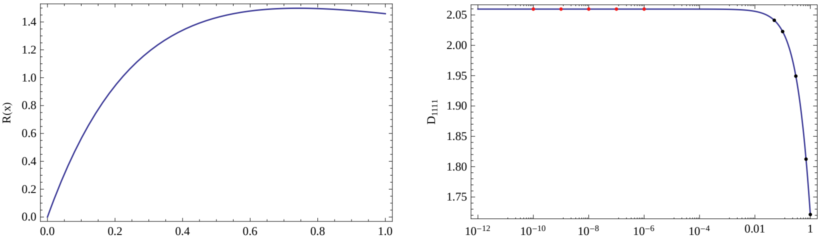
From (23) we see that a "small Gram determinant" expansion will be useful when the following dimensionless parameter becomes small:

$$R = \frac{()_4}{\binom{0}{0}_4} \times s, \tag{27}$$

where s is a typical scale of the process, e.g. we will choose $s = s_{\mu\bar{\nu}u}$.
Following [13], we further choose:

$$\begin{aligned} s_{\mu\bar{\nu}u} &= 2 \times 10^4 \mathrm{GeV}^2, \\ s_{\bar{\nu}u} &= 1 \times 10^4 \mathrm{GeV}^2, \\ t_{\bar{e}\mu} &= -4 \times 10^4 \mathrm{GeV}^2, \end{aligned}$$

and get $t_{ed,\mathrm{crit}} = -6 \times 10^4 \mathrm{GeV}^2$. For x=1, the Gram determinant becomes $()_4 = 4.8 \times 10^{13}$ GeV^3.
The small expansion parameter $R(x)$ and D_{1111} are shown in figure 2.



Small Gram expansion and Pade approximation I

[Fleischer,TR: PRD 2011 [6]]

Tables have been taken from there.

They were shown first at QCD@LHC@Trento2010

The use of appropriate Pade approximations is explained there. Convergence in the small Gram determinant region is considerably improved.

x		$\mathfrak{Re}\, D_{1111}$	$\mathfrak{Im}\, D_{1111}$
0.	[exp 0,0]	2.05969289730 E-10	1.55594910118 E-10
10^{-8}	[exp x,2]	2.05969289342 E-10	1.55594909187 E-10
	[exp 0,2]	2.05969289349 E-10	1.55594909187 E-10
10^{-4}	[exp x,5]	2.05965609497 E-10	1.55585605343 E-10
	[exp 0,5]	2.05965609495 E-10	1.55585605343 E-10
0.001	[exp 0,6]	2.05932484380 E-10	1.55501912433 E-10
	[exp x,6]	2.05932484381 E-10	1.55501912433 E-10
	$I_{4,2222}^{[d+]^4}$	2.02292295240 E-10	1.54974785467 E-10
	D_{1111}	2.01707671668 E-10	1.62587142251 E-10
0.005	[exp 0,6]	2.05786054801 E-10	1.55131031024 E-10
	[pade 0,3]	2.05785198947 E-10	1.55131031003 E-10
	[exp x,6]	2.05786364440 E-10	1.55131031024 E-10
	[pade x,3]	2.05785199805 E-10	1.55131030706 E-10
	$I_{4,2222}^{[d+]^4}$	2.05778894114 E-10	1.55135794453 E-10
	D_{1111}	2.05779811490 E-10	1.55136343923 E-10
0.01	[exp 0,6]	2.05703298143 E-10	1.54669910676 E-10
	[pade 0,3]	2.05600940065 E-10	1.54669907784 E-10
	[exp 0,10]	2.05600964693 E-10	1.54669910676 E-10
	[pade 0,5]	2.05600955381 E-10	1. 54669910676E-10
	[exp x,10]	2.05600963675 E-10	1.54669910676 E-10
	[pade x,5]	2.05600955381 E-10	1.54669910676 E-10
	$I_{4,2222}^{[d+]^4}$	2.05600013702 E-10	1.54670651917 E-10
	D_{1111}	2.05600239280 E-10	1.54670771210 E-10

Table: Numerical values for the tensor coefficient D_{1111}. Values marked by D_{1111} are evaluated with LoopTools, the $I_{4,2222}^{[d+]^4}$ corresponds to (29) The labels [exp 0,2n] and [pade 0,n] denote iteration $2n$ and Pade approximant $[n, n]$ when the small Gram determinant expansion starts at $x = 0$, and [exp x,2n] and [pade x,n] are the corresponding numbers for an expansion starting at x.

x		$\Re e\ D_{1111}$	$\Im m\ D_{1111}$
0.01	[exp 0,6]	2.05703298143 E-10	1.54669910676 E-10
	[pade 0,3]	2.05600940065 E-10	1.54669907784 E-10
	[exp 0,10]	2.05600964693 E-10	1.54669910676 E-10
	[pade 0,5]	2.05600955381 E-10	1. 54669910676E-10
	[exp x,10]	2.05600963675 E-10	1.54669910676 E-10
	[pade x,5]	2.05600955381 E-10	1.54669910676 E-10
	$I_{4,2222}^{[d+]^4}$	2.05600013702 E-10	1.54670651917 E-10
	D_{1111}	2.05600239280 E-10	1.54670771210 E-10
0.05	[exp 0,6]	4.83822963052 E-09	1.51077429118 E-10
	[pade 0,3]	2.01518061131 E-10	1.50591643209 E-10
	[exp 0,20]	2.04218962072 E-10	1.51077424143 E-10
	[pade 0,10]	2.04122727654 E-10	1.51077424149 E-10
	[exp x,20]	2.04190274030 E-10	1.51077424143 E-10
	[pade x,10]	2.04122727971 E-10	1.51077423985 E-10
	$I_{4,2222}^{[d+]^4}$	2.04122726387 E-10	1.51077422901 E-10
	D_{1111}	2.04122726601 E-10	1.51077423320 E-10
0.1	[exp 0,26]	2.20215264409 E-08	1.46815247004 E-10
	[pade 0,13]	2.01749674352 E-10	1.46681287362 E-10
	[exp x,26]	2.08190721550 E-08	1.46815247004 E-10
	[pade x,13]	2.03995221326 E-10	1.46785977364 E-10
	$I_{4,2222}^{[d+]^4}$	2.02269485177 E-10	1.46815247061 E-10
	D_{1111}	2.02269485217 E-10	1.46815247051 E-10
1.	$I_{4,2222}^{[d+]^4}$	1.72115440143 E-10	9.74550747662 E-11
	D_{1111}	1.72115440148 E-10	9.74550747662 E-11

Table: Numerical values for the tensor coefficient D_{1111}. Values marked by D_{1111} are evaluated with LoopTools, the $I_{4,2222}^{[d+]^4}$ corresponds to (29) The labels [exp 0,2n] and [pade 0,n] denote iteration $2n$ and Pade approximant $[n, n]$ when the small Gram determinant expansion starts at $x = 0$, and [exp x,2n] and [pade x,n] are

PJFry - an open source c++ program by V. Yundin I

PJFry 1.0.0 - one loop tensor integral library

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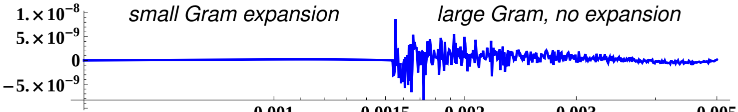
- More information and the latest source code:
  project page: https://github.com/Vayu/PJFry/
- $\rightarrow$ how to install
- $\rightarrow$ how to use
- $\rightarrow$ samples
- See also:
  V. Yundin's talk at last SFB meeting "One loop tensor reduction program PJFRY"
- Yundin's PhD thesis, defended February 2012 at Humboldt University

PJFry — numerical package [from V.Y. Valencia 2011] I

Numerical implementation of described algorithms:
C++ package **PJFry** by V. Yundin [see project webpage]

- Reduction of **5-point** 1-loop tensor integrals up to **rank 5**
- No limitations on internal/external masses combinations
- Small Gram determinants treatment by expansion
- Interfaces for C, C++, FORTRAN and MATHEMATICA

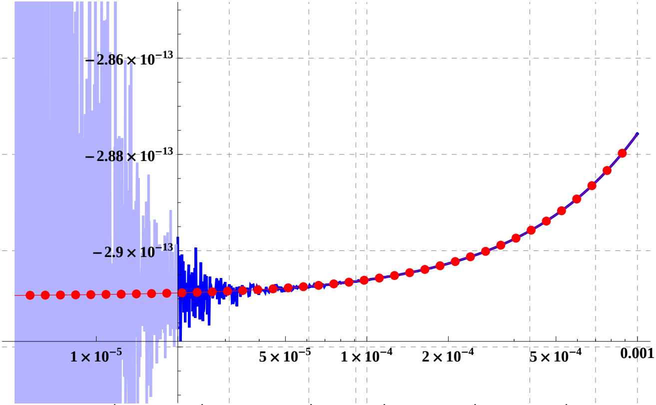
Example:
Relative accuracy of E_{3333} coef. around small Gram4 region



PJFry — small Gram region example [from V.Y. Valencia 2011]

Example: E_{3333} coefficient in small Gram region ($x = 0$)

Comparison of Regular and Expansion formulae:



$x{=}0$: $E_{3333}(0, 0, -6{\times}10^4, 0, 0, 10^4, -3.5{\times}10^4, 2{\times}10^4, -4{\times}10^4, 1.5{\times}10^4, 0, 6550, 0, 0, 8315)$

Contractions with external momenta (or with CHORDS) I

[Fleischer,TR: PLB 2011 [11]]
After having tensor reductions with basis functions I_n^D,
which are independent of the indices $i, j, k, ...$,
one may use contractions with external momenta in order to perform all the sums over $i, j, k, ...$
This leads to a significant simplification and shortening of calculations.
Reminder:
One option was to avoid the appearance of inverse Gram determinants $1/()_5$. For rank $R = 5$, e.g.,

$$I_5^{\mu\,\nu\,\lambda\rho\sigma} = \sum_{s=1}^{5} \Bigg[\sum_{i,j,k,l,m=1}^{5} q_i^{\mu} q_j^{\nu} q_k^{\lambda} q_l^{\rho} q_m^{\sigma} E_{ijklm}^{s} + \sum_{i,j,k=1}^{5} g^{[\mu\nu} q_i^{\lambda} q_j^{\rho} q_k^{\sigma]} E_{00ijk}^{s} + \sum_{i=1}^{5} g^{[\mu\nu} g^{\lambda\rho} q_i^{\sigma]} E_{0000i}^{s} \Bigg] \tag{28}$$

Contractions with external momenta (or with CHORDS) I

The tensor coefficients are expressed in terms of integrals $I_{4,i\cdots}^{[d+]^l,s}$, e.g.:

$$E_{ijklm}^{s} = -\frac{1}{\binom{0}{0}_5}\Bigg\{\Bigg[\binom{0l}{sm}_5 n_{ijk} I_{4,ijk}^{[d+]^4,s} + (i \leftrightarrow l) + (j \leftrightarrow l) + (k \leftrightarrow l)\Bigg] + \binom{0s}{0m}_5 n_{ijkl} I_{4,ijkl}^{[d+]^4,s}\Bigg\}.$$

Now, in a next step, one may avoid the appearance of inverse sub-Gram determinants $()_4$.
The complete dependence on the indices i of the tensor coefficients is contained now in the pre-factors with signed minors. One can say that the indices *decouple* from the integrals.
As an example, we reproduce the 4-point part of

$$n_{ijkl} I_{4,ijkl}^{[d+]^4} = \frac{\binom{0}{i}}{\binom{0}{0}}\frac{\binom{0}{j}}{\binom{0}{0}}\frac{\binom{0}{k}}{\binom{0}{0}}\frac{\binom{0}{l}}{\binom{0}{0}} d(d+1)(d+2)(d+3) I_4^{[d+]^4}$$
$$+ \frac{\binom{0i}{0j}\binom{0}{k}\binom{0}{l} + \binom{0i}{0k}\binom{0}{j}\binom{0}{l} + \binom{0j}{0k}\binom{0}{i}\binom{0}{l} + \binom{0i}{0l}\binom{0}{j}\binom{0}{k} + \binom{0j}{0l}\binom{0}{i}\binom{0}{k} + \binom{0k}{0l}\binom{0}{i}\binom{0}{j}}{\binom{0}{0}^3} d(d+1) I_4^{[d+]^3}$$
$$+ \frac{\binom{0i}{0l}\binom{0j}{0k} + \binom{0j}{0l}\binom{0i}{0k} + \binom{0k}{0l}\binom{0i}{0j}}{\binom{0}{0}^2} I_4^{[d+]^2} + \cdots [\textit{ simpler terms}] \tag{29}$$

Contractions with external momenta (or with CHORDS) II

In (29), one has to understand the 4-point integrals to carry the corresponding index s and the signed minors are $\binom{0}{k} \to \binom{0s}{ks}_5$ etc.

Contractions with external momenta (or with CHORDS) I

A chord is the momentum shift of an internal line due to external momenta, $D_i = (k - q_i)^2 - m_i^2 + i\epsilon$, and $q_i = (p_1 + p_2 + + p_i)$, with $q_n = 0$.

The tensor 5-point integral of rank $R = 1$ yields, when contracted with a chord,

$$q_{a\mu} I_5^{\mu} = -\frac{1}{\binom{0}{0}_5} \sum_{s=1}^{5} \left[\sum_{i=1}^{4} (q_a \cdot q_i) \begin{pmatrix} 0i \\ 0s \end{pmatrix}_5 \right] I_4^s. \qquad (30)$$

In fact, the sum over i may be performed explicitly:

$$\Sigma_a^{1,s} \equiv \sum_{i=1}^{4} (q_a \cdot q_i) \begin{pmatrix} 0s \\ 0i \end{pmatrix}_5 = +\frac{1}{2} \left\{ \begin{pmatrix} s \\ 0 \end{pmatrix}_5 (Y_{a5} - Y_{55}) + \begin{pmatrix} 0 \\ 0 \end{pmatrix}_5 (\delta_{as} - \delta_{5s}) \right\},$$

Contractions with external momenta I

We get immediately

$$q_{a\mu} I_5^{\mu} = -\frac{1}{\binom{0}{0}_5} \sum_{s=1}^{5} \Sigma_a^{1,s} \quad I_4^s. \qquad (31)$$

Contractions with external momenta I

The tensor 5-point integral of rank $R = 2$

$$I_5^{\mu\,\nu} = \sum_{i,j=1}^{4} q_i^{\mu}\, q_j^{\nu}\, E_{ij} + g^{\mu\nu} E_{00}, \tag{32}$$

has the following tensor coefficients free of $1/()_5$:

$$E_{00} = -\sum_{s=1}^{5} \frac{1}{2} \frac{1}{\binom{0}{0}_5} \binom{s}{0}_5 I_4^{[d+],s}, \tag{33}$$

$$E_{ij} = \sum_{s=1}^{5} \frac{1}{\binom{0}{0}_5} \left[\binom{0i}{sj}_5 I_4^{[d+],s} + \binom{0s}{0j}_5 I_{4,i}^{[d+],s} \right]. \tag{34}$$

Contractions with external momenta I

Equation (32) yields for the contractions with chords:

$$q_{a\mu} q_{b\nu} I_5^{\mu\,\nu} = \sum_{i,j=1}^{4} (q_a \cdot q_i)(q_b \cdot q_j) E_{ij} + (q_a \cdot q_b) E_{00}. \tag{35}$$

and finally (35) simply reads

$$\begin{aligned} q_{a\mu} q_{b\nu} I_5^{\mu\,\nu} \quad = \quad & \frac{1}{4} \sum_{s=1}^{5} \left\{ \frac{\binom{s}{0}_5}{\binom{0s}{0s}_5} (\delta_{ab}\delta_{as} + \delta_{5s}) + \frac{\binom{s}{s}_5}{\binom{0s}{0s}_5} \left[(\delta_{as} - \delta_{5s}) \left(Y_{b5} - Y_{55} \right) \right.\right. \\ & \left.\left. + \left(\delta_{bs} - \delta_{5s} \right) \left(Y_{a5} - Y_{55} \right) + \frac{\binom{s}{0}_5}{\binom{0}{0}_5} \left(Y_{a5} - Y_{55} \right) \left(Y_{b5} - Y_{55} \right) \right] \right\} I_4^{[d+],s} \\ & + \frac{1}{\binom{0}{0}_5} \sum_{s=1}^{5} \frac{\Sigma_b^{1,s}}{\binom{0s}{0s}_5} \sum_{t=1}^{5} \Sigma_a^{2,st} I_3^{st}, \end{aligned}$$

Contractions with external momenta I

with

$$\Sigma_a^{2,st} \equiv \sum_{i=1}^{4} (q_a \cdot q_i) \begin{pmatrix} 0st \\ 0si \end{pmatrix}_5$$
$$= \frac{1}{2}(1-\delta_{st}) \left\{ \begin{pmatrix} ts \\ 0s \end{pmatrix}_5 (Y_{a5} - Y_{55}) + \begin{pmatrix} 0s \\ 0s \end{pmatrix}_5 (\delta_{at} - \delta_{5t}) - \begin{pmatrix} 0s \\ 0t \end{pmatrix}_5 (\delta_{as} - \delta_{5s}) \right\}$$

This has been extended also to higher ranks.
We need at most double sums, e.g.:

$$\Sigma_{ab}^{2,s} \equiv \sum_{i,j=1}^{4} (q_a \cdot q_i)(q_b \cdot q_j) \begin{pmatrix} si \\ sj \end{pmatrix}_5$$
$$= \frac{1}{2}(q_a \cdot q_b) \begin{pmatrix} s \\ s \end{pmatrix}_5 - \frac{1}{4}()_5 (\delta_{ab}\delta_{as} + \delta_{5s}), \qquad (36)$$

Contractions with external momenta I

Many of the sums over signed minors, weighted with scalar products of chords are given in PLB 2011 [[11]], and an almost complete list may be obtained on request from J. Fleischer, T.R.

Modifications for 7- and higher point functions I

For N=6,7,8..., the Gram determinant vanishes, and also further determinants:
$()_n = 0$, $n > 5$
but also $\binom{0}{k}_7 = 0$
As a result, one has to reorganize the reductions, avoiding the $1/()_n$ completely.
This may be done, and one arrives at expressions with $I_6^{[d+]}$.
The problematic case is the integral $I_{6,i}^{[d+]}$ for which one can write

$$I_{6,i}^{[d+]} = \sum_{s=1,s\neq i}^{7} \frac{\binom{R}{s}_6}{\binom{R}{0}_6} I_{5,i}^{[d+],s} + \frac{\binom{R}{i}_6}{\binom{R}{0}_6} I_6^{[d+]}.$$

For $I_6^{[d+]}$ one finally needs 4-point functions up to the order ϵ in the basis.
A numerical alternative: See J. Fleischer, T. Riemann, the Radcor 2011 contribution

Modifications for 7- and higher point functions II

But:
We learned from Gudrun Heinrich et al. that there is a representation in terms just of ordinary 4-point functions:

Binoth, Guillet, Heinrich, hep-ph/0504267, hep-ph/9911342

$n > 6$: assumption in Bern,Dixon,Kosower, hep-ph/9306240
This we have worked out explicitely on our formalism in PLB 707 (2012) for $N = 7$ and $N = 8$.

Results for $N = 6, 7, 8, \ldots$ I

In [16] we solve analytically the generalized recursions for $n \geq 6$, which were derived in [17, 18]:

$$I_n^{\mu_1\mu_2\cdots\mu_R} = -\sum_{r=1}^{n} C_r^{\mu_1}(n) I_{n-1}^{\mu_2\cdots\mu_R,r}, \tag{37}$$

where in $I_{n-1}^{\mu,\cdots,r}$ the line r is scratched.
Assume a representation of the metric tensor in the form

$$\frac{1}{2} g^{\mu\nu} = \sum_{i,j=1}^{N-1} G_{ij} q_i^{\mu} q_j^{\nu} \tag{38}$$

is available.
Then, the vector

$$C_j^{\mu} = \sum_{i=1}^{N-1} G_{ij} q_i^{\mu} \tag{39}$$

Results for $N = 6, 7, 8, \ldots$ II

is a solution of (37).
An additional requirement has to be fulfilled for this vector:

$$\sum_{j=1}^{N} C_j^{\mu} = 0, \tag{40}$$

The coefficients for 6-point functions are:

$$C_r^{s,\mu}(6) = \sum_{i=1}^{5} \frac{1}{\binom{0}{s}_6} \binom{0r}{si}_6 q_i^{\mu_1}, \quad s = 0 \ldots 6, \tag{41}$$

where the q_i are chords, and $\binom{0r}{si}_6$ etc. are signed minors with arbitrary s. For the 7-point and 8-point functions, we found several representations, among them

$$C_r^{st,\mu}(7) = \sum_{i=1}^{6} \frac{1}{\binom{st}{st}_7} \binom{sti}{str}_7 q_i^{\mu} \tag{42}$$

Results for $N = 6, 7, 8, \ldots$ III

and

$$C_r^{stu,\mu}(8) = \sum_{i=1}^{7} \frac{1}{\left(\begin{smallmatrix} stu \\ stu \end{smallmatrix}\right)_8} \begin{pmatrix} stui \\ stur \end{pmatrix}_8 q_i^{\mu} \tag{43}$$

The upper indices s, t and u stand for the redundancy of the solutions and can be freely chosen.

Two 7-point examples for sums with external momenta:

The rank $R = 2, 3$ integrals become by contraction

$$q_{a,\mu} q_{b,\nu} \; I_7^{\mu\nu} = \sum_{r,t=1}^{7} K^{ab,rt} I_5^{rt}, \tag{44}$$

$$q_{a,\mu} q_{b,\nu} q_{c,\lambda} \; I_7^{\mu\nu\lambda} = \sum_{r,t,u=1}^{7} K^{abc,rtu} I_4^{rtu}, \tag{45}$$

Results for $N = 6, 7, 8, \ldots$ IV

where I_5^{rt} and I_4^{rtu} are scalar 5- and 4-point functions, arising from the 7-point function by scratching lines $r, t, \ldots$ In the general case, we have at this stage higher-dimensional integrals I_n^{d+2l}, $n = 2, \ldots, 5$, to be further reduced following the known scheme, if needed. Here, the I_5^{rt} have to be expressed by 4-point functions.
The expansion coefficients are factorizing here,

$$K^{ab,rt} = K^{a,r} K^{b,rt}, \tag{46}$$

$$K^{abc,rtu} = -K^{a,r} K^{b,rt} K^{c,rtu}, \tag{47}$$

and the sums over signed minors have been performed analytically:

$$K^{a,r} = \frac{1}{2} \left(\delta_{ar} - \delta_{7r} \right), \tag{48}$$

$$K^{b,rt} = \sum_{j=1}^{6} (q_b q_j) \frac{\binom{rst}{rsj}_7}{\binom{rs}{rs}_7} \equiv \frac{\Sigma_b^{1,stu}}{\binom{rs}{rs}_7} = \frac{1}{2} \left(\delta_{bt} - \delta_{7t} \right) - \frac{1}{2} \frac{\binom{rs}{ts}_7}{\binom{rs}{rs}_7} \left(\delta_{br} - \delta_{7r} \right), \tag{49}$$

Results for $N = 6, 7, 8, \ldots$ V

$$K^{a,stu} = \sum_{i=1}^{6} (q_a q_i) \binom{0stu}{0sti}_7 \equiv \Sigma_a^{2,stu} \tag{50}$$

$$= \frac{1}{2}\left\{ \binom{stu}{st0}_7 (Y_{a7} - Y_{77}) + \binom{0st}{0st}_7 (\delta_{au} - \delta_{7u}) - \binom{0st}{0su}_7 (\delta_{at} - \delta_{7t}) - \binom{0ts}{0tu}_7 (\delta_{as} - \delta_{7s}) \right\} \tag{51}$$

with

$$Y_{jk} = -(q_j - q_k)^2 + m_j^2 + m_k^2. \tag{52}$$

Conventionally, $q_7 = 0$.
The sums may be found in eqns. (A.15) and (A.16) of [11]. The s is redundant and fulfils $s \neq r, b, 7$ in $K^{b,rt}$. In $K_0^{a,stu}$ it is $s, t, u = 1, \ldots 7$ with $s \neq u, t \neq u$.

Summary

- Systematic derivation of expressions which are explicitely free of inverse Gram determinants $()_5$ until pentagons of rank $R = 5$
- Proper isolation of inverse Gram determinants of subdiagrams of the type $\binom{s}{s}_4$, which cannot be completely avoided
- Numerical C++ package PJFry (V. Yundin, open source) for C, c++, Mathematica, Fortran
- Perform multiple sums with signed minors and scalar products after contractions with chords or external momenta
- N=6,7,8 Analytical treatment worked out

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