

New results on tensor reduction

Tord Riemann, in collaboration with Jochem Fleischer
and with V. Yundin for the PJFry package

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A simple example

1-loop self-energy:

$$\begin{aligned}
 I_2^\mu &= \int \frac{d^d k}{i\pi^{d/2}} \frac{k^\mu}{[k^2 - M_1^2][(k+p)^2 - M_2^2]} \\
 &= p_\mu B_1
 \end{aligned}$$

Solve:

$$\begin{aligned}
 p_\mu I_2^\mu &= p^2 B_1(p, M_1, M_2) \\
 &= \int \frac{d^d k}{i\pi^{d/2}} \frac{pk}{[k^2 - M_1^2][(k+p)^2 - M_2^2]} = \int \frac{d^d k}{i\pi^{d/2}} \frac{pk}{D_1 D_2} \\
 &= \int \frac{d^d k}{i\pi^{d/2}} \left[\frac{D_2 - (p^2 - M_2^2 - M_1^2) - D_1}{D_1 D_2} \right],
 \end{aligned}$$

$$B_1(p, M_1, M_2) = \frac{1}{2p^2} \left[A_0(M_1) - A_0(M_2) - (p^2 - M_2^2 - M_1^2) B_0(p, M_1, M_2) \right]$$

A **tensor** Feynman integral is expressed in terms of **scalar** Feynman integrals.

This talk: Efficient reduction formulae in the algebraic Davydychev-Fleischer-Tarasov approach

Recent developments in the Fleischer-Davydychev-Tarasov approach:

- Get $n > 4$ tensor reduction with \dots :
- \dots **arbitrary** masses
- \dots **killed** pentagon Gram determinants
- \dots **treatment of** full kinematics, also with small sub-diagram Gram determinants
- \rightarrow **c++ code PJFry**, $n \leq 5$ by V. Yundin [\rightarrow GOSAM option]
see talk of Yundin at last SFB meeting
- **NEW** : **multiple sums over tensor coefficients** made efficient by contracting with external momenta

arXiv/1104.4067, PLB 701 (2011) 646

- **NEW** : $n \geq 7$ \rightarrow arXiv/1111.5821, PLB 707 (2012) 375

Outline

- [7] 1991 Davydychev, . . . *Reducing Feynman diagrams to scalar integrals*
- [8] 1996 Tarasov, *Connection [of] Feynman integrals [with] different . . . space-time dimensions*
- [9] 1999 Fleischer et al., *Algebraic reduction of one-loop Feynman graph amplitudes*

- 1 Introduction
- 2 Recursions
- 3 Simplifying
- 4 Numbers: D_{1111}
- 5 PJFry
- 6 External momenta
- 7 $N \geq 6$
- 8 Summary

References:

- [10] 2010 Diakonidis et al., PLB 683, . . . *recursive reduction of tensor Feynman integrals*
- [6] 2011 Fleischer, T.R., PRD 83, *Complete . . . reduction of . . . tensor Feynman integrals*
- [11] 2011 Fleischer, T.R., PLB 701, . . . *contracted tensor Feynman integrals*
- Aug. 2011: V. Yundin, PhD thesis [with PJFry code]
- [12] 2012 Fleischer, T.R., PLB 707, . . . *tensor Feynman integrals with rank ≥ 6*

Notations: Gram and modified Cayley determinant, signed minors [Melrose:1965]

Gram determinant G_n :

$$G_n = |2q_i q_j|, i, j = 1, \dots, n \quad (1)$$

Modified Cayley determinant $()_N$ of a diagram with N internal lines and chords q_j ; for a choice $q_n = 0$, both determinants are related:

$$()_N \equiv \begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & Y_{11} & Y_{12} & \dots & Y_{1N} \\ 1 & Y_{12} & Y_{22} & \dots & Y_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & Y_{1N} & Y_{2N} & \dots & Y_{NN} \end{vmatrix} = -G_{N-1}, \quad (2)$$

where $D_i = (k - q_i)^2 - m_i^2$ [with $q_i = \text{chord}$], and the matrix elements

$$Y_{ij} = -(q_i - q_j)^2 + m_i^2 + m_j^2, \quad (i, j = 1 \dots N) \quad (3)$$

⇒ The Gram determinant $()_N$ does not depend on the masses.

Notations: signed minors [Melrose:1965]

signed minors of $()_N$ are constructed by deleting m rows and m columns from $()_N$, and multiplying with a sign factor:

$$\begin{aligned} \left(\begin{array}{cccc} j_1 & j_2 & \cdots & j_m \\ k_1 & k_2 & \cdots & k_m \end{array} \right)_N &\equiv \\ &\equiv (-1)^{\sum_l (j_l + k_l)} \operatorname{sgn}_{\{j\}} \operatorname{sgn}_{\{k\}} \left| \begin{array}{c} \text{rows } j_1 \cdots j_m \text{ deleted} \\ \text{columns } k_1 \cdots k_m \text{ deleted} \end{array} \right| \end{aligned} \quad (4)$$

where $\operatorname{sgn}_{\{j\}}$ and $\operatorname{sgn}_{\{k\}}$ are the signs of permutations that sort the deleted rows $j_1 \cdots j_m$ and columns $k_1 \cdots k_m$ into ascending order.

Example:

$$\left(\begin{array}{c} 0 \\ 0 \end{array} \right)_N \equiv \left| \begin{array}{cccc} Y_{11} & Y_{12} & \cdots & Y_{1N} \\ Y_{12} & Y_{22} & \cdots & Y_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{1N} & Y_{2N} & \cdots & Y_{NN} \end{array} \right|, \quad (5)$$

Example: Getting a 4-point function from a six-point function I

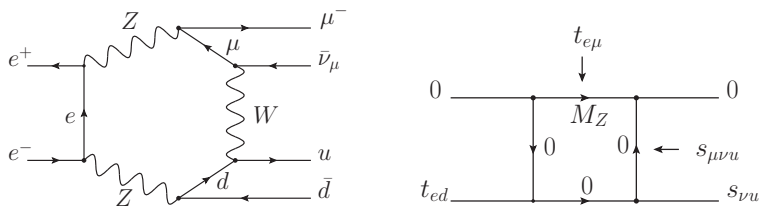


Figure: A six-point topology (a) leading to four-point functions (b) with realistically vanishing Gram determinants.

Example: Getting a 4-point function from a six-point function II

The example is taken from [13].

The corresponding 4-point tensor integrals are, in LoopTools [5, 14] notation:

$$D0i(\text{id}, 0, 0, s_{\bar{\nu}U}, t_{ed}, t_{\bar{e}\mu}, s_{\mu\bar{\nu}U}, 0, M_Z^2, 0, 0). \quad (6)$$

The Gram determinant is:

$$()_4 = -2t_{\bar{e}\mu}[s_{\mu\bar{\nu}U}^2 + s_{\bar{\nu}U}t_{ed} - s_{\mu\bar{\nu}U}(s_{\bar{\nu}U} + t_{ed} - t_{\bar{e}\mu})], \quad (7)$$

It vanishes if:

$$t_{ed} \rightarrow t_{ed,\text{crit}} = \frac{s_{\mu\bar{\nu}U}(s_{\mu\bar{\nu}U} - s_{\bar{\nu}U} + t_{\bar{e}\mu})}{s_{\mu\bar{\nu}U} - s_{\bar{\nu}U}}. \quad (8)$$

In terms of a dimensionless scaling parameter x ,

$$t_{ed} = (1 + x)t_{ed,\text{crit}}, \quad (9)$$

Example: Getting a 4-point function from a six-point function III

the Gram determinant becomes:

$$(\)_4 = 2 \times s_{\mu\bar{\nu}U} t_{\bar{\theta}\mu} (s_{\mu\bar{\nu}U} - s_{\bar{\nu}U} + t_{\bar{\theta}\mu}). \quad (10)$$

We will also need the modified Cayley determinant:

$$\begin{aligned} \begin{pmatrix} 0 \\ 0 \end{pmatrix}_4 &= \begin{pmatrix} 2M_Z^2 & M_Z^2 & M_Z^2 - s_{\mu\bar{\nu}U} & M_Z^2 \\ M_Z^2 & 0 & -s_{\bar{\nu}U} & M_Z^2 \\ M_Z^2 - s_{\mu\bar{\nu}U} & -s_{\bar{\nu}U} & 0 & -t_{ed} \\ M_Z^2 & -t_{\bar{\theta}\mu} & -t_{ed} & 0 \end{pmatrix} \\ &= s_{\mu\bar{\nu}U}^2 t_{\bar{\theta}\mu}^2 + 2 M_Z^2 t_{\bar{\theta}\mu} [-2s_{\bar{\nu}U} t_{ed} + s_{\mu\bar{\nu}U} (s_{\bar{\nu}U} + t_{ed} - t_{\bar{\theta}\mu})] \\ &\quad + M_Z^4 (s_{\bar{\nu}U}^2 + (t_{ed} - t_{\bar{\theta}\mu})^2 - 2s_{\bar{\nu}U} (t_{ed} + t_{\bar{\theta}\mu})). \end{aligned}$$

Dimensional shifts and recurrence relations for pentagons (I)

Following [Davydychev:1991 [7]]

Replace tensors by scalar integrals in higher dimensions:

Example $R = 3$:

$$\begin{aligned}
 I_5^{\mu\nu\lambda} &= \int \frac{d^{4-2\epsilon}k}{i\pi^{d/2}} \prod_{r=1}^5 c_r^{-1} k^\mu k^\nu k^\lambda & (11) \\
 &= - \sum_{i,j,k=1}^4 q_i^\mu q_j^\nu q_k^\lambda n_{ijk} I_{5,ijk}^{[d+]} + \frac{1}{2} \sum_{i=1}^{n-1} (g^{\mu\nu} q_i^\lambda + g^{\mu\lambda} q_i^\nu + g^{\nu\lambda} q_i^\mu) I_{5,i}^{[d+]}{}^2,
 \end{aligned}$$

and $n_{ijk} = (1 + \delta_{ij})(1 + \delta_{ik} + \delta_{jk})$.

$$[d+]^l = 4 - 2\epsilon + 2l$$

$I_{5,i}^{[d+]}{}^2$ – scratch the line i from $I_5^{[d+]}{}^2$.

Dimensional shifts and recurrence relations for pentagons (II)

'Naive', direct approach – just perform dimensional recurrences

Following [Tarasov:1996, Fleischer:1999 [8, 9]]

apply **recurrence relations**, relating scalar integrals of different dimensions, in order to get rid of the dimensionalities $[d+]^l = 4 - 2\epsilon + 2l$:

$$\nu_j \mathbf{j}^+ I_5^{[d+]} = \frac{1}{\binom{0}{5}} \left[-\binom{j}{0}_5 + \sum_{k=1}^5 \binom{j}{k}_5 \mathbf{k}^- \right] I_5 \quad (12)$$

$$\left(d - \sum_{i=1}^5 \nu_i + 1 \right) I_5^{[d+]} = \frac{1}{\binom{0}{5}} \left[\binom{0}{0}_5 - \sum_{k=1}^5 \binom{0}{k}_5 \mathbf{k}^- \right] I_5, \quad (13)$$

also:

$$\nu_j \mathbf{j}^+ I_5 = \frac{1}{\binom{0}{5}} \sum_{k=1}^5 \binom{0j}{0k}_5 \left[d - \sum_{i=1}^5 \nu_i (\mathbf{k}^- \mathbf{i}^+ + 1) \right] I_5 \quad (14)$$

where the operators $\mathbf{i}^\pm, \mathbf{j}^\pm, \mathbf{k}^\pm$ act by shifting the indices ν_i, ν_j, ν_k by ± 1 .

The result of simplifying manipulations

... and collecting all contributions, our final result for e.g. the tensor of rank $R = 3$ can be written as follows:

$$I_5^{\mu\nu\lambda} = \sum_{i,j,k=1}^4 q_i^\mu q_j^\nu q_k^\lambda E_{ijk} + \sum_{k=1}^4 g^{[\mu\nu} q_k^{\lambda]} E_{00k}, \quad (15)$$

with:

$$E_{00j} = \sum_{s=1}^5 \frac{1}{\binom{0}{0}_5} \left[\frac{1}{2} \binom{0s}{0j}_5 I_4^{[d+],s} - \frac{d-1}{3} \binom{s}{j}_5 I_4^{[d+]^2,s} \right], \quad (16)$$

$$E_{ijk} = - \sum_{s=1}^5 \frac{1}{\binom{0}{0}_5} \left\{ \left[\binom{0j}{sk}_5 I_{4,i}^{[d+]^2,s} + (i \leftrightarrow j) \right] + \binom{0s}{0k}_5 \nu_{ij} I_{4,ij}^{[d+]^2,s} \right\}. \quad (17)$$

✓ no scalar 5-point integrals in higher dimensions

✓ no inverse Gram det. $\binom{0}{0}_5$

We have yet:

† scalar 4-point integrals in higher dimensions: $I_{4,ij}^{[d+]^2,s}$ etc.

† inverse Gram det. $\binom{0}{0}_5 \equiv \binom{0}{0}_4$

Reduce $I_{4,ij\dots}^{[d+],s}$ to $I_4^{[d+],s}$ plus simpler objects I

By nontrivial manipulations we get e.g.:

$$I_{4,i}^{[d+],s} = \frac{1}{\binom{0s}{0s}_5} \left[-\binom{0s}{is}_5 (d-3) I_4^{[d+],s} + \sum_{t=1}^5 \binom{0st}{0si}_5 I_3^{st} \right] \quad (18)$$

$$\begin{aligned} \nu_{ij} I_{4,ij}^{[d+]} = & \frac{\binom{0}{i}_4 \binom{0}{j}_4}{\binom{0}{0}_4 \binom{0}{0}_4} (d-2)(d-1) I_4^{[d+]} + \frac{\binom{0i}{0j}_4}{\binom{0}{0}_4} I_4^{[d+]} \\ & - \frac{\binom{0}{j}_4}{\binom{0}{0}_4} \frac{d-2}{\binom{0}{0}_4} \sum_{t=1}^4 \binom{0t}{0i}_4 I_3^{[d+],t} + \frac{1}{\binom{0}{0}_4} \sum_{t=1}^4 \binom{0t}{0j}_4 I_{3,i}^{[d+],t} \quad (19) \end{aligned}$$

These equations are free of inverse Gram determinants $(\)_4$.

But they contain yet the generic 4-point and (partly indexed) 3-point functions in higher dimensions, $I_4^{[d+],s}$, $I_3^{[d+],t}$, etc.

Last step: evaluate the $I_4^{[d+],s}$, $I_3^{[d+],t}$, etc. |

Several strategies are now possible:

- Just evaluate them **analytically** in $d + 2l - 2\epsilon$ dimensions – if you may do that
- Just evaluate them **numerically** in $d + 2l - 2\epsilon$ dimensions
- **Reduce** them further by recurrences – buy the towers of $1/(\)_4 \rightarrow$ apply (13)
- Make a **small Gram determinant expansion** \rightarrow apply (13) another way round

Last two items are done here.

Reduction of scalars I_4^D to the generic dimension $\rightarrow I_4^d = D_0, I_3^d = C_0$ |

Non-small 4-point Gram determinants:

Direct, iterative use of (13) yields e.g.:

$$I_4^{[d+]'l} = \left[\frac{\binom{0}{0}_4}{\binom{}{0}_4} I_4^{[d+]'l-1} - \sum_{t=1}^4 \frac{\binom{t}{0}_4}{\binom{}{0}_4} I_3^{[d+]'l-1,t} \right] \frac{1}{d+2l-5} \quad (20)$$

$$I_3^{[d+]'l,t} = \left[\frac{\binom{0t}{0t}_4}{\binom{}{t}_4} I_3^{[d+]'l-1,t} - \sum_{u=1, u \neq t}^4 \frac{\binom{ut}{0t}_4}{\binom{}{t}_4} I_2^{[d+]'l-1,tu} \right] \frac{1}{d+2l-4} \quad (21)$$

And we are done.

This works fine if $\binom{}{0}_4$ is not small [and also the $\binom{t}{t}_4$].

Make a small Gram expansion I

Again use (13):

$$()_4 (d - \sum_{i=1}^4 \nu_i + 1) I_4^{[d+1]} = \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}_4 I_4 - \sum_{k=1}^4 \begin{pmatrix} 0 \\ k \end{pmatrix}_4 I_3^k \right]$$

If $()_4 = 0$, then it follows ($n = 4$):

$$I_n^D = \sum_k \frac{\begin{pmatrix} 0 \\ k \end{pmatrix}_n}{\begin{pmatrix} 0 \\ 0 \end{pmatrix}_n} I_{n-1}^{D,k} \quad (22)$$

If $()_4 \ll 1$, re-write (13), as follows:

$$I_n^D = \sum_k \frac{\begin{pmatrix} 0 \\ k \end{pmatrix}_n}{\begin{pmatrix} 0 \\ 0 \end{pmatrix}_n} I_{n-1}^{D,k} - \frac{()_n}{\begin{pmatrix} 0 \\ 0 \end{pmatrix}_n} \left[(D+1) - \sum_i \nu_i \right] I_n^{D+2}. \quad (23)$$

Effectively we may evaluate I_n^D in terms of simpler functions $I_{n-1}^{D,k}$ with a small correction depending on I_n^{D+2} .

We may go a step further, and insert into (23) for I_n^{D+2} the rhs. of (22), taken now at $D' = D + 2$:

$$\begin{aligned}
 I_n^D &= \sum_k^n \frac{\binom{0}{k}_n}{\binom{0}{0}_n} I_{n-1}^{D,k} \\
 &\quad - \frac{\binom{0}{0}_n}{\binom{0}{0}_n} \left[(D+1) - \sum_i^n \nu_i \right] \\
 &\quad \times \left[\sum_k^n \frac{\binom{0}{k}_n}{\binom{0}{0}_n} I_{n-1}^{D+2,k} - \frac{\binom{0}{0}_n}{\binom{0}{0}_n} \left[(D+3) - \sum_i^n \nu_i \right] I_n^{D+4} \right].
 \end{aligned}$$

The terms proportional to $[(0)_n / \binom{0}{0}_n]^a$, $a = 0, 1$ may be evaluated at the correct kinematics. They depend on three-point functions, and their reduction by normal recurrences will not introduce the unwanted powers of $1/(0)_4$. The last term, suppressed by the factor $[(0)_n / \binom{0}{0}_n]^2$, depends on I_n^{D+4} . It may either be taken approximately at $(0)_n = 0$, where it can also be represented by 3-point functions (and their reductions), or it may be evaluated more correctly by another iteration based on (22).

And so on and so on ...

In the numerical example – next section – we worked out up to 10 stable iterations.

Numbers: D_{1111} I

Following Davydychev, [7], one gets

$$I_4^{\mu\nu\lambda} = \int^d \frac{k^\mu k^\nu k^\lambda}{\prod_{r=1}^n c_r} = - \sum_{i,j,k=1}^n q_i^\mu q_j^\nu q_k^\lambda \nu_{ijk} I_{n,ijk}^{[d+]} + \frac{1}{2} \sum_{i=1}^n g^{[\mu\nu} q_i^{\lambda]} I_{n,i}^{[d+]} \quad (24)$$

We identify the tensor coefficients $D_{11\dots}$ a la LoopTools, e.g.:

$$D_{111} = I_{4,222}^{[d+]} \quad (25)$$

Similarly:

$$D_{1111} = I_{4,2222}^{[d+]} \quad (26)$$

Rank $R = 4$ tensor D_{1111} – Numerics with dimensional recurrences

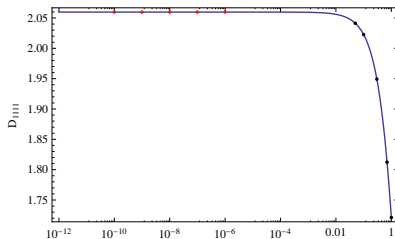
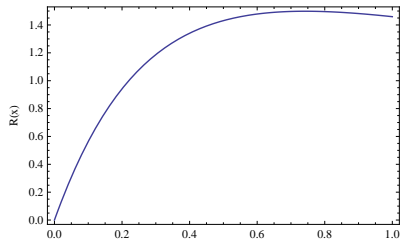
From (23) we see that a “small Gram determinant” expansion will be useful when the following dimensionless parameter becomes small:

$$R = \frac{\binom{4}{0}}{\binom{0}{0}} \times s, \quad (27)$$

where s is a typical scale of the process, e.g. we will choose $s = s_{\mu\bar{\nu}U}$.
Following [13], we further choose:

$$\begin{aligned} s_{\mu\bar{\nu}U} &= 2 \times 10^4 \text{ GeV}^2, \\ s_{\bar{\nu}U} &= 1 \times 10^4 \text{ GeV}^2, \\ t_{\bar{e}\mu} &= -4 \times 10^4 \text{ GeV}^2, \end{aligned}$$

and get $t_{ed,\text{crit}} = -6 \times 10^4 \text{ GeV}^2$. For $x=1$, the Gram determinant becomes $\binom{4}{0} = 4.8 \times 10^{13} \text{ GeV}^3$.
The small expansion parameter $R(x)$ and D_{1111} are shown in figure 2.



Small Gram expansion and Pade approximation I

[Fleischer,TR: PRD 2011 [6]]

Tables have been taken from there.

They were shown first at [QCD@LHC@Trento2010](#)

The use of appropriate **Pade approximations** is explained there.
Convergence in the small Gram determinant region is considerably improved.

x	$\Re e D_{1111}$	$\Im m D_{1111}$
0. [exp 0,0]	2.05969289730 E-10	1.55594910118 E-10
10^{-8} [exp x,2] [exp 0,2]	2.05969289342 E-10 2.05969289349 E-10	1.55594909187 E-10 1.55594909187 E-10
10^{-4} [exp x,5] [exp 0,5]	2.05965609497 E-10 2.05965609495 E-10	1.55585605343 E-10 1.55585605343 E-10
0.001 [exp 0,6] [exp x,6]	2.05932484380 E-10 2.05932484381 E-10	1.55501912433 E-10 1.55501912433 E-10
$I_{4,2222}^{[d+]}^4$ D_{1111}	2.02292295240 E-10 2.01707671668 E-10	1.54974785467 E-10 1.62587142251 E-10
0.005 [exp 0,6] [pade 0,3] [exp x,6] [pade x,3]	2.05786054801 E-10 2.05785198947 E-10 2.05786364440 E-10 2.05785199805 E-10	1.55131031024 E-10 1.55131031003 E-10 1.55131031024 E-10 1.55131030706 E-10
$I_{4,2222}^{[d+]}^4$ D_{1111}	2.05778894114 E-10 2.05779811490 E-10	1.55135794453 E-10 1.55136343923 E-10
0.01 [exp 0,6] [pade 0,3] [exp 0,10] [pade 0,5] [exp x,10] [pade x,5]	2.05703298143 E-10 2.05600940065 E-10 2.05600964693 E-10 2.05600955381 E-10 2.05600963675 E-10 2.05600955381 E-10	1.54669910676 E-10 1.54669907784 E-10 1.54669910676 E-10 1.54669910676 E-10 1.54669910676 E-10 1.54669910676 E-10
$I_{4,2222}^{[d+]}^4$ D_{1111}	2.05600013702 E-10 2.05600239280 E-10	1.54670651917 E-10 1.54670771210 E-10

Table: Numerical values for the tensor coefficient D_{1111} . Values marked by D_{1111} are evaluated with LoopTools, the $I_{4,2222}^{[d+]}^4$ corresponds to (29) The labels [exp 0,2n] and [pade 0,n] denote iteration 2n and Pade approximant $[n, n]$ when the small Gram determinant expansion starts at $x = 0$, and [exp x,2n] and [pade x,n] are the corresponding numbers for an expansion starting at x .

x	$\Re D_{1111}$	$\Im D_{1111}$
0.01 [exp 0,6] [pade 0,3] [exp 0,10] [pade 0,5] [exp x,10] [pade x,5] $I_{4,2222}^{[d+]}^4$ D_{1111}	2.05703298143 E-10	1.54669910676 E-10
	2.05600940065 E-10	1.54669907784 E-10
	2.05600964693 E-10	1.54669910676 E-10
	2.05600955381 E-10	1.54669910676 E-10
	2.05600963675 E-10	1.54669910676 E-10
	2.05600955381 E-10	1.54669910676 E-10
	2.05600013702 E-10	1.54670651917 E-10
	2.05600239280 E-10	1.54670771210 E-10
0.05 [exp 0,6] [pade 0,3] [exp 0,20] [pade 0,10] [exp x,20] [pade x,10] $I_{4,2222}^{[d+]}^4$ D_{1111}	4.83822963052 E-09	1.51077429118 E-10
	2.01518061131 E-10	1.50591643209 E-10
	2.04218962072 E-10	1.51077424143 E-10
	2.04122727654 E-10	1.51077424149 E-10
	2.04190274030 E-10	1.51077424143 E-10
	2.04122727971 E-10	1.51077423985 E-10
	2.04122726387 E-10	1.51077422901 E-10
	2.04122726601 E-10	1.51077423320 E-10
0.1 [exp 0,26] [pade 0,13] [exp x,26] [pade x,13] $I_{4,2222}^{[d+]}^4$ D_{1111}	2.20215264409 E-08	1.46815247004 E-10
	2.01749674352 E-10	1.46681287362 E-10
	2.08190721550 E-08	1.46815247004 E-10
	2.03995221326 E-10	1.46785977364 E-10
	2.02269485177 E-10	1.46815247061 E-10
	2.02269485217 E-10	1.46815247051 E-10
1. $I_{4,2222}^{[d+]}^4$ D_{1111}	1.72115440143 E-10	9.74550747662 E-11
	1.72115440148 E-10	9.74550747662 E-11

Table: Numerical values for the tensor coefficient D_{1111} . Values marked by D_{1111} are evaluated with LoopTools, the $I_{4,2222}^{[d+]}^4$ corresponds to (29) The labels [exp 0,2n] and [pade 0,n] denote iteration 2n and Pade approximant $[n, n]$ when the small Gram determinant expansion starts at $x = 0$, and [exp x,2n] and [pade x,n] are

PJFry - an open source c++ program by V. Yundin I

PJFry 1.0.0 - one loop tensor integral library

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- More information and the latest source code:
project page: <https://github.com/Vayu/PJFry/>
- → how to install
- → how to use
- → samples
- See also:
V. Yundin's talk at last SFB meeting "One loop tensor reduction program PJFRY"
- Yundin's PhD thesis, defended February 2012 at Humboldt University

PJFry — numerical package [from V.Y. Valencia 2011] |

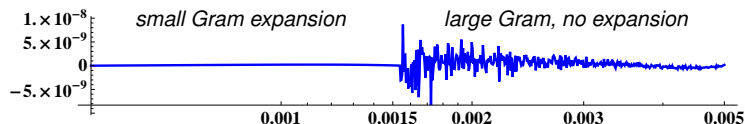
Numerical implementation of described algorithms:

C++ package **PJFry** by **V. Yundin** [see [project webpage](#)]

- Reduction of **5-point** 1-loop tensor integrals up to **rank 5**
- No limitations on internal/external masses combinations
- Small Gram determinants treatment by expansion
- Interfaces for C, C++, FORTRAN and MATHEMATICA

Example:

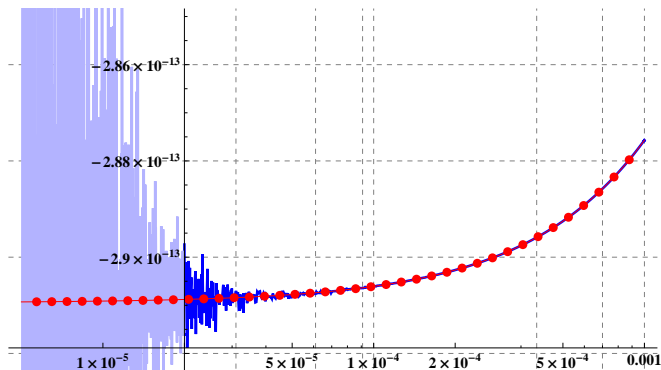
Relative accuracy of E_{3333} coef. around small Gram4 region



PJFry — small Gram region example [from V.Y. Valencia 2011] |

Example: E_{3333} coefficient in small Gram region ($x = 0$)

Comparison of **Regular** and **Expansion** formulae:



$x=0: E_{3333}(0, 0, -6 \times 10^4, 0, 0, 10^4, -3.5 \times 10^4, 2 \times 10^4, -4 \times 10^4, 1.5 \times 10^4, 0, 6550, 0, 0, 8315)$

Contractions with external momenta (or with CHORDS) I

[Fleischer,TR: PLB 2011 [11]]

After having tensor reductions with basis functions I_n^D , which are independent of the indices i, j, k, \dots , one may use **contractions with external momenta** in order to perform all the sums over i, j, k, \dots .

This leads to a **significant simplification and shortening** of calculations.

Reminder:

One option was to avoid the appearance of inverse Gram determinants $1/(\)_5$. For rank $R = 5$, e.g.,

$$I_5^{\mu\nu\lambda\rho\sigma} = \sum_{s=1}^5 \left[\sum_{i,j,k,l,m=1}^5 q_i^\mu q_j^\nu q_k^\lambda q_l^\rho q_m^\sigma E_{ijklm}^s + \sum_{i,j,k=1}^5 g^{[\mu\nu} q_i^\lambda q_j^\rho q_k^{\sigma]} E_{00ijk}^s + \sum_{i=1}^5 g^{[\mu\nu} g^{\lambda\rho} q_i^{\sigma]} E_{0000i}^s \right] \quad (28)$$

Contractions with external momenta (or with CHORDS) I

The tensor coefficients are expressed in terms of integrals $I_{4,i\dots}^{[d+],s}$, e.g.:

$$E_{ijklm}^S = -\frac{1}{\binom{0}{0}_5} \left\{ \left[\binom{0l}{sm}_5 n_{ijk} I_{4,ijk}^{[d+],s} + (i \leftrightarrow l) + (j \leftrightarrow l) + (k \leftrightarrow l) \right] + \binom{0s}{0m}_5 n_{ijkl} I_{4,ijkl}^{[d+],s} \right\}.$$

Now, in a next step, one may avoid the appearance of inverse sub-Gram determinants $\binom{0}{0}_4$.

The complete dependence on the indices i of the tensor coefficients is contained now in the pre-factors with signed minors. One can say that **the indices decouple from the integrals**.

As an example, we reproduce the 4-point part of

$$\begin{aligned} n_{ijkl} I_{4,ijkl}^{[d+],s} &= \frac{\binom{0}{i}}{\binom{0}{0}} \frac{\binom{0}{j}}{\binom{0}{0}} \frac{\binom{0}{k}}{\binom{0}{0}} \frac{\binom{0}{l}}{\binom{0}{0}} d(d+1)(d+2)(d+3) I_4^{[d+],4} \\ &+ \frac{\binom{0i}{0j} \binom{0}{k} \binom{0}{l} + \binom{0i}{0k} \binom{0}{j} \binom{0}{l} + \binom{0j}{0k} \binom{0}{i} \binom{0}{l} + \binom{0i}{0l} \binom{0}{j} \binom{0}{k} + \binom{0j}{0l} \binom{0}{i} \binom{0}{k} + \binom{0k}{0l} \binom{0}{i} \binom{0}{j}}{\binom{0}{0}^3} d(d+1) I_4^{[d+],3} \\ &+ \frac{\binom{0i}{0l} \binom{0j}{0k} + \binom{0j}{0l} \binom{0i}{0k} + \binom{0k}{0l} \binom{0i}{0j}}{\binom{0}{0}^2} I_4^{[d+],2} + \dots [\text{simpler terms}] \end{aligned} \quad (29)$$

Contractions with external momenta (or with CHORDS) II

In (29), one has to understand the 4-point integrals to carry the corresponding index s and the signed minors are

$$\binom{0}{k} \rightarrow \binom{0s}{ks}_5 \text{ etc.}$$

Contractions with external momenta (or with CHORDS) I

A chord is the momentum shift of an internal line due to external momenta, $D_i = (k - q_i)^2 - m_i^2 + i\epsilon$, and $q_i = (p_1 + p_2 + \dots + p_i)$, with $q_n = 0$.

The tensor 5-point integral of rank $R = 1$ yields, when contracted with a chord,

$$q_{a\mu} I_5^\mu = -\frac{1}{\binom{0}{0}_5} \sum_{s=1}^5 \left[\sum_{i=1}^4 (q_a \cdot q_i) \binom{0i}{0s}_5 \right] I_4^s. \quad (30)$$

In fact, the sum over i may be performed explicitly:

$$\Sigma_a^{1,s} \equiv \sum_{i=1}^4 (q_a \cdot q_i) \binom{0s}{0i}_5 = +\frac{1}{2} \left\{ \binom{s}{0}_5 (Y_{a5} - Y_{55}) + \binom{0}{0}_5 (\delta_{as} - \delta_{5s}) \right\},$$

Contractions with external momenta I

We get immediately

$$q_{a\mu} l_5^\mu = - \frac{1}{\binom{0}{0}_5} \sum_{s=1}^5 \Sigma_a^{1,s} l_4^s. \quad (31)$$

Contractions with external momenta I

The tensor 5-point integral of rank $R = 2$

$$I_5^{\mu\nu} = \sum_{i,j=1}^4 q_i^\mu q_j^\nu E_{ij} + g^{\mu\nu} E_{00}, \quad (32)$$

has the following tensor coefficients free of $1/(\)_5$:

$$E_{00} = - \sum_{s=1}^5 \frac{1}{2} \frac{1}{\binom{0}{0}_5} \binom{s}{0}_5 I_4^{[d+],s}, \quad (33)$$

$$E_{ij} = \sum_{s=1}^5 \frac{1}{\binom{0}{0}_5} \left[\binom{0i}{sj}_5 I_4^{[d+],s} + \binom{0s}{0j}_5 I_{4,i}^{[d+],s} \right]. \quad (34)$$

Contractions with external momenta I

Equation (32) yields for the contractions with chords:

$$q_{a\mu} q_{b\nu} l_5^{\mu\nu} = \sum_{i,j=1}^4 (q_a \cdot q_i)(q_b \cdot q_j) E_{ij} + (q_a \cdot q_b) E_{00}. \quad (35)$$

and finally (35) simply reads

$$\begin{aligned} q_{a\mu} q_{b\nu} l_5^{\mu\nu} &= \frac{1}{4} \sum_{s=1}^5 \left\{ \frac{\binom{s}{0}_5}{\binom{0s}{0s}_5} (\delta_{ab} \delta_{as} + \delta_{5s}) + \frac{\binom{s}{s}_5}{\binom{0s}{0s}_5} [(\delta_{as} - \delta_{5s})(Y_{b5} - Y_{55}) \right. \\ &\quad \left. + (\delta_{bs} - \delta_{5s})(Y_{a5} - Y_{55}) + \frac{\binom{s}{0}_5}{\binom{0}{0}_5} (Y_{a5} - Y_{55})(Y_{b5} - Y_{55})] \right\} l_4^{[d+],s} \\ &\quad + \frac{1}{\binom{0}{0}_5} \sum_{s=1}^5 \frac{\sum_b^{1,s}}{\binom{0s}{0s}_5} \sum_{t=1}^5 \Sigma_a^{2,st} l_3^{st}, \end{aligned}$$

Contractions with external momenta I

with

$$\begin{aligned}\Sigma_a^{2,st} &\equiv \sum_{i=1}^4 (q_a \cdot q_i) \begin{pmatrix} 0st \\ 0si \end{pmatrix}_5 \\ &= \frac{1}{2} (1 - \delta_{st}) \left\{ \begin{pmatrix} ts \\ 0s \end{pmatrix}_5 (Y_{a5} - Y_{55}) + \begin{pmatrix} 0s \\ 0s \end{pmatrix}_5 (\delta_{at} - \delta_{5t}) - \begin{pmatrix} 0s \\ 0t \end{pmatrix}_5 (\delta_{as} - \delta_{5s}) \right\}\end{aligned}$$

This has been extended also to higher ranks.

We need at most double sums, e.g.:

$$\begin{aligned}\Sigma_{ab}^{2,s} &\equiv \sum_{i,j=1}^4 (q_a \cdot q_i)(q_b \cdot q_j) \begin{pmatrix} si \\ sj \end{pmatrix}_5 \\ &= \frac{1}{2} (q_a \cdot q_b) \begin{pmatrix} s \\ s \end{pmatrix}_5 - \frac{1}{4} ()_5 (\delta_{ab}\delta_{as} + \delta_{5s}), \quad (36)\end{aligned}$$

Contractions with external momenta I

Many of the **sums over signed minors, weighted with scalar products of chords** are given in PLB 2011 [\[\[11\]\]](#), and an almost complete list may be obtained on request from J. Fleischer, T.R.

Modifications for 7- and higher point functions I

For $N=6,7,8, \dots$, the Gram determinant vanishes, and also further determinants:

$$\binom{0}{n} = 0, n > 5$$

$$\text{but also } \binom{0}{k}_7 = 0$$

As a result, one has to reorganize the reductions, avoiding the $1/\binom{0}{n}$ completely.

This may be done, and one arrives at expressions with $I_6^{[d+]}$.

The problematic case is the integral $I_{6,i}^{[d+]}$ for which one can write

$$I_{6,i}^{[d+]} = \sum_{s=1, s \neq i}^7 \frac{\binom{R}{s}_6}{\binom{R}{0}_6} I_{5,i}^{[d+],s} + \frac{\binom{R}{i}_6}{\binom{R}{0}_6} I_6^{[d+]}$$

For $I_6^{[d+]}$ one finally needs 4-point functions up to the order ϵ in the basis.
A numerical alternative: See J. Fleischer, T. Riemann, the Radcor 2011 contribution

Modifications for 7- and higher point functions II

But:

We learned from Gudrun Heinrich et al. that there is a representation in terms just of ordinary 4-point functions:

Binoth, Guillet, Heinrich, hep-ph/0504267, hep-ph/9911342

$n > 6$: assumption in Bern,Dixon,Kosower, hep-ph/9306240

This we have worked out explicitly on our formalism in PLB 707 (2012) for $N = 7$ and $N = 8$.

Results for $N = 6, 7, 8, \dots$ I

In [16] we solve analytically the generalized recursions for $n \geq 6$, which were derived in [17, 18]:

$$I_n^{\mu_1 \mu_2 \dots \mu_R} = - \sum_{r=1}^n C_r^{\mu_1} (n) I_{n-1}^{\mu_2 \dots \mu_R, r}, \quad (37)$$

where in $I_{n-1}^{\mu_2 \dots \mu_R, r}$ the line r is scratched.

Assume a representation of the metric tensor in the form

$$\frac{1}{2} g^{\mu\nu} = \sum_{i,j=1}^{N-1} G_{ij} q_i^\mu q_j^\nu \quad (38)$$

is available.

Then, the vector

$$C_j^\mu = \sum_{i=1}^{N-1} G_{ij} q_i^\mu \quad (39)$$

Results for $N = 6, 7, 8, \dots$ II

is a solution of (37).

An additional requirement has to be fulfilled for this vector:

$$\sum_{j=1}^N C_j^\mu = 0, \quad (40)$$

The coefficients for 6-point functions are:

$$C_r^{s,\mu}(6) = \sum_{i=1}^5 \frac{1}{\binom{0}{s}_6} \binom{0r}{si}_6 q_i^{\mu_1}, \quad s = 0 \dots 6, \quad (41)$$

where the q_i are chords, and $\binom{0r}{si}_6$ etc. are signed minors with arbitrary s . For the 7-point and 8-point functions, we found several representations, among them

$$C_r^{st,\mu}(7) = \sum_{i=1}^6 \frac{1}{\binom{st}{st}_7} \binom{sti}{str}_7 q_i^\mu \quad (42)$$

Results for $N = 6, 7, 8, \dots$ III

and

$$C_r^{stu,\mu}(8) = \sum_{i=1}^7 \frac{1}{\binom{stu}{stu}_8} \binom{stui}{stur}_8 q_i^\mu \quad (43)$$

The upper indices s , t and u stand for the redundancy of the solutions and can be freely chosen.

Two 7-point examples for sums with external momenta:

The rank $R = 2, 3$ integrals become by contraction

$$q_{a,\mu} q_{b,\nu} I_7^{\mu\nu} = \sum_{r,t=1}^7 K^{ab,rt} I_5^{rt}, \quad (44)$$

$$q_{a,\mu} q_{b,\nu} q_{c,\lambda} I_7^{\mu\nu\lambda} = \sum_{r,t,u=1}^7 K^{abc,rtu} I_4^{rtu}, \quad (45)$$

Results for $N = 6, 7, 8, \dots$ IV

where I_5^{rt} and I_4^{rtu} are scalar 5- and 4-point functions, arising from the 7-point function by scratching lines r, t, \dots . In the general case, we have at this stage higher-dimensional integrals I_n^{d+2l} , $n = 2, \dots, 5$, to be further reduced following the known scheme, if needed. Here, the I_5^{rt} have to be expressed by 4-point functions.

The expansion coefficients are factorizing here,

$$K^{ab,rt} = K^{a,r} K^{b,rt}, \quad (46)$$

$$K^{abc,rtu} = -K^{a,r} K^{b,rt} K^{c,rtu}, \quad (47)$$

and the sums over signed minors have been performed analytically:

$$K^{a,r} = \frac{1}{2} (\delta_{ar} - \delta_{7r}), \quad (48)$$

$$K^{b,rt} = \sum_{j=1}^6 (q_b q_j) \frac{\binom{rst}{rsj}_7}{\binom{rs}{rs}_7} \equiv \frac{\sum_b^{1,stu}}{\binom{rs}{rs}_7} = \frac{1}{2} (\delta_{bt} - \delta_{7t}) - \frac{1}{2} \frac{\binom{rs}{ts}_7}{\binom{rs}{rs}_7} (\delta_{br} - \delta_{7r}) \quad (49)$$

Results for $N = 6, 7, 8, \dots V$

$$\begin{aligned}
 K^{a,stu} &= \sum_{i=1}^6 (q_a q_i) \begin{pmatrix} 0stu \\ 0sti \end{pmatrix}_7 \equiv \Sigma_a^{2,stu} & (50) \\
 &= \frac{1}{2} \left\{ \begin{pmatrix} stu \\ st0 \end{pmatrix}_7 (Y_{a7} - Y_{77}) \right. \\
 &\quad \left. + \begin{pmatrix} 0st \\ 0st \end{pmatrix}_7 (\delta_{au} - \delta_{7u}) - \begin{pmatrix} 0st \\ 0su \end{pmatrix}_7 (\delta_{at} - \delta_{7t}) - \begin{pmatrix} 0ts \\ 0tu \end{pmatrix}_7 (\delta_{as} - \delta_{7s}) \right\}
 \end{aligned}$$

with

$$Y_{jk} = -(q_j - q_k)^2 + m_j^2 + m_k^2. \quad (52)$$

Conventionally, $q_7 = 0$.

The sums may be found in eqns. (A.15) and (A.16) of [11]. The s is redundant and fulfils $s \neq r, b, 7$ in $K^{b,rt}$. In $K_0^{a,stu}$ it is $s, t, u = 1, \dots, 7$ with $s \neq u, t \neq u$.

Summary

- Systematic derivation of expressions which are explicitly **free of inverse Gram determinants** $(\)_5$ until pentagons of rank $R = 5$
- Proper **isolation of inverse Gram determinants of subdiagrams of the type** $\binom{S}{s}_4$, which cannot be completely avoided
- Numerical **C++ package PJFry** (V. Yundin, open source) for C, c++, Mathematica, Fortran
- **Perform multiple sums with signed minors and scalar products** after contractions with chords or external momenta
- **N=6,7,8** Analytical treatment worked out

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