Simple Feynman diagrams and simple sums

Tord Riemann DESY, Zeuthen, Germany

in cooperation with J. Gluza, Katowice et al.

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1. Introduction

2. Mellin-Barnes representations

3. Evaluation

4. Summary

This is not a review ...

... and it is also not a lecture

For many references and a more balanced list of references see some older lectures located at my homepage in Zeuthen:
http://www-zeuthen.desy.de/~riemann/
e.g.

"Feynman Integrals and Mellin-Barnes representations"
Lecture, DESY CAPP School, March 2011, Zeuthen, Germany
http://www-zeuthen.desy.de/~riemann/Talks/riemann-capp-2011.pdf
"Feynman Integrals - Mellin-Barnes representations - Sums"
Two lectures, Humboldt-Universitt, Berlin, June 2010
http://www-zeuthen.desy.de/~riemann/Talks/HUB-Berlin-SS10-MB-lectures.pdf
"Evaluation of Feynman diagrams with Mellin-Barnes representations"
Lecture at Helmholtz-JINR CALC School, July 2009, JINR, Dubna, Russia
http://www-zeuthen.desy.de/~riemann/Talks/calc2009-riemann.pdf
"Evaluation of Feynman Integrals: Advanced Methods"
Three lectures, RECAPP 2009, February 2009, Allahabad, India
http://www-zeuthen.desy.de/~riemann/Talks/riemann-recapp-09.pdf

Introduction – Feynman integrals I

The perturbative expansion in a small coupling constant may be treated with Feynman integrals.

The Feynman integrals may be visualized by Feynman diagrams.

Higher orders in perturbation theory lead to Feynman diagrams with more and more loops - closed lines, repesenting internal virtual particles.

Normally, the lowest perturbative order has no Feynman diagrams with loops and is then called Born approximation.
The higher orders of perturbation theory are also called "radiative corrections".

The one-loop corrections are, at least in principle, easy to calculate.

For non-abelian gauge field theories with spontaneous symmetry breaking - e.g. the Standard Model -
the basics for one-loop calculations with dimensional regularization have been systematically worked out more than 30 years ago.

Seminal articles are:

All the scalar Feynman integrals for 4-particle interactions ...

... are expressed in terms of a basis of
1-point functions – tadpoles
2-point functions – self-energies
3-point functions – vertices
4-point functions – box diagrams
't Hooft, Veltman, "Scalar One Loop Integrals", [1, Nucl.Phys. B153, 1978]
All the scalar integrals lead to Euler Dilogarithms or simpler functions.

The tensor Feynman integrals ...

... stem from fermion and/or boson propagators and from vector boson couplings,
and they may be traced back to the basis of scalar Feynman integrals, see
Passarino, Veltman, "One Loop Corrections for e^+e^- Annihilation Into $\mu^+\mu^-$ in the Weinberg Model", [2, Nucl.Phys. B160, 1979]
Basic idea:
Derive a system of algebraic equations for the tensor integrals and solve it.

Not solved in the old papers:

more external particles

case of more external particles, i.e. N-point functions with $N > 4$

kinematical singularities

treatment of kinematical singularities or of spurious singularities, artificially introduced by the algorithm

higher terms in ϵ

higher terms in ϵ, stemming from the so-called ϵ-expansion in $d = 4 - 2\epsilon$ due to the dimensional regularization

The $\epsilon^0 =$ constant terms are physical, singular terms have to compensate each other, and higher powers of ϵ interact with higher order corrections from perturbation theory.

Some later generalizations I

Look at 1-loop Feynman integrals, but we are interested in expressions fulfilled for ...

- n external particles, $n = 5, \cdots 8$ and higher, with proper treatment of singularities
- higher orders in the ϵ-expansion
- arbitrary kinematical situations, including massive particles

All this leads to Feynman integrals with **many different scales**

e.g. a **4-point function**, with 4 external momenta $p_i, i = 1, \cdots 4$ may depend in general on:

- **2 kinematical variables**, usually called $s \sim p_1 p_2$ and $t \sim p_1 p_3$, where $s + t + u = 0, u \sim p_1 p_4$, but due to $p_1 + p_2 + p_3 = -p_4$ the u is not independent
- **plus 4 internal masses** M_i
- **plus 4 external "virtualities"** $p_i^2 \equiv m_i^2$

Some later generalizations II

It would be wonderful to have an algorithm for **automatic evaluation** of ll the scalar integrals by infinite sums.

One question:
How to systematically evaluate the 1-loop Feynman integrals

Can this be extended to *n* loop?

Rest of this talk:

Use of Mellin-Barnes integrals

Explain one approach with use of Mellin-Barnes integrals, leading to infinite sums to be evaluated somehow by somebody.

Somehow ... somebody = question to people at RISC $\cdots$

Tensor integrals: A simple example

1-loop self-energy:

$$
\begin{aligned}
I_2^\mu &= \int \frac{d^d k}{i\pi^{d/2}} \frac{k^\mu}{[k^2 - M_1^2]\,[(k+p)^2 - M_2^2]} \\
&= p_\mu B_1
\end{aligned}
$$

Solve:

$$
\begin{aligned}
p_\mu I_2^\mu &= p^2 B_1(p, M_1, M_2) \\
&= \int \frac{d^d k}{i\pi^{d/2}} \frac{pk}{[k^2 - M_1^2]\,[(k+p)^2 - M_2^2]} = \int \frac{d^d k}{i\pi^{d/2}} \frac{pk}{D_1\, D_2} \\
&= \int \frac{d^d k}{i\pi^{d/2}} \left[\frac{D_2 - (p^2 - M_2^2 - M_1^2) - D_1}{D_1\, D_2} \right], \\
B_1(p, M_1, M_2) &= \frac{1}{2p^2} \left[A_0(M_1) - A_0(M_2) - (p^2 - M_2^2 - M_1^2) B_0(p, M_1, M_2) \right]
\end{aligned}
$$

A tensor Feynman integral is expressed in terms of scalar Feynman integrals.

Feynman integrals: The Bhabha box diagram

In the rest of the talk we use as example, in $d = 4 - 2\epsilon$:

$$e^-(p_1) + e^+(p_2) \quad \rightarrow \quad e^-(p_3) + e^+(p_4)$$

with kinematics

$$\begin{aligned} p_1 + p_2 &= p_3 + p_4 \\ p_1^2 = p_2^2 = p_3^2 = p_4^2 &= m^2 \\ (p_1 + p_2)^2 &= s \\ (p_1 - p_3)^2 &= t \end{aligned}$$

Bhabha box, with photon exchange in s channel:

$$I_4 = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{[k^2 - m^2 + i\epsilon] \times [(k+p_1)^2] + i\epsilon]} \times \frac{1}{[(k+p_1+p_2)^2 - m^2 + i\epsilon] \times [(k+p_3)^2 + i\epsilon]}$$

The Feynman diagram I

$$I_4 = Box(s,t) =$$

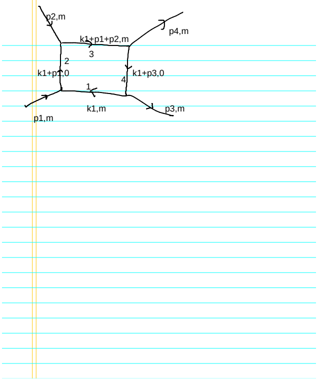


Figure: The topology of the massive QED box diagram (aka B4l2m).

There are many ways to evaluate the ϵ-expansion of I_4:

$$I_4 = \frac{1}{\epsilon} B_{-1}(s,t) + B_0(s,t) + \epsilon B_1(s,t) + \cdots$$

Already for $B_1(s,t)$, the direct integration of the Feynman parameter integral becomes difficult.

System of differential equations

One way, not further exemplified here, but very efficient:
Derive a system of differential equations for the original k-momentum integral and solve it iteratively
This leads to a solution in terms of e.g.
Generalized Harmonic Polylogarithms
and allows to determine arbitrary high terms $B_n(s,t)$.
See: Fleischer, Gluza, Lorca, Riemann, [3, EPJC48, 2006] and references therein, especially Bonciani et al., [4, NPB681, 2004].

The general analytical result in d dimensions ...

... was given already in 2003 by Fleischer, Jegerlehner, Tarasov in [5, NPB672, 2003] **in terms of two Appell hypergeometric functions F_1 and F_2**
and a Kampe de Feriet function.

The first ϵ-terms of the QED box diagram

$$I_4 = \frac{1}{\epsilon} \; B_{-1}(s,t) + B_0(s,t) + \epsilon \; B_1(s,t) + \cdots$$

with

$$B_{-1} = \frac{2xy}{(1-x^2)(1-y)^2} \; H(0,x)$$

where $H(0,x) \equiv \ln(x)$ **has been introduced, and**

$$x = \frac{\sqrt{1-4/s}-1}{\sqrt{1-4/s}+1}, \quad y = \frac{\sqrt{1-4/t}-1}{\sqrt{1-4/t}+1},$$

Further,

$$B_0 = \frac{2}{st\sqrt{1-4/s}} \; H(0,x) \; \Big[H(0,y) + 2H(1,y) \Big]$$

Finally, here the ϵ-term:

$$
\begin{aligned}
B_1 \quad = \quad & \frac{-2}{st\sqrt{1-4/s}} \Big\{ G(-\frac{1}{y},0,0,x) + G(-y,0,0,x) \\
& -2\Big(G(-\frac{1}{y},-1,0,x) + G(-y,-1,0,x)\Big) \\
& -\Big(G(-\frac{1}{y},0,x) + G(-y,0,x) - 2H(-1,0,x)\Big)[H(0,y)+2H(1,y)] \\
& -\Big(G(-\frac{1}{y},x) - G(-y,x) + H(0,x)\Big)[H(0,0,y)+2H(0,1,y)] \\
& -\Big(5G(-\frac{1}{y},x) - 3G(-y,x) - \frac{1}{2}H(0,x) - 2H(-1,x) - 4H(0,y)\Big)\zeta_2 \\
& -2\Big(H(1,y)H(0,0,y) - H(0,y)H(0,1,y)\Big) \\
& -2\Big(H(-1,0,0,x) - 2H(-1,-1,0,x)\Big) - 2H(0,x)[H(1,0,y)+2H(1,1,y)] \\
& +H(0,0,0,y) + 2H(1,0,0,y) - 2\zeta_3 \ \Big\}.
\end{aligned}
$$

The functions G are generalized harmonic polylogarithms, see Gehrmann and Remiddi, [6, NPB601, 2001], and [4, NPB681, 2004].

The 2-dimensional Harmonic Polylogarithms in Bonciani et al., [4, NPB681, 2004], are defined as the set of functions generated by the repeated integrations

$$\int_0^x dz \ \ \{g(j;z)\} \ \ G(m_w;z)$$

with

$$\begin{aligned} g(-1;x) &= \frac{1}{1+x} \\ g(0;x) &= \frac{1}{x} \\ g(-y;x) &= \frac{1}{y+x} \\ g(-1/y;x) &= \frac{1}{1/y+x} \end{aligned}$$

Introduce Feynman parameters I

... change product into sum ...

$$\frac{1}{D_1^{\nu_1} D_2^{\nu_2} \dots D_N^{\nu_N}} = \frac{\Gamma(\nu_1 + \dots + \nu_N)}{\Gamma(\nu_1) \dots \Gamma(\nu_N)} \int_0^1 dx_1 \dots \int_0^1 dx_N \frac{x_1^{\nu_1 -1} \dots x_N^{\nu_N -1} \delta(1 - x_1 \dots - x_N)}{(x_1 D_1 + \dots + x_N D_N)^{N_\nu}},$$

with $N_\nu = \nu_1 + \dots \nu_N$.

$$(m^2) \quad = \quad (x_1 D_1 + \dots + x_N D_N) \; = \; (k_i M_{ij} k_j - 2 Q_j k_j + J)$$

Here M is an (LxL)-matrix, $Q = Q(x_i, p_e)$ an L-vector and $J = J(x_i x_j, m_i^2, p_{e_j} p_{e_l})$.
M, Q, J are linear in x_i.

Manipulate (in order to diagonalize) and get for the momentum integral

For L-loop integrals ...

$$I_N(L) \quad = \quad (-1)^{N_\nu} (i)^L \int \frac{Dk_1 \ldots Dk_L}{\left(kMk + \mu^2\right)^{N_\nu}}.$$

$$\mu^2(x) \quad = \quad -(J - QM^{-1}Q) \qquad \rightarrow -J + Q^2 \quad \text{for 1-loop}$$

For 1-loop integrals it is $L = 1, M = 1$ and we are ready to do the k-integration.

The Feynman parameter representation

1-loop scalar integrals with N lines:

$$I(G(1)) = (-1)^N \Gamma\left(N-\frac{d}{2}\right) \int_0^1 \prod_{j=1}^{N} dx_j \delta\left(1-\sum_{i=1}^{N} x_i\right) \frac{1}{F(x)^{N-d/2}}$$

with

$$F(x) = \mu^2 = -J + Q^2$$

Trick for one-loop functions:
$U = \det M = 1 = \sum x_i$ and so the construct $F_1(x)$ can be made bilinear in $x_i x_j$:

$$F_1(x) = -J(\sum x_i) + Q^2 = \sum A_{ij} x_i x_j .$$

Examples for one-loop F-polynomials

One-loop vertex:

$$F(t, m^2) = m^2 \ (x_1+x_2)^2 + [-t] \ x_1 x_2$$

one-loop QED box $\rightarrow$ to be used here in the following:

$$\mathbf{F(s,t,m^2) = m^2 \ (x_1+x_2)^2 + [-t] \ x_1 x_2 + [-s] \ x_3 x_4}$$

one-loop QED pentagon:

$$F(s,t,t',v_1,v_2,m^2) = m^2 \ (x_1+x_3+x_4)^2 + [-t] \ x_1 x_3 + [-t'] \ x_1 x_4 + [-s] \ x_2 x_5 + [-v_1] \ x_3 x_5 + [-v_2] \ x_2 x_4$$

Mellin-Barnes representations I

Now using Mellin-Barnes Representations

Perform the x-integrations
Computercodes:

Derive Mellin-Barnes representations for Feynman integrals

Ambre.m, Gluza, Kajda, Riemann, Yundin, [7, CPC 2007]

and

Find an ϵ-expansion and evaluate numerically in Euclidean region

MB.m, Czakon, Smirnov, [8, CPC 2005]

How to define divergent MB-representations? I

We want to apply now:

$$\int_0^1 \prod_{j=1}^{N} dx_j \; x_j^{\alpha_j - 1} \; \delta\left(1 - \sum x_i\right) \quad = \quad \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\cdots\Gamma(\alpha_7 N)}{\Gamma\left(\alpha_1 + \alpha_2 + \cdots + \alpha_N\right)}$$

with coefficients α_i dependent on ν_i and on the structure of the F

For this, we have to apply one or several MB-integrals here.

Simplest cases: I

$$\int_0^1 dx_1 \; x_1^{\alpha_1 - 1} \; \delta\left(1 - x_1\right) = 1$$

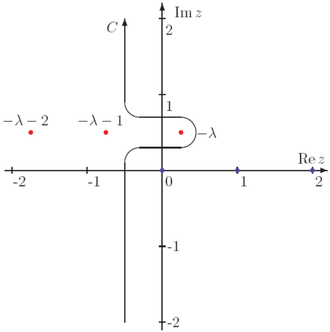
$$\int_0^1 \prod_{j=1}^{2} dx_j \; x_j^{\alpha_j - 1} \; \delta\left(1 - \sum_{i=1}^{N} x_i\right) = \int_0^1 dx_1 x_1^{\alpha_1 - 1}(1 - x_1)^{\alpha_2 - 1} = B(\alpha_1, \alpha_2)$$

$$= \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma\left(\alpha_1 + \alpha_2\right)}$$

Here we want to go: Mellin-Barnes transformation ...

$$\frac{1}{(A+B)^{\lambda}} = \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz \Gamma(\lambda + z)\Gamma(-z) \frac{B^z}{A^{\lambda+z}} \qquad (1)$$

Simplest cases: II



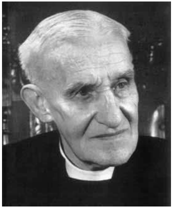
The integration path separates poles of $\Gamma[\lambda+z]$ and $\Gamma[-z]$.

Path may be closed to the right or to the left.

Integral may be evaluated numerically or by summing over poles inside the integration path.

One might well assume that these two gentlemen did not dream of so heavy use of their results in basic research · · ·

Mellin, Robert, Hjalmar, 1854-1933
Barnes, Ernest, William, 1874-1953



How can the Mellin-Barnes formula be made useful in the context of Feynman integrals?

Transform a massive propagator to a massless one, with index a of the line changed to $(a+\sigma)$)

$$\frac{1}{(p^2 - m^2)^a} = \frac{1}{2\pi i\, \Gamma(a)} \int_{-i\infty}^{i\infty} d\sigma \frac{(-m^2)^\sigma}{(p^2)^{a+\sigma}} \Gamma(a+\sigma)\Gamma(-\sigma)$$

Transform a sum of monomials into *one* monomial

$$\frac{1}{[A(s)x_1^{a_1} + B(s)x_1^{b_1}x_2^{b_2}]^a} = \frac{1}{2\pi i\, \Gamma(a)} \int_{-i\infty}^{i\infty} d\sigma [A(s)x_1^{a_1}]^\sigma [B(s)x_1^{b_1}x_2^{b_2}]^{a+\sigma}\, \Gamma(a+\sigma)\Gamma(-\sigma)$$

Both methods leave Mellin-Barnes (MB-) integrals to be performed afterwards.

A short remark on history

- N. Usyukina, 1975: "ON A REPRESENTATION FOR THREE POINT FUNCTION", Teor. Mat. Fiz. 22;
  a finite massless off-shell 3-point 1-loop function represented by 2-dimensional MB-integral
- E. Boos, A. Davydychev, 1990: "A Method of evaluating massive Feynman integrals", Theor. Math. Phys. 89 (1991);
  N-point 1-loop functions represented by n-dimensional MB-integral
- V. Smirnov, 1999: "Analytical result for dimensionally regularized massless on-shell double box", Phys. Lett. B460 (1999);
  treat UV and IR divergencies by analytical continuation: shifting contours and taking residues 'in an appropriate way'
- B. Tausk, 1999: "Non-planar massless two-loop Feynman diagrams with four on-shell legs", Phys. Lett. B469 (1999);
  nice algorithmic approach to that, starting from search for some unphysical space-time dimension d for which the MB-integral is finite and well-defined
- M. Czakon, 2005 (with experience from common work with J. Gluza and TR): "Automatized analytic continuation of Mellin-Barnes integrals", Comput. Phys. Commun. (2006);
  Tausk's approach realized in Mathematica program MB.m, published and available for use

B412m, the 1-loop QED box, with two photons in the s-channel; the Mellin-Barnes representation reads for finite ϵ:

$$I_4 = \mathtt{B412m} = \mathtt{Box}(t,s) = \frac{e^{\epsilon\gamma_E}}{\Gamma[-2\epsilon](-t)^{(2+\epsilon)}} \frac{1}{(2\pi i)^2} \int_{-i\infty}^{+i\infty} dz_1 \int_{-i\infty}^{+i\infty} dz_2$$
$$\frac{(-s)^{z_1}(m^2)^{z_2}}{(-t)^{z_1+z_2}} \Gamma[2+\epsilon+z_1+z_2]\Gamma^2[1+z_1]\Gamma[-z_1]\Gamma[-z_2]$$
$$\Gamma^2[-1-\epsilon-z_1-z_2]\frac{\Gamma[-2-2\epsilon-2z_1]}{\Gamma[-2-2\epsilon-2z_1-2z_2]}$$

Mathematica package MB used for analytical expansion $\epsilon \to 0$:[8]

$$I_4 = -\frac{1}{\epsilon}\mathrm{J}_1 + \ln(-s)\mathrm{J}_1 + \epsilon\left(\frac{1}{2}\left[\zeta(2) - \ln^2(-s)\right]\mathrm{J}_1 - 2\mathrm{J}_2\right) + \cdots$$

with J_1 being also the divergent part of the vertex function $C_0(t;m,0,m)/s = \mathtt{V312m}/s$ (as is well-known):

$$J_1 = \frac{e^{\epsilon\gamma_E}}{st}\frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dz_1 \left(\frac{m^2}{-t}\right)^{z_1} \frac{\Gamma^3[-z_1]\Gamma[1+z_1]}{\Gamma[-2z_1]} = \frac{1}{m^2 s}\frac{2y}{1-y^2}\ln(y)$$

with

$$y = \frac{\sqrt{1-4m^2/t}-1)}{(\sqrt{1-4m^2/t}+1)}$$

The J_2 is more complicated:

$$J_2 = \frac{e^{\epsilon\gamma_E}}{t^2}\frac{1}{(2\pi i)^2} \int_{-\frac{3}{4}-i\infty}^{-\frac{3}{4}+i\infty} dz_1 \left(\frac{s}{t}\right)^{z_1} \Gamma[-z_1]\Gamma[-2(1+z_1)]\Gamma^2[1+z_1]$$
$$\times \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dz_2 \left(-\frac{m^2}{t}\right)^{z_2} \Gamma[-z_2] \frac{\Gamma^2[-1-z_1-z_2]}{\Gamma[-2(1+z_1+z_2)]} \Gamma[2+z_1+z_2].$$

Problem of summations

Summing over residues and performing the limit $\epsilon \to 0$ may be exchanged.

In any case: When crossing a pole by letting $\epsilon \to 0$, take the residue and add it to the expression.
Generates lower dimensional sums.

Then look at the resulting Mellin-Barnes integrals and for the series of remaining poles
Take the sum over residues when closing the contour to the right or to the left.

Look for the results of this procedure for the ϵ-term of the expansion for the box diagram, $B_1(s,t)$

$$B_1(s,t) = (sAA2 + sAB2 + sBAeq2 + sBA12x + sBA21x + sBC2) + sd11$$

The $sd11$ is the 1-dimensional sum for $B_1(s,t)$, not exemplified here:

```
residued11 = -(((-1)^-M (-s)^M (M!)^2  / (4 t (1 + 2 M)!)

             (3 [Pi]^2  + 2 HarmonicNumber[M]^2

              -  2 HarmonicNumber[M, 2]

              + 4 HarmonicNumber[M] Log[-t])))

sd11 = Sum[residued11/. Li, {M,0,Infinity}] //N}
```

The 2-dimensional MB-integral as sum of several 1-folf and 2-fold sums I

```
(* The Mellin-Barnes integral for the QED 1-loop box with mass m=1 gives a one-dim. and a
two-dimensional MB-integral. The two-dimensional one over z1,z2 with Re[z1]=-1/4, Re[z2]=-1/2+
0<eta<<1
has the following integrand: *)

int2dim =((-s)^z2 (-t)^(-2 - eps - z1 - z2)
    Gamma[-z1] Gamma[-1 - eps - z1 - z2]^2 Gamma[-z2] Gamma[
    1 + z2]^2 Gamma[2 + eps + z1 + z2] Gamma[
    2 + 2 z1 + 2 z2])/(Gamma[-2 eps] Gamma[2 + 2 z2])

(* We take first the residues in z1 for closing the line to the right, i.e. with Re[z1]>-1/4.
are two series of them:  *)

res2dimA = Residue[int2dim, {z1, n1}] // Simplify   (*n1>=0*)
res2dimB = Residue[int2dim, {z1, n1 - 1 - z2 - eps}] // Simplify   (*n1>=1*)

(* The following residues in z2 for closing to the right are four series, the two AA and AB f
res2dimA, and the other two BA, BC from  res2dimB:  *)

res2dimAA = Residue[res2dimA, {z2, n2}] // Simplify  (*n2,n1>=0*)
res2dimAA = ((-s)^n2 (-t)^-eps t^(-2 - n1 - n2)
    Gamma[-1 - eps - n1 - n2]^2 Gamma[1 + n2]^2 Gamma[
    2 (1 + n1 + n2)] Gamma[
    2 + eps + n1 + n2])/(n1! n2! Gamma[-2 eps] Gamma[2 + 2 n2])
```

```
res2dimAB = Assuming[n2 > n1, Residue[res2dimA, {z2, n2 - n1 - 1 - eps}]] // Simplify  (*n2>n
res2dimAB = ...

res2dimBA = Residue[res2dimB, {z2, n2}] // Simplify(*n2>=0,n1>=1*)
res2dimBA = ...

res2dimBC = Assuming[n2 < n1, Residue[res2dimB, {z2, n1 - n2 - 1 - eps}]] // Simplify (*n1>n2
res2dimBC = ..

(* For the epsilon expansion, in res2dimBA there are three cases with different answer: n2>n1,
n2=n1, n2<n1. *)

rAAdef = Coefficient[Normal[Series[res2dimAA, {eps, 0, 1}]] // FullSimplify, eps]

rAA = -((-s)^n2 t^(-2 - n1 - n2)
     n2! Gamma[
     2 (1 + n1 + n2)] (4 EulerGamma^2 + 4 EulerGamma Log[-t] +
      Log[-t]^2 + 3 EulerGamma PolyGamma[0, 2 + n1 + n2] +
      HarmonicNumber[1 + n1 + n2] PolyGamma[0, 2 + n1 + n2] +
      2 Log[-t] PolyGamma[0, 2 + n1 + n2] -
      PolyGamma[1, 2 + n1 + n2]))/(n1! Gamma[2 + n1 + n2] Gamma[
     2 + 2 n2])


rABdef = Assuming[n2 > n1, Coefficient[Normal[Series[res2dimAB, {eps, 0, 1}]] // FullSimplify,
      eps]]

rAB = ...
```

```
rBAeqdef = Assuming[n2 == n1, Coefficient[Normal[Series[res2dimBA, {eps, 0, 1}]] // FullSimpli
   eps]]

rBAeq = (s^n2 (-t)^-n2 (HarmonicNumber[n2] + Log[-t] -
   2 (EulerGamma + PolyGamma[0, 2 n2])))/(n2 (1 + 2 n2) t)

rBA21def = Assuming[n2 > n1, Coefficient[Normal[Series[res2dimBA, {eps, 0, 1}]] // FullSimpli
   eps]]

rBA21 = 1/(t n1! Gamma[2 + 2 n2])
  2 s^n2 (-t)^-n1 n2! (-n1 + n2)! Gamma[
   2 n1] (HarmonicNumber[n1] + HarmonicNumber[-n1 + n2] + Log[-t] -
    2 (EulerGamma + PolyGamma[0, 2 n1]))

rBA12def = Assuming[n2 < n1, Coefficient[Normal[Series[res2dimBA, {eps, 0, 1}]] // FullSimpli
   eps]]

rBA12 = ((-s)^n2 t^(-1 - n1)
    n2! Gamma[
    2 n1] (4 EulerGamma^2 - \[Pi]^2 + 4 EulerGamma Log[-t] +
     4 HarmonicNumber[n1] PolyGamma[0, 2 n1] -
     4 HarmonicNumber[-1 + 2 n1] PolyGamma[0, 2 n1] +
     4 Log[-t] PolyGamma[0, 2 n1] +
     4 EulerGamma PolyGamma[0, 1 + n1] -
     2 HarmonicNumber[n1] PolyGamma[0, n1 - n2] +
     2 HarmonicNumber[-1 + 2 n1] PolyGamma[0, n1 - n2] -
     2 Log[-t] PolyGamma[0, n1 - n2] +
     2 PolyGamma[0, 2 n1] PolyGamma[0, n1 - n2] -
     PolyGamma[0, n1 - n2]^2 - 4 PolyGamma[1, 2 n1] +
     PolyGamma[1, n1 - n2]))/(n1! Gamma[n1 - n2] Gamma[2 + 2 n2])
```

```
rBCdef = Assuming[n2 < n1, Coefficient[Normal[Series[res2dimBC, {eps, 0, 1}]] // FullSimplify,
   eps]]

rBC = ...

The conditions for n1 and n2 are given.
Both are assumed to be >= 0.

See the definition of "Li" [= Liste der Kinematik] when doing numerics.

sAA = Sum[Sum[rAA /. Li, {n1, 0, m1 + 150}], {n2, 0, m2 + 150}] //N
sAB = Sum[Sum[rAB /. Li, {n2, n1 + 1, m2 + 150}], {n1, 0, m1 + 150}] //N
sBA21 = Sum[Sum[rBA21 /. Li, {n2, n1 + 1, m2 + 150}], {n1, 1, m1 + 150}] // N
sBAeq = Sum[Sum[rBAeq /. Li, {n1, 1, m1 + 150}], {n2, n1, n1}] // N
sBA12 = Sum[Sum[rBA12 /. Li, {n1, n2 + 1, m1 + 150}], {n2, 0, m2 + 150}] // N
sBC = Sum[Sum[rBC /. Li, {n1, n2 + 1, m1 + 150}], {n2, 0, m2 + 150}] // N

anasum1 = sAA + sAB + sBAeq + sBA12 + sBA21 + sBC

(*  2012-03-29: anasum1 = anasum12 but the latter has only HarmonicNumber, no PolyLog  *)

rAA2 = -(1/(6 n1! (1 + n1 + n2)! (1 + 2 n2)!))(-s)^n2 t^(-2 - n1 - n2)
   n2! (1 + 2 n1 + 2 n2)! (6 EulerGamma^2 - \[Pi]^2 +
    6 HarmonicNumber[1 + n1 + n2]^2 +
    6 HarmonicNumber[1 + n1 + n2, 2] + 12 EulerGamma Log[-t] +
    6 Log[-t]^2 +
    12 HarmonicNumber[1 + n1 + n2] (EulerGamma + Log[-t]))
```

```
rAB2 = 1/(6 n1! n2! (-1 - 2 n1 + 2 n2)!) (-s)^(-1 - n1 + n2)
   t^(-1 - n2) (-1 - n1 + n2)! (-1 + 2 n2)! (6 EulerGamma^2 -
    7 \[Pi]^2 - 6 HarmonicNumber[-1 - n1 + n2]^2 -
    24 HarmonicNumber[-1 + 2 n2]^2 +
    48 HarmonicNumber[-1 + 2 n2] HarmonicNumber[-1 - 2 n1 + 2 n2] -
    24 HarmonicNumber[-1 - 2 n1 + 2 n2]^2 +
    6 HarmonicNumber[-1 - n1 + n2, 2] +
    24 HarmonicNumber[-1 + 2 n2, 2] -
    24 HarmonicNumber[-1 - 2 n1 + 2 n2, 2] -
    24 HarmonicNumber[-1 + 2 n2] Log[-s] +
    24 HarmonicNumber[-1 - 2 n1 + 2 n2] Log[-s] - 6 Log[-s]^2 +
    12 HarmonicNumber[
      n2] (EulerGamma + HarmonicNumber[-1 - n1 + n2] +
       2 HarmonicNumber[-1 + 2 n2] -
       2 HarmonicNumber[-1 - 2 n1 + 2 n2] + Log[-s]) -
    12 HarmonicNumber[-1 - n1 + n2] (2 HarmonicNumber[-1 + 2 n2] -
       2 HarmonicNumber[-1 - 2 n1 + 2 n2] + Log[-s] - Log[-t]) +
    12 EulerGamma Log[-t] + 24 HarmonicNumber[-1 + 2 n2] Log[-t] -
    24 HarmonicNumber[-1 - 2 n1 + 2 n2] Log[-t] + 12 Log[-s] Log[-t])

rBA21x = ...

rBAeq2 = (s^n2 (-t)^-n2 (HarmonicNumber[n2] - 2 HarmonicNumber[-1 + 2 n2] +
   Log[-t]))/(n2 (1 + 2 n2) t)

rBA12x = ...

rBA12x = ...
```

```
rBC2 = 1/(6 n1! (-1 + 2 n1 - 2 n2)! n2!) (-1)^(n1 + n2) s^(-1 + n1 - n2)
   t^(-1 - n1) (-1 + 2 n1)! (-1 + n1 - n2)! (6 EulerGamma^2 -
    7 \[Pi]^2 - 24 HarmonicNumber[-1 + 2 n1]^2 -
    24 HarmonicNumber[-1 + 2 n1 - 2 n2]^2 +
    24 HarmonicNumber[-1 + 2 n1 - 2 n2] HarmonicNumber[-1 + n1 -
       n2] - 6 HarmonicNumber[-1 + n1 - n2]^2 +
    24 HarmonicNumber[-1 + 2 n1, 2] -
    24 HarmonicNumber[-1 + 2 n1 - 2 n2, 2] +
    6 HarmonicNumber[-1 + n1 - n2, 2] +
    24 HarmonicNumber[-1 + 2 n1 - 2 n2] Log[-s] -
    12 HarmonicNumber[-1 + n1 - n2] Log[-s] - 6 Log[-s]^2 +
    12 HarmonicNumber[n1] (EulerGamma + 2 HarmonicNumber[-1 + 2 n1] -
       2 HarmonicNumber[-1 + 2 n1 - 2 n2] +
       HarmonicNumber[-1 + n1 - n2] + Log[-s]) +
    12 EulerGamma Log[-t] -
    24 HarmonicNumber[-1 + 2 n1 - 2 n2] Log[-t] +
    12 HarmonicNumber[-1 + n1 - n2] Log[-t] + 12 Log[-s] Log[-t] +
    24 HarmonicNumber[-1 + 2 n1] (2 HarmonicNumber[-1 + 2 n1 - 2 n2] -
        HarmonicNumber[-1 + n1 - n2] - Log[-s] + Log[-t]))

m1=m2=3

sAA2 = Sum[Sum[rAA2 /. Li, {n1, 0, m1 + 150}], {n2, 0, m2 + 150}] //N          = -0.19992
sAB2 = Sum[Sum[rAB2 /. Li, {n2, n1 + 1, m2 + 150}], {n1, 0, m1 + 150}] //N     = -0.51828
sBA21x = Sum[Sum[rBA21x /. Li, {n2, n1 + 1, m2 + 150}], {n1, 1, m1 + 150}] // N = -0.00022
sBAeq2 = Sum[Sum[rBAeqx /. Li, {n1, 1, m1 + 150}], {n2, n1, n1}] // N          =  0.00275
sBA12x = Sum[Sum[rBA12x /. Li, {n1, n2 + 1, m1 + 150}], {n2, 0, m2 + 150}] // N = -0.06865
sBC2 = Sum[Sum[rBC2 /. Li, {n1, n2 + 1, m1 + 150}], {n2, 0, m2 + 150}] // N    =  0.51828

anasum12 = sAA2 + sAB2 + sBAeq2 + sBA12x + sBA21x + sBC2                       = -0.26605
```

$$B_1(s,t) = \textbf{anasum12} \quad + \textbf{1-dim. sum}$$

Summary

Although for the QED 1-loop box we know the result, it would be interesting to get it via summation techniques

This would allow an algorithmic solution with useful properties:

- only one diagram per calculation – no coupled system of equations
- derivations are relatively simple
- generalization to higher loop problems – in principle – easy

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