Tensor integrals

New solutions to old problems

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Based on work done in collaboration with Jochem Fleischer and Valery Yundin
talk held at German Japanese WS on “Modern Trends in Quantum Chromodynamics”, 3-5 October 2011, DESY, Zeuthen, Germany

A simple example

1-loop self-energy:

$$
\begin{aligned}
I_2^{\mu} &= \int \frac{d^dk}{i\pi^{d/2}} \frac{k^{\mu}}{[k^2 - M_1^2]\,[(k+p)^2 - M_2^2]} \\
&= p_{\mu} B_1
\end{aligned}
$$

Solve:

$$
\begin{aligned}
p_{\mu} I_2^{\mu} &= p^2 B_1(p, M_1, M_2) \\
&= \int \frac{d^dk}{i\pi^{d/2}} \frac{pk}{[k^2 - M_1^2]\,[(k+p)^2 - M_2^2]} = \int \frac{d^dk}{i\pi^{d/2}} \frac{pk}{D_1\ D_2} \\
&= \frac{1}{2} \int \frac{d^dk}{i\pi^{d/2}} \left[\frac{D_2 - (p^2 - M_2^2 - M_1^2) - D_1}{D_1\ D_2} \right], \\
B_1(p, M_1, M_2) &= \frac{1}{2p^2} \left[A_0(M_1) - A_0(M_2) - \frac{1}{2}(p^2 - M_2^2 - M_1^2) B_0(p, M_1, M_2) \right]
\end{aligned}
$$

A tensor Feynman integral is expressed in terms of scalar Feynman integrals.

Reduction of tensor integrals

Reduction of tensor integrals = expressing them by a (very) small set of scalar integrals – is needed for practically any Feynman diagram calculation.
Feynman diagrams with loops contain tensor integrals due to:

- fermion propagators
- three-gauge boson couplings
- e.g. unitary gauge propagators

Examples:

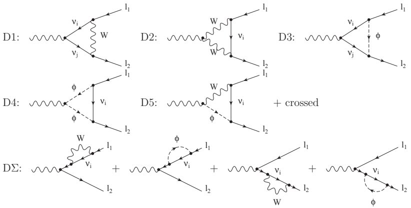
- LO (Lowest order) of $Z \to e + \mu$ is one-loop; processes forbidden in tree diagrams of Standard Model
- NLO: one-loop corrections to Born diagrams; prominent by now for LHC physics: $2 \to 3, 4, \cdots$
- NNLO: radiative loop corrections to Born diagrams: e.g. $e^+e^- \to e^+e^-\gamma$ with a loop (including 5-point functions)

My first tensor reduction

My first tensor reduction [Proc. Ahrenshoop Symposium, 1981] was for:

$$Z \to \mu + e, \quad b + s, \quad \text{etc.} \tag{1}$$

Use of the 'conventional' Passarino-Veltman reduction (Nucl.Phys.B160, 1979), invented not so much earlier.
The process is, in the Standard Model, possible due to one-loop vertex diagrams with internal flavor-changing W bosons and mixing heavy fermions:



J. Illana,TR, PRD 2000 study repeated for ILC applications for SM and models with heavy neutrinos
Later J. Illana et al. studied also supersymmetric cases.

Comparison with grace in 2002: weak corr.s to top pair production at ILC/NLC

Numerical Results Based on
J. Fleischer, A. Leike, T. Riemann, A. Werthenbach
and
J. Fujimoto, T. Ishikawa, Y. Shimizu

One-loop Corrections to the Process $e^+e^- \to t\bar{t}$
Including Hard Bremsstrahlung

hep-ph/0203220
(weak corrections and also hard bremsstrahlung, topfit compared to Grace)

Top-quark Pair-production Cross section

Conventions for comparisons

$$
\begin{aligned}
\Gamma_Z &= 0 \\
\alpha = \frac{e^2}{4\,\pi} &= 1/137.03599976 \\
E_\gamma^{\max} &= \sqrt{s}/10 \\
M_w &= 80.4514958 \ \mathrm{GeV} \\
M_z &= 91.1867 \ \mathrm{GeV} \\
M_h &= 120 \ \mathrm{GeV} \\
m_e &= 0.00051099907 \ \mathrm{GeV} \\
m_t &= 173.8 \ \mathrm{GeV} \\
m_b &= 4.7 \ \mathrm{GeV} \\
m_\mu &= 0.105658389 \ \mathrm{GeV} \\
m_u &= 0.062 \ \mathrm{GeV} \\
m_d &= 0.083 \ \mathrm{GeV} \\
m_\tau &= 1.77705 \ \mathrm{GeV} \\
m_c &= 1.5 \ \mathrm{GeV}, \qquad m_s = 0.215 \ \mathrm{GeV}
\end{aligned}
$$

$\cos\theta$	$\omega/\sqrt{s}$	$\left[\frac{d\sigma}{d\cos\theta}\right]_{\rm Born}$	$\left[\frac{d\sigma}{d\cos\theta}\right]_{\rm QED}$	$\left[\frac{d\sigma}{d\cos\theta}\right]_{\rm SM}$	$\left[\frac{d\sigma}{d\cos\theta}\right]_{\rm tot}$
−0.9	T : 0.1	0.108839194075	+0.098664253	+0.11408410	0.13144
	T : 0.00001	0.108839194075	−0.017474702	−0.002054858	0.13229
	G : 0.00001	0.108839194076		−0.002054859	0.13206(12)
−0.5	T : 0.1	0.142275069392	+0.12850790	+0.14308121	0.15973
	T : 0.00001	0.142275069392	−0.029702340	−0.015129038	0.16029
	G : 0.00001	0.142275069393		−0.015129039	0.16013(13)
+0.0	T : 0.1	0.225470464033	+0.20239167	+0.21718801	0.23638
	T : 0.00001	0.225470464033	−0.058010508	−0.043214169	0.23476
	G : 0.00001	0.225470464033		−0.043214168	0.23513(14)
+0.5	T : 0.1	0.354666470332	+0.31511723	+0.32933727	0.35651
	T : 0.00001	0.354666470332	−0.109721291	−0.095501257	0.35062
	G : 0.00001	0.354666470332		−0.095501252	0.35104(17)
+0.9	T : 0.1	0.491143715767	+0.43071437	+0.44290816	0.48796
	T : 0.00001	0.491143715767	−0.179672655	−0.16747886	0.47768
	G : 0.00001	0.491143715767		−0.16747886	0.47709(21)

Various differential cross sections (see also text). The upper and lower numbers correspond to the `topfit` (T) and GRACE (G) approach, respectively, $\sqrt{s} = 500$ GeV.

$\cos\theta$	$\omega/\sqrt{s}$	$\left[\frac{d\sigma}{d\cos\theta}\right]_{\rm Born}$	$\left[\frac{d\sigma}{d\cos\theta}\right]_{\rm QED}$	$\left[\frac{d\sigma}{d\cos\theta}\right]_{\rm SM}$	$\left[\frac{d\sigma}{d\cos\theta}\right]_{\rm tot}$
-0.9	T : 0.1	0.0227854232732	+0.020365844	+0.023101706	0.036334
	T : 0.00001	0.0227854232732	−0.004756230	−0.002020367	0.036461
	G : 0.00001	0.02278542327319		−0.002020369	0.036582(48
-0.5	T : 0.1	0.0297821311031	+0.026741663	+0.028823021	0.038888
	T : 0.00001	0.0297821311031	−0.008561495	−0.006480137	0.039055
	G : 0.00001	0.0297821311031		−0.006480139	0.038965(42
$+0.0$	T : 0.1	0.0611800674224	+0.054539344	+0.054950889	0.067789
	T : 0.00001	0.0611800674224	−0.021532420	−0.021120874	0.067801
	G : 0.00001	0.0611800674225		−0.021120874	0.068039(55
$+0.5$	T : 0.1	0.117746949888	+0.10311626	+0.099416999	0.12095
	T : 0.00001	0.117746949888	−0.050123708	−0.053822973	0.12051
	G : 0.00001	0.117746949888		−0.053822964	0.12064(07)
$+0.9$	T : 0.1	0.181122097086	+0.15403823	+0.14426232	0.19355
	T : 0.00001	0.181122097086	−0.096682759	−0.10645866	0.19272
	G : 0.00001	0.181122097086		−0.10645866	0.19057(10)

Same as other table, $\sqrt{s} = 1000$ GeV.

Systematic approach

to tensor reductions:
1- to 4-point functions: Passarino, Veltman 1978 [1]

Need in addition a library of scalar functions:
'tHooft, Veltman 1979 [2]
State of the art + open source programs:
K. Ellis and G. Zanderighi, QCDloop/FF [3, 4] 2007,1990
T. Hahn, LoopTools/FF [5, 4] 1998,1990

Open source programs for 5,6,7-point reductions:

- LoopTools/FF [T. Hahn et al.]
- Golem/OPP etc. [G. Heinrich; A. van Hameren, this conference]
- HELAC/PHEGAS/OPP etc. [A. van Hameren, this conference]

$\rightarrow$ c++ code PJFry by V. Yundin, PhD thesis + [6] 2010 [Fleischer/Riemann/Yundin group], released this Summer 2011

This talk: Efficient reduction formulae in the algebraic Fleischer-Davydychev-Tarasov approach

Recent developments in the Fleischer-Davydychev-Tarasov approach:

- Get $n > 4$ tensor reduction with $\cdots$:
- $\cdots$ arbitrary masses
- $\cdots$ killed pentagon Gram determinants
- $\cdots$ treatment of full kinematics, also with small sub-diagram Gram determinants
  $\rightarrow$ presented by J. Fleischer at QCD@LHC@Trento2010
- $\rightarrow$ c++ code PJFry, $n \leq 5$ by V. Yundin [$\rightarrow$ GOLEM option]
  see talk at LHCphenonet Meeting 02/2011
- $\cdots$ multiple sums over tensor coefficients made efficient by contracting with external momenta arXiv/1104.4067, PLB 701(2011)646
- $\cdots$ $n \geq 7$ – a fresh look

Outline

[7] 1991 Davydychev, . . . *Reducing Feynman diagrams to scalar integrals*
[8] 1996 Tarasov, *Connection [of] Feynman integrals [with] different . . . space-time dimensions*
[9] 1999 Fleischer et al., *Algebraic reduction of one-loop Feynman graph amplitudes*

References:
[10] 2010 Diakonidis et al., PLB 683, . . . *recursive reduction of tensor Feynman integrals*
[6] 2011 Fleischer, T.R., PRD 83, *Complete . . . reduction of . . . tensor Feynman integrals*
[11] 2011 Fleischer, T.R., PLB 701, . . . *contracted tensor Feynman integrals*
subm. Aug. 2011: V. Yundin, PhD thesis [with PJFry code]

Notations: Gram and modified Cayley determinant, signed minors [Melrose:1965]

Gram determinant G_n:

$$G_n = |2q_iq_j|, i,j = 1, \dots n \tag{2}$$

Modified Cayley determinant $()_N$ of a diagram with N internal lines and chords q_j;
for a choice $q_n = 0$, both determinants are related:

$$()_N \equiv \begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & Y_{11} & Y_{12} & \dots & Y_{1N} \\ 1 & Y_{12} & Y_{22} & \dots & Y_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & Y_{1N} & Y_{2N} & \dots & Y_{NN} \end{vmatrix} = -G_{N-1}, \tag{3}$$

where $D_i = (k - q_i)^2 - m_i^2$ [with q_i = chord], and the matrix elements

$$Y_{ij} = -(q_i - q_j)^2 + m_i^2 + m_j^2, \quad (i,j = 1 \dots N) \tag{4}$$

$\Rightarrow$ The Gram determinant $()_N$ does not depend on the masses.

Notations: signed minors [Melrose:1965]

signed minors of $()_N$ are constructed by deleting m rows and m columns from $()_N$, and multiplying with a sign factor:

$$\begin{pmatrix} j_1 & j_2 & \cdots & j_m \\ k_1 & k_2 & \cdots & k_m \end{pmatrix}_N \equiv$$
$$\equiv (-1)^{\sum_l (j_l+k_l)} \mathrm{sgn}_{\{j\}}\, \mathrm{sgn}_{\{k\}} \left| \begin{array}{c} \text{rows } j_1 \cdots j_m \text{ deleted} \\ \text{columns } k_1 \cdots k_m \text{ deleted} \end{array} \right| \qquad (5)$$

where $\mathrm{sgn}_{\{j\}}$ and $\mathrm{sgn}_{\{k\}}$ are the signs of permutations that sort the deleted rows $j_1 \cdots j_m$ and columns $k_1 \cdots k_m$ into ascending order.

Example:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}_N \equiv \begin{vmatrix} Y_{11} & Y_{12} & \dots & Y_{1N} \\ Y_{12} & Y_{22} & \dots & Y_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{1N} & Y_{2N} & \dots & Y_{NN} \end{vmatrix}, \qquad (6)$$

Example: Getting a 4-point function from a six-point function I

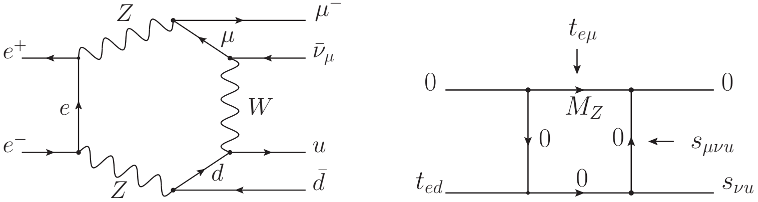


Figure: A six-point topology (a) leading to four-point functions (b) with realistically vanishing Gram determinants.

Example: Getting a 4-point function from a six-point function II

The example is taken from [12].

The corresponding 4-point tensor integrals are, in LoopTools [5, 13] notation:

$$\mathrm{D0i}(\mathrm{id}, 0, 0, s_{\bar{\nu}u}, t_{ed}, t_{\bar{e}\mu}, s_{\mu\bar{\nu}u}, 0, M_Z^2, 0, 0). \tag{7}$$

The Gram determinant is:

$$()_4 \quad = \quad -2t_{\bar{e}\mu}[s^2_{\mu\bar{\nu}u} + s_{\bar{\nu}u}t_{ed} - s_{\mu\bar{\nu}u}(s_{\bar{\nu}u} + t_{ed} - t_{\bar{e}\mu})], \tag{8}$$

It vanishes if:

$$t_{ed} \to t_{ed,\mathrm{crit}} = \frac{s_{\mu\bar{\nu}u}(s_{\mu\bar{\nu}u} - s_{\bar{\nu}u} + t_{\bar{e}\mu})}{s_{\mu\bar{\nu}u} - s_{\bar{\nu}u}}. \tag{9}$$

In terms of a dimensionless scaling parameter x,

$$t_{ed} \quad = \quad (1 + x)t_{ed,\mathrm{crit}}, \tag{10}$$

Example: Getting a 4-point function from a six-point function III

the Gram determinant becomes:

$$()_4 = 2\, x\, s_{\mu\bar{\nu}u} t_{\bar{e}\mu} (s_{\mu\bar{\nu}u} - s_{\bar{\nu}u} + t_{\bar{e}\mu}). \tag{11}$$

We will also need the modified Cayley determinant:

$$\begin{aligned}
\binom{0}{0}_4 &= \begin{pmatrix} 2M_Z^2 & M_Z^2 & M_Z^2 - s_{\mu\bar{\nu}u} & M_Z^2 \\ M_Z^2 & 0 & -s_{\bar{\nu}u} & M_Z^2 \\ M_Z^2 - s_{\mu\bar{\nu}u} & -s_{\bar{\nu}u} & 0 & -t_{ed} \\ M_Z^2 & -t_{\bar{e}\mu} & -t_{ed} & 0 \end{pmatrix} \\
&= s_{\mu\bar{\nu}u}^2 t_{\bar{e}\mu}^2 + 2\, M_Z^2 t_{\bar{e}\mu} [-2 s_{\bar{\nu}u} t_{ed} + s_{\mu\bar{\nu}u} (s_{\bar{\nu}u} + t_{ed} - t_{\bar{e}\mu})] \\
&\quad + M_Z^4 (s_{\bar{\nu}u}^2 + (t_{ed} - t_{\bar{e}\mu})^2 - 2 s_{\bar{\nu}u} (t_{ed} + t_{\bar{e}\mu})).
\end{aligned}$$

Dimensional shifts and recurrence relations for pentagons (I)

Following [Davydychev:1991 [7]]

Replace tensors by scalar integrals in higher dimensions:

Example $R = 3$:

$$I_5^{\mu\nu\lambda} = \int \frac{d^{4-2\epsilon}k}{i\pi^{d/2}} \prod_{r=1}^{5} c_r^{-1} k^\mu k^\nu k^\lambda \tag{12}$$

$$= -\sum_{i,j,k=1}^{4} q_i^\mu q_j^\nu q_k^\lambda n_{ijk} I_{5,ijk}^{[d+]^3} + \frac{1}{2}\sum_{i=1}^{n-1} (g^{\mu\nu} q_i^\lambda + g^{\mu\lambda} q_i^\nu + g^{\nu\lambda} q_i^\mu) I_{5,i}^{[d+]^2},$$

and $n_{ijk} = (1+\delta_{ij})(1+\delta_{ik}+\delta_{jk})$.

$$[d+]^l = 4 - 2\epsilon + 2l$$

$I_{5,i}^{[d+]^2}$ – scratch the line i from $I_5^{[d+]^2}$.

Dimensional shifts and recurrence relations for pentagons (II)

'Naive', direct approach – just perform dimensional recurrences

Following [Tarasov:1996,Fleischer:1999 [8, 9]]
apply recurrence relations, relating scalar integrals of different dimensions, in order to get rid of the dimensionalities $[d+]^l = 4 - 2\epsilon + 2l$:

$$\nu_j (\mathbf{j}^+ I_5^{[d+]}) = \frac{1}{()_5} \left[-\binom{j}{0}_5 + \sum_{k=1}^{5} \binom{j}{k}_5 \mathbf{k}^- \right] I_5 \qquad (13)$$

$$(d - \sum_{i=1}^{5} \nu_i + 1) I_5^{[d+]} = \frac{1}{()_5} \left[\binom{0}{0}_5 - \sum_{k=1}^{5} \binom{0}{k}_5 \mathbf{k}^- \right] I_5, \qquad (14)$$

also:

$$\nu_j \mathbf{j}^+ I_5 = \frac{1}{\binom{0}{0}_5} \sum_{k=1}^{5} \binom{0j}{0k}_5 \left[d - \sum_{i=1}^{5} \nu_i (\mathbf{k}^- \mathbf{i}^+ + 1) \right] I_5 \qquad (15)$$

where the operators $\mathbf{i}^{\pm}, \mathbf{j}^{\pm}, \mathbf{k}^{\pm}$ act by shifting the indices ν_i, ν_j, ν_k by ± 1.

The result of simplifying manipulations

...and collecting all contributions, our final result for e.g. the tensor of rank $R = 3$ can be written as follows:

$$I_5^{\mu\,\nu\,\lambda} = \sum_{i,j,k=1}^{4} q_i^{\mu}\, q_j^{\nu}\, q_k^{\lambda} E_{ijk} + \sum_{k=1}^{4} g^{[\mu\nu} q_k^{\lambda]} E_{00k}, \qquad (16)$$

with:

$$E_{00j} = \sum_{s=1}^{5} \frac{1}{\binom{0}{0}_5} \left[\frac{1}{2} \binom{0s}{0j}_5 I_4^{[d+],s} - \frac{d-1}{3} \binom{s}{j}_5 I_4^{[d+]^2,s} \right], \qquad (17)$$

$$E_{ijk} = -\sum_{s=1}^{5} \frac{1}{\binom{0}{0}_5} \left\{ \left[\binom{0j}{sk}_5 I_{4,i}^{[d+]^2,s} + (i \leftrightarrow j) \right] + \binom{0s}{0k}_5 \nu_{ij} I_{4,ij}^{[d+]^2,s} \right\}. \qquad (18)$$

✓ no scalar 5-point integrals in higher dimensions
✓ no inverse Gram det. $()_5$
We have yet:

† scalar 4-point integrals in higher dimensions: $I_{4,ij}^{[d+]^2,s}$ etc.

† inverse Gram det. $\binom{0}{0}_5 \equiv ()_4$

Reduce $I_{4,ij\cdots}^{[d+]^l,s}$ to $I_4^{[d+]^L,s}$ plus simpler objects I

By nontrivial manipulations we get e.g.:

$$I_{4,i}^{[d+],s} = \frac{1}{\binom{0s}{0s}_5}\left[-\binom{0s}{is}_5 (d-3) I_4^{[d+],s} + \sum_{t=1}^{5}\binom{0st}{0si}_5 I_3^{st}\right] \tag{19}$$

$$\begin{aligned}\nu_{ij} I_{4,ij}^{[d+]^2} = & \frac{\binom{0}{i}_4}{\binom{0}{0}_4}\frac{\binom{0}{j}_4}{\binom{0}{0}_4}(d-2)(d-1) I_4^{[d+]^2} + \frac{\binom{0i}{0j}_4}{\binom{0}{0}_4} I_4^{[d+]} \\ & - \frac{\binom{0}{j}_4}{\binom{0}{0}_4}\frac{d-2}{\binom{0}{0}_4}\sum_{t=1}^{4}\binom{0t}{0i}_4 I_3^{[d+],t} + \frac{1}{\binom{0}{0}_4}\sum_{t=1}^{4}\binom{0t}{0j}_4 I_{3,i}^{[d+],t}\end{aligned} \tag{20}$$

These equations are free of inverse Gram determinants $()_4$.
But they contain yet the generic 4-point and (partly indexed) 3-point functions in higher dimensions, $I_4^{[d+],s}$, $I_3^{[d+],t}$, etc.

Last step: evaluate the $I_4^{[d+],s}$, $I_3^{[d+],t}$, etc. I

Several strategies are now possible:

- Just evaluate them analytically in $d+2l-2\epsilon$ dimensions – if you may do that
- Just evaluate them numerically in $d+2l-2\epsilon$ dimensions
- Reduce them further by recurrences – buy the towers of $1/()_4$ $\rightarrow$ apply (14)
- Make a small Gram determinant expansion $\rightarrow$ apply (14) another way round

Last two items are done here.

Reduction of scalars I_4^D to the generic dimension $\rightarrow I_4^d = D_0,\, I_3^d = C_0$ I

Non-small 4-point Gram determinants:

Direct, iterative use of (14) yields e.g.:

$$I_4^{[d+]^l} = \left[\frac{\binom{0}{0}_4}{()_4} I_4^{[d+]^{l-1}} - \sum_{t=1}^{4} \frac{\binom{t}{0}_4}{()_4} I_3^{[d+]^{l-1},t} \right] \frac{1}{d+2l-5} \tag{21}$$

$$I_3^{[d+]^l,t} = \left[\frac{\binom{0t}{0t}_4}{\binom{t}{t}_4} I_3^{[d+]^{l-1},t} - \sum_{u=1,u\neq t}^{4} \frac{\binom{ut}{0t}_4}{\binom{t}{t}_4} I_2^{[d+]^{l-1},tu} \right] \frac{1}{d+2l-4} \tag{22}$$

And we are done.

This works fine if $()_4$ is not small [and also the $\binom{t}{t}_4$].

Make a small Gram expansion I

Again use (14):

$$()_4(d - \sum_{i=1}^{4} \nu_i + 1) I_4^{[d+]} = \left[\binom{0}{0}_4 I_4 - \sum_{k=1}^{4} \binom{0}{k}_4 I_3^k \right]$$

If $()_4 = 0$, then it follows ($n = 4$):

$$I_n^D = \sum_k^n \frac{\binom{0}{k}_n}{\binom{0}{0}_n} I_{n-1}^{D,k} \qquad (23)$$

If $()_4 << 1$, re-write (14), as follows:

$$I_n^D = \sum_k^n \frac{\binom{0}{k}_n}{\binom{0}{0}_n} I_{n-1}^{D,k} - \frac{()_n}{\binom{0}{0}_n} [(D+1) - \sum_i^n \nu_i] I_n^{D+2}. \qquad (24)$$

Effectively we may evaluate I_n^D in terms of simpler functions $I_{n-1}^{D,k}$ with a small correction depending on I_n^{D+2}.

We may go a step further, and insert into (24) for I_n^{D+2} the rhs. of (23), taken now at $D' = D+2$:

$$
\begin{aligned}
I_n^D \quad = \quad & \sum_k^n \frac{\binom{0}{k}_n}{\binom{0}{0}_n} I_{n-1}^{D,k} \\
& - \frac{()_n}{\binom{0}{0}_n} [(D+1) - \sum_i^n \nu_i] \\
& \times \left[\sum_k^n \frac{\binom{0}{k}_n}{\binom{0}{0}_n} I_{n-1}^{D+2,k} - \frac{()_n}{\binom{0}{0}_n} [(D+3) - \sum_i^n \nu_i] I_n^{D+4} \right].
\end{aligned}
$$

The terms proportional to $[()_n/\binom{0}{0}_n]^a, a = 0, 1$ may be evaluated at the correct kinematics. They depend on three-point functions, and their reduction by normal recurrences will not introduce the unwanted powers of $1/()_4$. The last term, suppressed by the factor $[()_n/\binom{0}{0}_n]^2$, depends on I_n^{D+4}. It may either be taken approximately at $()_n = 0$, where it can also be represented by 3-point functions (and their reductions), or it may be evaluated more correctly by another iteration based on (23).

And so on and so on ...

In the numerical example – next section – we worked out up to 10 stable iterations.

A quite similar attempt to perform such a series of approximations was undertaken in [14] (see equation (5) there), where a specific example, forward light-by-light scattering through a massless fermion loop, was studied. The approach was then not further followed.

W. Giele, E. W. N. Glover, and G. Zanderighi,
in: Proceedings of Loops ans Legs 2004:
Numerical evaluation of one-loop diagrams near exceptional momentum configurations,

Following Davydychev, [7], one gets

$$I_4^{\mu\nu\lambda} = \int^d \frac{k^\mu k^\nu k^\lambda}{\prod_{r=1}^n c_r} = -\sum_{i,j,k=1}^{n} q_i^\mu q_j^\nu q_k^\lambda \nu_{ijk} I_{n,ijk}^{[d+]^3} + \frac{1}{2}\sum_{i=1}^{n} g^{[\mu\nu} q_i^{\lambda]} I_{n,i}^{[d+]^2} \quad (25)$$

We identify the tensor coefficients $D_{11\dots}$ a la LoopTools, e.g.:

$$D_{111} = I_{4,222}^{[d+]^3} \quad (26)$$

Similarly:

$$D_{1111} = I_{4,2222}^{[d+]^4} \quad (27)$$

Rank $R = 4$ tensor D_{1111} – Numerics with dimensional recurrences

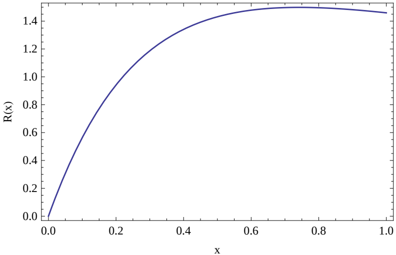
From (24) we see that a "small Gram determinant" expansion will be useful when the following dimensionless parameter becomes small:

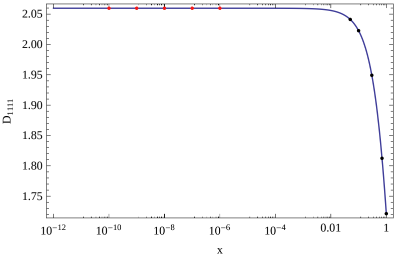
$$R = \frac{()_4}{\binom{0}{0}_4} \times s, \tag{28}$$

where s is a typical scale of the process, e.g. we will choose $s = s_{\mu\bar{\nu}u}$.
Following [12], we further choose:

$$\begin{aligned} s_{\mu\bar{\nu}u} &= 2 \times 10^4 \mathrm{GeV}^2, \\ s_{\bar{\nu}u} &= 1 \times 10^4 \mathrm{GeV}^2, \\ t_{\bar{e}\mu} &= -4 \times 10^4 \mathrm{GeV}^2, \end{aligned}$$

and get $t_{ed,\mathrm{crit}} = -6 \times 10^4 \mathrm{GeV}^2$. For x=1, the Gram determinant becomes $()_4 = 4.8 \times 10^{13}$ GeV^3.
The small expansion parameter $R(x)$ and D_{1111} are shown in figure 2.





New for QCD at LHC: small Gram expansion and Pade approximation I

[Fleischer,TR: PRD 2011 [6]]

Tables have been taken from there.

They were shown already at QCD@LHC@Trento2010

The use of appropriate Pade approximations is explained there.
Convergence in the small Gram determinant region is considerably improved.

x		$\Re e\ D_{1111}$	$\Im m\ D_{1111}$
0.	[exp 0,0]	2.05969289730 E-10	1.55594910118 E-10
10^{-8}	[exp x,2]	2.05969289342 E-10	1.55594909187 E-10
	[exp 0,2]	2.05969289349 E-10	1.55594909187 E-10
10^{-4}	[exp x,5]	2.05965609497 E-10	1.55585605343 E-10
	[exp 0,5]	2.05965609495 E-10	1.55585605343 E-10
0.001	[exp 0,6]	2.05932484380 E-10	1.55501912433 E-10
	[exp x,6]	2.05932484381 E-10	1.55501912433 E-10
	$I_{4,2222}^{[d+]^4}$	2.02292295240 E-10	1.54974785467 E-10
	D_{1111}	2.01707671668 E-10	1.62587142251 E-10
0.005	[exp 0,6]	2.05786054801 E-10	1.55131031024 E-10
	[pade 0,3]	2.05785198947 E-10	1.55131031003 E-10
	[exp x,6]	2.05786364440 E-10	1.55131031024 E-10
	[pade x,3]	2.05785199805 E-10	1.55131030706 E-10
	$I_{4,2222}^{[d+]^4}$	2.05778894114 E-10	1.55135794453 E-10
	D_{1111}	2.05779811490 E-10	1.55136343923 E-10
0.01	[exp 0,6]	2.05703298143 E-10	1.54669910676 E-10
	[pade 0,3]	2.05600940065 E-10	1.54669907784 E-10
	[exp 0,10]	2.05600964693 E-10	1.54669910676 E-10
	[pade 0,5]	2.05600955381 E-10	1. 54669910676E-10
	[exp x,10]	2.05600963675 E-10	1.54669910676 E-10
	[pade x,5]	2.05600955381 E-10	1.54669910676 E-10
	$I_{4,2222}^{[d+]^4}$	2.05600013702 E-10	1.54670651917 E-10
	D_{1111}	2.05600239280 E-10	1.54670771210 E-10

Table: Numerical values for the tensor coefficient D_{1111}. Values marked by D_{1111} are evaluated with LoopTools, the $I_{4,2222}^{[d+]^4}$ corresponds to (30) The labels [exp 0,2n] and [pade 0,n] denote iteration $2n$ and Pade approximant $[n, n]$ when the small Gram determinant expansion starts at $x = 0$, and [exp x,2n] and [pade x,n] are the corresponding numbers for an expansion starting at x.

x	$\Re e\, D_{1111}$	$\Im m\, D_{1111}$
0.01 [exp 0,6]	2.05703298143 E-10	1.54669910676 E-10
[pade 0,3]	2.05600940065 E-10	1.54669907784 E-10
[exp 0,10]	2.05600964693 E-10	1.54669910676 E-10
[pade 0,5]	2.05600955381 E-10	1. 54669910676E-10
[exp x,10]	2.05600963675 E-10	1.54669910676 E-10
[pade x,5]	2.05600955381 E-10	1.54669910676 E-10
$I_{4,2222}^{[d+]^4}$	2.05600013702 E-10	1.54670651917 E-10
D_{1111}	2.05600239280 E-10	1.54670771210 E-10
0.05 [exp 0,6]	4.83822963052 E-09	1.51077429118 E-10
[pade 0,3]	2.01518061131 E-10	1.50591643209 E-10
[exp 0,20]	2.04218962072 E-10	1.51077424143 E-10
[pade 0,10]	2.04122727654 E-10	1.51077424149 E-10
[exp x,20]	2.04190274030 E-10	1.51077424143 E-10
[pade x,10]	2.04122727971 E-10	1.51077423985 E-10
$I_{4,2222}^{[d+]^4}$	2.04122726387 E-10	1.51077422901 E-10
D_{1111}	2.04122726601 E-10	1.51077423320 E-10
0.1 [exp 0,26]	2.20215264409 E-08	1.46815247004 E-10
[pade 0,13]	2.01749674352 E-10	1.46681287362 E-10
[exp x,26]	2.08190721550 E-08	1.46815247004 E-10
[pade x,13]	2.03995221326 E-10	1.46785977364 E-10
$I_{4,2222}^{[d+]^4}$	2.02269485177 E-10	1.46815247061 E-10
D_{1111}	2.02269485217 E-10	1.46815247051 E-10
1. $I_{4,2222}^{[d+]^4}$	1.72115440143 E-10	9.74550747662 E-11
D_{1111}	1.72115440148 E-10	9.74550747662 E-11

Table: Numerical values for the tensor coefficient D_{1111}. Values marked by D_{1111} are evaluated with LoopTools, the $I_{4,2222}^{[d+]^4}$ corresponds to (30) The labels [exp 0,2n] and [pade 0,n] denote iteration $2n$ and Pade approximant $[n, n]$ when the small Gram determinant expansion starts at $x = 0$, and [exp x,2n] and [pade x,n] are the corresponding numbers for an expansion starting at x.

PJFry - an open source c++ program by V. Yundin I

PJFry 1.0.0 - one loop tensor integral library

- More information and the latest source code:
  project page: https://github.com/Vayu/PJFry/
- $\rightarrow$ how to install
- $\rightarrow$ how to use
- $\rightarrow$ samples
- See also:
  V. Yundin's talk at LHCphenoNet meeting, Valencia, Feb 2011:
  "One loop tensor reduction program PJFRY"

PJFry - an open source c++ program by V. Yundin II

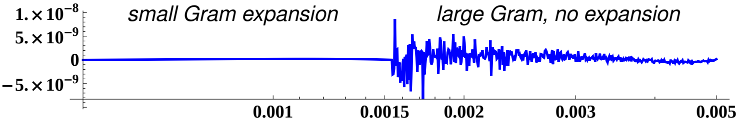
- Yundin's PhD thesis, submitted Aug 2011 at Humboldt University

PJFry — numerical package [from V.Y. Valencia 2011] I

Numerical implementation of described algorithms:
C++ package **PJFry** by V. Yundin [see project webpage]

- Reduction of **5-point** 1-loop tensor integrals up to **rank 5**
- No limitations on internal/external masses combinations
- Small Gram determinants treatment by expansion
- Interfaces for C, C++, FORTRAN and MATHEMATICA

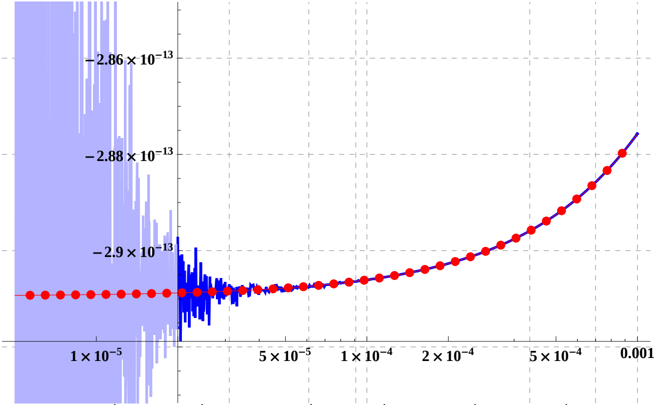
Example:
Relative accuracy of E_{3333} coef. around small Gram4 region



PJFry — small Gram region example [from V.Y. Valencia 2011]

Example: E_{3333} coefficient in small Gram region ($x = 0$)

Comparison of Regular and Expansion formulae:



$x{=}0$: $E_{3333}(0, 0, -6\times10^4, 0, 0, 10^4, -3.5\times10^4, 2\times10^4, -4\times10^4, 1.5\times10^4, 0, 6550, 0, 0, 8315)$

Applications of Formalism and of **PJFry**

Formalism [not PJFry] used by Laura Reina et al. [also Paulo Rottman, DESY]

An application of PJFry to
$e^+e^- \to \mu^+\mu^-\gamma$
at meson factories is being finished.

Due to the specific cuts of the experiments, the numerics is relatively unstable.

Option of the GOLEM package for automatized NLO calculations for LHC processes.

Contractions with external momenta [or with CHORDS] I

[Fleischer,TR: PLB 2011 [11]]

After having tensor reductions with basis functions I_n^D,
which are independent of the indices $i, j, k, ...$,
one may use contractions with external momenta in order to perform all the sums over $i, j, k, ...$
This leads to a significant simplification and shortening of calculations.

Reminder:

One option was to avoid the appearance of inverse Gram determinants $1/()_5$. For rank $R = 5$, e.g.,

$$I_5^{\mu\,\nu\,\lambda\rho\sigma} = \sum_{s=1}^{5}\left[\sum_{i,j,k,l,m=1}^{5} q_i^{\mu} q_j^{\nu} q_k^{\lambda} q_l^{\rho} q_m^{\sigma} E_{ijklm}^{s} + \sum_{i,j,k=1}^{5} g^{[\mu\nu} q_i^{\lambda} q_j^{\rho} q_k^{\sigma]} E_{00ijk}^{s} \right.$$

$$\left. + \sum_{i=1}^{5} g^{[\mu\nu} g^{\lambda\rho} q_i^{\sigma]} E_{0000i}^{s} \right] \qquad (29)$$

Contractions with external momenta [or with CHORDS] I

The tensor coefficients are expressed in terms of integrals $I_{4,i\cdots}^{[d+]^l,s}$, e.g.:

$$E_{ijklm}^{s} = -\frac{1}{\binom{0}{0}_5}\left\{\left[\binom{0l}{sm}_5 n_{ijk} I_{4,ijk}^{[d+]^4,s} + (i\leftrightarrow l) + (j\leftrightarrow l) + (k\leftrightarrow l)\right] + \binom{0s}{0m}_5 n_{ijkl} I_{4,ijkl}^{[d+]^4,s}\right\}.$$

Now, in a next step, one may avoid the appearance of inverse sub-Gram determinants $()_4$.
The complete dependence on the indices i of the tensor coefficients is contained now in the pre-factors with signed minors. One can say that the indices *decouple* from the integrals.
As an example, we reproduce the 4-point part of

$$\begin{aligned} n_{ijkl} I_{4,ijkl}^{[d+]^4} &= \frac{\binom{0}{i}}{\binom{0}{0}}\frac{\binom{0}{j}}{\binom{0}{0}}\frac{\binom{0}{k}}{\binom{0}{0}}\frac{\binom{0}{l}}{\binom{0}{0}} d(d+1)(d+2)(d+3) I_4^{[d+]^4} \\ &+ \frac{\binom{0i}{0j}\binom{0}{k}\binom{0}{l} + \binom{0i}{0k}\binom{0}{j}\binom{0}{l} + \binom{0j}{0k}\binom{0}{i}\binom{0}{l} + \binom{0i}{0l}\binom{0}{j}\binom{0}{k} + \binom{0j}{0l}\binom{0}{i}\binom{0}{k} + \binom{0k}{0l}\binom{0}{i}\binom{0}{j}}{\binom{0}{0}^3} d(d+1) I_4^{[d+]^3} \\ &+ \frac{\binom{0i}{0l}\binom{0j}{0k} + \binom{0j}{0l}\binom{0i}{0k} + \binom{0k}{0l}\binom{0i}{0j}}{\binom{0}{0}^2} I_4^{[d+]^2} + \cdots \end{aligned} \quad (30)$$

] II

Contractions with external momenta [or with CHORDS]

In (30), one has to understand the 4-point integrals to carry the corresponding index s and the signed minors are $\binom{0}{k} \to \binom{0s}{ks}_5$ etc.

Contractions with external momenta [or with CHORDS] I

A chord is the momentum shift of an internal line due to external momenta, $D_i = (k - q_i)^2 - m_i^2 + i\epsilon$, and $q_i = (p_1 + p_2 + + p_i)$, with $q_n = 0$.

The tensor 5-point integral of rank $R = 1$ yields, when contracted with a chord,

$$q_{a\mu} I_5^{\mu} \quad = \quad -\frac{1}{\binom{0}{0}_5} \sum_{s=1}^{5} \left[\sum_{i=1}^{4} (q_a \cdot q_i) \begin{pmatrix} 0i \\ 0s \end{pmatrix}_5 \right] I_4^s. \tag{31}$$

In fact, the sum over i may be performed explicitly:

$$\Sigma_a^{1,s} \quad \equiv \quad \sum_{i=1}^{4} (q_a \cdot q_i) \begin{pmatrix} 0s \\ 0i \end{pmatrix}_5 \; = \; +\frac{1}{2} \left\{ \begin{pmatrix} s \\ 0 \end{pmatrix}_5 (Y_{a5} - Y_{55}) + \begin{pmatrix} 0 \\ 0 \end{pmatrix}_5 (\delta_{as} - \delta_{5s}) \right\},$$

Contractions with external momenta I

We get immediately

$$q_{a\mu} I_5^{\mu} = -\frac{1}{\binom{0}{0}_5} \sum_{s=1}^{5} \Sigma_a^{1,s} \quad I_4^s. \tag{32}$$

Contractions with external momenta I

The tensor 5-point integral of rank $R = 2$

$$I_5^{\mu\,\nu} = \sum_{i,j=1}^{4} q_i^{\mu}\, q_j^{\nu} E_{ij} + g^{\mu\nu} E_{00}, \tag{33}$$

has the following tensor coefficients free of $1/()_5$:

$$E_{00} = -\sum_{s=1}^{5} \frac{1}{2} \frac{1}{\binom{0}{0}_5} \binom{s}{0}_5 I_4^{[d+],s}, \tag{34}$$

$$E_{ij} = \sum_{s=1}^{5} \frac{1}{\binom{0}{0}_5} \left[\binom{0i}{sj}_5 I_4^{[d+],s} + \binom{0s}{0j}_5 I_{4,i}^{[d+],s} \right]. \tag{35}$$

Contractions with external momenta I

Equation (33) yields for the contractions with chords:

$$q_{a\mu} q_{b\nu} I_5^{\mu\,\nu} = \sum_{i,j=1}^{4} (q_a \cdot q_i)(q_b \cdot q_j) E_{ij} + (q_a \cdot q_b) E_{00}. \tag{36}$$

and finally (36) simply reads

$$\begin{aligned} q_{a\mu} q_{b\nu} I_5^{\mu\,\nu} \quad &= \quad \frac{1}{4} \sum_{s=1}^{5} \Biggl\{ \frac{\binom{s}{0}_5}{\binom{0s}{0s}_5} (\delta_{ab}\delta_{as} + \delta_{5s}) + \frac{\binom{s}{s}_5}{\binom{0s}{0s}_5} \Bigl[(\delta_{as} - \delta_{5s}) \left(Y_{b5} - Y_{55} \right) \\ &+ \; (\delta_{bs} - \delta_{5s}) \left(Y_{a5} - Y_{55} \right) + \frac{\binom{s}{0}_5}{\binom{0}{0}_5} \left(Y_{a5} - Y_{55} \right) \left(Y_{b5} - Y_{55} \right) \Bigr] \Biggr\} I_4^{[d+],s} \\ &+ \frac{1}{\binom{0}{0}_5} \sum_{s=1}^{5} \frac{\Sigma_b^{1,s}}{\binom{0s}{0s}_5} \sum_{t=1}^{5} \Sigma_a^{2,st} I_3^{st}, \end{aligned}$$

Contractions with external momenta I

with

$$\Sigma_a^{2,st} \equiv \sum_{i=1}^{4}(q_a \cdot q_i)\begin{pmatrix}0st\\0si\end{pmatrix}_5$$
$$= \frac{1}{2}(1-\delta_{st})\left\{\begin{pmatrix}ts\\0s\end{pmatrix}_5 (Y_{a5}-Y_{55}) + \begin{pmatrix}0s\\0s\end{pmatrix}_5 (\delta_{at}-\delta_{5t}) - \begin{pmatrix}0s\\0t\end{pmatrix}_5 (\delta_{as}-\delta_{5s})\right\}$$

This has been extended also to higher ranks.
We need at most double sums, e.g.:

$$\Sigma_{ab}^{2,s} \equiv \sum_{i,j=1}^{4}(q_a \cdot q_i)(q_b \cdot q_j)\begin{pmatrix}si\\sj\end{pmatrix}_5 \frac{1}{2}(q_a \cdot q_b)\begin{pmatrix}s\\s\end{pmatrix}_5$$
$$= -\frac{1}{4}()_5\left(\delta_{ab}\delta_{as}+\delta_{5s}\right), \qquad (37)$$

Contractions with external momenta I

Many of the sums over signed minors, weighted with scalar products of chords are given in PLB 2011 [[11]], and an almost complete list may be obtained on request from J. Fleischer, T.R.

Modifications for 7- and higher point functions I

Here, the Gram determinant vanishes, and also further determinants:
$()_n = 0, n > 5$
but also $\binom{0}{k}_7 = 0$
As a result, one has to reorganize the reductions, avoiding the $1/()_n$ completely.
This may be done, and we are coming so far to expressions with $I_6^{[d+]}$.
The problematic case is the integral $I_{6,i}^{[d+]}$ for which we can write

$$I_{6,i}^{[d+]} = \sum_{s=1, s\neq i}^{7} \frac{\binom{R}{s}_6}{\binom{R}{0}_6} I_{5,i}^{[d+],s} + \frac{\binom{R}{i}_6}{\binom{R}{0}_6} I_6^{[d+]}.$$

Modifications for 7- and higher point functions II

For $I_6^{[d+]}$ we finally need 4-point functions up to the order ϵ in the basis.

But:
We know from Gudrun Heinrich et al. that there is a representation in terms just of ordinary 4-point functions:

Binoth, Guillet, Heinrich, hep-ph/0504267, hep-ph/9911342

$n > 6$: assumption in Bern,Dixon,Kosower, hep-ph/9306240

Summary

- Recursive treatment of hexagon and pentagon tensor integrals of rank R in terms of pentagons and boxes of rank $R-1$
- Systematic derivation of expressions which are explicitely free of inverse Gram determinants $()_5$ until pentagons of rank $R=5$
- Proper isolation of inverse Gram determinants of subdiagrams of the type $\binom{s}{s}_n 4$, which cannot be completely avoided
- Numerical C++ package PJFry (V. Yundin, open source) for C, c++, Mathematica, Fortran
- Perform multiple sums with signed minors and scalar products after contractions with chords or external momenta

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Miyamoto Musashi, Samurai, (1584 – 1645)

Miyamoto Musashi (1584 – 1645)

The Five Rings

The Broad Principles of Musashi's Strategy

- Do not think dishonestly.
- The Way is in training.
- Become aquainted with every art.
- Know the Ways of all professions.
- Distinguish between gain and loss in worldly matters.
- Develop intuitive judgement and understanding for everything.
- Perceive those things which cannot be seen.
- Pay attention even to trifles.
- Do nothing which is of no use.

http://commons.wikimedia.org/wiki/File:Miyamoto_Musas%EF%BD%88_Self-

Text source: old version of www.lucidcafe.com