

# Tensor integrals

## New solutions to old problems

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Based on work done in collaboration with Jochem Fleischer and Valery Yundin  
 talk held at German Japanese WS on "Modern Trends in Quantum Chromodynamics", 3-5 October 2011,  
 DESY, Zeuthen, Germany

## A simple example

### 1-loop self-energy:

$$\begin{aligned}
 I_2^\mu &= \int \frac{d^d k}{i\pi^{d/2}} \frac{k^\mu}{[k^2 - M_1^2][(k+p)^2 - M_2^2]} \\
 &= p_\mu B_1
 \end{aligned}$$

### Solve:

$$\begin{aligned}
 p_\mu I_2^\mu &= p^2 B_1(p, M_1, M_2) \\
 &= \int \frac{d^d k}{i\pi^{d/2}} \frac{pk}{[k^2 - M_1^2][(k+p)^2 - M_2^2]} = \int \frac{d^d k}{i\pi^{d/2}} \frac{pk}{D_1 D_2} \\
 &= \frac{1}{2} \int \frac{d^d k}{i\pi^{d/2}} \left[ \frac{D_2 - (p^2 - M_2^2 - M_1^2) - D_1}{D_1 D_2} \right],
 \end{aligned}$$

$$B_1(p, M_1, M_2) = \frac{1}{2p^2} \left[ A_0(M_1) - A_0(M_2) - \frac{1}{2}(p^2 - M_2^2 - M_1^2) B_0(p, M_1, M_2) \right]$$

A **tensor** Feynman integral is expressed in terms of **scalar** Feynman integrals.

## Reduction of tensor integrals

Reduction of tensor integrals = expressing them by a (very) small set of scalar integrals  
 – is needed for practically any Feynman diagram calculation.

Feynman diagrams with loops contain tensor integrals due to:

- fermion propagators
- three-gauge boson couplings
- e.g. unitary gauge propagators

Examples:

- LO (Lowest order) of  $Z \rightarrow e + \mu$  is one-loop; processes forbidden in tree diagrams of Standard Model
- NLO: one-loop corrections to Born diagrams; prominent by now for LHC physics:  $2 \rightarrow 3, 4, \dots$
- NNLO: radiative loop corrections to Born diagrams: e.g.  $e^+e^- \rightarrow e^+e^-\gamma$  with a loop (including 5-point functions)

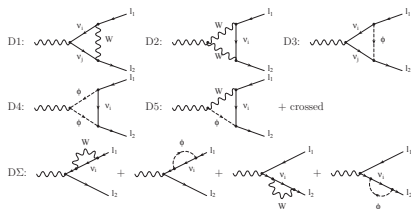
## My first tensor reduction

My first tensor reduction [Proc. Ahrenshoop Symposium, 1981] was for:

$$Z \rightarrow \mu + e, \quad b + s, \quad \text{etc.} \quad (1)$$

Use of the 'conventional' Passarino-Veltman reduction (Nucl.Phys.B160, 1979), invented not so much earlier.

The process is, in the Standard Model, possible due to one-loop vertex diagrams with internal flavor-changing  $W$  bosons and mixing heavy fermions:



J. Illana, TR, PRD 2000 study repeated for ILC applications for SM and models with heavy neutrinos

Later J. Illana et al. studied also supersymmetric cases.

## Comparison with grace in 2002: weak corr.s to top pair production at ILC/NLC

### Numerical Results Based on

J. Fleischer, A. Leike, T. Riemann, A. Werthenbach

and

J. Fujimoto, T. Ishikawa, Y. Shimizu

One-loop Corrections to the Process  $e^+ e^- \rightarrow t\bar{t}$   
Including Hard Bremsstrahlung

hep-ph/0203220

(weak corrections and also hard bremsstrahlung, topfit compared to Grace)

## Top-quark Pair-production Cross section

### Conventions for comparisons

$$\Gamma_Z = 0$$

$$\alpha = \frac{e^2}{4\pi} = 1/137.03599976$$

$$E_{\gamma}^{\max} = \sqrt{s}/10$$

$$M_W = 80.4514958 \text{ GeV}$$

$$M_Z = 91.1867 \text{ GeV}$$

$$M_h = 120 \text{ GeV}$$

$$m_e = 0.00051099907 \text{ GeV}$$

$$m_t = 173.8 \text{ GeV}$$

$$m_b = 4.7 \text{ GeV}$$

$$m_{\mu} = 0.105658389 \text{ GeV}$$

$$m_u = 0.062 \text{ GeV}$$

$$m_d = 0.083 \text{ GeV}$$

$$m_{\tau} = 1.77705 \text{ GeV}$$

$$m_c = 1.5 \text{ GeV}, \quad m_s = 0.215 \text{ GeV}$$

$\cos \theta$	$\omega/\sqrt{s}$	$\left[\frac{d\sigma}{d\cos\theta}\right]_{\text{Born}}$	$\left[\frac{d\sigma}{d\cos\theta}\right]_{\text{QED}}$	$\left[\frac{d\sigma}{d\cos\theta}\right]_{\text{SM}}$	$\left[\frac{d\sigma}{d\cos\theta}\right]_{\text{tot}}$
-0.9	T : 0.1	0.108839194075	+0.098664253	+0.11408410	0.13144
	T : 0.00001	0.108839194075	-0.017474702	-0.002054858	0.13229
	G : 0.00001	0.108839194076		-0.002054859	0.13206(12)
-0.5	T : 0.1	0.142275069392	+0.12850790	+0.14308121	0.15973
	T : 0.00001	0.142275069392	-0.029702340	-0.015129038	0.16029
	G : 0.00001	0.142275069393		-0.015129039	0.16013(13)
+0.0	T : 0.1	0.225470464033	+0.20239167	+0.21718801	0.23638
	T : 0.00001	0.225470464033	-0.058010508	-0.043214169	0.23476
	G : 0.00001	0.225470464033		-0.043214168	0.23513(14)
+0.5	T : 0.1	0.354666470332	+0.31511723	+0.32933727	0.35651
	T : 0.00001	0.354666470332	-0.109721291	-0.095501257	0.35062
	G : 0.00001	0.354666470332		-0.095501252	0.35104(17)
+0.9	T : 0.1	0.491143715767	+0.43071437	+0.44290816	0.48796
	T : 0.00001	0.491143715767	-0.179672655	-0.16747886	0.47768
	G : 0.00001	0.491143715767		-0.16747886	0.47709(21)

Various differential cross sections (see also text). The upper and lower numbers correspond to the  $\text{topfit}$  (T) and GRACE (G) approach, respectively,  $\sqrt{s} = 500$  GeV.

$\cos \theta$	$\omega/\sqrt{s}$	$\left[\frac{d\sigma}{d\cos\theta}\right]_{\text{Born}}$	$\left[\frac{d\sigma}{d\cos\theta}\right]_{\text{QED}}$	$\left[\frac{d\sigma}{d\cos\theta}\right]_{\text{SM}}$	$\left[\frac{d\sigma}{d\cos\theta}\right]_{\text{tot}}$
-0.9	T : 0.1	0.0227854232732	+0.020365844	+0.023101706	0.036334
	T : 0.00001	0.0227854232732	-0.004756230	-0.002020367	0.036461
	G : 0.00001	0.02278542327319		-0.002020369	0.036582(48)
-0.5	T : 0.1	0.0297821311031	+0.026741663	+0.028823021	0.038888
	T : 0.00001	0.0297821311031	-0.008561495	-0.006480137	0.039055
	G : 0.00001	0.0297821311031		-0.006480139	0.038965(42)
+0.0	T : 0.1	0.0611800674224	+0.054539344	+0.054950889	0.067789
	T : 0.00001	0.0611800674224	-0.021532420	-0.021120874	0.067801
	G : 0.00001	0.0611800674225		-0.021120874	0.068039(55)
+0.5	T : 0.1	0.117746949888	+0.10311626	+0.099416999	0.12095
	T : 0.00001	0.117746949888	-0.050123708	-0.053822973	0.12051
	G : 0.00001	0.117746949888		-0.053822964	0.12064(07)
+0.9	T : 0.1	0.181122097086	+0.15403823	+0.14426232	0.19355
	T : 0.00001	0.181122097086	-0.096682759	-0.10645866	0.19272
	G : 0.00001	0.181122097086		-0.10645866	0.19057(10)

Same as other table,  $\sqrt{s} = 1000 \text{ GeV}$ .



## Systematic approach

to **tensor reductions**:

1- to 4-point functions: Passarino, Veltman **1978** [1]

Need in addition a **library of scalar functions**:

'tHooft, Veltman **1979** [2]

**State of the art + open source programs**:

K. Ellis and G. Zanderighi, QCDloop/FF [3, 4] **2007, 1990**

T. Hahn, LoopTools/FF [5, 4] **1998, 1990**

Open source programs for 5,6,7-point reductions:

- LoopTools/FF [T. Hahn et al.]

- Golem/OPP etc. [G. Heinrich; A. van Hameren, this conference]

- HELAC/PHEGAS/OPP etc. [A. van Hameren, this conference]

→ c++ code PJFry by V. Yundin, PhD thesis + [6] **2010**

[Fleischer/Riemann/Yundin group], released this Summer **2011**

## This talk: Efficient reduction formulae in the algebraic Fleischer-Davydychev-Tarasov approach

Recent developments in the Fleischer-Davydychev-Tarasov approach:

- Get  $n > 4$  tensor reduction with  $\dots$ :
- $\dots$  **arbitrary** masses
- $\dots$  **killed** pentagon Gram determinants
- $\dots$  **treatment of** full kinematics, also with small sub-diagram Gram determinants  
→ presented by J. Fleischer at QCD@LHC@Trento2010
- → **c++ code PJFry**,  $n \leq 5$  by V. Yundin [ $\rightarrow$  GOLEM option]  
see talk at LHCphenonet Meeting 02/2011
- $\dots$  **multiple sums over tensor coefficients** made efficient by contracting with external momenta arXiv/1104.4067, PLB 701(2011)646
- $\dots$   $n \geq 7$  – a fresh look

## Outline

- [7] 1991 Davydychev, . . . *Reducing Feynman diagrams to scalar integrals*
- [8] 1996 Tarasov, *Connection [of] Feynman integrals [with] different . . . space-time dimensions*
- [9] 1999 Fleischer et al., *Algebraic reduction of one-loop Feynman graph amplitudes*

- 1 Introduction
- 2 Recursions
- 3 Simplifying
- 4  $D_{1111}$
- 5 Small Grams
- 6 PJFry
- 7 External momenta
- 8 Summary
- 9 Musashi

### References:

- [10] 2010 Diakonidis et al., PLB 683, . . . *recursive reduction of tensor Feynman integrals*
- [6] 2011 Fleischer, T.R., PRD 83, *Complete . . . reduction of . . . tensor Feynman integrals*
- [11] 2011 Fleischer, T.R., PLB 701, . . . *contracted tensor Feynman integrals*  
 subm. Aug. 2011: V. Yundin, PhD thesis [with PJFry code]

## Notations: Gram and modified Cayley determinant, signed minors [Melrose:1965]

Gram determinant  $G_n$ :

$$G_n = |2q_i q_j|, i, j = 1, \dots, n \quad (2)$$

Modified Cayley determinant  $()_N$  of a diagram with  $N$  internal lines and chords  $q_j$ ; for a choice  $q_n = 0$ , both determinants are related:

$$()_N \equiv \begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & Y_{11} & Y_{12} & \dots & Y_{1N} \\ 1 & Y_{12} & Y_{22} & \dots & Y_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & Y_{1N} & Y_{2N} & \dots & Y_{NN} \end{vmatrix} = -G_{N-1}, \quad (3)$$

where  $D_i = (k - q_i)^2 - m_i^2$  [with  $q_i = \text{chord}$ ], and the matrix elements

$$Y_{ij} = -(q_i - q_j)^2 + m_i^2 + m_j^2, \quad (i, j = 1 \dots N) \quad (4)$$

⇒ The Gram determinant  $()_N$  does not depend on the masses.

## Notations: signed minors [Melrose:1965]

signed minors of  $(\ )_N$  are constructed by deleting  $m$  rows and  $m$  columns from  $(\ )_N$ , and multiplying with a sign factor:

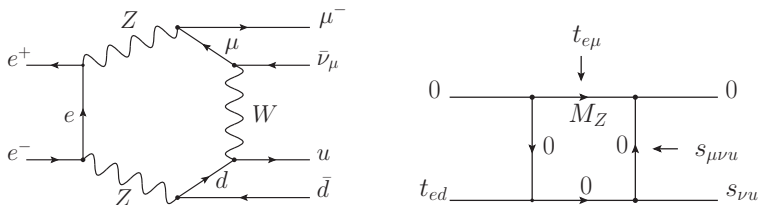
$$\begin{aligned} \begin{pmatrix} j_1 & j_2 & \cdots & j_m \\ k_1 & k_2 & \cdots & k_m \end{pmatrix}_N &\equiv \\ &\equiv (-1)^{\sum_i (j_i + k_i)} \operatorname{sgn}_{\{j\}} \operatorname{sgn}_{\{k\}} \left| \begin{array}{c} \text{rows } j_1 \cdots j_m \text{ deleted} \\ \text{columns } k_1 \cdots k_m \text{ deleted} \end{array} \right| \end{aligned} \quad (5)$$

where  $\operatorname{sgn}_{\{j\}}$  and  $\operatorname{sgn}_{\{k\}}$  are the signs of permutations that sort the deleted rows  $j_1 \cdots j_m$  and columns  $k_1 \cdots k_m$  into ascending order.

Example:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}_N \equiv \begin{vmatrix} Y_{11} & Y_{12} & \cdots & Y_{1N} \\ Y_{12} & Y_{22} & \cdots & Y_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{1N} & Y_{2N} & \cdots & Y_{NN} \end{vmatrix}, \quad (6)$$

## Example: Getting a 4-point function from a six-point function I



**Figure:** A six-point topology (a) leading to four-point functions (b) with realistically vanishing Gram determinants.

## Example: Getting a 4-point function from a six-point function II

The example is taken from [12].

The corresponding 4-point tensor integrals are, in LoopTools [5, 13] notation:

$$D0i(\text{id}, 0, 0, s_{\bar{\nu}U}, t_{ed}, t_{\bar{e}\mu}, s_{\mu\bar{\nu}U}, 0, M_Z^2, 0, 0). \quad (7)$$

The Gram determinant is:

$$(\ )_4 = -2t_{\bar{e}\mu}[s_{\mu\bar{\nu}U}^2 + s_{\bar{\nu}U}t_{ed} - s_{\mu\bar{\nu}U}(s_{\bar{\nu}U} + t_{ed} - t_{\bar{e}\mu})], \quad (8)$$

It vanishes if:

$$t_{ed} \rightarrow t_{ed,\text{crit}} = \frac{s_{\mu\bar{\nu}U}(s_{\mu\bar{\nu}U} - s_{\bar{\nu}U} + t_{\bar{e}\mu})}{s_{\mu\bar{\nu}U} - s_{\bar{\nu}U}}. \quad (9)$$

In terms of a dimensionless scaling parameter  $x$ ,

$$t_{ed} = (1 + x)t_{ed,\text{crit}}, \quad (10)$$

## Example: Getting a 4-point function from a six-point function III

the Gram determinant becomes:

$$(\cdot)_4 = 2 \times s_{\mu\bar{\nu}U} t_{\bar{\theta}\mu} (s_{\mu\bar{\nu}U} - s_{\bar{\nu}U} + t_{\bar{\theta}\mu}). \quad (11)$$

We will also need the modified Cayley determinant:

$$\begin{aligned} \begin{pmatrix} 0 \\ 0 \end{pmatrix}_4 &= \begin{pmatrix} 2M_Z^2 & M_Z^2 & M_Z^2 - s_{\mu\bar{\nu}U} & M_Z^2 \\ M_Z^2 & 0 & -s_{\bar{\nu}U} & M_Z^2 \\ M_Z^2 - s_{\mu\bar{\nu}U} & -s_{\bar{\nu}U} & 0 & -t_{ed} \\ M_Z^2 & -t_{\bar{\theta}\mu} & -t_{ed} & 0 \end{pmatrix} \\ &= s_{\mu\bar{\nu}U}^2 t_{\bar{\theta}\mu}^2 + 2 M_Z^2 t_{\bar{\theta}\mu} [-2s_{\bar{\nu}U} t_{ed} + s_{\mu\bar{\nu}U} (s_{\bar{\nu}U} + t_{ed} - t_{\bar{\theta}\mu})] \\ &\quad + M_Z^4 (s_{\bar{\nu}U}^2 + (t_{ed} - t_{\bar{\theta}\mu})^2 - 2s_{\bar{\nu}U} (t_{ed} + t_{\bar{\theta}\mu})). \end{aligned}$$



## Dimensional shifts and recurrence relations for pentagons (I)

Following [Davydychev:1991 [7]]

Replace tensors by scalar integrals in higher dimensions:

Example  $R = 3$ :

$$\begin{aligned}
 I_5^{\mu\nu\lambda} &= \int \frac{d^{4-2\epsilon}k}{i\pi^{d/2}} \prod_{r=1}^5 c_r^{-1} k^\mu k^\nu k^\lambda & (12) \\
 &= - \sum_{i,j,k=1}^4 q_i^\mu q_j^\nu q_k^\lambda n_{ijk} I_{5,ijk}^{[d+]} + \frac{1}{2} \sum_{i=1}^{n-1} (g^{\mu\nu} q_i^\lambda + g^{\mu\lambda} q_i^\nu + g^{\nu\lambda} q_i^\mu) I_{5,i}^{[d+]} ,
 \end{aligned}$$

and  $n_{ijk} = (1 + \delta_{ij})(1 + \delta_{ik} + \delta_{jk})$ .

$$[d+]^l = 4 - 2\epsilon + 2l$$

$I_{5,i}^{[d+]} -$  scratch the line  $i$  from  $I_5^{[d+]}.$

## Dimensional shifts and recurrence relations for pentagons (II)

'Naive', direct approach – just perform dimensional recurrences

Following [Tarasov:1996, Fleischer:1999 [8, 9]]

apply **recurrence relations**, relating scalar integrals of different dimensions, in order to get rid of the dimensionalities  $[d+]^l = 4 - 2\epsilon + 2l$ :

$$\nu_j(\mathbf{j}^+ l_5^{[d+]}) = \frac{1}{\binom{0}{5}_5} \left[ -\binom{j}{0}_5 + \sum_{k=1}^5 \binom{j}{k}_5 \mathbf{k}^- \right] l_5 \quad (13)$$

$$\left( d - \sum_{i=1}^5 \nu_i + 1 \right) l_5^{[d+]} = \frac{1}{\binom{0}{5}_5} \left[ \binom{0}{0}_5 - \sum_{k=1}^5 \binom{0}{k}_5 \mathbf{k}^- \right] l_5, \quad (14)$$

also:

$$\nu_j \mathbf{j}^+ l_5 = \frac{1}{\binom{0}{5}_5} \sum_{k=1}^5 \binom{0j}{0k}_5 \left[ d - \sum_{i=1}^5 \nu_i (\mathbf{k}^- \mathbf{i}^+ + 1) \right] l_5 \quad (15)$$

where the operators  $\mathbf{i}^\pm, \mathbf{j}^\pm, \mathbf{k}^\pm$  act by shifting the indices  $\nu_i, \nu_j, \nu_k$  by  $\pm 1$ .

## The result of simplifying manipulations

... and collecting all contributions, our final result for e.g. the tensor of rank  $R = 3$  can be written as follows:

$$I_5^{\mu\nu\lambda} = \sum_{i,j,k=1}^4 q_i^\mu q_j^\nu q_k^\lambda E_{ijk} + \sum_{k=1}^4 g^{[\mu\nu} q_k^{\lambda]} E_{00k}, \quad (16)$$

with:

$$E_{00j} = \sum_{s=1}^5 \frac{1}{\binom{0}{0}_5} \left[ \frac{1}{2} \binom{0s}{0j}_5 I_4^{[d+],s} - \frac{d-1}{3} \binom{s}{j}_5 I_4^{[d+]^2,s} \right], \quad (17)$$

$$E_{ijk} = - \sum_{s=1}^5 \frac{1}{\binom{0}{0}_5} \left\{ \left[ \binom{0j}{sk}_5 I_4^{[d+]^2,s} + (i \leftrightarrow j) \right] + \binom{0s}{0k}_5 \nu_{ij} I_4^{[d+]^2,s} \right\}. \quad (18)$$

✓ no scalar 5-point integrals in higher dimensions

✓ no inverse Gram det.  $\binom{0}{0}_5$

We have yet:

† scalar 4-point integrals in higher dimensions:  $I_{4,ij}^{[d+]^2,s}$  etc.

† inverse Gram det.  $\binom{0}{0}_5 \equiv \binom{0}{0}_4$

Reduce  $I_{4,ij\dots}^{[d+]',s}$  to  $I_4^{[d+]',s}$  plus simpler objects I

By nontrivial manipulations we get e.g.:

$$I_{4,i}^{[d+]',s} = \frac{1}{\binom{0s}{0s}_5} \left[ -\binom{0s}{is}_5 (d-3) I_4^{[d+]',s} + \sum_{t=1}^5 \binom{0st}{0si}_5 I_3^{st} \right] \quad (19)$$

$$\begin{aligned} \nu_{ij} I_{4,ij}^{[d+]^2} = & \frac{\binom{0}{i}_4 \binom{0}{j}_4}{\binom{0}{0}_4 \binom{0}{0}_4} (d-2)(d-1) I_4^{[d+]^2} + \frac{\binom{0j}{0j}_4}{\binom{0}{0}_4} I_4^{[d+]} \\ & - \frac{\binom{0}{j}_4}{\binom{0}{0}_4} \frac{d-2}{\binom{0}{0}_4} \sum_{t=1}^4 \binom{0t}{0i}_4 I_3^{[d+],t} + \frac{1}{\binom{0}{0}_4} \sum_{t=1}^4 \binom{0t}{0j}_4 I_{3,i}^{[d+],t} \quad (20) \end{aligned}$$

These equations are free of inverse Gram determinants  $(\ )_4$ .

But they contain yet the generic 4-point and (partly indexed) 3-point functions in higher dimensions,  $I_4^{[d+]',s}$ ,  $I_3^{[d+],t}$ , etc.

Last step: evaluate the  $I_4^{[d+],s}$ ,  $I_3^{[d+],t}$ , etc. |

Several strategies are now possible:

- Just evaluate them **analytically** in  $d + 2l - 2\epsilon$  dimensions – if you may do that
- Just evaluate them **numerically** in  $d + 2l - 2\epsilon$  dimensions
- **Reduce** them further by recurrences – buy the towers of  $1/()_4 \rightarrow$  apply (14)
- Make a **small Gram determinant expansion**  $\rightarrow$  apply (14) another way round

Last two items are done here.

Reduction of scalars  $I_4^D$  to the generic dimension  $\rightarrow I_4^d = D_0, I_3^d = C_0$  |

Non-small 4-point Gram determinants:

Direct, iterative use of (14) yields e.g.:

$$I_4^{[d+]' } = \left[ \frac{\binom{0}{0}_4}{\binom{t}{t}_4} I_4^{[d+]'-1} - \sum_{t=1}^4 \frac{\binom{t}{0}_4}{\binom{t}{t}_4} I_3^{[d+]'-1,t} \right] \frac{1}{d+2l-5} \quad (21)$$

$$I_3^{[d+]',t} = \left[ \frac{\binom{0t}{0t}_4}{\binom{t}{t}_4} I_3^{[d+]'-1,t} - \sum_{u=1, u \neq t}^4 \frac{\binom{ut}{0t}_4}{\binom{t}{t}_4} I_2^{[d+]'-1,tu} \right] \frac{1}{d+2l-4} \quad (22)$$

And we are done.

This works fine if  $\binom{t}{t}_4$  is not small [and also the  $\binom{t}{t}_4$ ].

## Make a small Gram expansion I

Again use (14):

$$({}_4)(d - \sum_{i=1}^4 \nu_i + 1)I_4^{[d+]} = \left[ \begin{pmatrix} 0 \\ 0 \end{pmatrix}_4 I_4 - \sum_{k=1}^4 \begin{pmatrix} 0 \\ k \end{pmatrix}_4 I_3^k \right]$$

If  $({}_4) = 0$ , then it follows ( $n = 4$ ):

$$I_n^D = \sum_k \frac{\binom{0}{k}_n}{\binom{0}{0}_n} I_{n-1}^{D,k} \quad (23)$$

If  $({}_4) \ll 1$ , re-write (14), as follows:

$$I_n^D = \sum_k \frac{\binom{0}{k}_n}{\binom{0}{0}_n} I_{n-1}^{D,k} - \frac{({}_4)_n}{\binom{0}{0}_n} \left[ (D+1) - \sum_i \nu_i \right] I_n^{D+2}. \quad (24)$$

Effectively we may evaluate  $I_n^D$  in terms of simpler functions  $I_{n-1}^{D,k}$  with a small correction depending on  $I_n^{D+2}$ .

We may go a step further, and insert into (24) for  $I_n^{D+2}$  the rhs. of (23), taken now at  $D' = D + 2$ :

$$\begin{aligned}
 I_n^D &= \sum_k^n \frac{\binom{0}{k}_n}{\binom{0}{0}_n} I_{n-1}^{D,k} \\
 &\quad - \frac{\binom{0}{0}_n}{\binom{0}{0}_n} [(D+1) - \sum_i^n \nu_i] \\
 &\quad \times \left[ \sum_k^n \frac{\binom{0}{k}_n}{\binom{0}{0}_n} I_{n-1}^{D+2,k} - \frac{\binom{0}{0}_n}{\binom{0}{0}_n} [(D+3) - \sum_i^n \nu_i] I_n^{D+4} \right].
 \end{aligned}$$

The terms proportional to  $[(\ )_n / \binom{0}{0}_n]^a$ ,  $a = 0, 1$  may be evaluated at the correct kinematics. They depend on three-point functions, and their reduction by normal recurrences will not introduce the unwanted powers of  $1/(\ )_4$ . The last term, suppressed by the factor  $[(\ )_n / \binom{0}{0}_n]^2$ , depends on  $I_n^{D+4}$ . It may either be taken approximately at  $(\ )_n = 0$ , where it can also be represented by 3-point functions (and their reductions), or it may be evaluated more correctly by another iteration based on (23).

And so on and so on ...

In the numerical example – next section – we worked out up to 10 stable iterations.



A quite similar attempt to perform such a series of approximations was undertaken in [14] (see equation (5) there), where a specific **example, forward light-by-light scattering through a massless fermion loop**, was studied. The approach was then not further followed.

W. Giele, E. W. N. Glover, and G. Zanderighi,  
in: Proceedings of Loops and Legs 2004:  
*Numerical evaluation of one-loop diagrams near exceptional momentum configurations,*

Following Davydychev, [7], one gets

$$I_4^{\mu\nu\lambda} = \int^d \frac{k^\mu k^\nu k^\lambda}{\prod_{r=1}^n c_r} = - \sum_{i,j,k=1}^n q_i^\mu q_j^\nu q_k^\lambda \nu_{ijk} I_{n,ijk}^{[d+]}{}^3 + \frac{1}{2} \sum_{i=1}^n g^{[\mu\nu} q_i^{\lambda]} I_{n,i}^{[d+]}{}^2 \quad (25)$$

We identify the tensor coefficients  $D_{11\dots}$  a la LoopTools, e.g.:

$$D_{111} = I_{4,222}^{[d+]}{}^3 \quad (26)$$

Similarly:

$$D_{1111} = I_{4,2222}^{[d+]}{}^4 \quad (27)$$

## Rank $R = 4$ tensor $D_{1111}$ – Numerics with dimensional recurrences

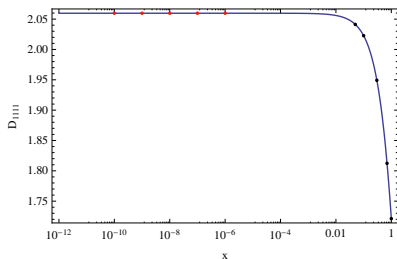
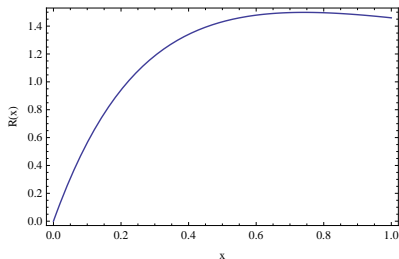
From (24) we see that a “small Gram determinant” expansion will be useful when the following dimensionless parameter becomes small:

$$R = \frac{()_4}{\binom{0}{0}_4} \times s, \quad (28)$$

where  $s$  is a typical scale of the process, e.g. we will choose  $s = s_{\mu\bar{\nu}U}$ .  
Following [12], we further choose:

$$\begin{aligned} s_{\mu\bar{\nu}U} &= 2 \times 10^4 \text{ GeV}^2, \\ s_{\bar{\nu}U} &= 1 \times 10^4 \text{ GeV}^2, \\ t_{\bar{e}\mu} &= -4 \times 10^4 \text{ GeV}^2, \end{aligned}$$

and get  $t_{ed,\text{crit}} = -6 \times 10^4 \text{ GeV}^2$ . For  $x=1$ , the Gram determinant becomes  $()_4 = 4.8 \times 10^{13} \text{ GeV}^3$ .  
The small expansion parameter  $R(x)$  and  $D_{1111}$  are shown in figure 2.



## New for QCD at LHC: small Gram expansion and Pade approximation I

[Fleischer,TR: PRD 2011 [6] ]

Tables have been taken from there.

They were shown already at QCD@LHC@Trento2010

The use of appropriate Pade approximations is explained there.  
Convergence in the small Gram determinant region is considerably improved.

x		$\Re D_{1111}$	$\Im D_{1111}$
0.	[exp 0,0]	2.05969289730 E-10	1.55594910118 E-10
$10^{-8}$	[exp x,2] [exp 0,2]	2.05969289342 E-10 2.05969289349 E-10	1.55594909187 E-10 1.55594909187 E-10
$10^{-4}$	[exp x,5] [exp 0,5]	2.05965609497 E-10 2.05965609495 E-10	1.55585605343 E-10 1.55585605343 E-10
0.001	[exp 0,6] [exp x,6]	2.05932484380 E-10 2.05932484381 E-10	1.55501912433 E-10 1.55501912433 E-10
	$l_{4,2222}^{[d+]}^4$	2.02292295240 E-10	1.54974785467 E-10
	$D_{1111}$	2.01707671668 E-10	1.62587142251 E-10
0.005	[exp 0,6] [pade 0,3] [exp x,6] [pade x,3]	2.05786054801 E-10 2.05785198947 E-10 2.05786364440 E-10 2.05785199805 E-10	1.55131031024 E-10 1.55131031003 E-10 1.55131031024 E-10 1.55131030706 E-10
	$l_{4,2222}^{[d+]}^4$	2.05778894114 E-10	1.55135794453 E-10
	$D_{1111}$	2.05779811490 E-10	1.55136343923 E-10
0.01	[exp 0,6] [pade 0,3] [exp 0,10] [pade 0,5] [exp x,10] [pade x,5]	2.05703298143 E-10 2.05600940065 E-10 2.05600964693 E-10 2.05600955381 E-10 2.05600963675 E-10 2.05600955381 E-10	1.54669910676 E-10 1.54669907784 E-10 1.54669910676 E-10 1.54669910676 E-10 1.54669910676 E-10 1.54669910676 E-10
	$l_{4,2222}^{[d+]}^4$	2.05600013702 E-10	1.54670651917 E-10
	$D_{1111}$	2.05600239280 E-10	1.54670771210 E-10

**Table:** Numerical values for the tensor coefficient  $D_{1111}$ . Values marked by  $D_{1111}$  are evaluated with LoopTools, the  $l_{4,2222}^{[d+]}^4$  corresponds to (30) The labels [exp 0,2n] and [pade 0,n] denote iteration 2n and Pade approximant  $[n, n]$  when the small Gram determinant expansion starts at  $x = 0$ , and [exp x,2n] and [pade x,n] are the corresponding numbers for an expansion starting at x.

$x$	$\Re D_{1111}$	$\Im D_{1111}$
0.01 [exp 0,6] [pade 0,3] [exp 0,10] [pade 0,5] [exp x,10] [pade x,5]	2.05703298143 E-10	1.54669910676 E-10
	2.05600940065 E-10	1.54669907784 E-10
	2.05600964693 E-10	1.54669910676 E-10
	2.05600955381 E-10	1.54669910676 E-10
	2.05600963675 E-10	1.54669910676 E-10
	2.05600955381 E-10	1.54669910676 E-10
$l_{4,2222}^{[d+]}^4$ $D_{1111}$	2.05600013702 E-10	1.54670651917 E-10
	2.05600239280 E-10	1.54670771210 E-10
	4.83822963052 E-09	1.51077429118 E-10
	2.01518061131 E-10	1.50591643209 E-10
0.05 [exp 0,6] [pade 0,3] [exp 0,20] [pade 0,10] [exp x,20] [pade x,10]	2.04218962072 E-10	1.51077424143 E-10
	2.04122727654 E-10	1.51077424149 E-10
	2.04190274030 E-10	1.51077424143 E-10
	2.04122727971 E-10	1.51077423985 E-10
	2.04122726387 E-10	1.51077422901 E-10
	2.04122726601 E-10	1.51077423320 E-10
0.1 [exp 0,26] [pade 0,13] [exp x,26] [pade x,13]	2.20215264409 E-08	1.46815247004 E-10
	2.01749674352 E-10	1.46681287362 E-10
	2.08190721550 E-08	1.46815247004 E-10
	2.03995221326 E-10	1.46785977364 E-10
$l_{4,2222}^{[d+]}^4$ $D_{1111}$	2.02269485177 E-10	1.46815247061 E-10
	2.02269485217 E-10	1.46815247051 E-10
1.	1.72115440143 E-10	9.74550747662 E-11
	1.72115440148 E-10	9.74550747662 E-11

**Table:** Numerical values for the tensor coefficient  $D_{1111}$ . Values marked by  $D_{1111}$  are evaluated with

LoopTools, the  $l_{4,2222}^{[d+]}^4$  corresponds to (30). The labels [exp 0,2n] and [pade 0,n] denote iteration  $2n$  and Pade approximant  $[n, n]$  when the small Gram determinant expansion starts at  $x = 0$ , and [exp x,2n] and [pade x,n] are the corresponding numbers for an expansion starting at  $x$ .

## PJFry - an open source c++ program by V. Yundin I

### PJFry 1.0.0 - one loop tensor integral library

- 
- More information and the latest source code:  
project page: <https://github.com/Vayu/PJFry/>
- → how to install
- → how to use
- → samples
- See also:  
V. Yundin's **talk** at LHCphenoNet meeting, Valencia, Feb 2011:  
“One loop tensor reduction program PJFRY”

## PJFry - an open source c++ program by V. Yundin II

- Yundin's PhD thesis, submitted Aug 2011 at Humboldt University



## PJFry — numerical package [from V.Y. Valencia 2011] I

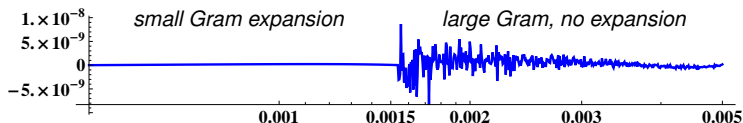
Numerical implementation of described algorithms:

C++ package **PJFry** by **V. Yundin** [see [project webpage](#)]

- Reduction of **5-point** 1-loop tensor integrals up to **rank 5**
- No limitations on internal/external masses combinations
- Small Gram determinants treatment by expansion
- Interfaces for C, C++, FORTRAN and MATHEMATICA

### Example:

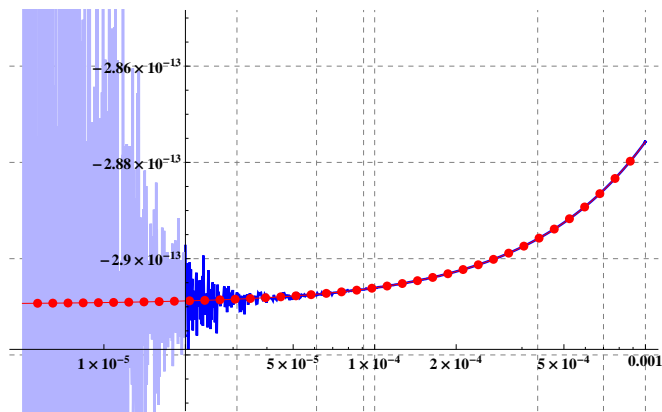
Relative accuracy of  $E_{3333}$  coef. around small Gram4 region



## PJFry — small Gram region example [from V.Y. Valencia 2011] |

**Example:**  $E_{3333}$  coefficient in small Gram region ( $x = 0$ )

Comparison of **Regular** and **Expansion** formulae:



$x=0: E_{3333}(0, 0, -6 \times 10^4, 0, 0, 10^4, -3.5 \times 10^4, 2 \times 10^4, -4 \times 10^4, 1.5 \times 10^4, 0, 6550, 0, 0, 8315)$

## Applications of Formalism and of PJFry

Formalism [not PJFry] used by Laura Reina et al. [also Paulo Rottman, DESY]

An application of PJFry to

$$e^+ e^- \rightarrow \mu^+ \mu^- \gamma$$

at meson factories is being finished.

Due to the specific cuts of the experiments, the numerics is relatively unstable.

Option of the GOLEM package for automatized NLO calculations for LHC processes.

## Contractions with external momenta [or with CHORDS] I

[Fleischer,TR: PLB 2011 [11]]

After having tensor reductions with basis functions  $I_n^D$ ,

which are independent of the indices  $i, j, k, \dots$ ,

one may use **contractions with external momenta** in order to perform all the sums over  $i, j, k, \dots$

This leads to a **significant simplification and shortening** of calculations.

**Reminder:**

One option was to avoid the appearance of inverse Gram determinants  $1/(\ )_5$ . For rank  $R = 5$ , e.g.,

$$\begin{aligned}
 I_5^{\mu\nu\lambda\rho\sigma} &= \sum_{s=1}^5 \left[ \sum_{i,j,k,l,m=1}^5 q_i^\mu q_j^\nu q_k^\lambda q_l^\rho q_m^\sigma E_{ijklm}^s + \sum_{i,j,k=1}^5 g^{[\mu\nu} q_i^\lambda q_j^\rho q_k^{\sigma]} E_{00ijk}^s \right. \\
 &\quad \left. + \sum_{i=1}^5 g^{[\mu\nu} g^{\lambda\rho} q_i^{\sigma]} E_{0000i}^s \right] \quad (29)
 \end{aligned}$$

## Contractions with external momenta [or with CHORDS] I

The tensor coefficients are expressed in terms of integrals  $I_{4,i\dots}^{[d+],s}$ , e.g.:

$$E_{ijkl}^s = -\frac{1}{\binom{0}{0}_5} \left\{ \left[ \binom{0l}{sm}_5 n_{ijk} I_{4,ijk}^{[d+],s} + (i \leftrightarrow l) + (j \leftrightarrow l) + (k \leftrightarrow l) \right] + \binom{0s}{0m}_5 n_{ijkl} I_{4,ijkl}^{[d+],s} \right\}.$$

Now, in a next step, one may avoid the appearance of inverse sub-Gram determinants  $(\ )_4$ .

The complete dependence on the indices  $i$  of the tensor coefficients is contained now in the pre-factors with signed minors. One can say that **the indices decouple from the integrals**.

As an example, we reproduce the 4-point part of

$$\begin{aligned} n_{ijkl} I_{4,ijkl}^{[d+],4} &= \frac{\binom{0}{i}}{\binom{0}{0}} \frac{\binom{0}{j}}{\binom{0}{0}} \frac{\binom{0}{k}}{\binom{0}{0}} \frac{\binom{0}{l}}{\binom{0}{0}} d(d+1)(d+2)(d+3) I_4^{[d+],4} \\ &+ \frac{\binom{0i}{0j} \binom{0}{k} \binom{0}{l} + \binom{0i}{0k} \binom{0}{j} \binom{0}{l} + \binom{0j}{0k} \binom{0}{i} \binom{0}{l} + \binom{0i}{0l} \binom{0}{j} \binom{0}{k} + \binom{0j}{0l} \binom{0}{i} \binom{0}{k} + \binom{0k}{0l} \binom{0}{i} \binom{0}{j}}{\binom{0}{0}^3} d(d+1) I_4^{[d+],3} \\ &+ \frac{\binom{0i}{0l} \binom{0j}{0k} + \binom{0j}{0l} \binom{0i}{0k} + \binom{0k}{0l} \binom{0i}{0j}}{\binom{0}{0}^2} I_4^{[d+],2} + \dots \end{aligned} \quad (30)$$

## Contractions with external momenta [or with CHORDS]

In (30), one has to understand the 4-point integrals to carry the corresponding index  $s$  and the signed minors are

$$\binom{0}{k} \rightarrow \binom{0s}{ks}_5 \text{ etc.}$$

## Contractions with external momenta [or with CHORDS] I

A chord is the momentum shift of an internal line due to external momenta,  $D_i = (k - q_i)^2 - m_i^2 + i\epsilon$ , and  $q_i = (p_1 + p_2 + \dots + p_i)$ , with  $q_n = 0$ .

The tensor 5-point integral of rank  $R = 1$  yields, when contracted with a chord,

$$q_{a\mu} I_5^\mu = -\frac{1}{\binom{0}{0}_5} \sum_{s=1}^5 \left[ \sum_{i=1}^4 (q_a \cdot q_i) \binom{0i}{0s}_5 \right] I_4^s. \quad (31)$$

In fact, the sum over  $i$  may be performed explicitly:

$$\Sigma_a^{1,s} \equiv \sum_{i=1}^4 (q_a \cdot q_i) \binom{0s}{0i}_5 = +\frac{1}{2} \left\{ \binom{s}{0}_5 (Y_{a5} - Y_{55}) + \binom{0}{0}_5 (\delta_{as} - \delta_{5s}) \right\},$$

## Contractions with external momenta I

We get immediately

$$q_{a\mu} l_5^\mu = - \frac{1}{\binom{0}{0}_5} \sum_{s=1}^5 \Sigma_a^{1,s} l_4^s. \quad (32)$$



## Contractions with external momenta I

The tensor 5-point integral of rank  $R = 2$

$$I_5^{\mu\nu} = \sum_{i,j=1}^4 q_i^\mu q_j^\nu E_{ij} + g^{\mu\nu} E_{00}, \quad (33)$$

has the following tensor coefficients free of  $1/(\ )_5$ :

$$E_{00} = - \sum_{s=1}^5 \frac{1}{2} \frac{1}{\binom{0}{0}_5} \binom{s}{0}_5 I_4^{[d+],s}, \quad (34)$$

$$E_{ij} = \sum_{s=1}^5 \frac{1}{\binom{0}{0}_5} \left[ \binom{0i}{sj}_5 I_4^{[d+],s} + \binom{0s}{0j}_5 I_{4,i}^{[d+],s} \right]. \quad (35)$$

## Contractions with external momenta I

Equation (33) yields for the contractions with chords:

$$q_{a\mu} q_{b\nu} I_5^{\mu\nu} = \sum_{i,j=1}^4 (q_a \cdot q_i)(q_b \cdot q_j) E_{ij} + (q_a \cdot q_b) E_{00}. \quad (36)$$

and finally (36) simply reads

$$\begin{aligned} q_{a\mu} q_{b\nu} I_5^{\mu\nu} &= \frac{1}{4} \sum_{s=1}^5 \left\{ \frac{\binom{s}{0}_5}{\binom{0s}{0s}_5} (\delta_{ab} \delta_{as} + \delta_{5s}) + \frac{\binom{s}{s}_5}{\binom{0s}{0s}_5} [(\delta_{as} - \delta_{5s})(Y_{b5} - Y_{55}) \right. \\ &\quad \left. + (\delta_{bs} - \delta_{5s})(Y_{a5} - Y_{55}) + \frac{\binom{s}{0}_5}{\binom{0}{0}_5} (Y_{a5} - Y_{55})(Y_{b5} - Y_{55}) \right\} I_4^{[d+],s} \\ &\quad + \frac{1}{\binom{0}{0}_5} \sum_{s=1}^5 \frac{\Sigma_b^{1,s}}{\binom{0s}{0s}_5} \sum_{t=1}^5 \Sigma_a^{2,st} I_3^{st}, \end{aligned}$$

## Contractions with external momenta I

with

$$\begin{aligned}\Sigma_a^{2,st} &\equiv \sum_{i=1}^4 (q_a \cdot q_i) \begin{pmatrix} 0st \\ 0si \end{pmatrix}_5 \\ &= \frac{1}{2} (1 - \delta_{st}) \left\{ \begin{pmatrix} ts \\ 0s \end{pmatrix}_5 (Y_{a5} - Y_{55}) + \begin{pmatrix} 0s \\ 0s \end{pmatrix}_5 (\delta_{at} - \delta_{5t}) - \begin{pmatrix} 0s \\ 0t \end{pmatrix}_5 (\delta_{as} - \delta_{5s}) \right\}\end{aligned}$$

This has been extended also to higher ranks.

We need at most double sums, e.g.:

$$\begin{aligned}\Sigma_{ab}^{2,s} &\equiv \sum_{i,j=1}^4 (q_a \cdot q_i)(q_b \cdot q_j) \begin{pmatrix} si \\ sj \end{pmatrix}_5 \frac{1}{2} (q_a \cdot q_b) \begin{pmatrix} s \\ s \end{pmatrix}_5 \\ &= -\frac{1}{4} ( )_5 (\delta_{ab} \delta_{as} + \delta_{5s}),\end{aligned}\tag{37}$$

## Contractions with external momenta I

Many of the **sums over signed minors, weighted with scalar products of chords** are given in PLB 2011 [[11]], and an almost complete list may be obtained on request from J. Fleischer, T.R.

## Modifications for 7- and higher point functions I

Here, the Gram determinant vanishes, and also further determinants:

$$\binom{0}{n} = 0, n > 5$$

$$\text{but also } \binom{0}{k}_7 = 0$$

As a result, one has to reorganize the reductions, avoiding the  $1/\binom{0}{n}$  completely.

This may be done, and we are coming so far to expressions with  $I_6^{[d+]}$ .

The problematic case is the integral  $I_{6,i}^{[d+]}$  for which we can write

$$I_{6,i}^{[d+]} = \sum_{s=1, s \neq i}^7 \frac{\binom{R}{s}_6}{\binom{R}{0}_6} I_{5,i}^{[d+],s} + \frac{\binom{R}{i}_6}{\binom{R}{0}_6} I_6^{[d+]}$$

## Modifications for 7- and higher point functions II

For  $I_6^{[d+]}$  we finally need 4-point functions up to the order  $\epsilon$  in the basis.

But:

We know from Gudrun Heinrich et al. that there is a representation in terms just of ordinary 4-point functions:

Binoth, Guillet, Heinrich, hep-ph/0504267, hep-ph/9911342

$n > 6$ : assumption in Bern, Dixon, Kosower, hep-ph/9306240

## Summary

- **Recursive treatment** of hexagon and pentagon tensor integrals of rank  $R$  in terms of pentagons and boxes of rank  $R - 1$
- Systematic derivation of expressions which are explicitly **free of inverse Gram determinants**  $(\ )_5$  until pentagons of rank  $R = 5$
- Proper **isolation of inverse Gram determinants of subdiagrams of the type**  $\binom{s}{s}_n 4$ , which cannot be completely avoided
- Numerical **C++ package PJFry** (V. Yundin, open source) for C, c++, Mathematica, Fortran
- **Perform multiple sums with signed minors and scalar products** after contractions with chords or external momenta

## References I



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## Miyamoto Musashi, Samurai, (1584 – 1645)



### Miyamoto Musashi (1584 – 1645)

#### The Five Rings

#### The Broad Principles of Musashi's Strategy

- Do not think dishonestly.
- The Way is in training.
- Become acquainted with every art.
- Know the Ways of all professions.
- Distinguish between gain and loss in worldly matters.
- Develop intuitive judgement and understanding for everything.
- Perceive those things which cannot be seen.
- Pay attention even to trifles.
- Do nothing which is of no use.

[http://commons.wikimedia.org/wiki/File:Miyamoto\\_Musashi-Self-Portrait](http://commons.wikimedia.org/wiki/File:Miyamoto_Musashi-Self-Portrait)

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