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NNLO contributions to Bhabha scattering

Tord Riemann, DESY, Zeuthen

based on work with:
S. Actis (RWTH Aachen), M. Czakon (U. Würzburg)
and J. Gluza (Sileasian U. Katowice)

10 Dec 2008, Seminar U. Edinburgh

- Bhabha scattering – Born cross-section and experimental aspects
- Electroweak one-loop contributions
- LEP and later: Meson factories, ILC, GigaZ
- Overview: Two-loop contributions to Bhabha Scattering
- Heavy fermion and hadronic contributions with $m_e^2 << m_f^2, s, t$
  ACGR: Phys. Rev. Letters 100 (2008) [arXiv:0711..3847]
  ACGR: Phys. Rev. D78 (2008) 085019 [arXiv:0807.4691]
- Summary and outlook

Born cross-section and experimental aspects

H. Bhabha,

"The Scattering of Positrons by Electrons with Exchange on Dirac's Theory of the Positron",

Proc. Roy. Soc. A154 (1936) 195

$$\frac{d\sigma_0}{d\Omega} = \frac{\alpha^2}{s}\left(\frac{s}{t} + 1 + \frac{t}{s}\right)^2$$

where the relations between beam energy, scattering angle and s, t are:

$$s = 4E^2$$

$$t = \frac{s}{2}(1 - \cos\vartheta)$$

- $|\mathcal{M}_s + \mathcal{M}_t|^2$
- simple process with zero [one] mass scales
- strong forward peak
- But: μ-pair production advantageous ?

It is not completely clear [to us] whether one may make the paper public in the web. Here is the first page:

molecules have orientations similar to the dibenzyl orientation, and the other two can approximately be derived from them by a rotation of 180° about the a axis, and a translation of $\frac{1}{2}c$. The resulting structure explains the pseudo-orthorhombic properties, the approximate halvings, and the principal X-ray intensities. It is contrary to a structure previously deduced from magnetic measurements by Krishnan, Guha, and Banerjee, who predicted a twisted and distorted molecule; but it is shown that the new structure is equally capable of explaining the magnetic data. Detailed measurements have not yet been made on tolane and azobenzene, but the preliminary data are sufficient to show that they are both closely similar to the stilbene structure.

The Scattering of Positrons by Electrons with Exchange on Dirac's Theory of the Positron

By H. J. Bhabha, Ph.D., Gonville and Caius College

(*Communicated by R. H. Fowler, F.R.S.—Received October* 20, 1935)

It has been shown by Mott† that exchange effects play a considerable part in the collision and consequent scattering of one electron by another. Mott's original calculation was non-relativistic, and there the exchange effect vanishes when the two electrons have their spins pointing in opposite directions. Møller‡ later developed relativistically invariant expressions for the collision of two charged particles with spin, and it may be seen directly from Møller's general formula for the collision cross-section that, in the collision of two identical particles, the effect of exchange does not in general vanish even when the two colliding particles initially have their spins pointing in opposite directions. It tends however to zero in this case as the relative velocity of the particles becomes small compared to c, the velocity of light, in agreement with the calculation of Mott.

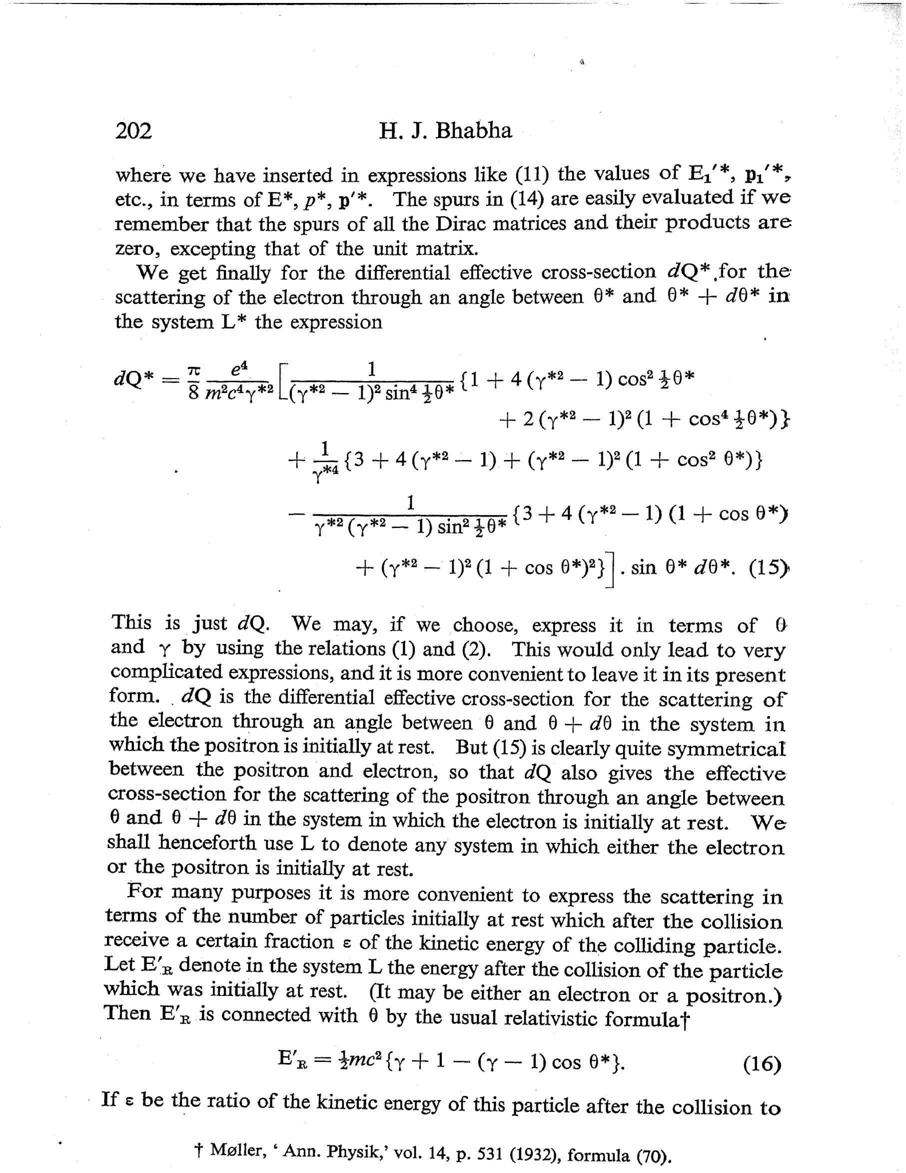
The effect of exchange in the general relativistic case will still be considerable if one of the two electrons be initially (and therefore finally) in a state of negative energy. (If one of the electrons be initially in a negative energy state, then it follows from the conservation of energy

† 'Proc. Roy. Soc.,' A, vol. 126, p. 259 (1930).
‡ 'Ann. Physik,' vol. 14, p. 531 (1932).

O 2

Here some formula:

It would take a while to discover that the paper derives what we call now Bhabha scattering



where we have inserted in expressions like (11) the values of $E_1'^*$, $\mathbf{p}_1'^*$, etc., in terms of E^*, p^*, $\mathbf{p}'^*$. The spurs in (14) are easily evaluated if we remember that the spurs of all the Dirac matrices and their products are zero, excepting that of the unit matrix.

We get finally for the differential effective cross-section dQ^* for the scattering of the electron through an angle between θ^* and $\theta^* + d\theta^*$ in the system L^* the expression

$$dQ^* = \frac{\pi}{8}\frac{e^4}{m^2c^4\gamma^{*2}}\Big[\frac{1}{(\gamma^{*2}-1)^2\sin^4\frac{1}{2}\theta^*}\{1+4(\gamma^{*2}-1)\cos^2\tfrac{1}{2}\theta^* + 2(\gamma^{*2}-1)^2(1+\cos^4\tfrac{1}{2}\theta^*)\} + \frac{1}{\gamma^{*4}}\{3+4(\gamma^{*2}-1)+(\gamma^{*2}-1)^2(1+\cos^2\theta^*)\} - \frac{1}{\gamma^{*2}(\gamma^{*2}-1)\sin^2\frac{1}{2}\theta^*}\{3+4(\gamma^{*2}-1)(1+\cos\theta^*) + (\gamma^{*2}-1)^2(1+\cos\theta^*)^2\}\Big] \,.\, \sin\theta^*\, d\theta^*. \quad (15)$$

This is just dQ. We may, if we choose, express it in terms of θ and γ by using the relations (1) and (2). This would only lead to very complicated expressions, and it is more convenient to leave it in its present form. dQ is the differential effective cross-section for the scattering of the electron through an angle between θ and $\theta + d\theta$ in the system in which the positron is initially at rest. But (15) is clearly quite symmetrical between the positron and electron, so that dQ also gives the effective cross-section for the scattering of the positron through an angle between θ and $\theta + d\theta$ in the system in which the electron is initially at rest. We shall henceforth use L to denote any system in which either the electron or the positron is initially at rest.

For many purposes it is more convenient to express the scattering in terms of the number of particles initially at rest which after the collision receive a certain fraction ε of the kinetic energy of the colliding particle. Let E'_R denote in the system L the energy after the collision of the particle which was initially at rest. (It may be either an electron or a positron.) Then E'_R is connected with θ by the usual relativistic formula†

$$E'_R = \tfrac{1}{2}mc^2\{\gamma + 1 - (\gamma - 1)\cos\theta^*\}. \quad (16)$$

If ε be the ratio of the kinetic energy of this particle after the collision to

† Møller, 'Ann. Physik,' vol. 14, p. 531 (1932), formula (70).

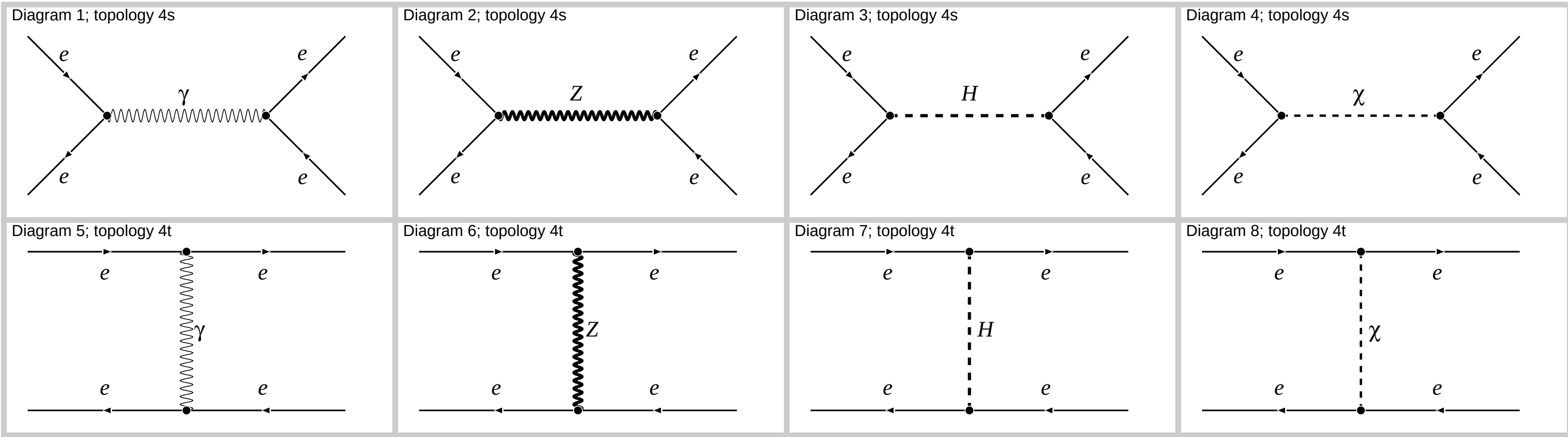


Diagram 1; topology 4s
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Diagram 2; topology 4s
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Diagram 8; topology 4t
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Electroweak Born and 1-loop contributions

The Born cross-section is:

$$\frac{d\sigma_{ew}}{d\Omega} = \frac{\alpha^2}{4s}\left(T_s + T_{st} + T_t\right),$$

with

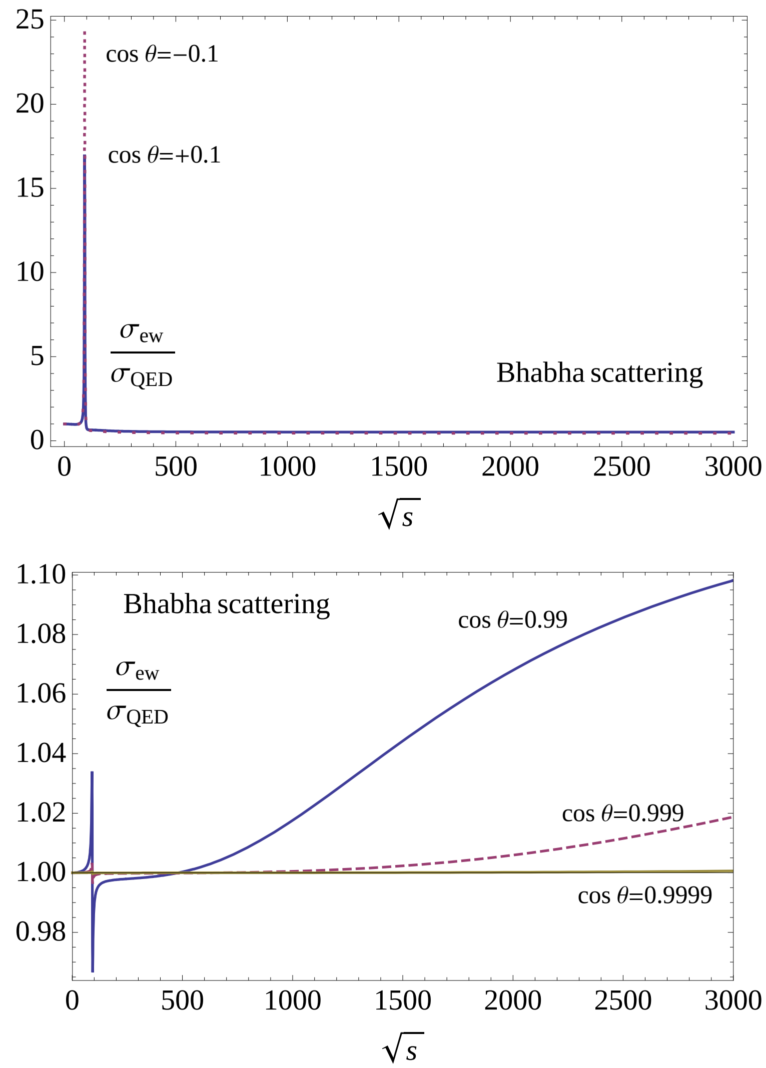
$$\begin{aligned}
T_s &= (1+\cos^2\theta)\left[1 + 2\mathbf{Re}\chi(s)\left(v^2\right) + |\chi(s)|^2\left(1+v^2\right)^2\right] + 2\cos\theta\left[2\mathbf{Re}\chi(s) + |\chi(s)|^2\left(4v^2\right)\right],\\
T_{st} &= -2\frac{(1+\cos\theta)^2}{(1-\cos\theta)}\left\{1 + \left[\chi(t) + \mathbf{Re}\chi(s)\right]\left(1+v^2\right) + \chi(t)\mathbf{Re}\chi(s)\left[(1+v^2)^2 + 4v^2\right]\right\},\\
T_t &= 2\,\frac{(1+\cos\theta)^2}{(1-\cos\theta)^2}\left\{1 + 2\chi(t)\left(1+v^2\right) + \chi(t)^2\left[(1+v^2)^2 + 4v^2\right]\right\}\\
&+ \frac{8}{(1-\cos\theta)^2}\left[1 - \chi(t)\left(1-v^2\right)\right]^2.
\end{aligned}$$

We choose the following conventions:

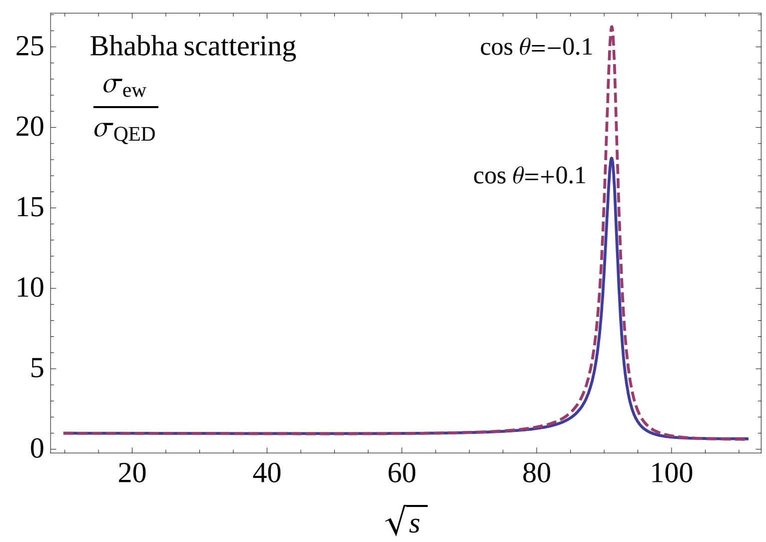
$$\begin{aligned}
v &= 1 - 4s_w^2,\\
\chi(s) &= \frac{G_F}{\sqrt{2}}\frac{M_Z^2}{8\pi\alpha}\frac{s}{s - M_Z^2 + iM_Z\Gamma_Z},\\
\chi(t) &= \frac{G_F}{\sqrt{2}}\frac{M_Z^2}{8\pi\alpha}\frac{t}{t - M_Z^2}.
\end{aligned}$$

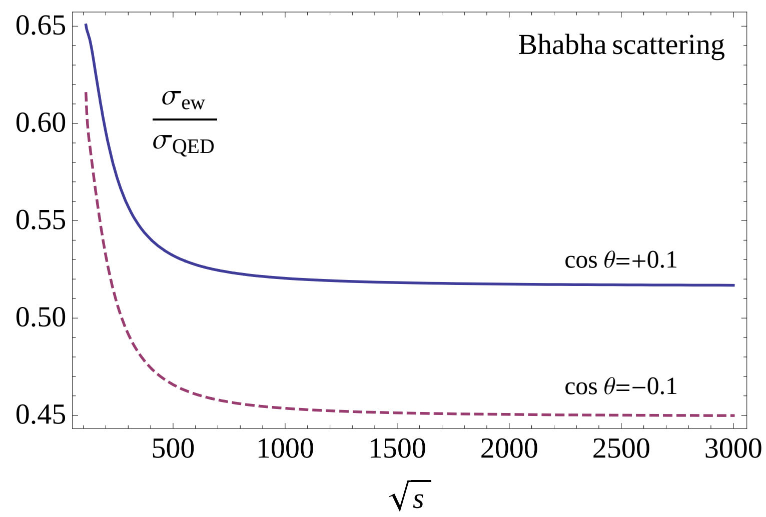
Among the quantities α, G_F, s_w^2, M_Z there are only three independent, and Γ_Z is predicted by the theory as well.
The phrasing *effective Born cross-section* **means here that we use, besides α, the following input variables:**

$$
\begin{aligned}
s_w^2 &= 0.23, \\
M_Z &= 91.1876 \pm 0.0021 \text{ GeV}, \\
\Gamma_Z &= 2.4952 \pm 0.0023 \text{ GeV}, \\
G_F &= (1.16637 \pm 0.00001) \times 10^{-5} \text{ GeV}^{-2}.
\end{aligned}
$$



Ratio of electroweak to QED Bhabha scattering cross-section at large angles (up) and small angles (down) as a function of $\sqrt{s}$.





Ratio of electroweak to QED Bhabha scattering cross-section at large angles in the energy ranges of LEP1/GigaZ (up) and ILC (down).

DIANA OneLoop_page5.ps

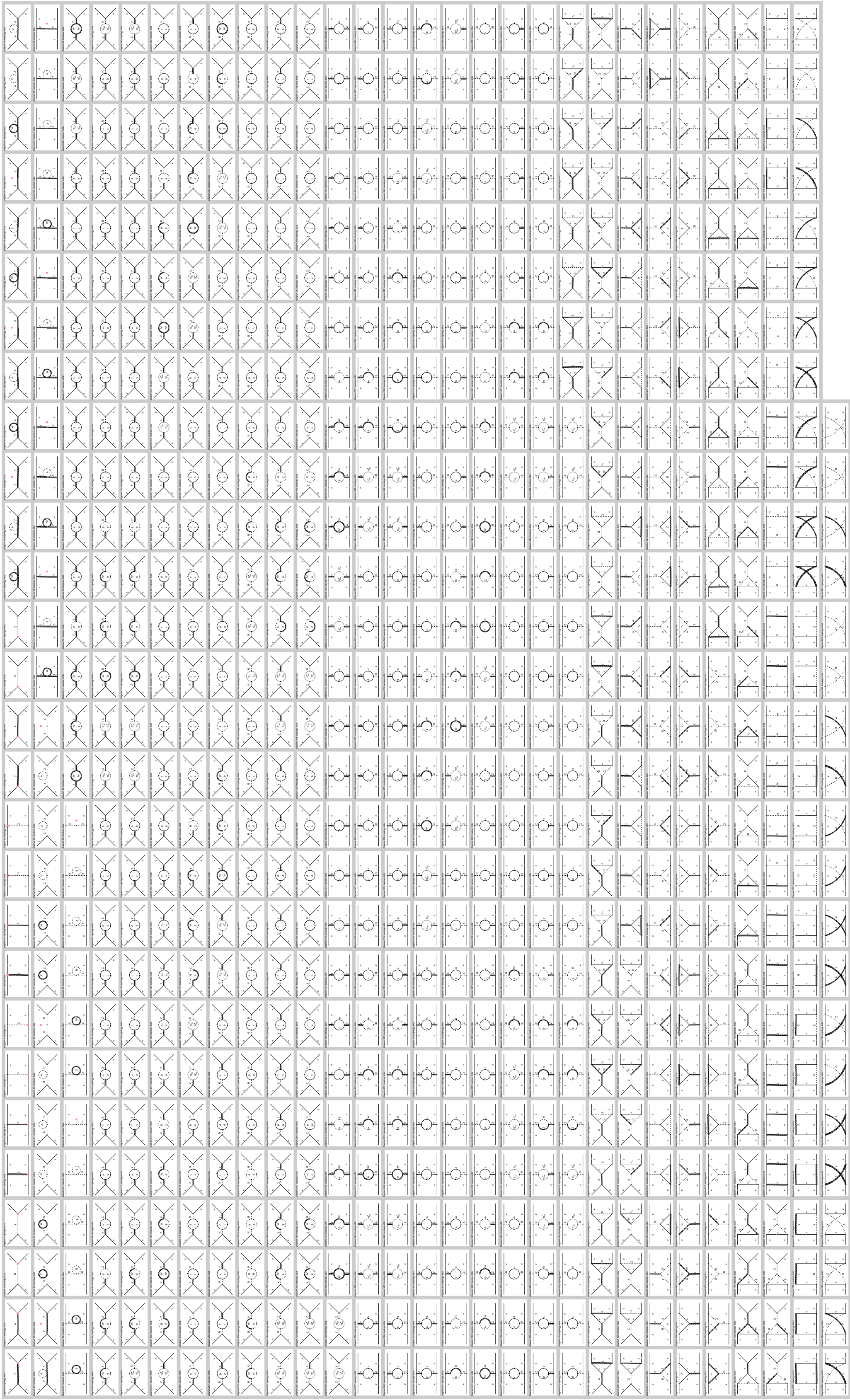


Table 2:
The differential Bhabha cross section in nbarn as function of the scattering angle and the cms-energy.
$M_Z = 91.16\ GeV$, $m_t = 150\ GeV$, $M_H = 100\ GeV$.
Upper rows: DZ, lower rows: H.
δ_m: *largest relative deviation in per mille.*

$\sqrt{s}$ (GeV) θ	60	89	91.16	93	200
15°	129.6	65.11	57.93	49.00	11.82
	129.6	65.11	57.93	49.00	11.82
45°	1.451	1.376	1.755	.4833	11.67
	1.451	1.377	1.756	.4837	11.68
60°	.4303	.6124	1.125	.2697	.03075
	.4305	.6129	1.126	.2699	.03077
75°	.1717	.3627	.8718	.2232	.01072
	.1718	.3630	.8720	.2233	.01072
90°	.08873	.2768	.7790	.2088	.004862
	.08876	.2769	.7787	.2087	.004855
105°	.05917	.2690	.8082	.2157	.002858
	.05918	.2690	.8074	.2157	.002853
120°	.04906	.3053	.9323	.2429	.002077
	.04906	.3051	.9309	.2426	.002074
135°	.04671	.3626	1.111	.2838	.001743
	.04672	.3624	1.109	.2833	.001742
165°	.04839	.4638	1.425	.3590	.001539
	.04839	.4635	1.422	.3584	.001540
δ_m	0.6	0.8	1.8	2.0	1.7

Bhabha scattering

Bardin,Hollik,T.R., Z.PhysikC49(1991)485

The 1991 result is yet the state of the art in e.g. the programs ZFITTER and BHWIDE.
Now, such calculations of O(1000) diagrams are better than to 10 digits.

Results: Numerical comparison in all $f\bar{f}$

Bhabha $e^-e^+ \to e^-e^+(\gamma)$ **at LC:** $\sqrt{s} = 500$ GeV, $E_{\max}(\gamma_{\rm soft}) = \frac{\sqrt{s}}{10}$

$\cos\theta$	$\left[\frac{d\sigma}{d\cos\theta}\right]_{\rm Born}$ (pb)	$\left[\frac{d\sigma}{d\cos\theta}\right]_{\mathcal{O}(\alpha^3)=\text{Born+QED+weak+soft}}$	Group
−0.9999	0.21482 70434 05632 5	0.14889 12125 78083 7	aITALC
−0.9999	0.21482 70434 05632 6	0.14889 12189 28404 0	*FeynArts*
−0.9	0.21699 88288 10920 5	0.19344 50785 26863 6	aITALC
−0.9	0.21699 88288 10920 0	0.19344 50785 26862 2	*FeynArts*
−0.9	0.21699 88288 41513 1	0.19344 50785 62637 9	$m_e = 0$
+0.0	0.59814 23072 50330 3	0.54667 71794 69423 1	aITALC
+0.0	0.59814 23072 50329 4	0.54667 71794 69421 8	*FeynArts*
+0.0	0.59814 23072 88584 4	0.54667 71794 99961 4	$m_e = 0$
+0.9	$0.18916\ 03223\ 32270\ 6 \cdot 10^3$	$0.17292\ 83490\ 66507\ 2 \cdot 10^3$	aITALC
+0.9	$0.18916\ 03223\ 32270\ 6 \cdot 10^3$	$0.17292\ 83490\ 66508\ 0 \cdot 10^3$	*FeynArts*
+0.9	$0.18916\ 03223\ 31848\ 5 \cdot 10^3$	$0.17292\ 83490\ 61347\ 4 \cdot 10^3$	$m_e = 0$
+0.9999	$0.20842\ 90676\ 46142\ 9 \cdot 10^9$	$0.19140\ 17861\ 11341\ 6 \cdot 10^9$	aITALC
+0.9999	$0.20842\ 90676\ 46436\ 4 \cdot 10^9$	$0.19140\ 17861\ 11979\ 0 \cdot 10^9$	*FeynArts*

Great independent agreement up to 14 digits! : limit in double precision

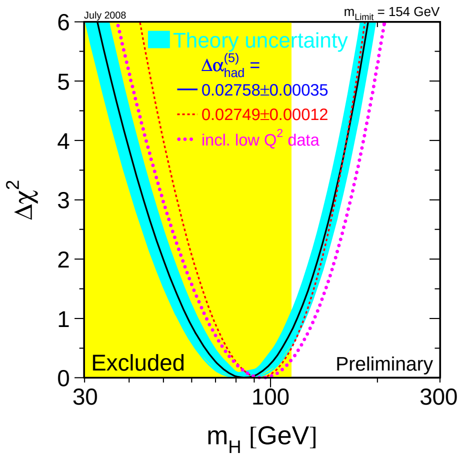
Previous agreement with *FeynArts*: 11 digits hep-ph/0307132**, SANC: 10 digits** hep-ph/0207156

Thanks to T. Hahn, numbers supplied with *FeynArts + FormCalc + LoopTools*

ZFITTER and Higgs physics

Fortran package ZFITTER for: $e^+e^- \to \mu\mu, \bar{q}q, e^+e^-$
has been used for many experimental and phenomenological studies.
The perhaps most important applications for the elementary particle physics community may be found at the webpage of the
LEP electroweak working group LEPEWWG.

The weak one-loop library of ZFITTER is used in the Monte-Carlo [where appropriate].



See also:

ZFITTER news page at DESY Zeuthen:
http://www-zeuthen.desy.de/theory/research/zfitter/

We will now look for really precise predictions . . .

for both

- **Small angle Bhabha scattering**
- **Large angle Bhabha scattering**

– why?

$\ldots 10^{-4}$

– what does it mean?

The physics needs at high energies

ILC, $\sqrt{s} = 90$ GeV – 1 TeV

For more details see e.g.:
K. Mönig, "Bhabha scattering at the ILC"
talk at Mini-WS on Bhabha scattering, Univ. Karlsruhe, April 2005
/afs/ifh.de/user/m/moenig/public/www/bhabha_ilc.pdf

ILC – Need Bhabha cross-sections with 3–4 significant digits.

Why?

- **ILC:** $e^+e^- \to W^+W^-, f\bar{f}$ **with** $O(10^6)$ **events** $\to 10^{-3}$
- **GigaZ: relevant physics derived from** $Z \to$ **hadrons**, l^+l^-, **the latter with** $O(10^8)$ **events** $\to 10^{-4}$, **the systematic errors (luminosity!) influence this**
- **ILC:**$e^+e^- \to e^+e^-$, **a probe for New Physics with** $O(10^5)$ **events/year** $\to 10^{-3}$

Conclude: will need $\Delta\mathcal{L}/\mathcal{L} \approx 2 \times 10^{-4}$

The luminosity comes from very forward Bhabha scattering.

October 20, 2003

MEMO
Luminosity Measurement via Bhabha Scattering: Requirements on Position Reconstruction to Achieve a 10^{-4} Precision

Achim Stahl
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Abstract

This memo is based on Monte Carlo simulations with the BHLUMI generator of Jadach and Was. It adresses the question how accurately electrons and positrons have to be reconstructed in the TESLA LumCal in order to achieve a precision of 10^{-4} on the luminosity measurement.

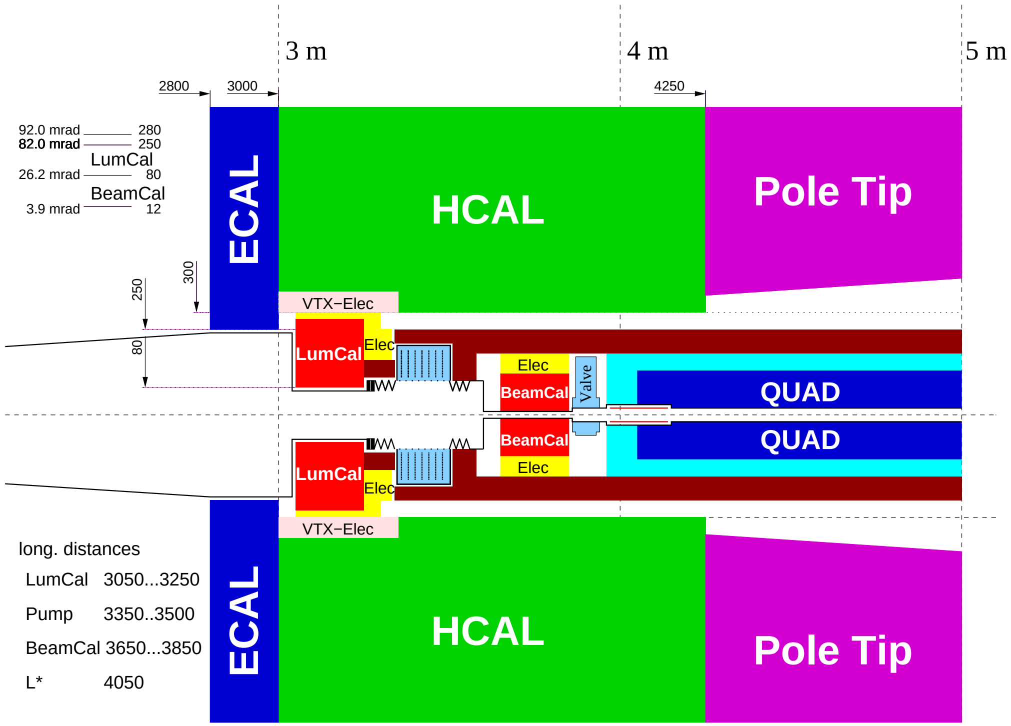


Figure 13: Preliminary redesign of the forward region for a focal length of the final focus between 4 and 5 meters.

13

1 Method

Bhabha events are simulated with BHLUMI[1] in the phase space region of the TESLA luminosity calorimeter. The simulation is based on a redesign of the forward region for an l^* that allows to place the luminosity calorimeter behind the ECal endcap. The luminosity calorimeter is called LumCal to distinguish it from the LAT and LCal of the TDR design. The design is sketched in fig. 13.

A simple selection is applied to the simulated events (see below). In a first step the position of the electron and positron on the front face of LumCal is calculated. The scattering angles are determined from these positions. The energy of the particles is taken from the tree. In subsequent steps the position and energy are subjected to systematic misreconstructions. The change in the number of accepted events with respect to step 1 gives the systematic error introduced by the respective systematic error. The size of the effects are varied in order to determine the level which is acceptable in order to achieve a 10^{-4} precision on the luminosity measurement. The same event sample is used for all steps so that statistical fluctuations largely cancel.

BHLUMI version 4.04 was used with the following parameters:

Type of generator	BHLUM4
Photon Removal	on
Event weights	off
Random generator	RANMAR
Z^0-contribution	on
QED matrix elements	from BHLUM4
Vacuum polarisation	from ref. 1
center-of-mass energy	250 GeV
min. scattering angle	25 mrad
max. scattering angle	90 mrad
photon infrared cut-off	10^{-4}

The following cuts are applied

- energy: $E\,(e^+) > 0.8\,E_{\text{beam}}$
  $E\,(e^-) > 0.8\,E_{\text{beam}}$
- scattering angle of positron: $30\,\text{mrad} < \theta^+ < 75\,\text{mrad}$
- accollinearity: $\cos\theta_{\text{acol}} > 0.98$

Some Kinematics at GigaZ and ILC

Need a cross-section prediction with 5 significant digits.

Perturbative orders:

$$\left(\frac{\alpha}{\pi}\right) = 2 \times 10^{-3}$$

$$\left(\frac{\alpha}{\pi}\right)^2 = 0.6 \times 10^{-5}$$

Kinematics:

$$\begin{aligned}
\sqrt{s} &= 90 \ldots 1000 \text{ GeV} \\
\vartheta &= 26 \ldots 82 \text{ mrad} \\
\cos\vartheta &\sim 0.999\,66 \ldots 0.996\,64 \\
T &= \frac{s}{2}(1 - \beta^2 \cos\vartheta) > 1.36 \text{ GeV}^2|_{GigaZ}, \qquad 42.2 \text{ GeV}^2|_{ILC500}
\end{aligned}$$

Conclude:

- t**-channel exchange of** γ **dominates everything else**
- $m_e^2/s \ < \ m_e^2/T \ \leq \ 10^{-5} \ldots 10^{-7}$
- **Calculate: 1-loop EWRC + 2-loop QED + corresp. bremsstrahlung**

.

The 1-loop electroweak corrections (plus some leading higher order terms) are well-known, with rising technical precision, since about 1988/91.
Böhm, Denner, Hollik; Bardin, TR 1991 $\rightarrow$ **Fig. 2004 Lorca, TR**

2-loop Bhabha scattering: What to be done?

- **Calculate:**

$$\sigma \quad = \quad (2 \to 2) \quad + \quad (2 \to 3) \quad + \quad (2 \to 4)$$

$$\begin{aligned} \sigma \quad = \quad & |\textbf{Born} \; + \; \textbf{1-loop} \; + \; \textbf{2-loop}|^2 \\ & + \; |(\textbf{Born} + \textbf{1-}\gamma) \; + \; (\textbf{1-loop} + \textbf{1-}\gamma)|^2 \\ & + \; |(\textbf{Born} + \textbf{2-}\gamma)|^2 \end{aligned}$$

- **Do not include:** $|\textbf{2-loop}|^2$
  $|(\textbf{1-loop} + \textbf{1-}\gamma)|^2$

Status by end of 2004

Established: 10^{-3} MC programs for LEP, ILC

Introduction to NLLBHA by Trentadue and to BHLUMI by Jadach in:
Proc. of Loops and Legs, Rheinsberg, Germany, 1996

Recent mini-review: Jadach, "Theoretical error of luminosity cross section at LEP", hep-ph/0306083 [1]

- BHLUMI v.4.04: Jadach, Placzek, Richter-Was, Was: CPC 1997
- see also: Jadach, Melles, Ward, Yost: PLB 1996, thesis Melles 1996 [2]
- NLLBHA: Arbuzov, Fadin, Kuraev, Lipatov, Merenkov, Trentadue: NPB 1997, CERN 96-01
- SAMBHA: Arbuzov, Haidt, Matteuzzi, Paganoni, Trentadue: hep-ph/0402211

See e.g.: Table 1 of [1] and Figure 3.1 of [2] $\rightarrow$ Conclude:
The nonlogarithmic $O(\alpha^2)$ terms, originating from pure QED radiative 1-loop and from 2-loop diagrams are not completely covered.

They have to be calculated and integrated into the MC programs.

Beware:

$$m_e, m_\gamma, (d-4), E_\gamma$$

Offer by drinking-a-beer Frits Berends, NL, in 1989

Okey, I need

One million Swiss francs

and

five – or ten? – Russians

and I will do this

Until 2003 (i.e. in more than 15 years), there were about a dozen articles in spires with "Bhabha ... two-loop" in the title.

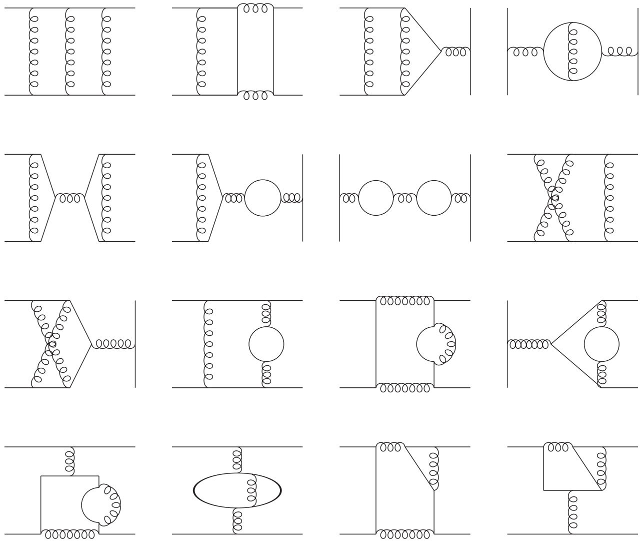
And Frits Berends retired recently.

Now we have about 30 articles of this kind.

Two Loop Bhabha Scattering

To calculate Bhabha scattering it is best to first compute $e^+e^- \to \mu^+\mu^-$, since it's closely related but has less diagrams.

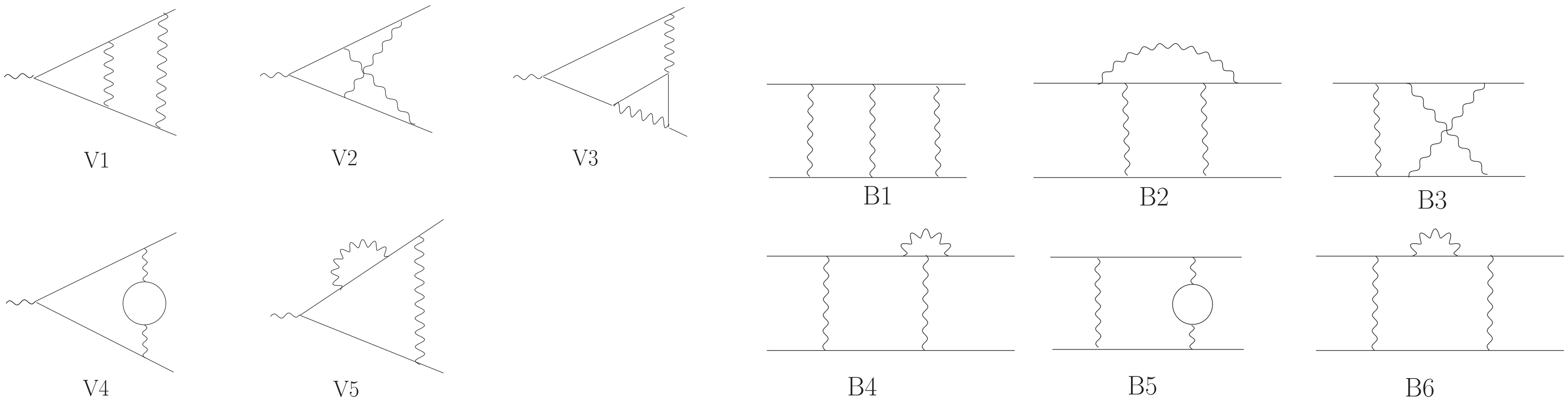
There are 47 QED diagrams contributing to $e^+e^- \to \mu^+\mu^-$.



The Bhabha scattering amplitude can be obtained from $e^+e^- \to \mu^+\mu^-$ simply by summing it with the crossed amplitude (including fermi minus sign).

Adapted from Z.Bern, LoopFest 2002

The technical target: photonic and fermionic $N_f = 1, 2$ topologies



- The unsolved problem, even in the limit $m_e^2 << s, t$:
  The non-planar photonic 2-loop boxes B3
- Finally the photonic corrections were derived from massless case by A. Penin ...
- ...and the $n_f = 1$ electron loops V4, B5 by R. Bonciani et al., and later also by ACGR
- We think we know how to do the non-planar boxes, but it is not easy

.

Status 2005

Know the constant term ($m_e = 0$) from 2-loop Bhabha scattering

A. Penin, Two-Loop Corrections to Bhabha Scattering, hep-ph/0501120 v.3, $\rightarrow$ PRL

Transform the massless 2-loop results of Bern, Dixon, Ghinculov (2002) with InfraRed (IR) regulation by $D = 4 - 2\epsilon$ into the on-mass-shell renormalization with $m_e \rightarrow 0$ and IR regulation by $\lambda = m_\gamma \neq 0$

Use IR-properties of amplitudes (see Penin):

[A] Exponentiation of the IR logarithms (Sudakov 1956,...)

[B] Factorization of the collinear logarithms into expernal legs (Frenkel, Taylor 1976)

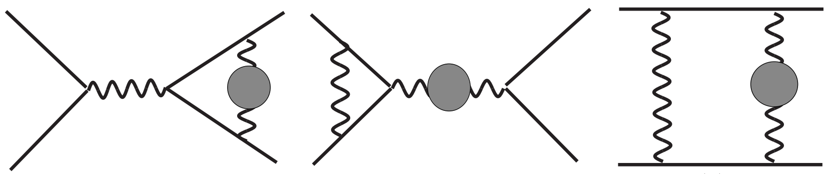
[C] Non-renormalization of the IR exponents (YFS 1961, ...)

Isolate the closed fermion loop contribution (does not fulfil [C]) and add it separately (Burgers 1985, Bonciani et al. 2005, Penin)

If all this is correct, the constant term in m_e is known for the MCs (but the radiative one-loops with 5-point functions).

The diagrams with electrons and photons define an $n_f = 1$ problem.

But there are additional ones with heavier fermions.
So we have to investigate an $n_f = 2$ problem
For self-energies starting with 1-loops, and for vertices and boxes starting with 2-loops:



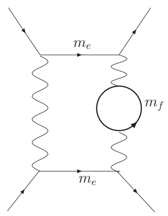
(a) (b) (c)

.

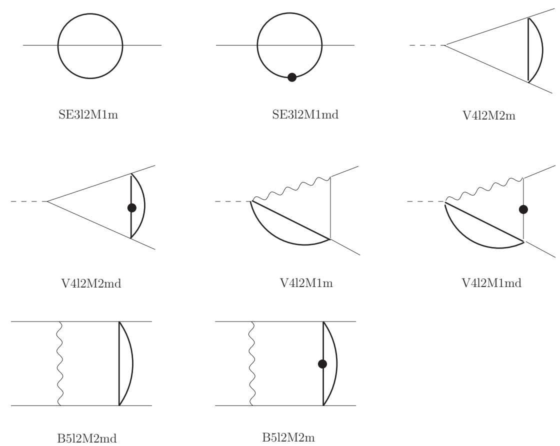
The $n_f = 2$ contributions have been determined in 2007

- Self-energies are not a two-masses-problem
- 2-vertices are known (for $m_e^2 = m_f^2$ and $m_e^2 << m_f^2$): G. Burgers PLB 164 (1885), Kniehl, Krawczyk, Kühn, Stuart PLB 209 (1988)
- What is really new: the 2-boxes with two different fermions involved

The 8 box-master integrals were identified in ACGR, PRD 71 (2005) [hep-ph/0412164]



$\longrightarrow$



- $m_e^2 << m_f^2 << s, t$: Becher,Melnikov JHEP 6 (2007) and ACGR NPB 786 (2007)
- $m_e^2 << m_f^2, s, t$: ACGR 0710.5111 $- >$ APP B38 (2007)
  and Bonciani,Ferroglia,Penin 0710.4775 (2007)
- $m_e^2 << m_{hadrons}^2, s, t$: ACGR 0711.3847 $- >$ PRL 100 (2008)

The Box Corrections

The contribution of the renormalized two-loop box diagrams of class 2e is given by

$$\frac{d\sigma^{\rm 2e\times tree}}{d\Omega} = \frac{\alpha^2}{2\,s}\left[\frac{1}{s}\,A_1^{\rm 2e\times tree}(s,t) + \frac{1}{t}\,A_2^{\rm 2e\times tree}(s,t)\right]$$

Here the auxiliary functions can be conveniently expressed through three independent form factors $B^{(2)}_{\rm I,f}(x,y)$, where $\mathrm{i} = \mathrm{A, B, C}$,

$$A_1^{\rm 2e\times tree}(s,t) = F_\epsilon^2 \sum_f Q_f^2\, \mathrm{Re}\left[B^{(2)}_{{\rm A},f}(s,t) + B^{(2)}_{{\rm B},f}(t,s) + B^{(2)}_{{\rm C},f}(u,t) - B^{(2)}_{{\rm B},f}(u,s)\right],$$

$$A_2^{\rm 2e\times tree}(s,t) = F_\epsilon^2 \sum_f Q_f^2\, \mathrm{Re}\left[B^{(2)}_{{\rm B},f}(s,t) + B^{(2)}_{{\rm A},f}(t,s) - B^{(2)}_{{\rm B},f}(u,t) + B^{(2)}_{{\rm C},f}(u,s)\right].$$

The normalization factor is

$$F_\epsilon = \left(\frac{m_e^2\,\pi\,e^{\gamma_E}}{\mu^2}\right)^{-\epsilon}$$

.

How to evaluate the $N_f = 2$ diagrams?

We did it in 2 ways

- Decompose the 2-loop integrals to master integrals, solve them.
  Here: In the limit $m_e^2 << m_f^2 << s, t, u$
  This was done in hep-ph/07042400v2 $\longrightarrow$ ACGR, NPB 786 (2007)
- Alternatively, rewrite the 2-loop integrals as dispersion integrals.
  Decompose the loop integrals afterwards into master integrals
  The master integrals are simpler, of one-loop type, but the numerical dispersion integration remains then.

Advantages of the dispersion integrals:

- get easily the range $m_e^2 << m_f^2, s, t, u$
- method applies also to hadronic insertions

.

Dispersion Integrals

$$\frac{g_{\mu\nu}}{q^2 + i\,\delta} \rightarrow \frac{g_{\mu\alpha}}{q^2 + i\,\delta} \left(q^2\, g^{\alpha\beta} - q^\alpha\, q^\beta\right) \Pi_{\rm had}(q^2)\, \frac{g_{\beta\nu}}{q^2 + i\,\delta},$$

the once-subtracted dispersion integral

$$\Pi_{\rm had}(q^2) = -\frac{q^2}{\pi} \int_{4M_\pi^2}^{\infty} \frac{dz}{z} \frac{\textbf{Im}\, \Pi_{\rm had}(z)}{q^2 - z + i\,\delta}.$$

Finally, one relates Im $\Pi_{\rm had}$ **to the hadronic cross-section ratio** $R_{\rm had}$**,**

$$\textbf{Im}\Pi_{\rm had}(z) = -\frac{\alpha}{3} R_{\rm had}(z) = -\frac{\alpha}{3} \frac{\sigma_{e^+e^-\to {\rm hadrons}}(z)}{(4\pi\alpha^2)/(3z)},$$

For heavy fermion insertions, we have instead of $R_{\rm had}(z)$**:**

$$R_f(z) = Q_f^2 C_f (1 + 2m_f^2/z)\sqrt{1 - 4m_f^2/z},$$

Replacing the $\Pi_{\rm had}(q^2)$ in a vertex or in box diagram by the z-dispersion integral and exchanging the $\int d^4k$ with the $\int dz$ creates one-loop diagrams with a subsequent z-integration.

.

The kernel functions for the dispersion integrals

$$\Delta\alpha(x) = \Delta\alpha^{(5)}_{\text{had}}(x) + \Pi_e(x) + \sum_{f=\mu,\tau,t} \Pi_f(x)$$

$$\Delta\alpha^{(5)}_{\text{had}}(x) = \frac{\alpha}{\pi}\,\frac{x}{3}\int_{4m_\pi^2}^{\infty} dz\,\frac{R^{(5)}_{\text{had}}(z)}{z}\,\frac{1}{x - z + i\,\delta}$$

$$V_2(x) = V_{2e}(x) + V_{2\text{rest}}(x)$$

$$V_{2\text{rest}}(x) = \int_{4M^2}^{\infty} dz\,\frac{R(z)}{z}\,K_V(x + i\delta; z)$$

$$K_V(x;z) = \frac{1}{3}\left\{-\frac{7}{8} - \frac{z}{2\,x} + \left(\frac{3}{4} + \frac{z}{2\,x}\right)\ln\left(-\frac{x}{z}\right) - \frac{1}{2}\left(1 + \frac{z}{x}\right)^2\left[\zeta_2 - \mathbf{Li}_2\left(1 + \frac{x}{z}\right)\right]\right\}$$

$$B_i(x,y) = \int_{4M^2}^{\infty} dz\,\frac{R(z)}{z}\,K_{box,i}(x + i\delta, y + i\delta; z)$$

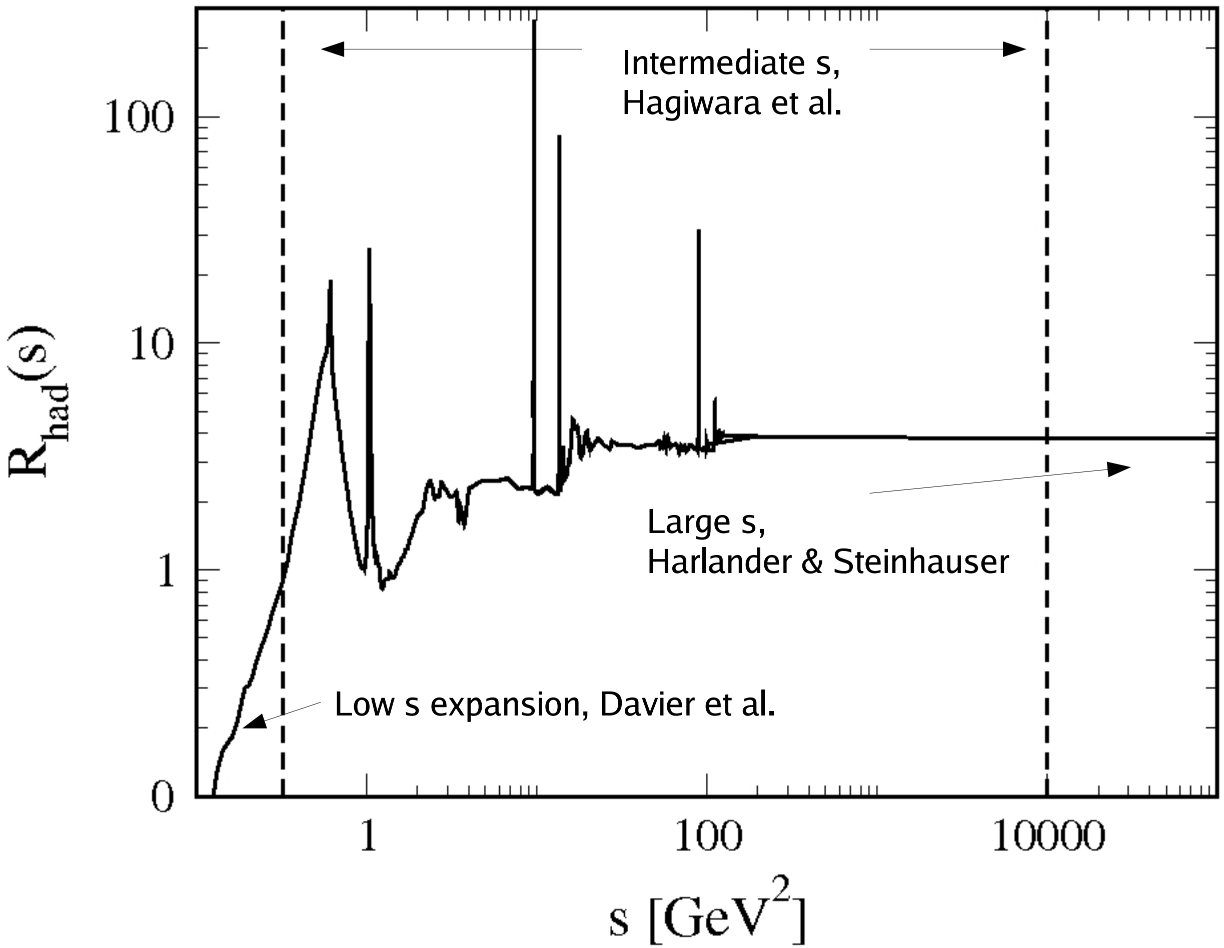
The $K_{box,i}(x,y;z)$ are determined as linear combinations of one-loop integrals with mass $z = M^2$.

Using R_{had}

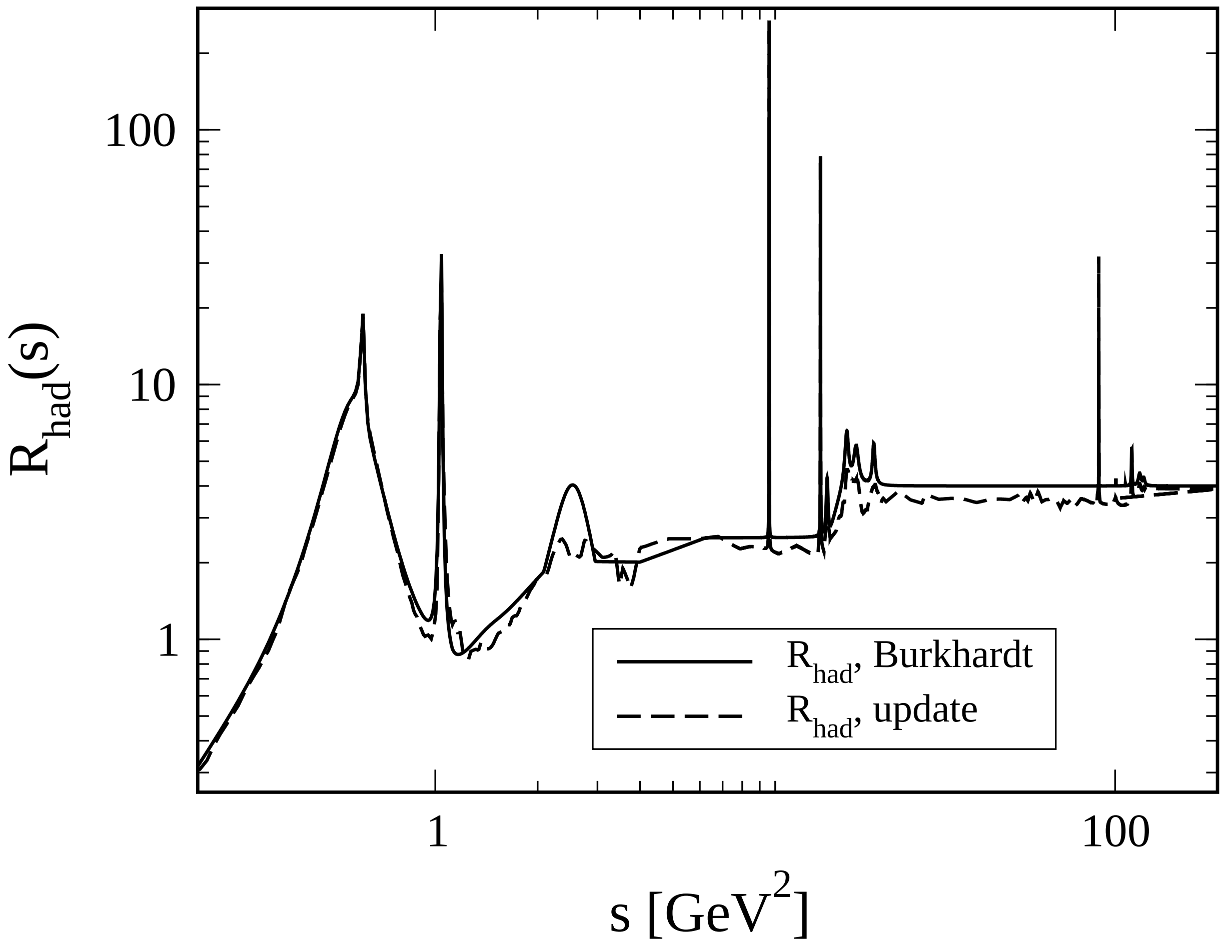
This is a topic by itself, because R_{had} is basically unpublished.

```
N.N.1:
Fuer R(s) mit Fehlern, Kontinuum + Resonanzen haben wir nur unsere interne
Arbeitsversion.

N.N.2:
This procedure is a follow up of complicated programs, which unfortunately
do not exist in a really user-friendly form.

N.N.3:
I understand that for your problem it is probably too
cumbersome (and time-consuming) to use the data.

N.N.4:
es hat etwas gedauert, bis ich in meinen alten Verzeichnissen auf einer
1994er Vax am MPI fuendig geworden bin.
```

So, finally, we might reproduce the old estimates given for the vertex dispersion relation in Kniehl, Krawczyk, Kühn, Stuart (1988) $\longrightarrow$ finally we have numerics, but with larger errors than necessary



The implementation of $R_{\rm had}$ used for the numerical evaluation of irreducible two-loop corrections.



A comparison of the parametrizations from [Burkhardt:1981jk] *and* [rintpl:2008AA] *.*

Final formula and results

We distinguish 3 different categories of 2-loop contributions:

- **Running α**
- **the irreducible 2-loop vertices**
- **the 'rest': irreducible vertices and boxes plus 2-loop boxes**

$$
\begin{aligned}
\frac{d\overline{\sigma}}{d\Omega} &= c \int_{4M_\pi^2}^{\infty} dz \frac{R_{\mathrm{had}}(z)}{z} \frac{1}{t-z} F_1(z) \qquad (1) \\
&+ \quad c \int_{4M_\pi^2}^{\infty} \frac{dz}{z\,(s-z)} \Big\{ R_{\mathrm{had}}(z) \Big[F_2(z) + F_3(z) \ln\Big|1-\frac{z}{s}\Big| \Big] \\
&- \quad R_{\mathrm{h}}(s) \Big[F_2(s) + F_3(s) \ln\Big|1-\frac{z}{s}\Big| \Big] \Big\} \\
&+ \quad c\, \frac{R_{\mathrm{h}}(s)}{s} \Big\{ F_2(s) \ln\Big(\frac{s}{4M_\pi^2}-1\Big) - 6\zeta_2 F_a(s) \\
&+ \quad F_3(s) \Big[2\zeta_2 + \frac{1}{2}\ln^2\Big(\frac{s}{4M_\pi^2}-1\Big) + \mathbf{Li}_2\Big(1-\frac{s}{4M_\pi^2}\Big) \Big] \Big\},
\end{aligned}
$$

with $c = \alpha^4/(\pi^2 s)$ **and** $R_{\mathrm{h}}(s) = \theta(s-4M_\pi^2)\; R_{\mathrm{had}}(s)$**.**

$$
\begin{aligned}
F_1(z) &= \frac{1}{3}\Big\{ 9\,\bar{c}(s,t)\ln\Big(\frac{s}{m_e^2}\Big) + \Big[-z^2\Big(\frac{1}{s}+\frac{2}{t}+2\,\frac{s}{t^2}\Big) + z\,\Big(4+4\,\frac{s}{t}+2\,\frac{t}{s}\Big) + \frac{1}{2}\,\frac{t^2}{s}+6\,\frac{s^2}{t} \\
&+ 5\,s+4\,t\Big]\,\ln\Big(-\frac{t}{s}\Big) + s\,\Big(-\frac{z}{t}+\frac{3}{2}\Big)\,\ln\Big(1+\frac{t}{s}\Big) + \Big[\frac{1}{2}\,\frac{z^2}{s}+2\,z\,\Big(1+\frac{s}{t}\Big)-\frac{11}{4}\,s-2\,t\Big]\,\ln^2\Big(-\frac{t}{s}\Big) \\
&- \Big[\frac{1}{2}\,\frac{z^2}{t}-z\,\Big(1+\frac{s}{t}\Big)+\frac{t^2}{s}+2\,\frac{s^2}{t}+\frac{9}{2}\,s+\frac{15}{4}\,t\Big]\,\ln^2\Big(1+\frac{t}{s}\Big) + \Big[\frac{z^2}{t}-2\,z\,\Big(1+\frac{s}{t}\Big)+2\,\frac{s^2}{t}+5\,s+\frac{5}{2}\,t\Big] \\
&\times \ln\Big(-\frac{t}{s}\Big)\,\ln\Big(1+\frac{t}{s}\Big) - 4\,\Big[\frac{t^2}{s}+2\,\frac{s^2}{t}+3\,\big(s+t\big)\Big]\,\Big[1+\mathrm{Li}_2\Big(-\frac{t}{s}\Big)\Big] - \Big[\frac{t^2}{s}+2\,\frac{s^2}{t}+3\,\big(s+t\big)\Big]\,\ln\Big(\frac{z}{s}\Big)\,\ln\Big(1+\frac{t}{s}\Big) \\
&- \Big[2\,\frac{z^2}{t}-4\,z\,\Big(1+\frac{s}{t}\Big)-4\,\frac{t^2}{s}-2\,\frac{s^2}{t}+s-\frac{11}{2}\,t\Big]\,\zeta_2 + \Big[z^2\,\Big(\frac{1}{s}+2\,\frac{s}{t^2}+\frac{2}{t}\Big)-z\,\Big(\frac{t}{s}+2\frac{s}{t}+2\Big)\Big]\,\ln\Big(\frac{z}{s}\Big) \\
&- \Big[z^2\,\Big(\frac{1}{s}+\frac{1}{t}\Big)+2\,z\,\Big(1+\frac{s}{t}\Big)+s+2\,\frac{s^2}{t}\Big]\,\ln\Big(\frac{z}{s}\Big)\,\ln\Big(1+\frac{z}{s}\Big) + \Big[\frac{z^2}{s}+4\,z\,\Big(1+\frac{s}{t}\Big)-\frac{t^2}{s}-4\,\big(s+t\big)\Big] \\
&\times \ln\Big(\frac{z}{s}\Big)\ln\Big(1-\frac{z}{t}\Big) - \Big[z^2\,\Big(\frac{1}{s}+2\frac{s}{t^2}+\frac{2}{t}\Big)-2\,z\,\Big(\frac{t}{s}+2\,\frac{s}{t}+2\Big)+\frac{t^2}{s}+2\,\big(s+t\big)\Big]\,\ln\Big(1-\frac{z}{t}\Big) \\
&+ \Big[\frac{z^2}{t}-2\,z\,\Big(1+\frac{s}{t}\Big)+2\,\frac{t^2}{s}+8\,s+4\,\frac{s^2}{t}+7\,t\Big]\,\ln\Big(1-\frac{z}{t}\Big)\,\ln\Big(1+\frac{t}{s}\Big) + \Big[\frac{z^2}{s}+4\,z\,\Big(1+\frac{s}{t}\Big)-\frac{t^2}{s}-4\,\big(s+t\big)\Big] \\
&\times \mathrm{Li}_2\,\Big(\frac{z}{t}\Big) - \Big[z^2\,\Big(\frac{1}{s}+\frac{1}{t}\Big)+2\,z\,\Big(1+\frac{s}{t}\Big)+s+2\,\frac{s^2}{t}\Big]\,\mathrm{Li}_2\,\Big(-\frac{z}{s}\Big) - \Big[\,\frac{z^2}{t}-2\,z\,\Big(1+\frac{s}{t}\Big)+\frac{t^2}{s}+5\,s+2\,\frac{s^2}{t}+4\,t\Big] \\
&\times \mathrm{Li}_2\,\Big(1+\frac{z}{u}\Big)\Big\} + 4\,\bar{c}(s,t)\ln\Big(\frac{2\,\omega}{\sqrt{s}}\Big)\,\Big[\ln\Big(\frac{s}{m_e^2}\Big)+\ln\Big(-\frac{t}{s}\Big)-\ln\Big(1+\frac{t}{s}\Big)-1\Big],
\end{aligned}
$$

and similarly for $F_2(z)$ and $F_3(z)$.

The $\int_{4M^2} dzF_i(z)$ gives from the lower integration bound the logarithmically enhanced terms $\ln(= M^2)^n$, e.g. from terms like $A(z)\ \ln(z/s)$ or from $B(z)\mathrm{Li}_2\Big(\frac{z}{s}\Big)$.

Some numerical results

We will now discuss the numerical net effects arising from the $N_f = 2$ vertex plus box diagrams (i.e. excluding the pure running coupling effects):

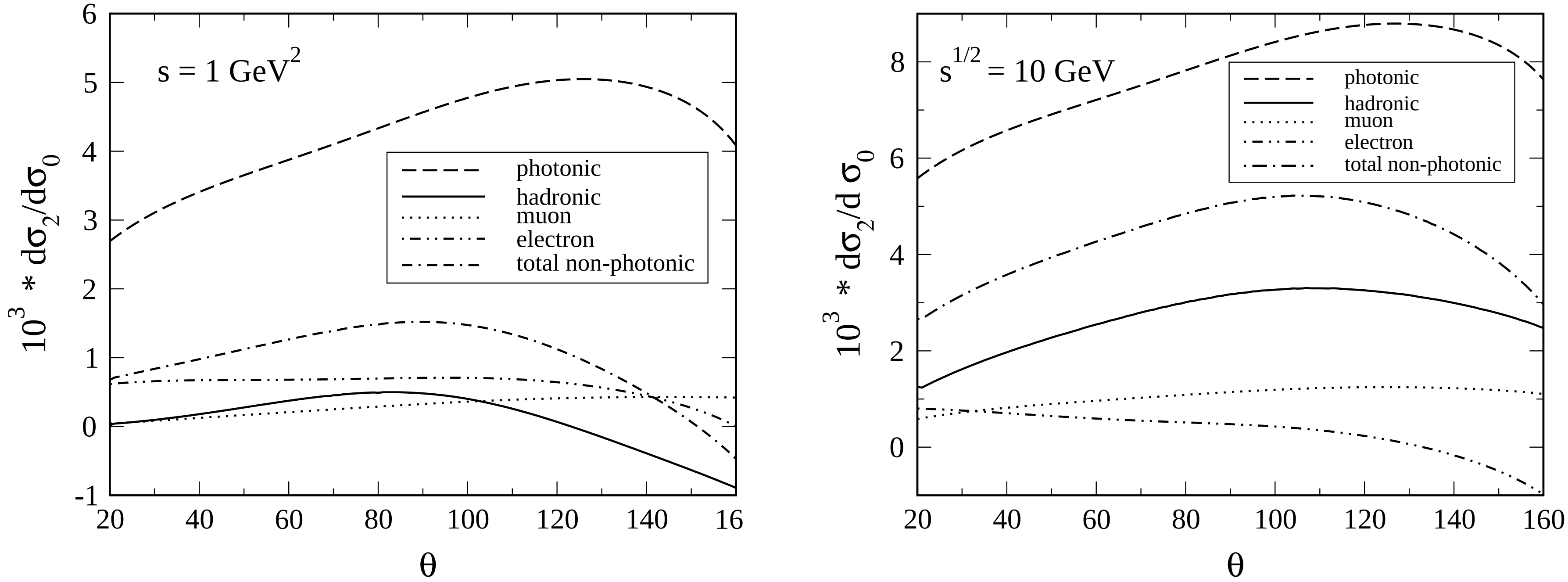
$$\frac{d\sigma_2}{d\Omega} = \frac{d\overline{\sigma}}{d\Omega} + \frac{d\sigma_v}{d\Omega},$$

with $d\overline{\sigma}/d\Omega$ from Eqn. (1). The expression for the irreducible vertex term $d\sigma_v/d\Omega$ derives directly from

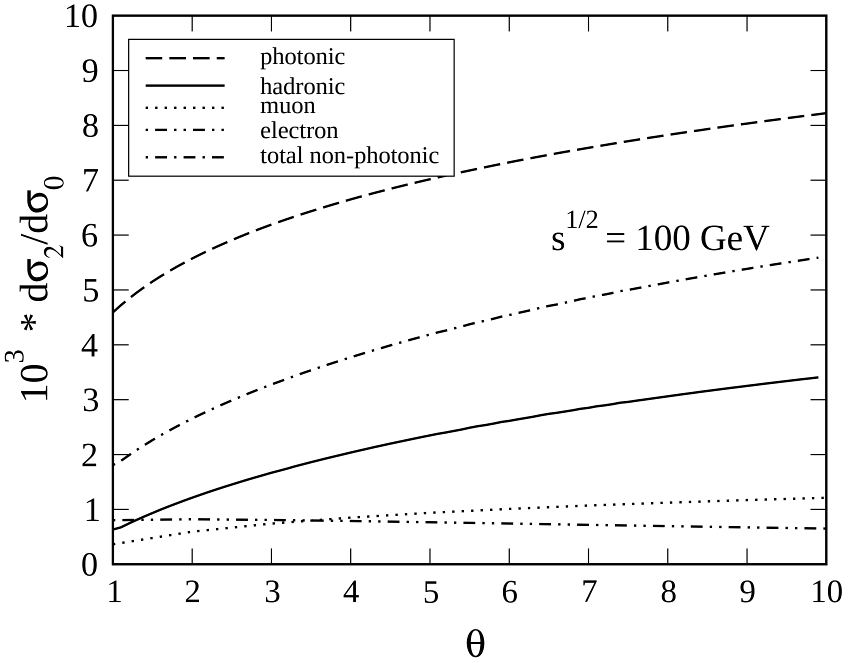
[Kniehl:1988id,webPage:2007x3]

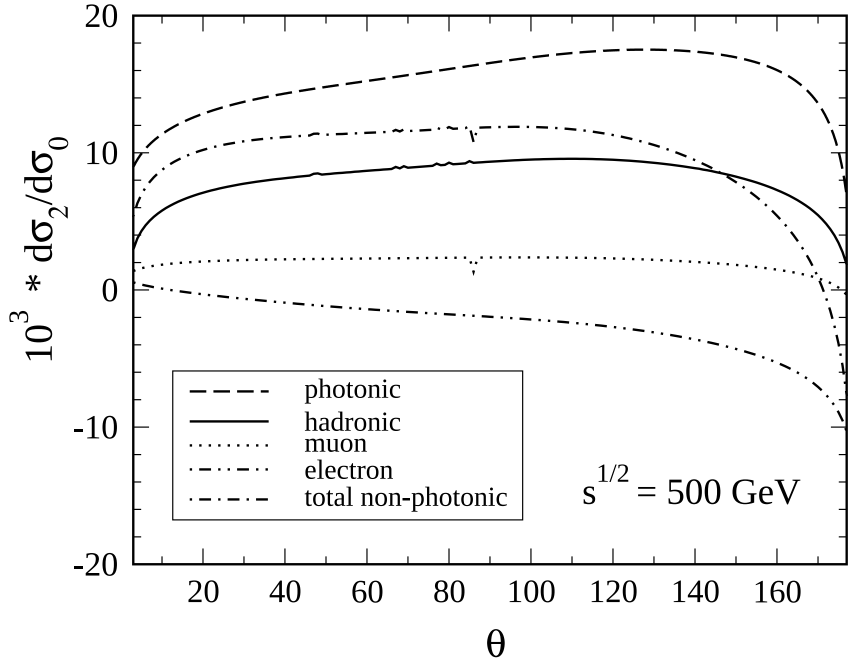
. The $d\sigma_2/d\Omega$ is normalized to the pure photonic Bhabha Born cross section $d\sigma_0/d\Omega$:

$$\frac{d\sigma_0}{d\Omega} = \frac{\alpha^2}{s}\left(\frac{s}{t} + 1 + \frac{t}{s}\right)^2 .$$

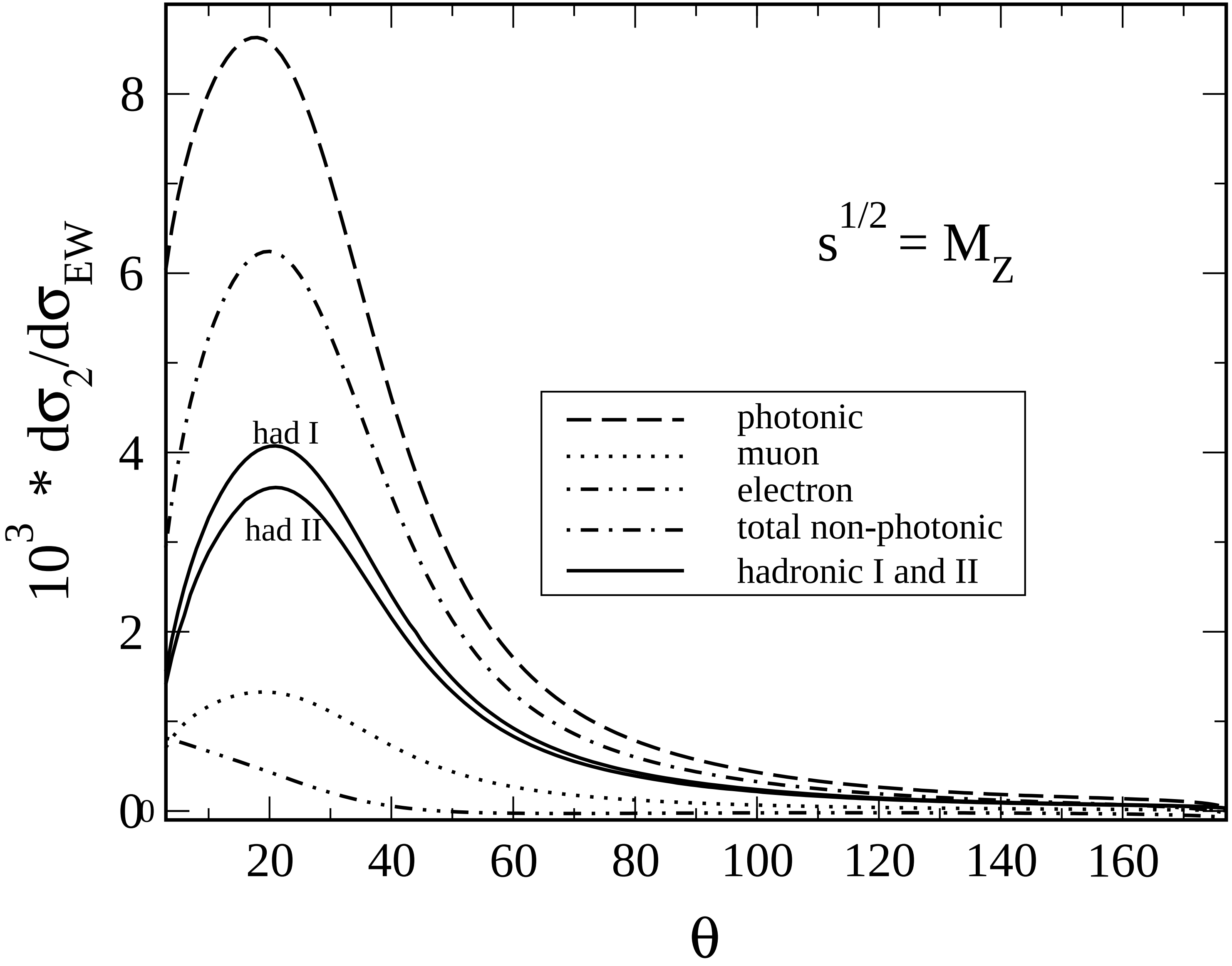


Two-loop $N_f = 2$ vertex and box corrections $d\sigma_2$ to Bhabha scattering in units of $10^{-3} d\sigma_0$ at meson factories, $\sqrt{s} = 1$ GeV (a) and $\sqrt{s} = 10$ GeV (b).





Two-loop $N_f = 2$ vertex and box corrections $d\sigma_2$ to Bhabha scattering in units of $10^{-3} d\sigma_0$ at ILC energies of $\sqrt{s} = 100$ GeV (GigaZ option) and $\sqrt{s} = 500$ GeV.



Two-loop corrections to Bhabha scattering at $\sqrt{s} = M_Z$*, normalized to the effective weak Born cross section.*

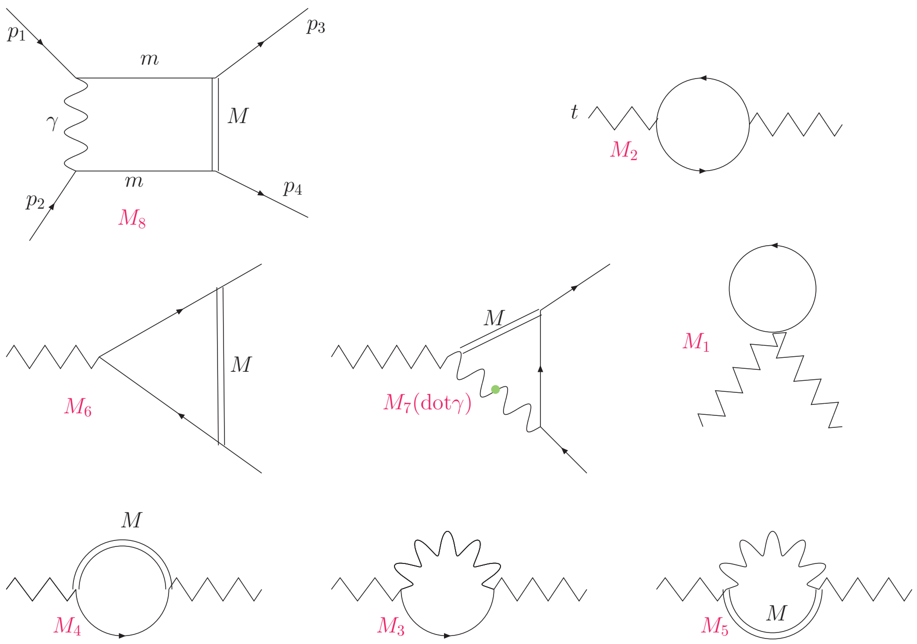
Summary

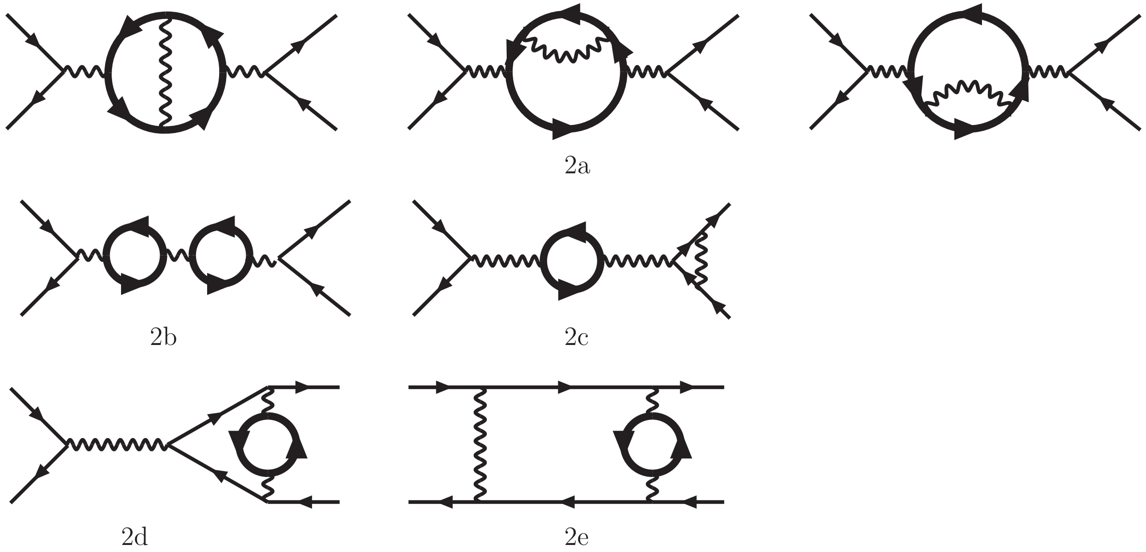
- We determine the $N_f = 2$ contributions to 2-loop Bhabha scattering, including the hadronic corrections
- They are small, but non-negligible at the scale 10^{-3} ($\rightarrow$ No LEP influencing)
- Agreement for $m_e^2 << m_l^2 << s, t, u$ with:
  "Two-loop QED corrections to Bhabha scattering"
  Thomas Becher, Kirill Melnikov, arXiv:0704.3582 [hep-ph], JHEP
- Agreement for $m_e^2 << m_l^2, s, t, u$ with:
  "Two-Loop Heavy-Flavor Contribution to Bhabha Scattering",
  Roberto Bonciani, Andrea Ferroglia, Sacha Penin, arXiv:0710.4775v3 [hep-ph]
- This year also for hadronic corrections with Kuehn, Uccirati
- To be evaluated yet:
  $\longrightarrow$ 1-loop diagrams with real photon emission, interfering with real (Born) radiation, including 5-point functions
- Also: $\longrightarrow$ Real heavy pair emission corrections
- Both items were studied already by Andrei Arbuzov, Kuraev, Shaitchatdenov (1998, small photon mass)

.

The dispersion master integrals for the $N_f = 2$ contributions

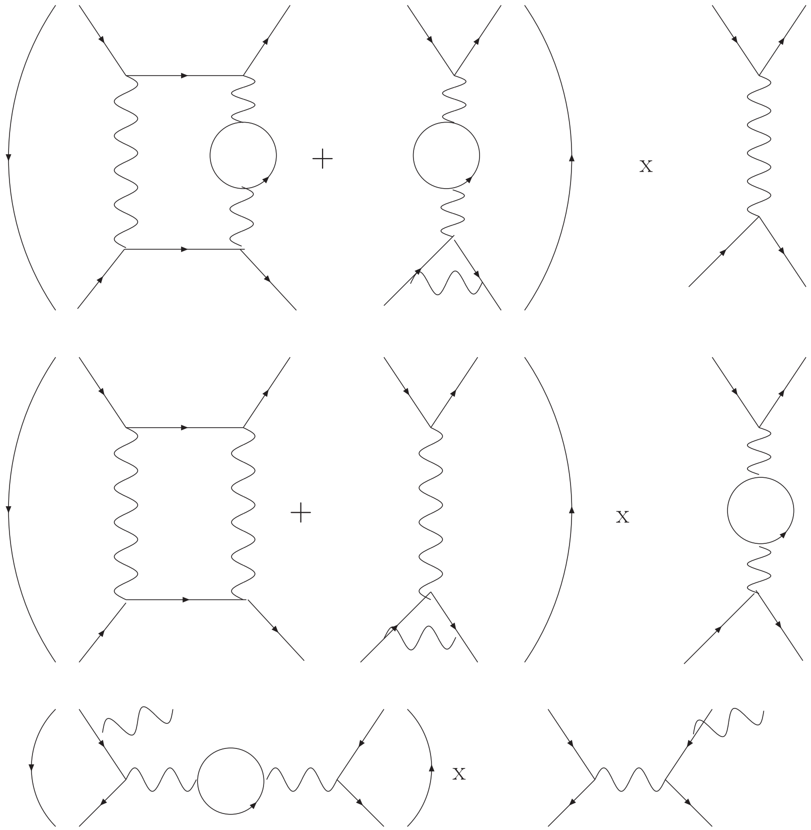
There are three box kernel functions, depending on m_e, m_f, s, t.with $m_e^2 << z = m_f^2, s, t$.
They are IR-divergent.
The eight master integrals for the 2-loop boxes are:





Classes of Bhabha-scattering 2-loop diagrams containing at least one fermion loop.

The 4 direct and 4 crossed fermionic 2-loop box diagrams have to be combined with other diagrams for an IR-finite contribution:



After combining the 2-loop terms with the loop-by-loop terms and with soft real corrections:

$$\frac{d\sigma^{\rm NNLO}}{d\Omega}+\frac{d\sigma_\gamma^{\rm NLO}}{d\Omega} = \frac{d\sigma^{\rm NNLO,e}}{d\Omega}+\sum_{f\neq e} Q_f^2\,\frac{d\sigma^{\rm NNLO,f^2}}{d\Omega}+\sum_{f\neq e} Q_f^4\,\frac{d\sigma^{\rm NNLO,f^4}}{d\Omega}$$
$$+\sum_{f_1,f_2\neq e} Q_{f_1}^2 Q_{f_2}^2\,\frac{d\sigma^{\rm NNLO,2f}}{d\Omega}.$$