New results for 5-point functions

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based on work with:
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- Introduction: 5-point functions for Bhabha scattering
- AMBRE and Mellin-Barnes integrals (arXiv:0704.2423)
- Novel approach to mixed IR-divergencies from loops and real photon emission (new)
- A new reduction to 3- and 4-point functions (new)
- Summary

Introduction: 5-point functions for Bhabha scattering and LHC processes

- Since 2004 we collected some experience in using Mellin-Barnes (MB) representations for massive 2-loop diagrams (see e.g. Czakon,Gluza,Riemann, NPB2006 (hep-ph/0604101) and CGR+Actis, arXiv:0704.2400)
  This was for QED, Bhabha scattering.
- The mathematica packages MB.m (Czakon, CPC 2005) and AMBRE.m (Gluza, Kajda, Riemann, arXiv:0704.2423) were developed for that.
- In parallel, Anastasiou/Daleo (2005) developed an (unpublished) MB-package for the complete numerical evaluation of Feynman diagrams.
  They stress the use for n-point functions with tensor structure.
- It is often said that massive 5-point and 6-point functions tend to be unstable in numerical evaluations.
  They are important for e.g. QCD LHC-background
- In Bhabha scattering, the radiative 1-loop contributions (interfering with lowest order real emission) include diagrams with 5-point functions (massless case, small photon mass: Arbuzov,Kuraev,Shaichatdenov 1998 et al.)
- So we decided to have a look at all this and try to see a workplace.

We found two interesting facts, and I want to report on them here:
The naive application of numerical Mellin-Barnes evaluation of the 5-point diagrams seems not to be competitive to Denner/Dittmaier/LoopTools.
But: We see a very interesting way to treat their IR-divergencies – they have two types of them: from the virtual IR-divergencies due to the loop, and also from the real emission of a massless particle, showing up in an endpoint singularity of the final phase space integral of that

- It was completely unclear, how mixed IR singularities have to be identified and treated with MB-integrals. Now we know.
- Related.
  Usually one may perform something like the Passarino-Veltman tensor reduction, representing the tensor 5-point functions by simpler ones: 4- and 3-point scalar integrals.
- We see a very efficient way to do the algebraic reduction. It was found to be interesting for applications during numerical tests of the MB-ansatz.
- This reduction is starts from earlier work: Melrose:1965kb, Davydychev:1991va, Tarasov:1996br Fleischer[Jegerlehner,Tarasov]:1999hq, Denner[Dittmaier]:2002ii
- On both these recent, unpublished developments I want to report here.
- They may be efficiently combined in large calculations with (semi-)automatization.
  We are exploring this right now in two applications.

Feynman integrals

$$I_5[A(q)] \quad = \quad e^{\epsilon\gamma_E} \int \frac{d^d q}{i\pi^{d/2}} \frac{A(q)}{c_1 c_2 c_3 c_4 c_5}.$$

$$c_i = (q - q_i)^2 - m_i^2$$

unique after chosing one of the chords, for e.g. the 5-point function:

$$q_5 = 0.$$

The numerator $A(q)$ contains the tensor structure,

$$A(q) = \{1, q^\mu, q^\mu q^\nu, q^\mu q^\nu q^\rho, \cdots\}$$

or may be used to define pinched diagrams (a shrinking of line 5 leads to a box diagram corresponding to

$$I_5[c_5] = e^{\epsilon\gamma_E} \int \frac{d^d q}{i\pi^{d/2}} \frac{1}{c_1 c_2 c_3 c_4}.$$

Example: The 5-point function of Bhabha scattering (I)

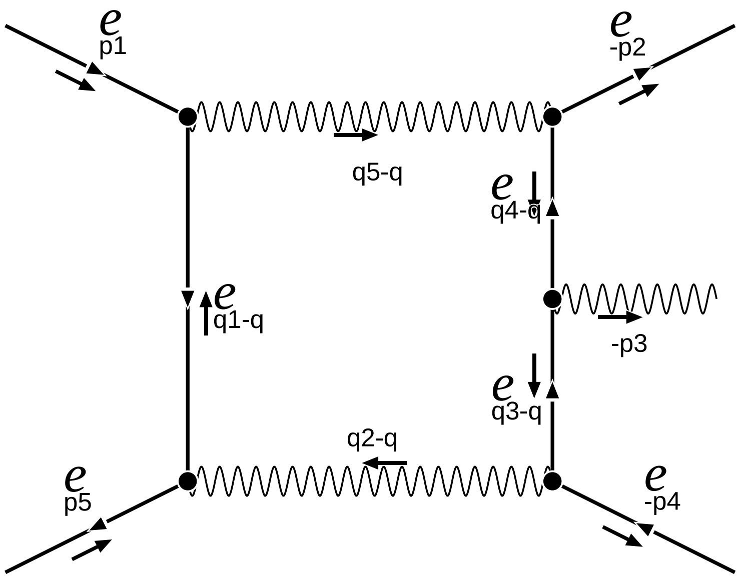


Figure 1: The pentagon topology of Bhabha scattering

$$I_5[A(q)] \quad = \quad -e^{\epsilon\gamma_E} \int_0^1 \prod_{j=1}^{5} dx_j \; \delta\left(1 - \sum_{i=1}^{5} x_i\right) \frac{\Gamma\left(3+\epsilon\right)}{F(x)^{3+\epsilon}} \; B(q),$$

with $B(1) = 1, B(q^\mu) = Q^\mu, B(q^\mu q^\nu) = Q^\mu Q^\nu - \frac{1}{2} g^{\mu\nu} F(x)/(2+\epsilon)$**, and** $Q^\mu = \sum x_i q_i^\mu$**.**

The diagram depends on five variables and the F-form is:

$$F(x) \quad = \quad m_e^2(x_1+x_3+x_4)^2+[-s]x_2x_5+[-z_4]x_2x_4+[-t]x_1x_3+[-t']x_1x_4+[-z_2]x_3x_5. \tag{1}$$

Henceforth, $m_e=1$. Photon momentum is p_3.

$$\begin{aligned} s &= (p_1+p_5)^2, \\ t &= (p_4+p_5)^2, && (2) \\ t' &= (p_1+p_2)^2, && (3) \\ z_2 &= 2p_2p_3, && (4) \\ z_4 &= 2p_4p_3. && (5) \\ & && (6) \end{aligned}$$

The MB-representation,

$$\frac{1}{[A(x)+Bx_ix_j]^R} \quad = \quad \frac{1}{2\pi i}\int_{\mathcal{C}} dz[A(x)]^z[Bx_ix_j]^{-R-z}\frac{\Gamma(R+z)\Gamma(-z)}{\Gamma(R)},$$

is used several times for replacing in $F(x)$ the sum over x_ix_j by products of monomials in the x_ix_j, thus allowing the subsequent x-integrations in a simple manner.

Short remarks on Mellin-Barnes integrals

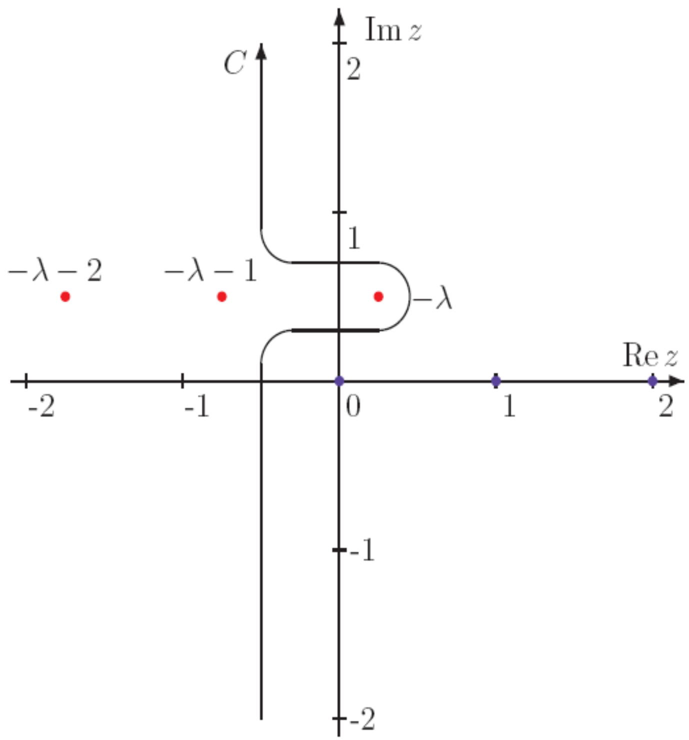
We want to apply now a simple formula for integrating over the x_i:

$$\int_0^1 \prod_{j=1}^{N} dx_j \ x_j^{\alpha_j - 1} \ \delta\left(1 - x_1 - \cdots - x_N\right) \quad = \quad \frac{\Gamma(\alpha_1) \cdots \Gamma(\alpha_N)}{\Gamma\left(\alpha_1 + \cdots + \alpha_N\right)}$$

with coefficients α_i dependent on F

For this, we have to apply several MB-integrals here.

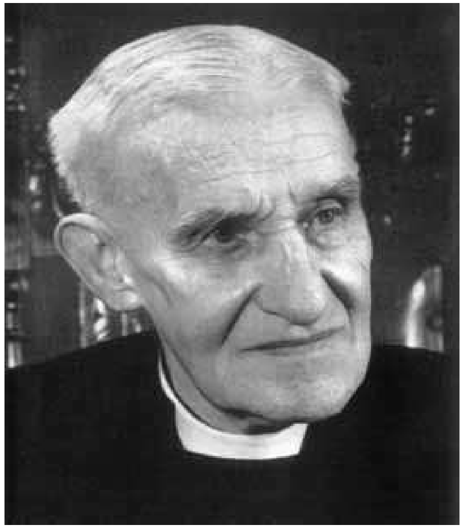
$$\frac{1}{[A(s)x_1^{a_1} + B(s)x_1^{b_1}x_2^{b_2}]^a} = \frac{1}{2\pi i\ \Gamma(a)} \int_{-i\infty}^{i\infty} d\sigma [A(s)x_1^{a_1}]^\sigma [B(s)x_1^{b_1}x_2^{b_2}]^{a+\sigma}\ \Gamma(a+\sigma)\Gamma(-\sigma)$$



The $\Gamma(z)$ has poles at the negative real axis, at $z=-n,\ n=0,-1,\cdots$, with residues

$$Res\Gamma(z)|_{z=-n} = \frac{(-1)^n}{n!}$$

Mellin, Robert, Hjalmar, 1854-1933
Barnes, Ernest, William, 1874-1953



A little history

- N. Usyukina, 1975: "ON A REPRESENTATION FOR THREE POINT FUNCTION", Teor. Mat. Fiz. 22;
  a finite massless off-shell 3-point 1-loop function represented by 2-dimensional MB-integral
- E. Boos, A. Davydychev, 1990: "A Method of evaluating massive Feynman integrals", Theor. Math. Phys. 89 (1991);
  N-point 1-loop functions represented by n-dimensional MB-integral
- V. Smirnov, 1999: "Analytical result for dimensionally regularized massless on-shell double box", Phys. Lett. B460 (1999);
  treat UV and IR divergencies by analytical continuation: shifting contours and taking residues 'in an appropriate way'
- B. Tausk, 1999: "Non-planar massless two-loop Feynman diagrams with four on-shell legs", Phys. Lett. B469 (1999);
  nice algorithmic approach to that, starting from search for some unphysical space-time dimension d for which the MB-integral is finite and well-defined
- M. Czakon, 2005 (with experience from common work with J. Gluza and TR): "Automatized analytic continuation of Mellin-Barnes integrals", Comput. Phys. Commun. (2006);
  Tausk's approach realized in Mathematica program MB.m, published and available for use

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AMBRE – **Automatic Mellin-Barnes Representations** for Feynman diagrams

For the Mathematica package AMBRE, many examples, and the program description, see:

http://prac.us.edu.pl/∼gluza/ambre/
http://www-zeuthen.desy.de/theory/research/CAS.html

Authors: J. Gluza, K. Kajda, T. Riemann

See also here:
http://www-zeuthen.desy.de/∼riemann/Talks/capp07/

with additional material presented at the CAPP – School on Computer Algebra in Particle Physics, DESY, Zeuthen, March 2007

A AMBRE functions list

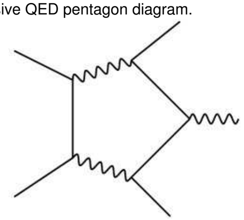
The basic functions of `AMBRE` are:

- **Fullintegral**[**{numerator},{propagators},{internal momenta}**] – is the basic function for input Feynman integrals
- **invariants** – is a list of invariants, e.g. **invariants = {p1*p1 → s}**
- **IntPart**[**iteration**] – prepares a subintegral for a given internal momentum by collecting the related numerator, propagators, integration momentum
- **Subloop**[**integral**] – determines for the selected subintegral the U and F polynomials and an MB-representation
- **ARint**[**result,i_**] – displays the MB-representation number i for Feynman integrals with numerators
- **Fauto**[**0**] – allows user specified modifications of the F polynomial **fupc**
- **BarnesLemma**[**repr,1,Shifts->True**] – function tries to apply Barnes' first lemma to a given MB-representation; when **Shifts->True** is set, `AMBRE` will try a simplifying shift of variables
  BarnesLemma[**repr,2,Shifts->True**] – function tries to apply Barnes' second lemma

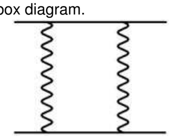
AMBRE - Automatic Mellin-Barnes REpresentation
(arXiv:0704.2423)

To download 'right click' and 'save target as'.

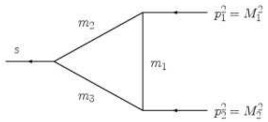
- The package AMBRE.m
- Kinematics generator for 4- 5- and 6- point functions with any external legs KinematicsGen.m
- Tarball with examples given below examples.tar.gz
  - example1.nb, example2.nb - Massive QED pentagon diagram.



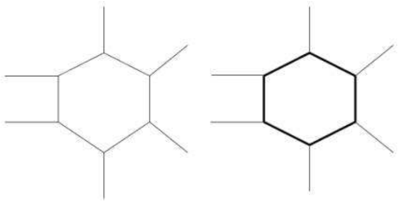
  - example3.nb - Massive QED one-loop box diagram.



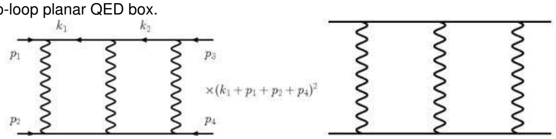
  - example4.nb - General one-loop vertex.



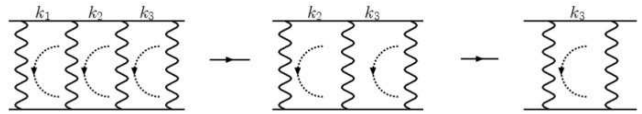
  - example5.nb - Six-point scalar functions;
    left: massless case,
    right: massive case.



  - example6.nb - **left**, example7.nb - **right**
    Massive two-loop planar QED box.



  - example8.nb - The loop-by-loop iterative procedure.



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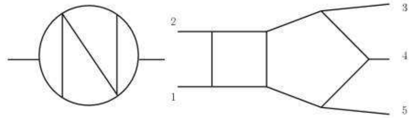

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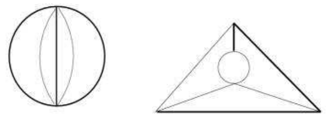

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Example: The 5-point function of Bhabha scattering (II)

In our example we get seven-fold MB-representations, one integral for each additive term in (1), and finally five-fold representations after twice applying Barnes' lemma in order to eliminate the spurious integrations from the mass term.

We have to consider scalar, vector, and degree-two tensor integrals. Explicitely, the scalar MB-representation is:

$$I_5[1] \quad = \quad \frac{-e^{\epsilon\gamma_E}}{(2\pi i)^5} \prod_{i=1}^{5} \int_{-i\infty+u_i}^{+i\infty+u_i} dr_i (-s)^{-3-\epsilon-r_1} (-t)^{r_2} (-t')^{r_3} \left(\frac{z_2}{s}\right)^{r_4} \left(\frac{z_4}{s}\right)^{r_5} \frac{\prod_{j=1..12} \Gamma_j}{\Gamma_0 \Gamma_{13}}, \tag{7}$$

$$\epsilon = -17/16$$

The real shifts u_i of the integration strips r_i are:

$$\begin{aligned} u_1 &= -89/64 = -1 - \delta \\ u_2 &= -1/4 && (8) \\ u_3 &= -3/8 && (9) \\ u_4 &= -1/8 && (10) \\ u_5 &= -1/32 && (11) \end{aligned}$$

with a normalization $\Gamma_0 = \Gamma[-1-2\epsilon]$, and the other Γ-functions are:

$$\begin{aligned}
\Gamma_1 &= \Gamma[-r_2], \\
\Gamma_2 &= \Gamma[-r_3], \\
\Gamma_3 &= \Gamma[1+r_2+r_3], \\
\Gamma_4 &= \Gamma[-r_1+r_2+r_3], \\
\Gamma_5 &= \Gamma[-2-\epsilon-r_1-r_4], \\
\Gamma_6 &= \Gamma[-r_4], \\
\Gamma_7 &= \Gamma[1+r_2+r_4], \\
\Gamma_8 &= \Gamma[-2-\epsilon-r_1-r_5], \\
\Gamma_9 &= \Gamma[-r_5], \\
\Gamma_{10} &= \Gamma[1+r_3+r_5], \\
\Gamma_{11} &= \Gamma[3+\epsilon+r_1+r_4+r_5], \\
\Gamma_{12} &= \Gamma[3+2r_1+r_4+r_5],
\end{aligned} \tag{12}$$

and

$$\Gamma_{13} = \Gamma[3+2(r_2+r_3)+r_4+r_5].$$

Analytical continuation in ϵ and deformation of integration contours

A well-defined MB-integral was found with the finite parameter ϵ and the strips parallel to the imaginary axis.

Now look at the real parts of arguments of Γ-functions (in the numerator only) and find out, which of them change sign (become negative) when $\epsilon \rightarrow o$

Rule:
Moving $\epsilon \longrightarrow o$ corresponds to a stepwise analytical continuation of the contour integral ($dimension = n$) and so we have to add or subtract the residues at these values of the integration varables.

The residues have the dimension of integration $n-1, n-2, \cdots$.

This procedure may be automatized "easily" and it is done in the mathematica package MB.m of M. Czakon.

Example: The 5-point function of Bhabha scattering (III)

After the analytical continuation in ϵ, the scalar pentagon function is

$$\begin{aligned} I_5[1] &= I_5^{IR}(s,z_2,t',t) + I_5^{IR}(s,z_4,t,t') + \quad \textbf{finite terms}, \\ I_5^{IR}(s,z_2,t',t) &= \frac{1}{2sz_2}\left\{\left(\frac{1}{\epsilon} + 2\ln\frac{t}{z_2}\right) S_{-1}(t') + S_{0,1}(t') - 2S_{0,2}(t,t')\right\}. \end{aligned} \tag{13}$$

Among the vector and tensor integrals, only two are infrared singular:

$$\begin{aligned} I_5[q^\mu] &= q_1^\mu \; I_5^{IR}(s,z_4,t,t') + \quad \textbf{finite terms}, \\ I_5[q^\mu q^\nu] &= q_1^\mu q_1^\nu \; I_5^{IR}(s,z_4,t,t') + \quad \textbf{finite terms}. \end{aligned} \tag{14}$$

If looking for the IR-divergent terms, we are ready now. The problem is solved. Of course, one has to evaluate the MB-integrals yet (wait a minute for that).

But: What means finite terms here?
The meaning is two-fold:

- **free of poles in ϵ**
- **non-singular when cross-section will be calculated**

The second point is here the crucial one. It leads us to the question of the endpoint singularities due to real photon emission.
After evaluation of the MB-integrals we come to this.

IR-divergencies as inverse binomial sums

$$S_{-1}(t) = \frac{1}{2\pi i} \int_{-i\infty+u}^{+i\infty+u} dr (-t)^{-1-r} \frac{\Gamma[-r]^3 \Gamma[1+r]}{\Gamma[-2r]}.$$

The integration contour in the complex r plane extends parallel to the imaginary axis with $\Re r = u \in (-\frac{1}{2}, 0)$. The integral may be evaluated by closing the contour to the left and taking residua, resulting in an inverse binomial series. The sum may be obtained in this simple case with `Mathematica`:

$$S_{-1}(t) = \sum_{n=0}^{\infty} \frac{(t)^n}{\binom{2n}{n}(2n+1)} = \frac{4\arcsin(\sqrt{t/2})}{\sqrt{4-t}\sqrt{t}} = -\frac{2y\ln(y)}{1-y^2},$$

with

$$y \equiv y(t) = \frac{\sqrt{1-4/t}-1}{\sqrt{1-4/t}+1}.$$

The $\frac{1}{2}S_{-1}(t)$ agrees with the infrared divergent part of the one-loop QED vertex function $I_5[c_5 c_4]$, and for finite z_i the infrared structure is completely explored by knowing this function.

Our 5-point function

$$I_5(s,t,t',z_1,z_2)$$

contributes to cross-sections after interfering with another diagram with real emission, and one has to integrate over the phase space.
This includes the soft photon integration, and thus (in 4 dimensions, no log-terms shown here):

$$\begin{aligned} \int_0^{\omega} dE_\gamma \left[M_1^{lowestorder} \times M_2(I_5) \right] &\sim \int_0^{\omega} dE_\gamma \left[\frac{A}{E_\gamma} + F(E_\gamma) \right] \\ &\sim \int_0^{Y_{cut}} dz_{1,2} \left[\frac{B}{z_{1,2}} + F(z_{1,2}) \right] \end{aligned} \tag{15}$$

This has to be regularized e.g. by dimensional regularization of the photon phase space ($4 \to d$).
Remember:

$$\begin{aligned} \int_0^{z_{max}} dz/z \sim \int_0^{\omega} dE/E &= \ln(E)|_0^{\omega} = \ln(\omega) - \ln(0) = divergent \\ \int_0^{z_{max}} dz/z^{5-d} \sim \int_0^{\omega} dE/E^{5-d} &= \frac{1}{d-4} E^{d-4}|_0^{\omega} = \frac{\omega^{2\epsilon} - 0}{2\epsilon} = finite \end{aligned} \tag{16}$$

We have to safely control the dependence on z_1, z_2 as part of the mixed infrared problem due to the common existence of virtual and real IR-sources.

Example: The 5-point function of Bhabha scattering (IV)

So come back to further sums $S_{0,1}(t)$, $S_{0,2}(t,t')$ besides $S_{-1}(t)$, get:

$$
\begin{aligned}
S_{0,1}(t) &= \frac{1}{2\pi i}\int_{-i\infty+u}^{+i\infty+u} dr(-t)^{-1-r}\,\frac{\Gamma[-r]^3\Gamma[1+r]}{\Gamma[-2r]}\left(\gamma_E+3\Psi[0,-r]-2\Psi[0,-2r]\right)\\
&= \sum_{n=0}^{\infty}\frac{t^n}{\begin{pmatrix}2n\\ n\end{pmatrix}(2n+1)}[3S_1(n)-2S_1(2n+1)],
\end{aligned}
\tag{17}
$$

where $\Psi[n,z]$ is the Polygamma function $\Psi^{(n)}(z)$ and $S_1(k)$ are Harmonic Numbers:

$$
\begin{aligned}
\textbf{Polygamma}[n+1] &\equiv \textbf{Polygamma}[0,n+1]\\
&= \Psi(n+1) = \frac{\Gamma'(n+1)}{\Gamma(n+1)} = S_1(n)-\gamma_E\\
S_k(n) &= \sum_{i=1}^{n}\frac{1}{i^k},
\end{aligned}
\tag{18}
$$

The third contribution is a two-dimensional integral (needed at $t = t'$):

$$\begin{aligned} S_{0,2}(t,t'=t) &= \frac{1}{(2\pi i)^2}\prod_{i=1}^{2}\int_{-i\infty+w_i}^{+i\infty+w_i} dr_i(-t)^{-1-r_1}\left(\frac{t'}{t}\right)^{r_2}\frac{\Gamma[-r_1]^2}{\Gamma[-2r_1]} \\ &\quad \Gamma[-r_2]\Gamma[1+r_2]\Gamma[1+r_1+r_2]\Gamma[-(1+r_1+r_2)] \\ &= \sum_{n=0}^{\infty}\frac{t^n}{\binom{2n}{n}(2n+1)}[\ln(-t)+3S_1(n)-2S_1(2n+1)] \\ &= \ln(-t)\ S_{-1}(t)+S_{0,1}(t). \end{aligned} \tag{19}$$

Here we might finish this introductory discussion.

But sometimes things are a little more involved.

Go back to the very beginning of evaluating I_5, set in the MB-integral immediately $t = t'$ and get something different after analytical continuation in ϵ:

$$
\begin{aligned}
I_5[1] &= I_5^{IR}(s,z_2,z_4,t) + I_5^{IR}(s,z_4,z_2,t) + \text{finite terms}, \\
I_5^{IR}(s,z_2,z_4,t) &= \frac{1}{2sz_2}\left\{\left[\frac{1}{\epsilon} + 2\ln\frac{z_4}{s} - \ln(-t)\right] S_{-1}(t) + S_{0,1}(t)\right\} - S_{0,3}(s,z_2,z_4,t),
\end{aligned}
\tag{20}
$$

with

$$
\begin{aligned}
S_{0,3}(s,z_2,z_4,t) &= \frac{1}{(2\pi i)^2}\prod_{i=1}^{2}\int_{-i\infty+w_i}^{+i\infty+w_i} dr_i (-s)^{r_2}(-t)^{-r_1+r_2}(-z_2)^{-2-r2}(-z_4)^{-1-r_2} \\
&\quad \Gamma[-r_1]\Gamma[-1-r_2]\Gamma[-1-r_1-r_2]\Gamma[r_1-r_2]\Gamma[-r_2]^2\Gamma[1+r_2] \\
&\quad \frac{\Gamma[2+r_2]\Gamma[1-r_1+r_2]}{\Gamma[-2r_1]\Gamma[-1-2r_2]},
\end{aligned}
\tag{21}
$$

with $w_i \in (-1,-\frac{1}{2}) = -1+\delta+iIm(w_i)$.

$S_{0,3}$ **is singular if $z_2 \to 0$: $(-z_2)^{-2-(-1+\delta)}) = (-z_2)^{-1-\delta}) \to$ not integrable at $z_2 = 0$.**

By shifting the integration strip in r_2 across $r_2 = -1$ (to the left) and taking the residue there, one remains with a finite MB-integral (now with $w_2 \in (-2, -1-\frac{1}{2})$) and a one-dimensional MB-integral containing the singularity:

$$\begin{aligned} S_{0,3}(s, z_2, z_4, t) &= \frac{1}{s z_2} \frac{1}{(2\pi i)} \int_{-i\infty+w_1}^{+i\infty+w_1} dr_1 (-t)^{-1-r_1} \frac{\Gamma[-r_1]^3}{\Gamma[-2r_1]} \Gamma[1+r_1] \\ &\quad \left[\ln \frac{z_4}{s} + \ln \frac{z_2}{t} + \gamma_E + \Psi[0, r_1+1] \right] + \text{finite terms.} \\ &= \frac{1}{s z_2} \left[\left(\ln \frac{z_4}{s} + \ln \frac{z_2}{t} \right) S_{-1}(t) + S_{0,1}(t) \right] + \text{finite terms.} \end{aligned} \qquad (22)$$

finite terms **is the original $S_{0,3}(s, z_2, z_4, t)$ but with shifted curve.**

Combining equations (20) to (22), we get again $I_5^{IR}(s, z_2, t', t) + I_5^{IR}(s, z_4, t, t')$.

The essential point here is that the applied method is general: It may happen hat the integration strip, after using `MB.m`, is such that some endpoint singularity is not explicit, but is contained in an MB-integral. Then, one has to shift the strip across the singular point and the residue exhibits the singularity searched for. This is, again, a simple recipy for this kind of problem.

Evidently, it is "trivial" to automatize the complete procedure.

Interlude

For the scalar and vector integrals, we have at most 3-dimensional MB-integrals.

For the tensor integrals, we stay with up to 5-dimensional MB-representations. I have doubt that this is competative in a numerical application compared to Denner/Dittmaier/LoopTools.

Thus we intend to apply the MB-approach after a reduction to simpler sclar one-loop functions. We had observed that the basis of 4-point and 3-point functions needed consists of at most 2-dimensional MB-integrals, which are relatively easy to treat.

Reduction of 5-point tensor functions

We all know the Passarino-Veltman reduction scheme for self-energies, vertices, boxes as it is e.g. realized in the FF-package (van Oldenborgh) and in LoopTools (Hahn).

This may be extended to 5-point functions in a straightforward way. Result lenghty, but reliable.

Reduction of 5-point scalar functions

Melrose:1965kb

$$I_5 = \frac{1}{\binom{0}{0}_5} \sum_{r=1}^{5} \binom{0}{r}_5 I_4^r, \tag{23}$$

The I_4^r are 4-point function obtained by shrinking lines. Further, there are the coefficients, 'signed minors', process-specific determinants.

With

$$Y_{ij} = -(q_i - q_j)^2 + m_i^2 + m_j^2,$$

the "modified Cayley determinant" of the diagram with internal lines 1 $\ldots n$ **is**

$$()_n \;\equiv\; \left| \begin{array}{ccccc} 0 & 1 & 1 & \ldots & 1 \\ 1 & Y_{11} & Y_{12} & \ldots & Y_{1n} \\ 1 & Y_{12} & Y_{22} & \ldots & Y_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & Y_{1n} & Y_{2n} & \ldots & Y_{nn} \end{array} \right|,$$

labelling elements $0, \ldots, n$**. This object is called the Gram determinant.**

Cutting from $()_n$ **rows** $j_1, j_2, \ldots$ **and columns** $k_1, k_2, \ldots$ **, get the "Signed minors" (Melrose:1965kb). They are denoted by**

$$\left(\begin{array}{ccc} j_1 & j_2 & \ldots \\ k_1 & k_2 & \ldots \end{array} \right)_n ,$$

$$\Delta_n = \begin{vmatrix} Y_{11} & Y_{12} & \dots & Y_{1n} \\ Y_{12} & Y_{22} & \dots & Y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{1n} & Y_{2n} & \dots & Y_{nn} \end{vmatrix} = \binom{0}{0}_n .$$

We introduce the following notations:

$$\begin{aligned} \binom{s}{j}_5 &= D[s,j], \\ \binom{ts}{ij}_5 &= D[t,i,s,j]. \end{aligned} \tag{24}$$

Explicitely, we get the following coefficients for our Bhabha pentagon::

$$D[0,0] = -2stt'v_1v_2 + 2m_e^2(s^2(t-t')^2 + 2s(t+t')v_1v_2 + v_1^2v_2^2),$$

and

$$\begin{aligned} D[0,1] &= s^2tt' - s^2t'^2 + st'^2v_1 - stt'v_2 - st'v_1v_2 \\ &\quad + m_e^2(-4st'v_1 + 2stv_2 + 2st'v_2 + 2v_1v_2^2), \end{aligned} \tag{25}$$

$$D[0,2] = D[0,1] \text{ (\textbf{with} } t, v_1 \leftrightarrow t', v_2) \tag{26}$$

$$\begin{aligned} D[0,3] &= -(tv_2(s(-t+t') + t'v_1 + (t-v_1)v_2)) \\ &\quad + m_e^2(2st^2 - 4stt' + 2st'^2 + 2tv_1v_2 + 2t'v_1v_2 + 4tv_2^2), \end{aligned} \tag{27}$$

$$\begin{aligned} D[0,4] &= -(v_1v_2(s(t+t') - t'v_1 + (-t+v_1)v_2)) \\ &\quad + m_e^2(-2stv_1 + 2st'v_1 + 2stv_2 - 2st'v_2 - 2v_1^2v_2 - 2v_1v_2^2), \end{aligned} \tag{28}$$

$$D[0,5] = D[0,3] \text{ (\textbf{with} } t, v_1 \leftrightarrow t', v_2).. \tag{29}$$

For the massless QED case, the reduction may also be found in (Arbuzov:1998ax), where it was derived with a technique developed in (vanNeerven:1983vr).

The Mellin-Barnes integrals for the 4-point functions are derived from the following Feynman parameter integrals:

$$I_4^n = e^{\epsilon\gamma_E} \int \frac{d^d q}{i\pi^{d/2}} \frac{c_n}{c_1 c_2 c_3 c_4 c_5}. = \int_0^1 \prod_{j=1}^{4} dx_j \; \delta\left(1 - \sum_{i=1}^{4} x_i\right) \frac{\Gamma(2+\epsilon)}{F_4^n(x)^{2+\epsilon}}.$$

with

$$\begin{aligned}
F_4^1 &= m^2(x_1+x_2)^2 - t^{'} x_1 x_2 - V_2 x_1 x_4 - s x_3 x_4, \\
F_4^2 &= F_4^1(\text{ **with** } t, v_1 \leftrightarrow t', v_2), \\
F_4^3 &= m^2(x_1+x_2+x_3)^2 - t x_1 x_2 - t^{'} x_1 x_3 - V_2 x_3 x_4, \\
F_4^4 &= m^2(x_1+x_2)^2 - V_1 x_1 x_3 - V_2 x_2 x_4 - s x_3 x_4, \\
F_4^5 &= F_4^3(\text{ **with** } t, v_1 \leftrightarrow t', v_2). \qquad (30)
\end{aligned}$$

Vector 5-point functions

For the vector integrals one may use (Chetyrkin/Tkachov 1981, Davydychev 1991, Tarasov 1996, Fleischer/Jegerlehner/Tarasov 2000):

$$I_5^{\mu} = \sum_{i=1}^{4} \bar{q}_i^{\mu} \frac{-1}{\binom{0}{0}_5} \sum_{r=1}^{5} \binom{0i}{0r}_5 I_4^r. \tag{31}$$

The five four-point functions I_4^r are those introduced for the scalar pentagon, and the new coefficients $\binom{0i}{0r}_5$ are signed minors (24), which have the following explicit form here:

$$
\begin{aligned}
D[0,0,1,1] &= s^2t'^2 - 4m_e^2 s^2 t', \\
D[0,0,1,2] &= -(s^2tt') + m_e^2 s(2st + 2st' + 2v_1v_2), && (32) \\
D[0,0,1,3] &= stt'v_2 - m_e^2 v_2(2st + 2st' + 2v_1v_2), && (33) \\
D[0,0,1,4] &= st'v_1v_2 - m_e^2 s(2st - 2st' + 2v_1v_2), && (34) \\
D[0,0,1,5] &= -st'^2v_1 + 4m_e^2 st'v_1, && (35) \\
D[0,0,2,2] &= s^2t^2 - 4m_e^2 s^2 t, && (36) \\
D[0,0,2,3] &= -st^2v_2 + 4m_e^2 stv_2, && (37) \\
D[0,0,2,4] &= stv_1v_2 + m_e^2 s(2st - 2st' - 2v_1v_2), && (38) \\
D[0,0,2,5] &= stt'v_1 - m_e^2 v_1(2st + 2st' + 2v_1v_2), && (39) \\
D[0,0,3,3] &= +t^2v_2^2 - 4m_e^2 tv_2^2, && (40) \\
D[0,0,3,4] &= -(tv_1v_2^2) - m_e^2 v_2(2st - 2st' - 2v_1v_2), && (41) \\
D[0,0,3,5] &= tt'v_1v_2 + m_e^2(-2st^2 + 4stt' - 2st'^2 - 2tv_1v_2 - 2t'v_1v_2), && (42) \\
D[0,0,4,4] &= +v_1^2v_2^2 + 4m_e^2 sv_1v_2, && (43) \\
D[0,0,4,5] &= -(t'v_1^2v_2) + m_e^2 v_1(2st - 2st' + 2v_1v_2), && (44) \\
D[0,0,5,5] &= +t'^2v_1^2 - 4m_e^2 t'v_1^2. && (45)
\end{aligned}
$$

Tensor 5-point integral reduction

The tensor integral of degree 2 can be written without a $g_{\mu\nu}$-term ($I_{00}=0$):

$$I_5^{\mu\,\nu} = I_{00} g^{\mu\,\nu} + \sum_{i,j=1}^{4} q_i^{\mu}\, q_j^{\nu} I_{5,ij}$$

due to (assuming $q_1 \cdots q_4$ 4-dimensional and independent)

$$g^{\mu\,\nu} = 2 \sum_{i,j=1}^{4} \frac{\binom{i}{j}_5}{()_5} q_i^{\mu}\, q_j^{\nu}$$

Then one may derive (Fleischer,Jegerlehner,Tarasov 2000):

$$I_{5,ij} \quad = \quad \frac{1}{()_5} \left\{ -\frac{\binom{0}{j}_5}{\binom{0}{0}_5} \sum_{s=1}^{5} \binom{0i}{0s}_5 I_4^s - \sum_{s=1,s\neq i}^{5} \frac{\binom{s}{j}_5 \binom{0s}{is}_5}{\binom{s}{s}_5} I_4^s + \sum_{s,t=1,s\neq i,t}^{5} \frac{\binom{s}{j}_5 \binom{ts}{is}_5}{\binom{s}{s}_5} I_3^{st} \right\} \tag{46}$$

One may eliminate the $1/\,()_5$ by an appropriate ansatz (Denner,Dittmaier 2002). Here this means to rewrite the bracket and cancel therein the inverse Gram determinant explicitely.:

$$I_5^{\mu\nu} \quad = \quad [I_5^{\mu\nu} - E_{00} g^{\mu\nu}] + E_{00} g^{\mu\nu}.$$

The coefficient E_{00} of the $g_{\mu\nu}$ term is (Denner:2002ii):

$$E_{00} = -\frac{1}{2} \frac{1}{\binom{0}{0}_5} \sum_{s=1}^{5} \frac{\binom{0}{s}_5}{\binom{s}{s}_5} \left[\binom{0s}{0s}_5 I_4^s - \sum_{t=1}^{5} \binom{0s}{ts}_5 I_3^{st} \right]$$

The result contains 4-point functions I_4^s and 3-point functions I_3^{st} (preliminary):

$$\begin{aligned} I_5^{\mu\,\nu} &= \sum_{s=1}^{5} \frac{1}{\binom{0}{0}_5 \binom{s}{s}_5} \left[-\frac{1}{2} \binom{0}{s}_5 \binom{0s}{0s}_5 g^{\mu\nu} + \sum_{i,j=1}^{4} X_{ij}^{s0} q_i^\mu \, q_j^\nu \right] I_4^s \\ &\quad + \sum_{s=1}^{5} \frac{1}{\binom{0}{0}_5 \binom{s}{s}_5} \sum_{t=1}^{5} \left[\frac{1}{2} \binom{0}{s}_5 \binom{0s}{ts}_5 g^{\mu\nu} - \sum_{i,j=1}^{4} X_{ij}^{st} q_i^\mu \, q_j^\nu \right] I_3^{st}, \qquad (47) \\ X_{ij}^{st} &= -\binom{0s}{0i}_5 \binom{ts}{js}_5 + \binom{0i}{sj}_5 \binom{ts}{0s}_5 .. \qquad (48) \end{aligned}$$

The signed minors are defined in (24), (24) and (24). If there is no symmetry at all in the original Feynman integral, we have five four-point functions I_4^s with one off-shell leg and ten three-point functions $I_3 st$, five of them with one and five with two off-shell legs. For the Bhabha QED case, we have e.g. three different four-point functions and six three-point functions (three plus three).

For tensors of degree 3 and 4, inverse powers two and three of the Gram determinant have to be cancelled. This is difficult, but may be done. Not shown here.

Summary on new results for 5-point functions

- AMBRE.m (May 2007): Derive a Mellin-Barnes representation of L-loop N-point planar Feynman integrals
- MB.m (2005): The determination of the UV- and IR-related ϵ-poles is generally solved
- The remaining (often non-trivial) problem is the evaluation of the multi-dimensional, finite MB-Integrals.
  We present a general algorithm for the evaluation of mixed IR-divergencies from virtual and real emission in terms of inverse binomial sums.
- This method may be combined with a preceding reduction of tensor integrals in order to get lower-dimensional MB-representations.
  For 5-point functions we re-analyzed older reductions and see an opportunity to get improved algebraic expressions compared to those published so far.
- The example is the pentagon of Bhabha scattering, but it is quite evident that the results hold also for 5-point functions from multi-particle production in ILC- and LHC-problems.