

Two-loop Heavy Fermion Corrections to Bhabha Scattering

Tord Riemann, DESY, Zeuthen

based on work with:

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Würzburg), Janusz Gluza (Katow-
ice)



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- See also: • <http://www-zeuthen.desy.de/theory/research/bhabha/>
and • [hep-ph/0412164](#) (List of all masters)
and • [hep-ph/0604101](#) (planar box masters), [hep-ph/0609051](#) ($N_f=2$ masters)
- Introduction: Two-Loop corrections to Bhabha Scattering
 - The Heavy Fermion Contributions [[arXiv:0704.2400](#), [hep-ph](#)], $\rightarrow$ NPB
 - Results
 - Summary

Status by end of 2004

Established: 10^{-3} MC programs for LEP, ILC

Introduction to **NLLBHA** by Trentadue and to **BHLUMI** by Jadach in:
Proc. of Loops and Legs, Rheinsberg, Germany, 1996

Recent mini-review: Jadach, "Theoretical error of luminosity cross section at LEP",
hep-ph/0306083 [1]

- **BHLUMI** v.4.04: Jadach, Placzek, Richter-Was, Was: CPC 1997
- see also: Jadach, Melles, Ward, Yost: PLB 1996, thesis Melles 1996 [2]
- **NLLBHA**: Arbuzov, Fadin, Kuraev, Lipatov, Merenkov, Trentadue: NPB 1997, CERN 96-01
- **SAMBHA**: Arbuzov, Haidt, Matteuzzi, Paganoni, Trentadue: hep-ph/0402211

See e.g.: Table 1 of [1] and Figure 3.1 of [2] → Conclude:

The nonlogarithmic $O(\alpha^2)$ terms, originating from pure QED radiative 1-loop and from 2-loop diagrams are not completely covered.

They have to be calculated and integrated into the MC programs.

Beware:

$$m_e, m_\gamma, (d-4), E_\gamma$$

Status 2005

Know the constant term ($m_e = 0$)
from 2-loop Bhabha scattering

A. Penin, **Two-Loop Corrections to Bhabha Scattering**, hep-ph/0501120 v.3, → PRL

Transform the **massless 2-loop results** of Bern, Dixon, Ghinculov (2002) with InfraRed (IR) regulation by $D = 4 - 2\epsilon$ into the **on-mass-shell renormalization** with $m_e \rightarrow 0$ and IR regulation by $\lambda = m_\gamma \neq 0$

Use **IR-properties of amplitudes** (see Penin):

[A] **Exponentiation** of the IR logarithms (Sudakov 1956,...)

[B] **Factorization** of the collinear logarithms into external legs (Frenkel, Taylor 1976)

[C] **Non-renormalization** of the IR exponents (YFS 1961,)

Isolate the closed fermion loop contribution (does not fulfil [C]) and add it separately (Burgers 1985, Bonciani et al. 2005, Penin)

If all this is correct, the constant term in m_e is known for the MCs (but the radiative one-loops with 5-point functions).

▪

Status 2006

Determination of master integrals
from 2-loop Bhabha scattering

[A] All planar box masters for $m_e^2 \ll s, t, u$

[B] All masters for $N_f = 2$ and $m_e^2 \ll m_f^2 \ll s, t, u$

We had to develop for this

Technique of semi-automatized derivation of Mellin-Barnes integrals (→ AMBRE package, 2007)

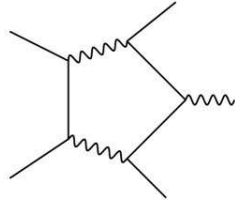
Automatized small-mass expansion for Mellin-Barnes integrals

AMBRE - Automatic Mellin-Barnes REpresentation (arXiv:0704.2423)

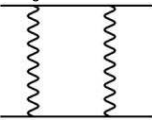
To download 'right click' and 'save target as'.

- The package [AMBRE.m](#)
- Kinematics generator for 4- 5- and 6- point functions with any external legs [KinematicsGen.m](#)
- Tarball with examples given below [examples.tar.gz](#)

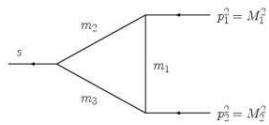
■ [example1.nb](#), [example2.nb](#) - Massive QED pentagon diagram.



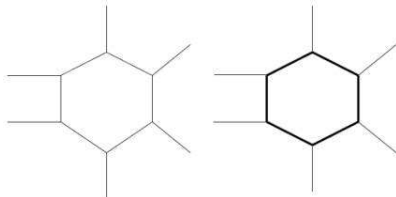
■ [example3.nb](#) - Massive QED one-loop box diagram.



■ [example4.nb](#) - General one-loop vertex.

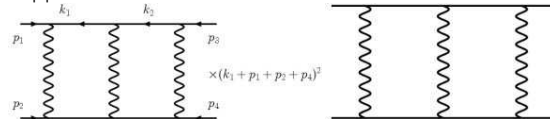


■ [example5.nb](#) - Six-point scalar functions;
left: massless case,
right: massive case.

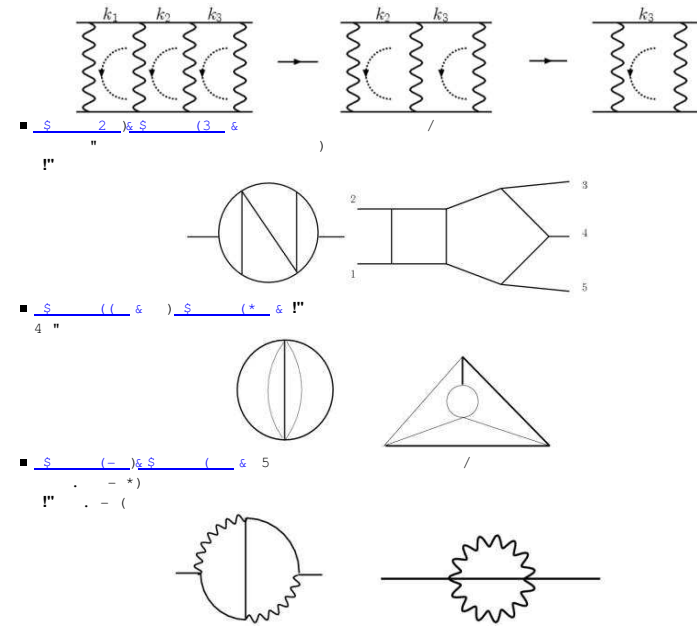


■ [example6.nb](#) - left, [example7.nb](#) - right

Massive two-loop planar QED box.



■ [example8.nb](#) - The loop-by-loop iterative procedure.

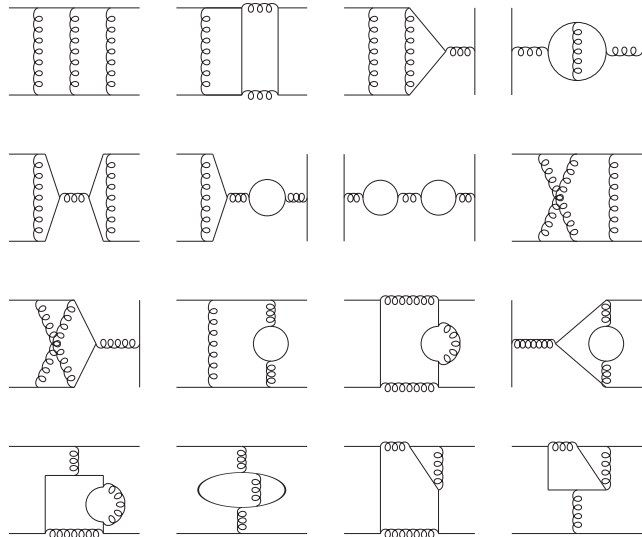


$$m = 0$$

Two Loop Bhabha Scattering

To calculate Bhabha scattering it is best to first compute $e^+e^- \rightarrow \mu^+\mu^-$, since it's closely related but has less diagrams.

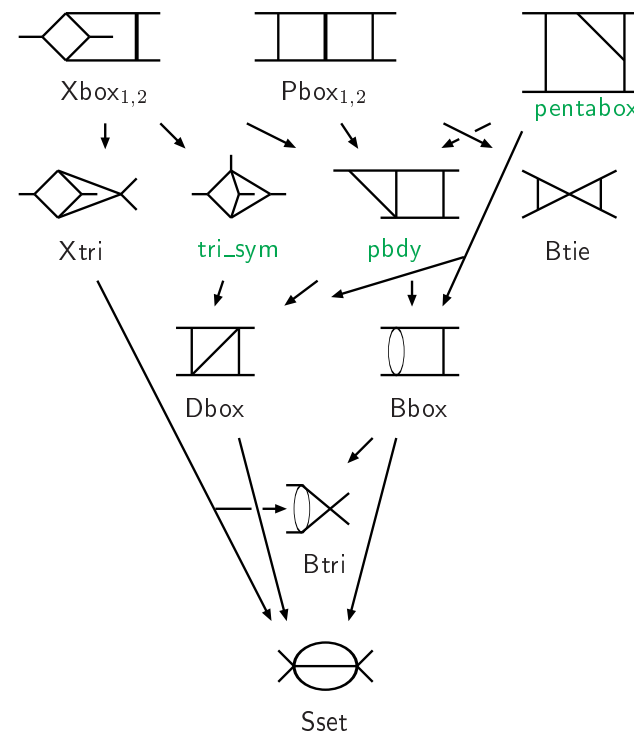
There are 47 QED diagrams contributing to $e^+e^- \rightarrow \mu^+\mu^-$.



In this calculation all particles massless.

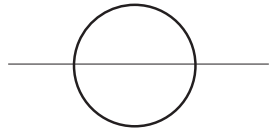
The Bhabha scattering amplitude can be obtained from $e^+e^- \rightarrow \mu^+\mu^-$ simply by summing it with the crossed amplitude (including fermi minus sign).

Two-loop integral inheritance chart

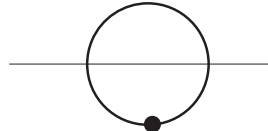


The master integrals for the $N_f = 2$ contributions

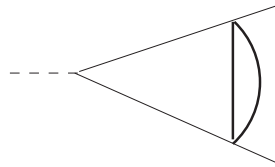
S. Actis, M. Czakon, J. Gluza, TR, 2006/2007



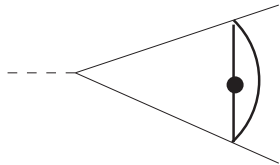
SE3l2M1m



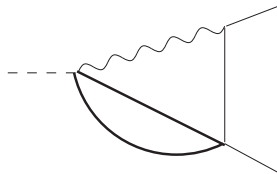
SE3l2M1md



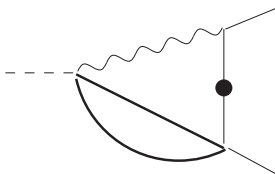
V4l2M2m



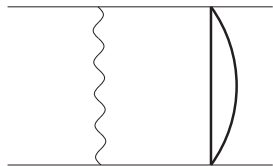
V4l2M2md



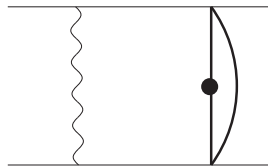
V4l2M1m



V4l2M1md



B5l2M2md



B5l2M2m

There are eight additional master integrals with two different mass scales.

The 2-box-diagrams represent a three-scale problem: $s/m_e^2, t/m_e^2, M^2/m_e^2$

Self-energies:

Actis,Czakon,Gluza,TR, NPB(PS) 160 (2006) 91, hep-ph/0609051

$$L(R) = \ln \left(\frac{m_e^2}{M^2} \right)$$

$$\begin{aligned} \text{SE3l2M1m[on shell]} &= M^2 m^{-4\epsilon} \left\{ R \left[\frac{1}{2\epsilon^2} + \frac{5}{4\epsilon} - \frac{3}{8} + \frac{\zeta_2}{2} + \frac{3}{2}L(R) - \frac{1}{2}L^2(R) \right] \right. \\ &+ R^2 \left[\frac{11}{18} - \frac{1}{3}L(R) \right] + \epsilon \left[R \left(\frac{45}{16} + \frac{5}{4}\zeta_2 - \frac{\zeta_3}{3} - \frac{7}{4}L(R) + L^2(R) \right. \right. \\ &\left. \left. - \frac{1}{2}L^3(R) \right) + R^2 \left(-\frac{3}{4} + \frac{8}{9}L(R) - \frac{1}{2}L^2(R) \right) \right] \left. \right\}, \end{aligned}$$

$$\begin{aligned} \text{SE3l2M1md[on shell]} &= m^{-4\epsilon} \left\{ \frac{1}{2\epsilon^2} + \frac{1}{2\epsilon} \left[1 + 2L(R) \right] + \frac{1}{2} (1 + \zeta_2) + L(R) + L^2(R) \right. \\ &+ \epsilon \left[\frac{1}{6} (3 + 3\zeta_2 - 2\zeta_3) + (1 + \zeta_2) L(R) + L^2(R) + \frac{2}{3}L^3(R) \right] \\ &+ R \left[-\frac{3}{4} + \frac{1}{2}L(R) + \epsilon \left(\frac{7}{8} - L(R) + \frac{3}{4}L^2(R) \right) \right] \\ &\left. + R^2 \left[-\frac{5}{36} + \frac{1}{6}L(R) + \epsilon \left(-\frac{5}{72} + \frac{1}{18}L(R) + \frac{1}{4}L^2(R) \right) \right] \right\}. \end{aligned}$$

Vertices

Actis,Czaron,Gluza,TR, NPB(PS) 160 (2006) 91, hep-ph/0609051

$L_m(x) = \ln(-m^2/x)$ and $L_M(x) = \ln(-M^2/x)$,

$$\begin{aligned} \text{V412M1m}[\mathbf{x}] &= m^{-4\epsilon} \left\{ \frac{1}{2\epsilon^2} + \frac{5}{2\epsilon} + \frac{1}{2} \left[19 - 3\zeta_2 - L_m^2(x) \right] \right. \\ &+ \frac{M^2}{x} \left[-2 + 4\zeta_2 - 4\zeta_3 - 2L_m(x) + 2L_M(x) - 4\zeta_2 L_M(x) \right. \\ &+ \left. \left. 2L_m(x)L_M(x) - L_M^2(x) - L_m(x)L_M^2(x) + \frac{1}{3}L_M^3(x) \right] \right\}, \end{aligned}$$

$$\begin{aligned} \text{V412M1md}[\mathbf{x}] &= \frac{m^{-4\epsilon}}{m^2} \left\{ \frac{1}{2\epsilon^2} + \frac{1}{\epsilon} \left[1 + \frac{1}{2}L_m(x) \right] + 2 - \zeta_2 + L_m(x) + \frac{1}{4}L_m^2(x) \right. \\ &+ \frac{M^2}{x} \left[\frac{1}{\epsilon} - \frac{1}{\epsilon}L_M(x) - 1 + 3\zeta_2 + L_m(x) + L_M(x) \right. \\ &- \left. \left. L_m(x)L_M(x) - \frac{1}{2}L_M^2(x) \right] \right\}, \end{aligned}$$

$$\text{V412M2m}[\mathbf{x}] = m^{-4\epsilon} \left\{ \frac{1}{2\epsilon^2} + \frac{1}{\epsilon} \left[\frac{5}{2} + L_m(x) \right] + \frac{1}{2}(19 + \zeta_2) + 5L_m(x) + L_m^2(x) \right\},$$

$$\text{V412M2md}[\mathbf{x}] = \frac{m^{-4\epsilon}}{6x} \left[12\zeta_3 - 6\zeta_2 L_M(x) - L_M^3(x) \right],$$

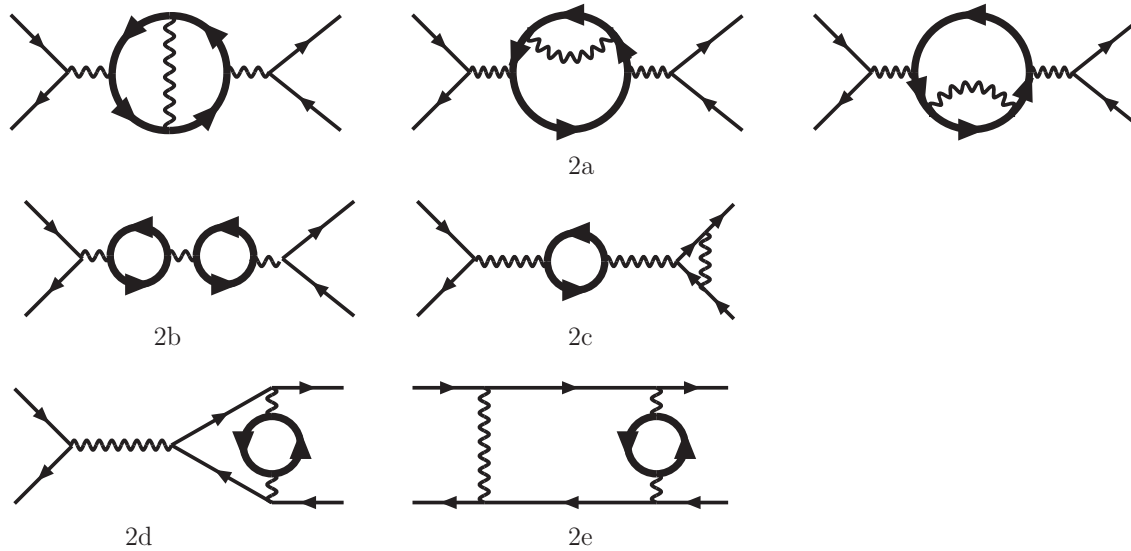
Box master integrals

Correct Mellin-Barnes representations in Actis et al., NPB(PS) 160 (2006) 91, hep-ph/0609051

But wrong mass expansion there! Correct results are:

$$\begin{aligned}
 \text{B512M2m}[x, y] &= \frac{m^{-4\epsilon}}{x} \left\{ \frac{1}{\epsilon^2} L_m(x) + \frac{1}{\epsilon} \left(-\zeta_2 + 2L_m(x) + \frac{1}{2} L_m^2(x) + L_m(x)L_m(y) \right) \right. \\
 &- 2\zeta_2 - 2\zeta_3 + 4L_m(x) + L_m^2(x) + \frac{1}{3} L_m^3(x) - 4\zeta_2 L_m(y) \\
 &+ 2L_m(x)L_m(y) + L_m(x)L_m^2(y) - \frac{1}{6} L_m^3(y) \\
 &- \left(3\zeta_2 + \frac{1}{2} L_m^2(x) - L_m(x)L_m(y) + \frac{1}{2} L_m^2(y) \right) \ln \left(1 + \frac{y}{x} \right) \\
 &\left. - \left(L_m(x) - L_m(y) \right) \text{Li}_2 \left(-\frac{y}{x} \right) + \text{Li}_3 \left(-\frac{y}{x} \right) \right\},
 \end{aligned}$$

$$\begin{aligned}
 \text{B512M2md}[x, y] &= \frac{m^{-4\epsilon}}{xy} \left\{ \frac{1}{\epsilon} \left[-L_m(x)L_m(y) + L_m(x)L(R) \right] - 2\zeta_3 + \zeta_2 L_m(x) + 4\zeta_2 L_m(y) \right. \\
 &- 2L_m(x)L_m^2(y) + \frac{1}{6} L_m^3(y) - 2\zeta_2 L(R) + 2L_m(x)L_m(y)L(R) - \frac{1}{6} L^3(R) \\
 &+ \left(3\zeta_2 + \frac{1}{2} L_m^2(x) - L_m(x)L_m(y) + \frac{1}{2} L_m^2(y) \right) \ln \left(1 + \frac{y}{x} \right) \\
 &\left. + \left(L_m(x) - L_m(y) \right) \text{Li}_2 \left(-\frac{y}{x} \right) - \text{Li}_3 \left(-\frac{y}{x} \right) \right\}.
 \end{aligned}$$



Classes of Bhabha-scattering **-loop diagrams** containing at least one fermion loop.

After combining the **2-loop** terms with the **loop-by-loop** terms and with **real** corrections:

$$\begin{aligned} \frac{d\sigma^{\text{NNLO}}}{d\Omega} + \frac{d\sigma_{\gamma}^{\text{NLO}}}{d\Omega} = & \frac{d\sigma^{\text{NNLO},e}}{d\Omega} + \sum_{f \neq e} Q_f^2 \frac{d\sigma^{\text{NNLO},f^2}}{d\Omega} + \sum_{f \neq e} Q_f^4 \frac{d\sigma^{\text{NNLO},f^4}}{d\Omega} \\ & + \sum_{f_1, f_2 \neq e} Q_{f_1}^2 Q_{f_2}^2 \frac{d\sigma^{\text{NNLO},2f}}{d\Omega}. \end{aligned}$$

$$\frac{d\sigma^{\text{NNLO},f^2}}{d\Omega} = \frac{\alpha^2}{s} \left\{ \sigma_1^{\text{NNLO},f^2} + \sigma_2^{\text{NNLO},f^2} \ln \left(\frac{2\omega}{\sqrt{s}} \right) \right\}$$

$$\begin{aligned} \sigma_1^{\text{NNLO},f^2} = & \frac{(1-x+x^2)^2}{3x^2} \left\{ -\frac{1}{3} \left[\ln^3 \left(\frac{s}{m_e^2} \right) + \ln^3 (R_f) \right] + \ln^2 \left(\frac{s}{m_e^2} \right) \left[\frac{55}{6} - \ln (R_f) \right. \right. \\ & + \ln (1-x) - \ln (x) \left. \right] + \ln \left(\frac{s}{m_e^2} \right) \left[-\frac{589}{18} + \frac{37}{3} \ln (R_f) - \ln^2 (R_f) \right. \\ & - 2 \ln (R_f) (\ln (x) - \ln (1-x)) - 8 \text{Li}_2 (x) \left. \right] + \frac{4795}{108} - \frac{409}{18} \ln (R_f) + \frac{19}{6} \ln^2 (R_f) \\ & - \ln^2 (R_f) (\ln (x) - \ln (1-x)) - 8 \ln (R_f) \text{Li}_2 (x) + \frac{40}{3} \text{Li}_2 (x) \left. \right\} \\ & + \ln \left(\frac{s}{m_e^2} \right) \left[\zeta_2 \left(-\frac{2}{3x^2} + \frac{4}{3x} + \frac{11}{2} - \frac{23}{3}x + \frac{16}{3}x^2 \right) + \ln^2 (x) \left(-\frac{1}{3x^2} + \frac{17}{12x} \right. \right. \\ & - \frac{5}{4} - \frac{x}{12} + \frac{2}{3}x^2 \left. \right) + \ln^2 (1-x) \left(-\frac{2}{3x^2} + \frac{11}{6x} - \frac{5}{2} + \frac{11}{6}x - \frac{2}{3}x^2 \right) \\ & + \ln (x) \ln (1-x) \left(\frac{2}{3x^2} - \frac{4}{3x} - \frac{1}{2} + \frac{5}{3}x - \frac{4}{3}x^2 \right) + \ln (x) \left(\frac{55}{9x^2} - \frac{83}{9x} + \frac{65}{6} \right. \\ & - \frac{85}{18}x + \frac{10}{9}x^2 \left. \right) + \frac{1}{3} \ln (1-x) \left(-\frac{10}{3x^2} + \frac{31}{6x} - 10 + \frac{31}{6}x - \frac{10}{3}x^2 \right) \left. \right] \\ & + \frac{1}{3} \ln^3 (x) \left(-\frac{1}{3x^2} + \frac{31}{12x} - \frac{11}{6} - \frac{x}{6} + \frac{x^2}{3} \right) + \frac{1}{3} \ln^3 (1-x) \left(-\frac{1}{3x^2} + \frac{1}{x} \right. \\ & - \frac{4}{3} + x - \frac{x^2}{3} \left. \right) + \ln^2 (x) \ln (1-x) \left(-\frac{1}{3x^2} + \frac{1}{3x} - \frac{4}{3} + x - \frac{x^2}{3} \right) \\ & + \frac{1}{3} \ln (x) \ln^2 (1-x) \left(-\frac{1}{x^2} + \frac{2}{x} - \frac{7}{4} + \frac{x}{2} \right) + \ln^2 (x) \left[\frac{55}{18x^2} - \frac{46}{9x} + \dots \right] \end{aligned}$$

$$\begin{aligned}
& \dots + \frac{14}{3} - \frac{4}{9}x - \frac{10}{9}x^2 + \ln(R_f) \left(-\frac{1}{3x^2} + \frac{17}{12x} - \frac{5}{4} - \frac{x}{12} + \frac{2}{3}x^2 \right) \\
& + \ln^2(1-x) \left[\frac{10}{9x^2} - \frac{29}{9x} + \frac{9}{2} - \frac{29}{9}x + \frac{10}{9}x^2 + \ln(R_f) \left(-\frac{2}{3x^2} + \frac{11}{6x} \right. \right. \\
& - \left. \left. \frac{5}{2} + \frac{11}{6}x - \frac{2}{3}x^2 \right) \right] + \ln(x) \ln(1-x) \left[-\frac{10}{9x^2} + \frac{37}{18x} + \frac{1}{2} - \frac{25}{9}x \right. \\
& + \left. \frac{20}{9}x^2 + \ln(R_f) \left(\frac{2}{3x^2} - \frac{4}{3x} - \frac{1}{2} + \frac{5}{3}x - \frac{4}{3}x^2 \right) \right] + \ln(x) \left[-\frac{589}{54x^2} + \frac{1753}{108x} \right. \\
& - \left. \frac{701}{36} + \frac{925}{108}x - \frac{56}{27}x^2 + \text{Li}_2(x) \left(-\frac{4}{x^2} + \frac{19}{3x} - 7 + 3x - \frac{2}{3}x^2 \right) \right. \\
& + \left. \ln(R_f) \left(\frac{37}{9x^2} - \frac{56}{9x} + \frac{47}{6} - \frac{67}{18}x + \frac{10}{9}x^2 \right) + \zeta_2 \left(-\frac{2}{3x^2} + \frac{4}{x} - \frac{1}{6} \right. \right. \\
& - \left. \left. \frac{10}{3}x + 2x^2 \right) \right] + \ln(1-x) \left[\frac{56}{27x^2} - \frac{161}{54x} + \frac{56}{9} - \frac{161}{54}x + \frac{56}{27}x^2 \right. \\
& + \left. \ln(R_f) \left(-\frac{10}{9x^2} + \frac{31}{18x} - \frac{10}{3} + \frac{31}{18}x - \frac{10}{9}x^2 \right) + \zeta_2 \left(-\frac{2}{x^2} + \frac{20}{3x} - \frac{32}{3} + \frac{20}{3}x \right. \right. \\
& - \left. \left. 2x^2 \right) \right] + \text{Li}_3(x) \left(\frac{4}{3x^2} - \frac{7}{3x} + 3 - \frac{5}{3}x + \frac{2}{3}x^2 \right) + \frac{2}{3}S_{1,2}(x) \left(-\frac{1}{x^2} + \frac{1}{x} \right. \\
& - \left. x + x^2 \right) + \zeta_2 \left[\frac{19}{9x^2} - \frac{13}{18x} - \frac{43}{3} + \frac{311}{18}x - \frac{98}{9}x^2 + \ln(R_f) \left(-\frac{2}{3x^2} + \frac{4}{3x} \right. \right. \\
& + \left. \left. \frac{11}{2} - \frac{23}{3}x + \frac{16}{3}x^2 \right) \right] + \zeta_3 \left(-\frac{4}{3x^2} + \frac{3}{x} - 5 + \frac{11}{3}x - 2x^2 \right)
\end{aligned}$$

The Box Corrections

The contribution of the renormalized two-loop box diagrams of class 2e is given by

$$\frac{d\sigma^{2e \times \text{tree}}}{d\Omega} = \frac{\alpha^2}{2s} \left[\frac{1}{s} A_1^{2e \times \text{tree}}(s, t) + \frac{1}{t} A_2^{2e \times \text{tree}}(s, t) \right]$$

Here the auxiliary functions can be conveniently expressed through three independent form factors $B_{i,f}^{(2)}(x, y)$, where $i = A, B, C$,

$$A_1^{2e \times \text{tree}}(s, t) = F_\epsilon^2 \sum_f Q_f^2 \text{Re} \left[B_{A,f}^{(2)}(s, t) + B_{B,f}^{(2)}(t, s) + B_{C,f}^{(2)}(u, t) - B_{B,f}^{(2)}(u, s) \right],$$

$$A_2^{2e \times \text{tree}}(s, t) = F_\epsilon^2 \sum_f Q_f^2 \text{Re} \left[B_{B,f}^{(2)}(s, t) + B_{A,f}^{(2)}(t, s) - B_{B,f}^{(2)}(u, t) + B_{C,f}^{(2)}(u, s) \right].$$

The normalization factor is

$$F_\epsilon = \left(\frac{m_e^2 \pi e^{\gamma_E}}{\mu^2} \right)^{-\epsilon}$$

Look e.g. at $B_{A,f}^{(2)}(t, s)$

The interference of the box diagram of class 2e with the s-channel tree-level amplitude,

$$B_{2e,f} = \frac{\alpha^2}{4s^2} \text{Re} \left[B_{A,f}^{(2)}(s, t) \right]$$

$$\begin{aligned}
B_{A,f}^{(2)}(x, y) = & \frac{1}{\epsilon} \frac{2}{3} \left(\frac{x^2}{y} + 2x + y \right) \left[\frac{5}{3} - L(R_f) + L_e(y) \right] L_e(x) \\
& + \frac{1}{3} \frac{x^2}{y} \left\{ 2 \left(\frac{131}{27} - 10\zeta_2 - 2\zeta_3 \right) - 2 \left(\frac{25}{9} - 6\zeta_2 \right) L(R_f) + \frac{7}{6} L^2(R_f) \right. \\
& - \frac{1}{3} L^3(R_f) + \left[\frac{82}{9} - 2\zeta_2 - \frac{4}{3} L(R_f) \right] L_e(x) - 2 \left[\frac{1}{3} + 8\zeta_2 - \frac{1}{2} L(R_f) \right] L_e(y) \\
& - \left[\frac{23}{6} - 2L(R_f) \right] L_e^2(y) + 4 \left[2 - L(R_f) \right] L_e(x) L_e(y) - 4 \left[\frac{5}{12} L_e^3(y) \right. \\
& - \left. L_e(x) L_e^2(y) \right] - \left[6\zeta_2 + \ln^2 \left(\frac{y}{x} \right) \right] \ln \left(1 + \frac{y}{x} \right) - 2 \ln \left(\frac{y}{x} \right) \text{Li}_2 \left(-\frac{y}{x} \right) \\
& + 2 \text{Li}_3 \left(-\frac{y}{x} \right) \left. \right\} + \frac{x}{3} \left\{ 2 \left(\frac{262}{27} - 9\zeta_2 - 4\zeta_3 \right) - 4 \left(\frac{25}{9} - 3\zeta_2 \right) L(R_f) \right. \\
& + \frac{7}{3} L^2(R_f) - \frac{2}{3} L^3(R_f) + 2 \left[\frac{121}{9} - \frac{10}{3} L(R_f) \right] L_e(x) - 2 \left[\frac{10}{3} + 12\zeta_2 \right. \\
& - \left. 2L(R_f) \right] L_e(y) + \left[\frac{13}{3} - 2L(R_f) \right] L_e^2(x) - \left[\frac{16}{3} - 2L(R_f) \right] L_e^2(y) \\
& + 2 \left[\frac{17}{3} - 2L(R_f) \right] L_e(x) L_e(y) + \frac{2}{3} L_e^3(x) \\
& + 6 L_e(x) L_e^2(y) - 2 L_e^3(y) - 2 \left[6\zeta_2 + \ln^2 \left(\frac{y}{x} \right) \right] \ln \left(1 + \frac{y}{x} \right) \\
& - 4 \ln \left(\frac{y}{x} \right) \text{Li}_2 \left(-\frac{y}{x} \right) + 4 \text{Li}_3 \left(-\frac{y}{x} \right) \left. \right\} + \frac{y}{3} \left\{ 2 \left(\frac{131}{27} - 7\zeta_2 - 2\zeta_3 \right) \right. \\
& - 2 \left(\frac{25}{9} - 3\zeta_2 \right) L(R_f) + \frac{7}{6} L^2(R_f) - \frac{1}{3} L^3(R_f) + \left[\frac{130}{9} - \frac{10}{3} L(R_f) \right] L_e(x) \\
& - \left[6 + 12\zeta_2 - 3L(R_f) \right] L_e(y) + \left[\frac{5}{3} - L(R_f) \right] L_e^2(x) - \left[\frac{25}{6} - L(R_f) \right] L_e^2(y) \\
& + 2 \left[\frac{10}{3} - L(R_f) \right] L_e(x) L_e(y) + \frac{1}{3} L_e^3(x) - L_e^3(y) + 3 L_e(x) L_e^2(y) \\
& - \left. \left[6\zeta_2 + \ln^2 \left(\frac{y}{x} \right) \right] \ln \left(1 + \frac{y}{x} \right) - 2 \ln \left(\frac{y}{x} \right) \text{Li}_2 \left(-\frac{y}{x} \right) + 2 \text{Li}_3 \left(-\frac{y}{x} \right) \right\}
\end{aligned}$$

(1)

(2)

$B_{2e,f}$ [nb] / $\sqrt{s}$ [GeV]	10	91	500
e	188758	5200.08	284.711
μ	1635.62	1686.88	130.579
τ			39.5554

Table 1: Finite part of $B_{2e,f}$ in nanobarns at a scattering angle $\theta = 3^\circ$.

$B_{2e,f}$ [nb] / $\sqrt{s}$ [GeV]	10	91	500
e	143.162	3.23102	0.160582
μ	61.3875	1.79381	0.0995184
τ	10.0105	0.935319	0.0639576
t			-0.00256757

Table 2: Finite part of $B_{2e,f}$ in nanobarns at a scattering angle $\theta = 90^\circ$.

$\sqrt{s}$ [GeV]	10	91	500
e	-124.237	-254.293	-400.574
μ	-4.8036	-29.1057	-70.1032
τ		-2.08719	-13.4901

Table 3: Real part for the vertex form factor.

$d\sigma / d\Omega$ [nb] $\sqrt{s}$ [GeV]	10	91	500
LO QED	440873	5323.91	176.349
LO Zfitter	440875	5331.5	176.283
NNLO (e)	-1397.35	-35.8374	-1.88151
NNLO ($e + \mu$)	-1394.74	-43.1888	-2.41643
NNLO ($e + \mu + \tau$)			-2.55179
NNLO photonic	9564.09	251.661	12.7943

$d\sigma / d\Omega$ [nb] $\sqrt{s}$ [GeV]	10	91	500
LO QED [Eq. (??)]	0.466409	0.00563228	0.000186564
LO Zfitter	0.468499	0.127292	0.0000854731
NNLO (e)	-0.00453987	-0.0000919387	$-4.28105 \cdot 10^{-6}$
NNLO ($e + \mu$)	-0.00570942	-0.000122796	$-5.90469 \cdot 10^{-6}$
NNLO ($e + \mu + \tau$)	-0.00586082	-0.000135449	$-6.7059 \cdot 10^{-6}$
NNLO ($e + \mu + \tau + t$)			$-6.6927 \cdot 10^{-6}$
NNLO photonic	0.0358755	0.000655126	0.0000284063

Table 4: Numerical values for the NNLO corrections to the differential cross section respect to the solid angle. Results are expressed in nanobarns for a scattering angle $\theta = 3^\circ$ and $\theta = 90^\circ$. Empty entries are related to cases where the high-energy approximation cannot be applied.

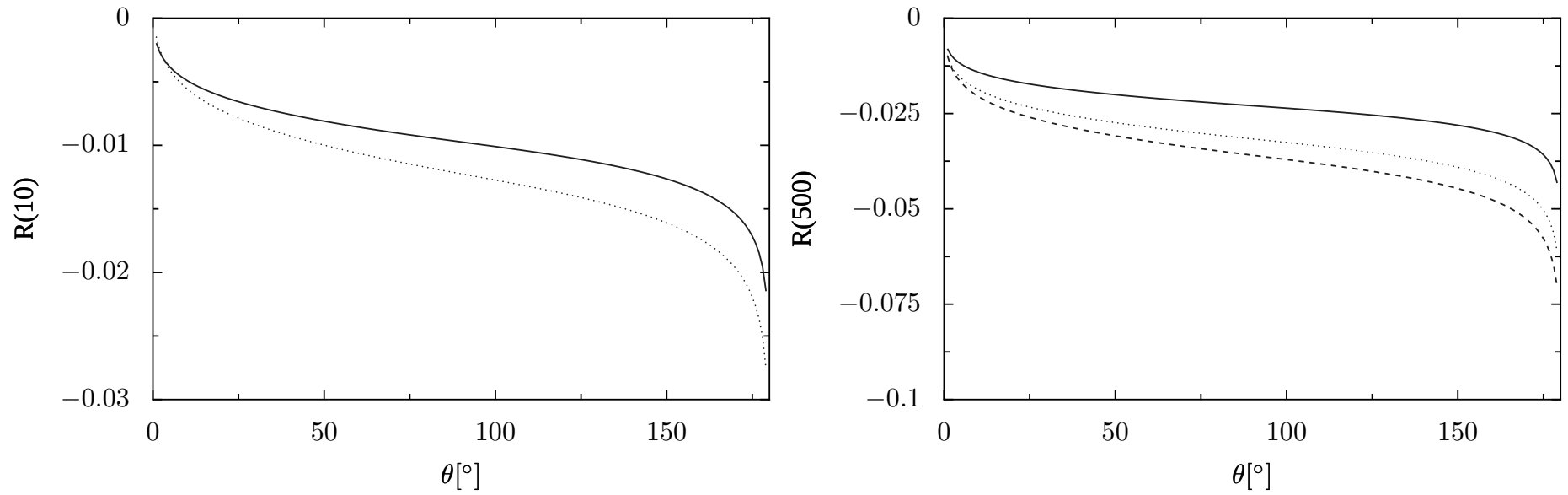


Figure 1: Ratio of the fermionic NNLO corrections to the differential cross section respect to the tree-level result for $\sqrt{s} = 10$ GeV and $\sqrt{s} = 500$ GeV. Solid line: electron-loop contributions, a dotted one the sum of electron- and muon-loop ones, and a dashed one includes also τ leptons.

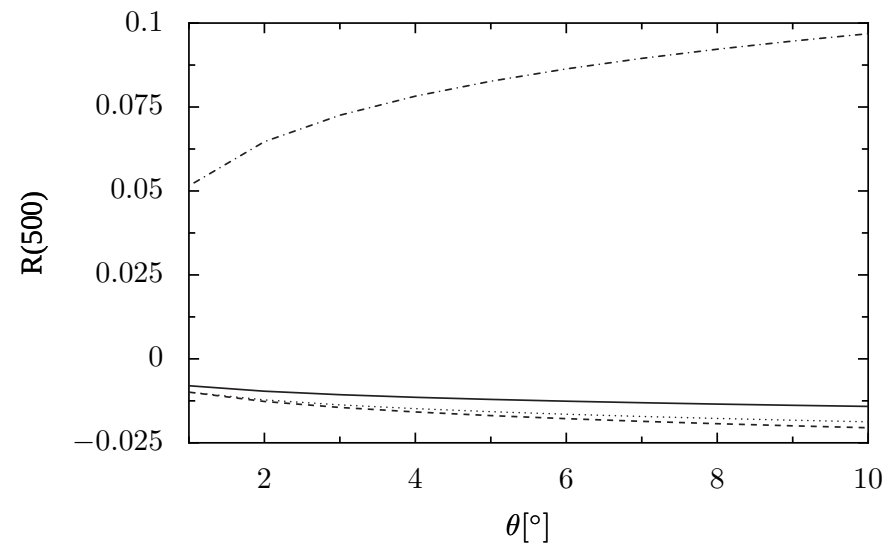
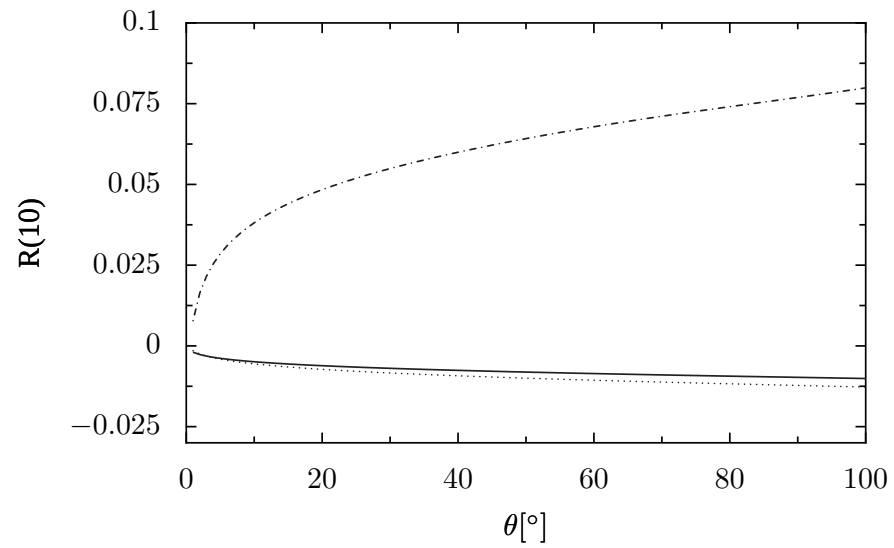


Figure 2: Here also with the photonic contributions of Arbuzov et al., Glover et al., Penin (dash-dotted lines).

Summary

- We derived the $N_f = 2$ contribution to 2-loop Bhabha scattering
- For forward scattering, it is small
- Agreement with:
 - ”Two-loop QED corrections to Bhabha scattering”
 - Thomas Becher (Fermilab) , Kirill Melnikov (Hawaii U.), arXiv:0704.3582 [hep-ph],
subm. to JHEP
 - They pointed out to us an error which we had to find then, unfortunately ...
- Status: Nearly complete knowledge of the 2-loop corrections to Bhabha scattering
To be determined yet:
 - quarkonic $N_f = 2$ contributions (non-perturbative)
 - radiative loops, interfering with real (Born) radiation, including 5-point functions.The latter was studied already by Arbuzov, Kuraev, Shaitchatdenov (1998, massless case, small photon mass)