

Some variations of the reduction of one-loop Feynman tensor integrals

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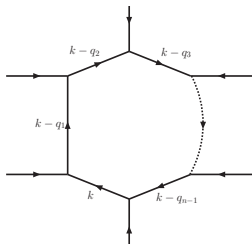
Introduction

n -point tensor integrals of rank R : (n,R) -integrals

$$I_n^{\mu_1 \dots \mu_R} = \int \frac{d^d k}{i\pi^{d/2}} \frac{\prod_{r=1}^R k^{\mu_r}}{\prod_{j=1}^n c_j^{\nu_j}}, \quad (1)$$

$d = 4 - 2\epsilon$ and denominators c_j have *indices* ν_j and *chords* q_j

$$c_j = (k - q_j)^2 - m_j^2 + i\epsilon \quad (2)$$



tensor integrals due to:

- fermion propagators
- three-gauge boson couplings
- e.g. unitary gauge propagators

Reduction of tensor integrals \Rightarrow express them by a (very) small set of scalar integrals

Presently needed for massive processes: $n \leq 6$ and rank $R \leq n$
 For box diagrams and simpler ones: Use of the 'conventional'
 Passarino-Veltman reduction [[Passarino:1978 \[1\]](#)]

Examples:

- LO (Lowest order) of e.g. $Z \rightarrow e + \mu$ is one-loop
[\[Riemann:1981, Mann:1983 \[2\]\[3\]\]](#)
- NLO: one-loop corrections to e.g. $H \rightarrow \tau^+ \tau^-$, WW , ZZ
[\[Fleischer:1980 \[4\]\]](#)
- NNLO: e.g. radiative loop corrections $e^+ e^- \rightarrow e^+ e^- \gamma$
 (here with 5-point functions)

Status of opensource packages - hopefully complete

- package **FF** [vanOldenborgh:1990 [5]] ,
- package **LoopTools/FF** [Hahn:1998,2006 [6]] – covers also 5-point functions, rank $R \leq 4$ $1/\epsilon^2$ not covered, and we observed sometimes problems in certain configurations with light-like external particles
- package **Golem95** [Binoth:2008 [7]] for $n \leq 6$, but only massless propagators
- Mathematica package **hexagon.m** [Diakonidis:2008 [8, 9]] for $n \leq 6$, $rank R \leq 4$
- package for all $n \leq 4$ scalar integrals: **QCDloop** [Ellis:2007 [10]]
- see also: review **A.Denner**, DESY TH workshop 2009

So, if you want to evaluate something less trivial, you have to create your own tensor reduction library.

This talk: derive efficient reduction formulae in the algebraic Fleischer-Davydychev-Tarasov approach

The original Passarino-Veltman reduction allows to express tensor integrals by a small set of scalar 4-,3-,2-,1-point functions integrals in d dimensions.

- Extensions: Need of reduction of n -point functions with $n > 4$
- Improvements: Avoid the break-down in certain kinematical configurations

A crucial step was to avoid for $n = 5$ – pentagons – the towers of inverse Gram determinants.

They may get small or vanishing and may appear by reducing algebraically the tensors to scalars.

Crucial contributions [of course, list is incomplete ...] \Rightarrow

- [Campbell:1996 [11]]
- [Denner:2002,2005 [12, 13]]
- [Binoth:1999,2005 [14, 15]]
- [Bern:1993 [16]]
- [Ossola:2006 [17]]

In the following, I will describe recent developments in the Fleischer-Davydychev-Tarasov approach.

- [Davydychev:1991, Tarasov:1996, Fleischer:1999, Diakonidis:2008,2009 [18, 19, 20, 9, 21]]

- get tensor reduction
- kill pentagon-Gram det's
- treat sub-Gram det's

Recursions for hexagons I

Express any hexagon by pentagons

[Fleischer:1999,Binoth:2005,Denner:2005,Diakonidis:2008 [20, 15, 13, 9]]

$$I_6^{\mu_1 \dots \mu_{R-1} \rho} = - \sum_{s=1}^6 I_5^{\mu_1 \dots \mu_{R-1}, s} \bar{Q}_s^\rho \quad (3)$$

auxiliary vectors

$$\bar{Q}_s^\rho = \sum_{i=1}^6 q_i^\rho \frac{\begin{pmatrix} 0s \\ 0i \end{pmatrix}_6}{\begin{pmatrix} 0 \\ 0 \end{pmatrix}_6}, \quad s = 1 \dots 6. \quad (4)$$

Scalar case [N=5: Melrose:1965 [22]]

$$I_N = - \sum_{s=1}^6 \frac{\begin{pmatrix} 0 \\ s \end{pmatrix}_N}{\begin{pmatrix} 0 \\ 0 \end{pmatrix}_N} I_{N-1}^s, \quad N = 5, 6 \quad (5)$$

Notations: $I_{n-1}^{\{\mu_1, \dots\}, s}$ etc.

$$I_{n-1, ab}^{\{\mu_1, \dots\}, s}$$

is obtained from

$$I_n^{\{\mu_1, \dots\}}$$

by

- shrinking line s
- raising the powers of inverse propagators a, b .

Notations: modified Cayley determinant [\[Melrose:1965\]](#)

Modified Cayley determinant $(\)_N$ of a diagram with N internal lines and chords q_j :

$$(\)_N \equiv \begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & Y_{11} & Y_{12} & \dots & Y_{1N} \\ 1 & Y_{12} & Y_{22} & \dots & Y_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & Y_{1N} & Y_{2N} & \dots & Y_{NN} \end{vmatrix}, \quad (6)$$

with matrix elements

$$Y_{ij} = -(q_i - q_j)^2 + m_i^2 + m_j^2, \quad (i, j = 1 \dots N) \quad (7)$$

Gram determinant G_n : $G_n = |2q_i q_j|, i, j = 1, \dots, n$

For a choice $q_n = 0$, both determinants are related: $(\)_N = -G_{N-1}$

⇒ The determinant $(\)_N$ does not depend on the masses.

Notations: signed minors [\[Melrose:1965\]](#)

We also need **signed minors of $()_N$** , constructed by deleting m rows and m columns from $()_N$, and multiplying with a sign factor:

$$\begin{aligned} & \begin{pmatrix} j_1 & j_2 & \cdots & j_m \\ k_1 & k_2 & \cdots & k_m \end{pmatrix}_N \equiv \\ \equiv & (-1)^{\sum_l (j_l + k_l)} \operatorname{sgn}_{\{j\}} \operatorname{sgn}_{\{k\}} \left| \begin{array}{c} \text{rows } j_1 \cdots j_m \text{ deleted} \\ \text{columns } k_1 \cdots k_m \text{ deleted} \end{array} \right| \end{aligned} \quad (8)$$

where $\operatorname{sgn}_{\{j\}}$ and $\operatorname{sgn}_{\{k\}}$ are the signs of permutations that sort the deleted rows $j_1 \cdots j_m$ and columns $k_1 \cdots k_m$ into ascending order.

Dimensional shifts and recurrence relations for pentagons I

'Naive', direct approach.

Following [\[Davydychev:1991 \[18\]\]](#)

replace tensors by scalar integrals in higher dimensions:

Example $R = 3$:

$$\begin{aligned}
 I_5^{\mu\nu\lambda} &= \int \frac{d^{4-2\epsilon}k}{\pi^{d/2}} \prod_{r=1}^5 c_r^{-1} k^\mu k^\nu k^\lambda & (9) \\
 &= - \sum_{i,j,k=1}^4 q_i^\mu q_j^\nu q_k^\lambda n_{ijk} I_{5,ijk}^{[d+]} + \frac{1}{2} \sum_{i=1}^{n-1} (g^{\mu\nu} q_i^\lambda + g^{\mu\lambda} q_i^\nu + g^{\nu\lambda} q_i^\mu) I_{n,i}^{[d+]}{}^2,
 \end{aligned}$$

and $n_{ijk} = (1 + \delta_{ij})(1 + \delta_{ik} + \delta_{jk})$.

Following [Tarasov:1996, Fleischer:1999 [19, 20]]

apply recurrence relations, some of them relating scalar integrals of different dimensions, in order to get rid of the dimensionalities $D = d - 2\epsilon + 2l$:

$$\nu_j \mathbf{j}^+ l_5^{[d+]} = \frac{1}{\binom{0}{5}} \left[-\binom{j}{0}_5 + \sum_{k=1}^5 \binom{j}{k}_5 \mathbf{k}^- \right] l_5 \quad (10)$$

$$(d - \sum_{i=1}^n \nu_i + 1) l_n^{[d+]} = \frac{1}{\binom{0}{n}} \left[\binom{0}{0}_n - \sum_{k=1}^n \binom{0}{k}_n \mathbf{k}^- \right] l_n, \quad n = 5 \text{ here} \quad (11)$$

$$\binom{0}{0}_5 \nu_j \mathbf{j}^+ l_5 = \sum_{k=1}^5 \binom{0j}{0k}_5 \left[d - \sum_{i=1}^5 \nu_i (\mathbf{k}^- \mathbf{i}^+ + 1) \right] l_5 \quad (12)$$

where the operators $\mathbf{i}^\pm, \mathbf{j}^\pm, \mathbf{k}^\pm$ act by shifting the indices ν_i, ν_j, ν_k by ± 1 .

At the end one may arrange a representation of e.g. $I_5^{\mu\nu\lambda}$ in terms of a collection of scalar 1-point to 4-point functions in generic dimension d : A_0, B_0, C_0, D_0 .

A problem for applications are the powers of the inverse Gram determinants

$$1/(\)_5$$

in recursions with dimensional shifts.

Alternative: Recursions for pentagons I

Express any $(5, R)$ pentagon by a $(5, R - 1)$ pentagon plus $(4, R - 1)$ boxes [Fleischer et al., Diakonidis:2010 [23]]

$$I_5^{\mu_1 \dots \mu_{R-1} \mu} = I_5^{\mu_1 \dots \mu_{R-1}} Q_0^\mu - \sum_{s=1}^5 I_4^{\mu_1 \dots \mu_{R-1}, s} Q_s^\mu \quad (13)$$

auxiliary vectors with inverse Gram determinants

$$Q_s^\mu = \sum_{i=1}^5 q_i^\mu \frac{\binom{s}{i}_5}{\binom{\mu}{i}_5}, \quad s = 0, \dots, 5 \quad (14)$$

For e.g. $R = 3$, again $[1/\binom{\mu}{i}_5]^3$ will occur.

Interlude: Further recursion steps

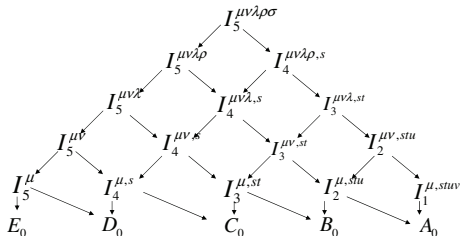
The recursions may be set forth, see [\[Diakonidis:2009 \[21\]\]](#); they get additional terms, e.g.:

$$I_4^{\mu\nu\lambda\rho} = I_4^{\mu\nu\lambda} Q_0^\rho - \sum_{t=1}^4 I_3^{\mu\nu\lambda,t} Q_t^\rho - G^{\mu\rho} T^{\nu\lambda} - G^{\nu\rho} T^{\mu\lambda} - G^{\lambda\rho} T^{\mu\nu}, \quad (15)$$

with the additional tensor and vector components:

$$T^{\mu\nu} = I_4^{\mu,[d+]} Q_0^\nu - \sum_{t=1}^4 I_3^{\mu,[d+],t} Q_t^\nu - G^{\mu\nu} I_4^{[d+]}{}^2 \quad (16)$$

$$G^{\mu\lambda} = \frac{1}{2} g^{\mu\lambda} - \sum_{i,j=1}^4 q_i^\mu q_j^\lambda \frac{\binom{i}{j}_4}{\binom{i}{j}_4}$$



Fleischer's triangle

Algebraic simplifications, 1st step

With the identity

$$\binom{0}{0}_5 \binom{s}{i}_5 = \binom{0s}{0i}_5 \binom{0}{i}_5 + \binom{0}{i}_5 \binom{s}{0}_5 \quad (17)$$

we eliminate the inverse Gram determinant from all terms with exclusion of Q_0^μ :

$$I_5^{\mu_1 \dots \mu_{R-1} \mu} = \left[I_5^{\mu_1 \dots \mu_{R-1}} - \sum_{s=1}^5 \frac{\binom{s}{0}_5}{\binom{0}{0}_5} I_4^{\mu_1 \dots \mu_{R-1}, s} \right] Q_0^\mu - \sum_{s=1}^5 I_4^{\mu_1 \dots \mu_{R-1}, s} \bar{Q}_s^\mu \quad (18)$$

The auxiliary vectors \bar{Q}_s^μ were introduced already for $n = 6$:

$$Q_0^\mu = \sum_{i=1}^5 q_i^\mu \frac{\binom{0}{i}_5}{\binom{0}{0}_5} \quad \text{while} \quad Q_0^\mu = \sum_{i=1}^5 q_i^\mu \frac{\binom{0}{i}_5}{\binom{0}{0}_5} \bar{Q}_s^\mu = \sum_{i=1}^5 q_i^\mu \frac{\binom{0s}{0i}_5}{\binom{0}{0}_5} \quad (19)$$

Algebraic simplifications, 2nd step I

Have to show for the product $T^{\mu_1 \dots \mu_{R-1}} \times Q_0^\mu$ that the Gram determinant cancels.

This came out to be a complicated task.

$$T^{\mu_1 \dots \mu_{R-1}} = \left[\begin{array}{c} \binom{0}{0} \\ \binom{0}{0} \end{array} \right]_5 I_5^{\mu_1 \dots \mu_{R-1}} - \sum_{s=1}^5 \binom{s}{0} \left[\begin{array}{c} s \\ 0 \end{array} \right]_5 I_4^{\mu_1 \dots \mu_{R-1}, s} \quad (20)$$

Example: For $R = 3$ pentagons need rank 2 tensor:

$$T^{\mu\nu} = \left[\begin{array}{c} \binom{0}{0} \\ \binom{0}{0} \end{array} \right]_5 I_5^{\mu\nu} - \sum_{s=1}^5 \binom{s}{0} \left[\begin{array}{c} s \\ 0 \end{array} \right]_5 I_4^{\mu\nu} \quad (21)$$

Algebraic simplifications, 2nd step I

Example $R = 3$: the building blocks are here $I_5^{\mu\nu}$ and $I_4^{\mu\nu}$:

$$\begin{aligned}
 I_5^{\mu\nu} &= \sum_{i,j=1}^5 q_i^\mu q_j^\nu \left[(1 + \delta_{ij}) I_{5,ij}^{[d+]} \right] + g^{\mu\nu} \left[-\frac{1}{2} I_5^{[d+]} \right] \\
 &\Rightarrow \text{work!!!} \sum_{i,j=1}^4 q_i^\mu q_j^\nu \frac{1}{\binom{0}{0}_5} \sum_{s=1}^5 \left[\binom{0i}{sj}_5 I_4^{[d+],s} + \binom{0s}{0j}_5 I_{4,i}^{[d+],s} \right] \\
 &\quad + g^{\mu\nu} \left[-\frac{1}{2} \frac{1}{\binom{0}{0}_5} \sum_{s=1}^5 \binom{s}{0}_5 I_4^{[d+],s} \right] \tag{22}
 \end{aligned}$$

See: The $I_5^{\mu\nu}$ had already been made free of $1/\binom{0}{0}_5$.

Davydychev's higher dimensional integrals

The second term with $I_4^{\mu\nu}$ is a typical example of [Davydychev:1991 [18]] : tensor \Rightarrow scalars in $d + 2l$

$$I_4^{\mu\nu} = \sum_{i,j=1}^4 q_i^\mu q_j^\nu \left[(1 + \delta_{ij}) I_{4,ij}^{[d+]^2} \right] + g^{\mu\nu} \left[-\frac{1}{2} I_4^{[d+]} \right] \quad (23)$$

Further, at this point, we have to reduce the scalar integrals $I_{4,ij}^{[d+]^2}$ etc. to generic dimension d with Tarasov's recurrence relations, see next slide.

The $I_4^{\mu\nu}$ is naturally free of $1/(\epsilon)_5$.

Algebraic simplifications, 2nd step

Work out the red part in

$$\binom{0}{0}_5 I_5^{\mu\nu\lambda} = \left[\binom{0}{0}_5 I_5^{\mu\nu} - \sum_{s=1}^5 \binom{s}{0}_5 I_4^{\mu\nu,s} \right] Q_0^\lambda - \sum_{s=1}^5 I_4^{\mu\nu,s} \overline{Q}_s^{0,\lambda}$$

⇒ Use of identities for the determinants
work!!!

$$\binom{0}{0}_5 \binom{s}{i}_5 = \binom{0s}{0i}_5 \binom{0}{0}_5 + \binom{0}{i}_5 \binom{s}{0}_5 \quad (24)$$

$$\binom{s}{i}_5 \frac{\binom{0}{j}_5}{\binom{0}{0}_5} = -\binom{0i}{sj}_5 + \binom{s}{0}_5 \frac{\binom{j}{0}_5}{\binom{0}{0}_5}, \quad g^{\mu\nu} = 2 \sum_{i,j=1}^4 \frac{\binom{j}{0}_5}{\binom{0}{0}_5} q_i^\mu q_j^\nu \quad (25)$$

$$\binom{s}{0}_5 \binom{0s}{is}_5 = \binom{s}{i}_5 \binom{0s}{0s}_5 - \binom{s}{s}_5 \binom{0s}{0i}_5 \quad (26)$$

$$\binom{s}{0}_5 \binom{ts}{js}_5 = \binom{s}{j}_5 \binom{ts}{0s}_5 - \binom{s}{s}_5 \binom{ts}{0j}_5 \quad (27)$$

Algebraic simplifications, 2nd step

⇒ Use of identities for the determinants
work!!!

$$\binom{s}{0}_5 \binom{is}{js}_5 = \binom{s}{i}_5 \binom{0s}{js}_5 + \binom{s}{s}_5 \binom{0i}{sj}_5 \quad (28)$$

$$\binom{s}{s}_5 \binom{0st}{0st}_5 = \binom{0s}{0s}_5 \binom{st}{st}_5 - \binom{ts}{0s}_5^2 \quad (29)$$

$$\left[\binom{ts}{0s}_5 \binom{ust}{jst}_5 - \binom{ts}{js}_5 \binom{ust}{0st}_5 \right] \binom{s}{s}_5 = \left[\binom{ts}{0s}_5 \binom{us}{js}_5 - \binom{ts}{js}_5 \binom{us}{0s}_5 \right] \binom{st}{st}_5 \quad (30)$$

$$\sum_{t=1}^5 \binom{ts}{is}_5 = 0 \quad (31)$$

Express pentagons I_5^μ , $I_5^{\mu\nu}$, $I_5^{\mu\nu\lambda}$ etc. by d -shifted scalar boxes I

Intermediate result with $I_4^{[d+],s}$, $I_{4,ij}^{[d+]^2,s}$ etc.

$$I_5^\mu = - \sum_{i=1}^4 \left[\frac{1}{\binom{0}{0}_5} \sum_{s=1}^5 \binom{0i}{0s}_5 I_4^s \right] q_i^\mu \quad (32)$$

$$I_5^{\mu\nu} = \sum_{i,j=1}^4 q_i^\mu q_j^\nu E_{ij} + g^{\mu\nu} E_{00} \quad (33)$$

$$E_{ij} = \frac{1}{\binom{0}{0}_5} \sum_{s=1}^5 \left[\binom{0i}{sj}_5 I_4^{[d+],s} + \binom{0s}{0j}_5 I_{4,i}^{[d+],s} \right] \quad (34)$$

$$E_{00} = -\frac{1}{2} \frac{1}{\binom{0}{0}_5} \sum_{s=1}^5 \binom{s}{0}_5 I_4^{[d+],s} \quad (35)$$

Express pentagons $I_5^\mu, I_5^{\mu\nu}, I_5^{\mu\nu\lambda}$ etc. by d -shifted scalar boxes II

$$I_5^{\mu\nu\lambda} = \sum_{i,j,k=1}^4 q_i^\mu q_j^\nu q_k^\lambda E_{ijk} + \sum_{k=1}^4 g^{[\mu\nu} q_k^{\lambda]} E_{00k} \quad (36)$$

$$E_{ijk} = -\frac{1}{\binom{0}{0}_5} \sum_{s=1}^5 \left\{ \left[\binom{0j}{sk}_5 I_{4,j}^{[d+]^2,s} + (i \leftrightarrow j) \right] + \binom{0s}{0k}_5 \nu_{ij} I_{4,jj}^{[d+]^2,s} \right\} \quad (37)$$

$$E_{00j} = \frac{1}{\binom{0}{0}_5} \sum_{s=1}^5 \left[\frac{1}{2} \binom{0s}{0j}_5 I_4^{[d+],s} - \frac{d-1}{3} \binom{s}{j}_5 I_4^{[d+]^2,s} \right] \quad (38)$$

These presentations are evidently free of inverse Gram determinants.

Isolation of inverse sub-Gram $\det^s ()_4$ I

We have now two kinds of objects in higher \dim^s to be evaluated:

$$I_4^s, I_4^{[d+],s}, I_4^{[d+]^2,s} \quad \text{boxes} \quad (39)$$

$$I_{4,i}^{[d+],s}, I_{4,i}^{[d+]^2,s}, I_{4,ij}^{[d+]^2,s} \quad \text{boxes with higher indices} \quad (40)$$

Application of dimension-shifting recurrence relations produces powers of $1/()_4$.

They are the **unwanted sub-Gram-determinants** $()_4$.

Simplest:

In I_5^μ (32) appears I_4^s .

Isolation of inverse sub-Gram $\det^s ()_4$ II

One of Tarasov's recurrence relations (11) applies ($n = 4$):

$$()_n(d-3)I_4^{[d+]} = \left[\binom{0}{0}_4 - \sum_{k=1}^4 \binom{0}{k}_4 \mathbf{k}^- \right] I_4 \quad (41)$$

So that we express:

$$I_4^D = \sum_{k=1}^4 \frac{\binom{k}{0}_4}{\binom{0}{0}_4} I_3^{D,k}, \quad \text{for } ()_n = 0 \quad (42)$$

If the dimension $D = d + 2l > d$, one has to reduce the dimension of $I_3^{D,k}$ by Tarasov's dim-recurrences, but now without $1/()_4$, because now $1/()_3$ appear.

Isolation of inverse sub-Gram $\det^s (\)_4$ III

For the other boxes, with higher indices, the game is more involved.

Try to have inverse Gram determinants only in front of boxes l_4 , and hold the l_3, l_2, l_1 free of them

Introduce:

$$\lim_{\binom{s}{s}_5 \rightarrow 0} l_4^{d,s} = \sum_{t=1}^5 \frac{\binom{ts}{0s}_5}{\binom{0s}{0s}_5} l_3^{d,st} \equiv Z_4^{d,s} \quad (43)$$

A typical example appearing in (34) is $l_{4,i}^{[d+],s}$.

With the aid of relations like

$$\binom{0s}{is}_5 \binom{ts}{0s}_5 - \binom{0s}{0s}_5 \binom{ts}{is}_5 = -\binom{s}{s}_5 \binom{0st}{0si}_5, \quad (44)$$

Isolation of inverse sub-Gram $\det^s (\)_4$ IV

one may cancel out the factor $\binom{s}{s}_5$ from the combination

$$\frac{\binom{0s}{is}_5}{\binom{s}{s}_5} Z_4^s - \sum_{t=1}^5 \frac{\binom{ts}{is}_5}{\binom{s}{s}_5} I_3^{st} = \dots = -\frac{1}{\binom{0s}{0s}_5} \sum_{t=1}^5 \binom{0st}{0si}_5 I_3^{st} \quad (45)$$

This allows to rewrite the recurrence relation

$$I_{4,i}^{[d+],s} = -\frac{\binom{0s}{is}_5}{\binom{s}{s}_5} I_4^s + \sum_{t=1}^4 \frac{\binom{ts}{is}_5}{\binom{s}{s}_5} I_3^{st}$$

into one with $1/\binom{s}{s}_5 = 1/(\)_4$ ONLY in front of boxes:

$$I_{4,i}^{[d+],s} = -\binom{0s}{is}_5 \frac{[I_4^s - Z_4^s]}{\binom{s}{s}_5} + \frac{1}{\binom{0s}{0s}_5} \sum_{t=1}^5 \binom{0st}{0si}_5 I_3^{st} \quad (46)$$

Isolation of inverse sub-Gram $\det^s \binom{s}{s}_4 V$

This is a nice expression, because it contains an explicit difference quotient for the case that $\binom{s}{s}_5$ becomes zero or if it becomes small but finite.

The numerical strategy will be:

- The $\binom{s}{s}_4 \neq 0$: Just evaluate what you like.
- The $\binom{s}{s}_4 = 0$: Replace

$$\frac{[I_4^s - Z_4^s]}{\binom{s}{s}_5} \Rightarrow \frac{d-3}{\binom{0}{0}_4} I_4^{[d+],s}$$

and the $I_4^{[d+],s}$ may be written, in that case $\binom{s}{s}_4 = 0$, by

$$\sum_t c_k I_3^{[d+],st}.$$

Further reducing to scalars in d dimensions, no $1/\binom{s}{s}_4$ will re-appear.

Isolation of inverse sub-Gram $\det^s ()_4$ VI

This is, what one may do if not explicitly work on expansions of the scalar integrals for small $()_n$.

Example: 4-point tensor of rank 3 I

Following Davydychev, [18], one gets

$$I_4^{\mu\nu\lambda} = \int^d \frac{k^\mu k^\nu k^\lambda}{\prod_{r=1}^n c_r} = - \sum_{i,j,k=1}^n q_i^\mu q_j^\nu q_k^\lambda \nu_{ijk} I_{n,ijk}^{[d+]}{}^3 + \frac{1}{2} \sum_{i=1}^n g^{[\mu\nu} q_i^{\lambda]} I_{n,i}^{[d+]}{}^2 \quad (47)$$

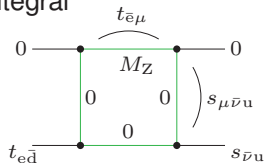
We identify the tensor coefficient D_{111} :

$$D_{111} \sim \nu_{ijk} I_{4,ijk}^{[d+]}{}^3 \quad \text{for } ijk = 222$$

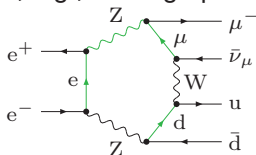
An example from A. Denner: D_{111}

Next figures copied from: [A.Denner](#), plenary talk DESY Theory Workshop 2009, p.69 (backup transparency)

box integral



appears, e.g., in subgraph of diagram



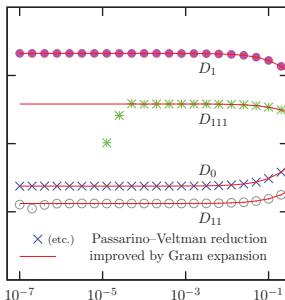
Gram det.: $\Delta^{(N)} \rightarrow 0$ if $t_{e\bar{d}} \rightarrow t_{\text{crit}} \equiv \frac{s_{\mu\bar{\nu}u}(s_{\mu\bar{\nu}u} - s_{\bar{\nu}u} + t_{e\bar{\mu}})}{s_{\mu\bar{\nu}u} - s_{\bar{\nu}u}}$

The figure demonstrates the effects of careful treatment of vanishing Gram determinant (Δ)₄.

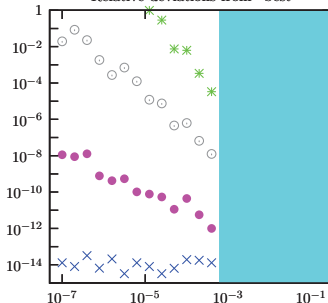
Gram det.: $\Delta^{(N)} \rightarrow 0$ if $t_{e\bar{d}} \rightarrow t_{\text{crit}} \equiv \frac{s_{\mu\bar{\nu}u}(s_{\mu\bar{\nu}u} - s_{\bar{\nu}u} + t_{\bar{e}\mu})}{s_{\mu\bar{\nu}u} - s_{\bar{\nu}u}}$

numerical comparison: maximal tensor rank = 6 (similar to $ee \rightarrow 4f$ application)

Absolute predictions



Relative deviations from "best"



$$x \equiv \frac{t_{e\bar{d}}}{t_{\text{crit}}} - 1$$

$$s_{\mu\bar{\nu}u} = +2 \times 10^4 \text{ GeV}^2$$

$$s_{\bar{\nu}u} = +1 \times 10^4 \text{ GeV}^2$$

$$t_{\bar{e}\mu} = -4 \times 10^4 \text{ GeV}^2$$

$$t_{\text{crit}} = -6 \times 10^4 \text{ GeV}^2$$

PV reduction breaks down, but Gram exp. stable for $\Delta^{(N)} \rightarrow 0$!

D_{111} – numerics with our reductions

Now rewrite:

$$\nu_{ijk} l_{4,ijk}^{[d+]}{}^3 = -\frac{\binom{0}{k}}{\binom{0}{4}} l_{4,ij}^{[d+]}{}^2 + \sum_{t=1, t \neq i, j}^4 \frac{\binom{t}{k}}{\binom{t}{4}} l_{3,ij}^{[d+]}{}^{2,t} + \frac{\binom{i}{k}}{\binom{i}{4}} l_{4,j}^{[d+]}{}^2 + \frac{\binom{j}{k}}{\binom{j}{4}} l_{4,i}^{[d+]}{}^2 \quad (48)$$

$$\begin{aligned} \nu_{ij} \nu_{ijk} l_{4,ijk}^{[d+]}{}^3, s, \text{reg} &= -\frac{\binom{0s}{ks}}{\binom{s}{5}} \nu_{ij} \left[l_{4,ij}^{[d+]}{}^{2,s} - Z_{4,ij}^{[d+]}{}^{2,s} \right] \\ &+ \frac{1}{\binom{0s}{0s}} \left[\frac{\binom{0si}{0sk}}{\binom{0s}{5}} l_{4,j}^{[d+]}{}^{2,s} + \frac{\binom{0sj}{0sk}}{\binom{0s}{5}} l_{4,i}^{[d+]}{}^{2,s} + \sum_{t=1, t \neq i, j}^5 \frac{\binom{0st}{0sk}}{\binom{0s}{5}} \nu_{ij} l_{3,ij}^{[d+]}{}^{2,st} \right] \quad (49) \end{aligned}$$

$$\begin{aligned} \nu_{ij} \nu_{ijk} l_{4,ijk}^{[d+]}{}^3, s &= -\frac{\binom{0}{k}}{\binom{0}{0}} \left[d \frac{\binom{0}{i}}{\binom{0}{0}} \frac{\binom{0}{j}}{\binom{0}{0}} (d-3)(d+1) l_4^{[d+]}{}^3 + (d-3) \frac{1}{\binom{0}{0}} \binom{0i}{0j} l_4^{[d+]}{}^2 + 2\nu_{ij} l_{4,ij}^{[d+]}{}^3 \right. \\ &- \frac{5d-16}{\binom{0}{0}} \frac{\binom{0}{j}}{\binom{0}{0}} \sum_{t=1}^4 \binom{0t}{0i} l_3^{[d+]}{}^{2,t} + \frac{d-3}{\binom{0}{0}} \sum_{t=1}^4 \binom{0t}{0j} l_{3,i}^{[d+]}{}^{2,t} \left. \right] \\ &+ \frac{1}{\binom{0}{0}} \left[\binom{0i}{0k} l_{4,j}^{[d+]}{}^2 + \binom{0j}{0k} l_{4,i}^{[d+]}{}^2 + \sum_{t=1, t \neq i, j}^4 \binom{0t}{0k} \nu_{ij} l_{3,ij}^{[d+]}{}^{2,t} \right] \quad (50) \end{aligned}$$

Numerics: LoopTools versus our approach

Using LoopTools call and our math numerics (preliminary):

```
x      D111

-7 : -0.007106204244698895      +0.0046539807850273325 I  D0i[dd111]
      -3.15345811639208  -10      -3.318373348243635  -10 I  Z4d30,Z4d20,I4id20

-6 : -3.2313079078584034-06      -2.8963160014947846-06 I  D0i[dd111]
      -3.1479286753545824-10      -3.318332145498356  -10 I  Z4d30,Z4d20,I4id20

-5 : -5.5231182028025025-09      +3.4832284324178667-09 I  D0i[dd111]
      -3.0926394107374516-10      -3.3179201270079527-10 I  Z4d30,Z4d20,I4id20

x< -4:      LoopTools dies out

-4 : -3.1544928789869657-10      -3.33218368329059  -10 I  D0i[dd111]
      -3.0798250216856066-10      -3.3447698103297804-10 I  flei

x < -3:      loss of accuracy

-3 : -3.153742175665908  -10      -3.31639655233478  -10 I  D0i[dd111]
      -3.1537481925176414-10      -3.3164147721227693-10 I  flei

-2 : -3.1500799889469005-10      -3.29915924109457  -10 I  D0i[dd111]
      -3.150080001830792  -10      -3.2991592067243136-10 I  flei

-1 : -3.112267506942415  -10      -3.135823319774082  -10 I  D0i[dd111]
      -3.1122675069507063-10      -3.1358233197649007-10 I  flei
```

Summary

- Recursive treatment of pentagon tensors integrals of rank R in terms of pentagons and boxes of rank $R - 1$
- Systematic derivation of expressions which are explicitly free of inverse Gram determinants $(\)_5$ until pentagons of rank $R = 5$
- Properly isolation of inverse Gram determinants of subdiagrams of the type $\binom{s}{s}_n$, which cannot be completely avoided.
- Some [preliminary] numerics, so far Mathematica, and C++ under way

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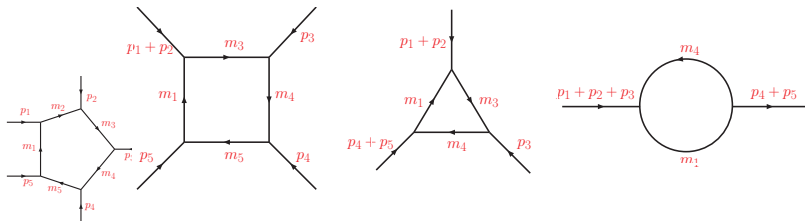


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Numbers (I) – Pentagons

Randomly chosen phase space point with massive and massless internal particles

p_1	5.0000000000 E+00	0.0000000000 E+00	0.0000000000 E+00	4.0000000000 E+00
p_2	5.0000000000 E+00	0.0000000000 E+00	0.0000000000 E+00	-4.0000000000 E+00
p_3	-0.30770034895 E+01	0.5359484673 E+00	-0.37447035150 E+00	-0.20120057390 E+00
p_4	-0.34048537280 E+01	0.2184763540 E-01	-0.10479394969 E+01	0.12224460727 E+01
p_5	-0.35181427825 E+01	-0.5577961027 E+00	0.14224098484 E+01	-0.10212454988 E+01
$m_1 = 0.0, \quad m_2 = 2.0, \quad m_3 = 3.0, \quad m_4 = 4.0, \quad m_5 = 5.0$				



Selected pentagon components

Shown are the constant terms of the tensor components

	<i>Pentagon.F</i>
E^2	(2.80450709388539E-05, -1.08461817406464E-05)
E^{12}	(-5.41333978667301E-06, 6.26985967678899E-06)
E^{232}	(-1.20374858970726E-04, 4.07974751672555E-04)
E^{0321}	(-9.11194535703727E-06, 4.39187998675819E-05)
E^{01230}	(4.37928367160152E-05, -2.18183151665913E-04)

<i>Box.F</i>	<i>LoopTools</i>
(6.81403420828588E-03, -5.74298462683219E-03)	(6.8140342082847463E-03, -5.7429846268324187E-03)
(2.40138809967981E-03, 1.11591328775015E-02)	(2.4013880996803092E-03, 1.1159132877500448E-02)
(-1.69702786278243E-03, -2.83731121595478E-03)	(-1.6970278627700630E-03, -2.8373112159962330E-03)
(-1.92190388316994E-04, -4.04730302413490E-04)	(-1.9219038693301300E-04, -4.0473030187772325E-04)

	<i>Triangle.F</i>	<i>LoopTools</i>
C^2	(2.44757827793318E-04, -7.50688449850356E-03)	(2.4475782779342707E-04, -7.5068844985030472E-03)
C^{01}	(-1.28259813172255E-02, -6.73809718907549E-02)	(-1.2825981317215014E-02, -6.7380971890795340E-02)
C^{133}	(-7.00360822297110E-02, 7.24628606014397E-02)	(-7.0036082229746830E-02, 7.2462860601566081E-02)

	<i>Bubble.F</i>	<i>LoopTools</i>
B^3	(-0.141525070262337E+00, 0.138870631815383E+00)	(-0.1415250702623366, 0.1388706318153829)
B^{12}	(0.102490343329085E+00, -6.12154531068256E-02)	(0.1024903433290848, -6.1215453106825706E-02)

here some text 1.

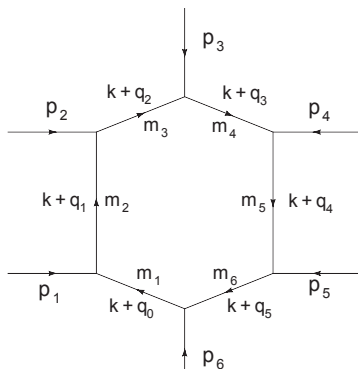


Figure: Momenta flow for the massive six-point topology.

here some text 2.

Numbers (II) – Hexagons

ρ_1	0.21774554 E+03	0.0	0.0	0.21774554 E+03
ρ_2	0.21774554 E+03	0.0	0.0	-0.21774554 E+03
ρ_3	-0.20369415 E+03	-0.47579512 E+02	0.42126823 E+02	0.84097181 E+02
ρ_4	-0.20907237 E+03	0.55215961 E+02	-0.46692034 E+02	-0.90010087 E+02
ρ_5	-0.68463308 E+01	0.53063195 E+01	0.29698267 E+01	-0.31456871 E+01
ρ_6	-0.15878244 E+02	-0.12942769 E+02	0.15953850 E+01	0.90585932 E+01
$m_1 = 110.0, m_2 = 120.0, m_3 = 130.0, m_4 = 140.0, m_5 = 150.0, m_6 = 160.0$				

		F_0
		-0.223393 E-18 - i 0.396728 E-19
μ	F^μ	
0	0.192487 E-17 + i 0.972635 E-17	
1	-0.363320 E-17 - i 0.11940 E-17	
2	0.365514 E-17 + i 0.106928 E-17	
3	0.239793 E-16 + i 0.341928 E-17	
μ	ν	$F^{\mu\nu}$
0	0	0.599459 E-14 - i 0.114601 E-14
0	1	0.323869 E-15 + i 0.423754 E-15
0	2	-0.294252 E-15 - i 0.375481 E-15
0	3	-0.255450 E-14 - i 0.195640 E-14
1	1	-0.164562 E-14 - i 0.993796 E-16
1	2	0.920944 E-16 + i 0.706487 E-17
1	3	0.347694 E-15 - i 0.127190 E-16
2	2	-0.163339 E-14 - i 0.994148 E-16
2	3	-0.341773 E-15 + i 0.818678 E-17
3	3	-0.413909 E-14 + i 0.670676 E-15

μ	ν	λ	$F^{\mu\nu\lambda}$
0	0	0	-0.227754 E-11 - i 0.267244 E-12
0	0	1	0.140271 E-13 - i 0.119448 E-12
0	0	2	-0.201270 E-13 + i 0.101968 E-12
0	0	3	0.102976 E-12 + i 0.624467 E-12
0	1	1	0.183904 E-12 + i 0.142429 E-12
0	1	2	-0.131028 E-13 - i 0.610343 E-14
0	1	3	-0.543316 E-13 - i 0.158809 E-13
0	2	2	0.181352 E-12 + i 0.141686 E-12
0	2	3	0.506408 E-13 + i 0.163568 E-13
0	3	3	0.600542 E-12 + i 0.130733 E-12
1	1	1	-0.563539 E-13 + i 0.178403 E-13
1	1	2	0.210641 E-13 - i 0.584990 E-14
1	1	3	0.120482 E-12 - i 0.574688 E-13
1	2	2	-0.201182 E-13 + i 0.620591 E-14
1	2	3	-0.686164 E-14 + i 0.205457 E-14
1	3	3	-0.447329 E-13 + i 0.193180 E-13
2	2	2	0.582201 E-13 - i 0.163889 E-13
2	2	3	0.119659 E-12 - i 0.570084 E-13
2	3	3	0.457464 E-13 - i 0.181141 E-13
3	3	3	0.557081 E-12 - i 0.374359 E-12

Table: Tensor components for a massive rank $R = 3$ six-point function

μ	ν	λ	ρ	$F^{\mu\nu\lambda\rho}$
0	0	0	0	0.666615 E-09 + i 0.247562 E-09
0	0	0	1	-0.200049 E-10 + i 0.294036 E-10
0	0	0	2	0.200975 E-10 - i 0.237333 E-10
0	0	0	3	0.645477 E-10 - i 0.162236 E-09
0	0	1	1	-0.116956 E-10 - i 0.516760 E-10
0	0	1	2	0.160357 E-11 + i 0.222284 E-11
0	0	1	3	0.792692 E-11 + i 0.729502 E-11
0	0	2	2	-0.111838 E-10 - i 0.513133 E-10
0	0	2	3	-0.681086 E-11 - i 0.708933 E-11
0	0	3	3	-0.804454 E-10 - i 0.801909 E-10
0	1	1	1	0.100498 E-10 - i 0.151735 E-13
0	1	1	2	-0.348984 E-11 - i 0.195436 E-12
0	1	1	3	-0.211111 E-10 + i 0.295212 E-11
0	1	2	2	0.357455 E-11 + i 0.662809 E-14
0	1	2	3	0.121595 E-11 - i 0.807388 E-13
0	1	3	3	0.825803 E-11 - i 0.142086 E-11
0	2	2	2	-0.958961 E-11 - i 0.585948 E-12
0	2	2	3	-0.209232 E-10 + i 0.289031 E-11
0	2	3	3	-0.802359 E-11 + i 0.994701 E-12
0	3	3	3	-0.102576 E-09 + i 0.378476 E-10
1	1	1	1	-0.246426 E-10 + i 0.276326 E-10
1	1	1	2	0.915670 E-12 - i 0.660629 E-12
1	1	1	3	0.303529 E-11 - i 0.287480 E-11
1	1	2	2	-0.822697 E-11 + i 0.919635 E-11
1	1	2	3	-0.116294 E-11 + i 0.100024 E-11
1	1	3	3	-0.146918 E-10 + i 0.183799 E-10
1	2	2	2	0.908296 E-12 - i 0.654735 E-12
1	2	2	3	0.109510 E-11 - i 0.100875 E-11
1	2	3	3	0.717342 E-12 - i 0.557293 E-12
1	3	3	3	0.450661 E-11 - i 0.485065 E-11
2	2	2	2	-0.245154 E-10 + i 0.274313 E-10
2	2	2	3	-0.318500 E-11 + i 0.279750 E-11
2	2	3	3	-0.146317 E-10 + i 0.182912 E-10
2	3	3	3	-0.477335 E-11 + i 0.477368 E-11
2	3	3	3	0.720168 E-10 - i 0.112865 E-09

p_1	0.5	0.0	0.0	0.5
p_2	0.5	0.0	0.0	-0.5
p_3	-0.19178191	-0.12741180	-0.08262477	-0.11713105
p_4	-0.33662712	0.06648281	0.31893785	0.08471424
p_5	-0.21604814	0.20363139	-0.04415762	-0.05710657
$p_6 = -(p_1 + p_2 + p_3 + p_4 + p_5)$				

Table: Phase space point of massless six-point functions taken from [Binoth:2008 [7]] . Golem95: Binoth, Guillet, Heinrich, Pilon, Reiter [arXiv:hep-ph/0810.0992]

Shown are only the constant terms of the tensor components.

	<i>Hexagon.F</i>	<i>Golem95</i>
F_{03121}	(0.158428986740235E+00 , 0.416706979843194E-01)	(0.158428980552600E+00 , 0.416706995132716E-01)
F_{11020}	(-0.143913859903552E+01 , -0.164647048275408E+00)	(-0.143913852754709E+01 , -0.164647075385477E+00)
F_{20200}	(0.242928799509288E+02 , 0.555041844207877E+02)	(0.242928775936564E+02 , 0.555041824180155E+02)
F_{22130}	(0.225563941055782E+00 , 0.231928571404353E+00)	(0.225563949300093E+00 , 0.231928509918651E+00)
F_{33333}	(0.244568134868438E+00 , 0.740146041525474E+00)	(0.244568138432017E+00 , 0.740146095196997E+00)