7. Plane-parallel atmospheres

Reading: Shu, Vol.I, Ch.4

7.1 LTE atmospheres

We have seen that the photon mean free path in stars is many orders of magnitude smaller that the system size, the stellar radius R_*. In the preceding chapter we have used the near isotropy of radiation thus implied to study the properties of the radiative energy transport in stars. When we now turn our attention to the stellar surface, we can use other simplifications that are justified if $l_R \ll R_*$.

• When integrating from infinity (the observer's position) inwards, the transition zone from optically thin to optically thick regions will be a narrow layer in comparison with R_*. In fact the stellar radius is defined by the location of the transition zone, where $F_{\rm rad} = b\,T^4 = a/4\,T^4$ after integration over the outward-oriented half-sphere. The resulting temperature, at which the star radiates as if it were a black-body, is called the effective temperature $T_{\rm eff}$.

• The frequency integrated flux $F_{\rm rad}$ must be independent of depth (or radius of the star), if there is no energy transport by the gas, otherwise excess heating or cooling would occur.

• A thin transition region implies that the curvature of the stellar surface can be neglected. We can study a one-dimensional system of a plane-parallel atmosphere. Our line-of-sight $\vec{e}_k$ may not be normal to the atmosphere, so we need to introduce the angle between the direction of radiation with path length coordinate s and the normal with path length coordinate $-z$.

$$\mu = \cos\theta = \vec{e}_k \cdot \vec{e}_n \qquad dz = -\mu\, ds \qquad \vec{e}_k \cdot \vec{\nabla} = -\mu\,\frac{d}{dz} \tag{7.1}$$

The aspect angle is thus only a parameter in a one-dimensional equation of transfer.

Let us assume the scattering to be isotropic $\sigma(\Omega_i, \Omega_a) = \sigma/4\pi$ and LTE to apply. The source function then becomes

$$S_\nu = \frac{1}{\kappa_\nu}\left[j_\nu + \oint d\Omega'\ I_\nu(\Omega')\,\sigma(\Omega', \Omega)\right] = \frac{1}{\kappa_\nu}\left[\alpha_\nu\, B_\nu(T) + \sigma\, J_\nu\right]$$

$$\Rightarrow \qquad S_\nu = (1 - A_\nu)\, B_\nu(T) + A_\nu\, J_\nu \qquad A_\nu = \frac{\sigma}{\kappa_\nu} = \frac{\sigma}{\alpha_\nu + \sigma} \quad \text{scattering albedo} \tag{7.2}$$

The total optical depth of the system is so large that we can neglect the background term in the solution of the radiation transport equation.

$$I_\nu(\tau) = \int_0^\tau d\tau'\ S_\nu(\tau')\exp(-\tau + \tau') \qquad \tau_{max} = \int_0^{s_{max}} ds\ \kappa_\nu\ \rightarrow \infty \tag{7.3}$$

We now use the atmospheric depth z as new coordinate and integrate the vertical optical depth inward.

$$\tau_z \equiv \int_0^z dz'\ \kappa_\nu = \mu\,(\tau_{max} - \tau) \qquad d\tau_z = -\mu\, d\tau \qquad \tau_z\,(\tau = 0) = \tau_{max} \rightarrow \infty \qquad \tau_z\,(\tau = \tau_{max}) = 0$$

$$\Rightarrow \qquad I_\nu(\mu,\tau_z) = \frac{1}{\mu}\int_{\tau_z}^{\infty} d\tau_z'\; S_\nu\;\exp\left(\frac{\tau_z-\tau_z'}{\mu}\right)$$

$$\Rightarrow \qquad I_\nu(\mu,\tau_z=0) = \frac{1}{\mu}\int_{0}^{\infty} d\tau_z'\; S_\nu\;\exp\left(-\frac{\tau_z'}{\mu}\right) \tag{7.4}$$

7.2 The Eddington approximation

Near the surface the intensity will not be isotropic, so the Rosseland approximation may not apply. However, we may assume that the anisotropy is linear in μ, which for small anisotropies corresponds to a first-order Taylor expansion.

$$I_\nu(\mu) = a_\nu + b_\nu\,\mu \tag{7.5}$$

The moments of that radiation field with respect to the stellar surface are obviously

$$u_\nu = \frac{4\pi}{c}\,a_\nu \qquad F_\nu = \frac{4\pi}{c}\,b_\nu \qquad P_\nu = \frac{4\pi}{3\,c}\,a_\nu \tag{7.6}$$

Surprisingly, the relation $P_\nu = u_\nu/3$, known as the Eddington approximation, is apparently valid also in situations with small anisotropy.

Let us calculate the moments of our modified radiation transport equation

$$\mu\,\frac{dI_\nu}{d\tau_z} = I_\nu - S_\nu \tag{7.7}$$

$$\frac{dF_\nu}{d\tau_z} = c\,u_\nu - 4\pi\,S_\nu \qquad\qquad c\,\frac{dP_\nu}{d\tau_z} = F_\nu = \frac{c}{3}\,\frac{du_\nu}{d\tau_z} \tag{7.8}$$

Combining these two equations we obtain

$$\frac{1}{3}\,\frac{d^2u_\nu}{d\tau_z^2} = u_\nu - \frac{4\pi}{c}\,S_\nu \overset{\text{Eq.7.3}}{=} (1-A_\nu)\,\left[u_\nu - \frac{4\pi}{c}\,B_\nu(T)\right] \tag{7.9}$$

This radiation diffusion equation can be solved, once the temperature profile and the scattering albedo is known. This yields S_ν at $\tau_z = 0$, which can be used in 7.4 to calculate the angular dependence of the emerging radiation.

The radiation diffusion equation is a second-order differential equation. We therefore need to specify two boundary conditions, which is a difficult problem. One can use frequency-integrated quantities assuming a specific behaviour of the scattering albedo, called gray opacity, to make use of the constancy of the total flux F_{rad}.

One may also neglect scattering, so

$$A_\nu = 0 \overset{Eq.7.3}{\Rightarrow} S_\nu = B_\nu \qquad \overset{\text{Eq.7.4}}{\Rightarrow} I_\nu(\mu,\tau_z=0) = \frac{1}{\mu}\int_0^\infty d\tau_z'\;B_\nu(T)\;\exp\left(-\frac{\tau_z'}{\mu}\right) \tag{7.10}$$