Quantum Chromodynamics

lecture II

Sven-Olaf Moch

Universität Hamburg & DESY, Zeuthen

Belgian Dutch German summer school (BND 2012), Bonn, Sep 22, 2012

Plan

- Introduction to QCD
  Friday, September 21, 2012
- *QCD at work: infrared safety, factorization and evolution*
  Saturday, September 22, 2012
- Higgs boson production
  Sunday, September 23, 2012
- Gauge boson production and QCD jets
  Monday, September 24, 2012
- Top quark production
  Tuesday, September 25, 2012

Perturbative QCD at Work

- QCD – the gauge theory of the strong interactions
- QCD covers dynamics in a large range of scales
  - asymptotically free theory of quarks and gluons at short distances
  - confining theory of hadrons at long distances
- Essential and established part of toolkit for discovering new physics
  - Tevatron and LHC
  - we no longer “test” QCD

Basic concepts of perturbative QCD

- Theoretical framework for QCD predictions at high energies relies on few basic concepts
  - infrared safety
  - factorization
  - evolution

Infrared safety

- Small class of cross sections at high energies and decay rates directly calculable in perturbation theory
- Infrared safe quantities
  - free of long range dependencies at leading power in large momentum scale Q Kinoshita '62; Lee, Nauenberg '64
- General structure of cross section
  - large momentum scale Q, renormalization scale μ

$$Q^2\,\hat{\sigma}\left(Q^2,\mu^2,\alpha_s(\mu^2)\right) = \sum_n \alpha_s^n\, c^{(n)}(Q^2/\mu^2)$$

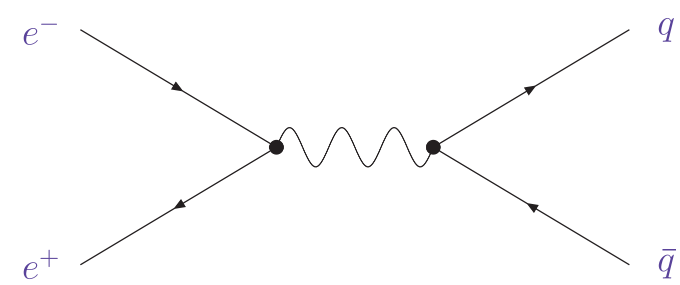
- Examples
  - total cross section in $e^+\,e^-$-annihilation

$$R^{\text{had}}(s) = \frac{\sigma(e^+\,e^- \to \text{hadrons})}{\sigma(e^+\,e^- \to \mu^+\,\mu^-)}$$

  - jet cross sections in $e^+\,e^-$-annihilation
  - total width of Z-boson

Soft and collinear singularities

- e^+e^--annihilation (massless quarks)
  - Born cross section $\sigma^{(0)} = \dfrac{4\pi\alpha^2}{3s}$



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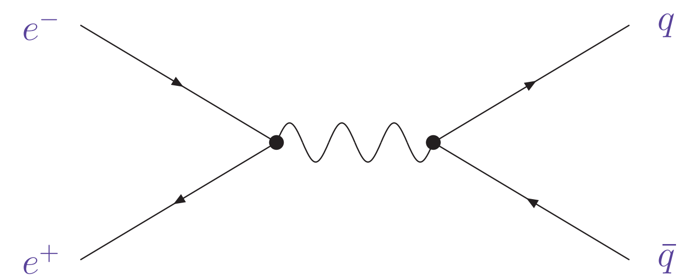
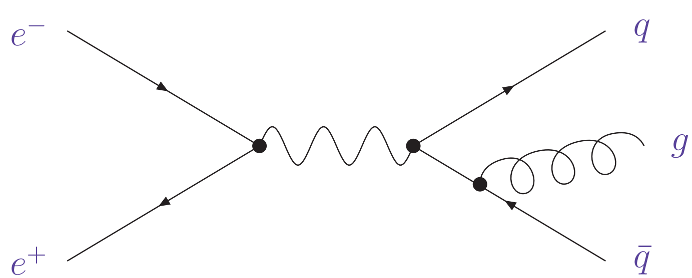
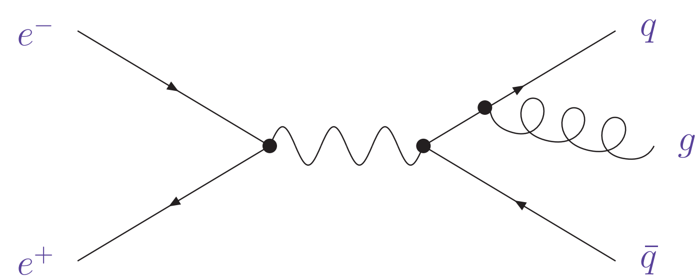


- Study QCD corrections (real emissions)



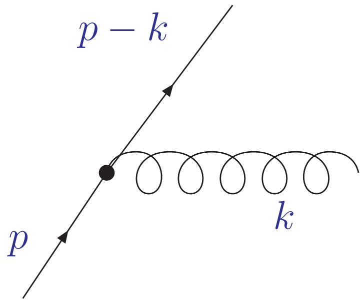


- Cross section
  - dimensional regularization $D = 4 - 2\epsilon$ (with $f(\epsilon) = 1 + \mathcal{O}(\epsilon^2)$)

$$\sigma^{q\bar{q}g} \quad = \quad \sigma^{(0)}\, 3 \sum_q e_q^2\, f(\epsilon)\, C_F\, \frac{\alpha_s}{2\pi} \int dx_1 dx_2\, \frac{x_1^2 + x_2^2 - \epsilon(2 - x_1 - x_2)}{(1-x_1)^{1+\epsilon}\,(1-x_2)^{1+\epsilon}}$$

- scaled energies $x_1 = 2\dfrac{E_q}{\sqrt{s}}$ and $x_2 = 2\dfrac{E_{\bar{q}}}{\sqrt{s}}$

- Soft and collinear divergencies ($0 \leq x_1, x_2 \leq 1$ and $x_1 + x_2 \geq 1$)



$$1 - x_1 = x_2 \frac{E_g}{\sqrt{s}}(1 - \cos\theta_{2g}) \text{ and}$$

$$1 - x_2 = x_1 \frac{E_g}{\sqrt{s}}(1 - \cos\theta_{1g})$$

- Integrate over phase space for real emission contributions

$$\sigma^{q\bar{q}g} \quad = \quad \sigma^{(0)}\, 3 \sum_q e_q^2\, f(\epsilon)\, C_F\, \frac{\alpha_s}{2\pi} \left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2} + \mathcal{O}(\epsilon) \right)$$

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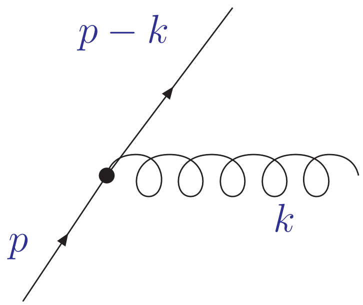

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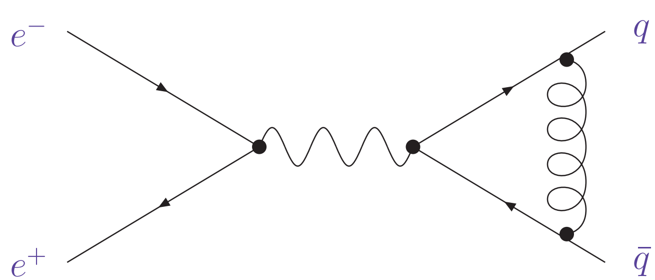
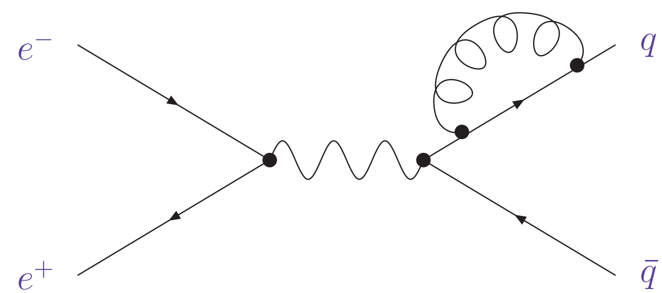
- Divergencies cancel against virtual contributions





$\times 2$

$$\sigma^{q\bar{q}(g)} \quad = \quad \sigma^{(0)}\, 3 \sum_q e_q^2\, f(\epsilon)\, C_F\, \frac{\alpha_s}{2\pi} \left(-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \mathcal{O}(\epsilon) \right)$$

Infrared safety

- Total cross section ($R(s)$) is directly calculable in perturbation theory (finite)

$$R(s) \quad = \quad 3 \sum_q e_q^2 \left\{ 1 + \frac{\alpha_s}{\pi} + \mathcal{O}(\alpha_s^2) \right\}$$

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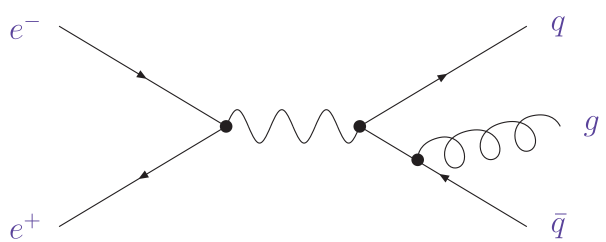
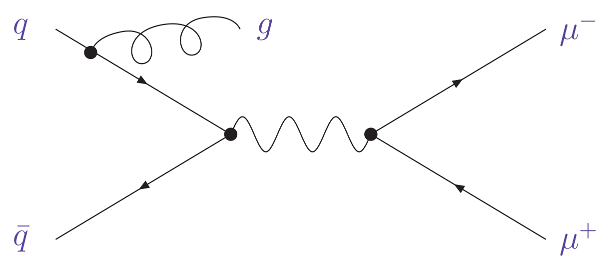
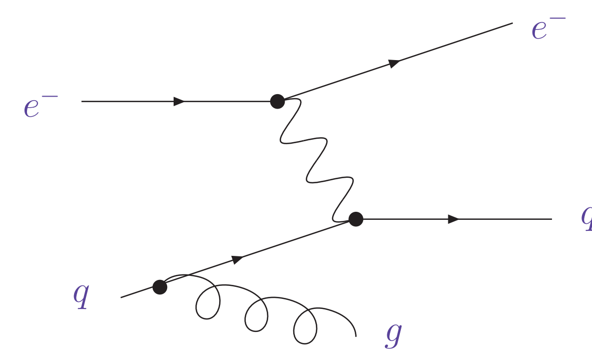
Collinear singularities

- Collinear divergencies remain for hadronic observables
  $\longrightarrow$ factorization







- Left: single-hadron inclusive e^+e^--annihilation (time-like kinematics)
- Center: Drell-Yan process in pp-scattering (space-like kinematics)
- Right: Deep-inelastic e^-p-scattering (space-like kinematics)

Factorization

- Large class of hard-scattering reactions with initial state hadrons
  - cross section not infrared safe
  - dependent on quark and gluon degrees of freedom in hadron
  - sensitive to nonperturbative processes at long distances
- Factorization of cross section
  - infrared safe hard part $\hat{\sigma}_{\mathrm{pt}}$ calculable in perturbative QCD
  - nonperturbative function f determined from data
  - f parametrizes hadron structure
- General structure of cross section
  - large momentum scale Q, factorization scale μ

$$Q^2\,\sigma_{\mathrm{phys}}\,(Q) = \hat{\sigma}_{\mathrm{pt}}\,(Q/\mu, \alpha_s(\mu))\;\otimes\; f\,(\mu)$$

  - convolution $\otimes$ in suitable kinematical variables
- Factorization
  - generalization of operator product expansion

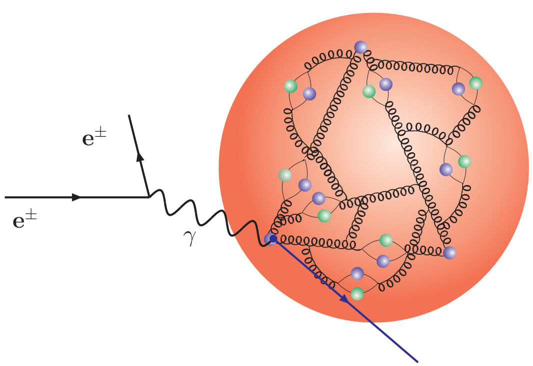
Classic example

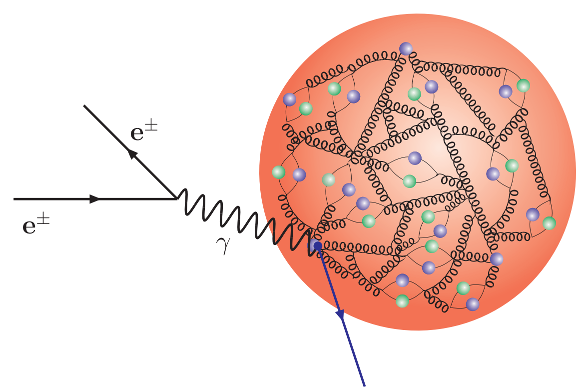
- Deep-inelastic scattering
  - test parton dynamics at factorization scale μ

$$\sigma_{\gamma p \to X} = \sum_i f_i(\mu^2) \otimes \hat{\sigma}_{\gamma i \to X} \left(\alpha_s(\mu^2), Q^2, \mu^2\right)$$

Physics picture

- QCD factorization
  - constituent partons from proton interact at short distance
  - photon momentum $Q^2 = -q^2$, Bjorken's $x = Q^2/(2p \cdot q)$
  - low resolution   high resolution

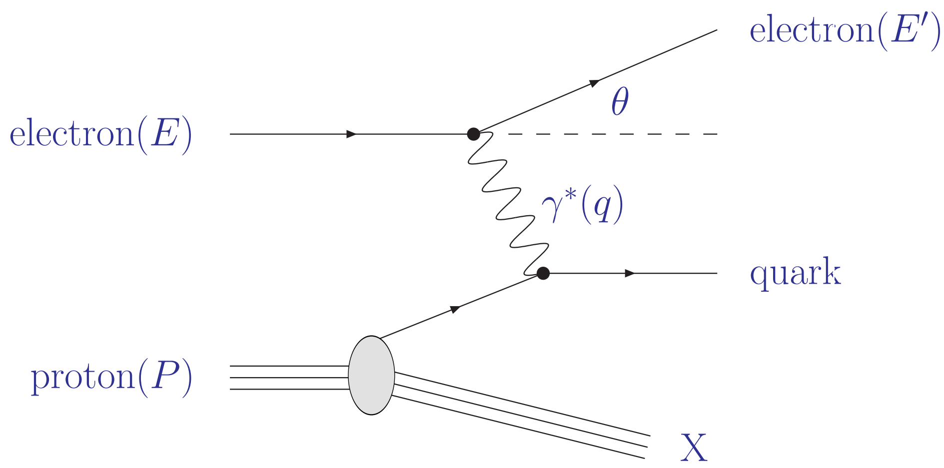




Once upon a time in the north . . .

- HERA: deep structure of proton at highest Q^2 and smallest x

Inelastic electron-proton scattering



- Virtuality of photon: resolution
  $$Q^2 \equiv -q^2 = 4EE' \sin^2(\theta/2)$$
- Bjorken variable: inelasticity
  $$x = \frac{Q^2}{2P\cdot q} < 1$$
- Cross section (X inclusive): proton structure function F_i^p

$$(E - E')\frac{d\sigma}{d\Omega\, dE'} \stackrel{\text{lab}}{=} \underbrace{\frac{\alpha^2 \cos^2\frac{\theta}{2}}{4E^2 \sin^4\frac{\theta}{2}}}_{\text{Mott-scattering (point-like)}} \left\{ F_2^p(x, Q^2) + \tan^2\frac{\theta}{2}\, F_1^p(x, Q^2) \right\}$$

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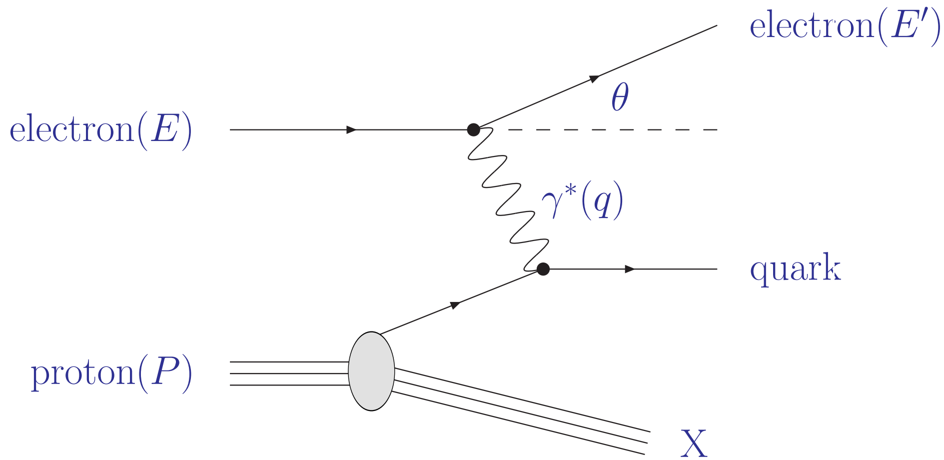

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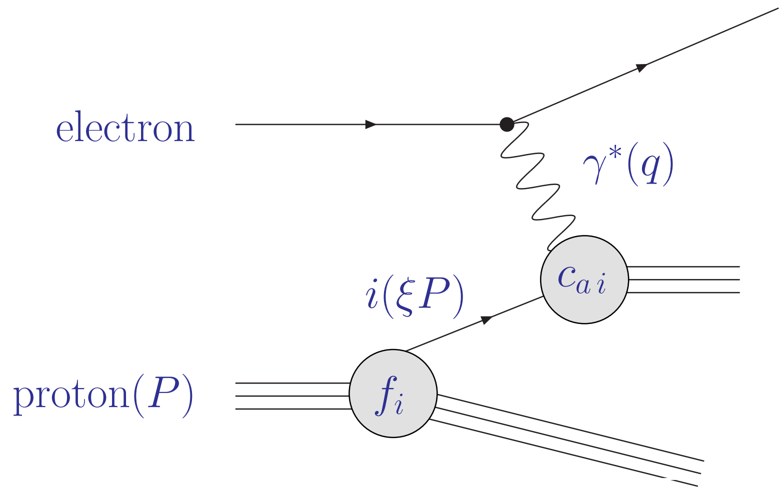
- Deep-inelastic scattering (Bjorken limit: $Q^2 \to \infty$ and x fixed)
  Parton modell (quasi-free point-like constituents, incoherence)
  $$F_2(x,Q^2) \simeq F_2(x) = \sum_i e_i^2\, x f_i(x)$$
  - $x f_i(x)$ distribution for momentum fraction x of parton i

QCD corrections in deep-inelastic scattering



- Structure function F_2 (up to terms $\mathcal{O}(1/Q^2)$)
  - Renormalization/factorization scale $\mu = \mathcal{O}(Q)$

$$x^{-1} F_2^p(x,Q^2) \;=\; \sum_i \int_x^1 \frac{d\xi}{\xi}\, c_{2,i}\left(\frac{x}{\xi}, \alpha_s(\mu^2), \frac{\mu^2}{Q^2}\right) f_i^p(\xi,\mu^2)$$

- Coefficient functions c_a

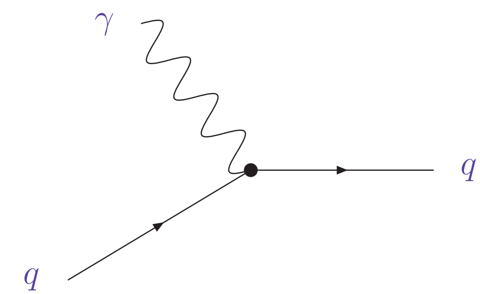
$$c_a \;=\; \underbrace{\alpha_s^{n_a}\Big[c_a^{(0)} + \alpha_s\, c_a^{(1)}}\; + \; \alpha_s^2\, c_a^{(2)} \; + \; \ldots \Big]$$

NLO: standard approximation (large uncertainties)

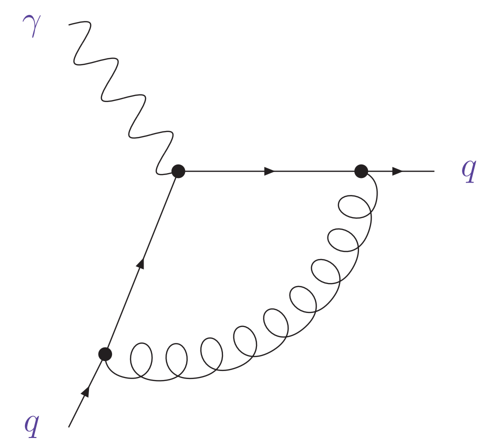
Radiative corrections in a nutshell

- Leading order
  - partonic structure function

$$\hat{F}^{(0)}_{2,q} \;=\; e_q^2 \delta(1-x)$$

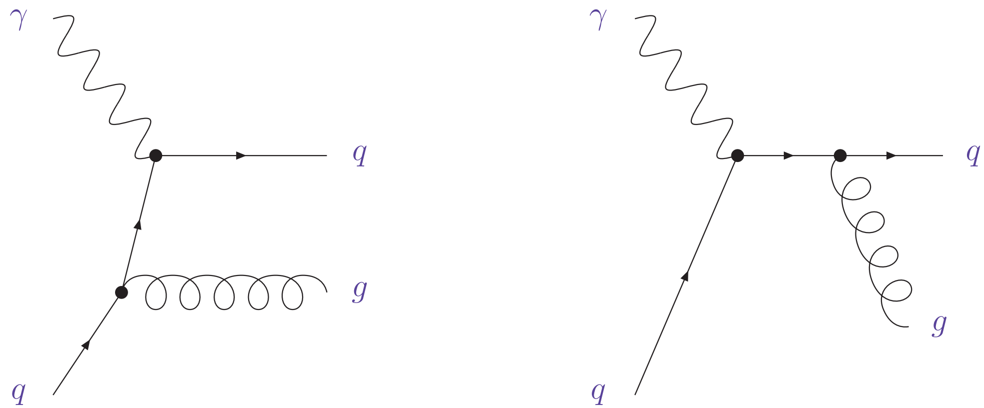


- Next-to-leading order
  - virtual correction
    (infrared divergent; proportional to Born)
  - dimensional regularization $D = 4 - 2\epsilon$



$$\hat{F}^{(1),v}_{2,q} \quad = \quad e_q^2\, C_F\, \frac{\alpha_s}{4\pi}\, \delta(1-x) \left(\frac{\mu^2}{Q^2}\right)^{\epsilon} \left(-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \zeta_2 + \mathcal{O}(\epsilon)\right)$$

- Next-to-leading order



  - add real and virtual corrections $\hat{F}^{(1)}_{2,q} = \hat{F}^{(1),r}_{2,q} + \hat{F}^{(1),v}_{2,q}$
  - collinear divergence remains splitting functions $P^{(0)}_{qq}$

$$
\begin{aligned}
\hat{F}^{(1)}_{2,q} \quad = \quad & e_q^2\, C_F\, \frac{\alpha_s}{4\pi} \left(\frac{\mu^2}{Q^2}\right)^{\epsilon} \left\{ \frac{1}{\epsilon} \left(\frac{4}{1-x} - 2 - 2x + 3\delta(1-x) \right) \right. \\
& +4\frac{\ln(1-x)}{1-x} - 3\frac{1}{1-x} - (9+4\zeta_2)\delta(1-x) \\
& \quad -2(1+x)(\ln(1-x) - \ln(x)) - 4\frac{1}{1-x}\ln(x) + 6 + 4x \\
& \left. +\mathcal{O}(\epsilon) \right\}
\end{aligned}
$$

- Structure of NLO correction
  - absorb collinear divergence $P_{qq}^{(0)}$ in renormalized parton distributions

$$\hat{F}_{2,q}^{(1),\mathrm{bare}} = e_q^2 \, \frac{\alpha_s}{4\pi} \left(\frac{\mu^2}{Q^2}\right)^{\epsilon} \left\{ \frac{1}{\epsilon} \, P_{qq}^{(0)}(x) + c_{2,q}^{(1)}(x) + \mathcal{O}(\epsilon) \right\}$$

$$q^{\mathrm{ren}}(\mu_F^2) = q^{\mathrm{bare}} - \frac{\alpha_s}{4\pi} \, \frac{1}{\epsilon} \, P_{qq}^{(0)}(x) \left(\frac{\mu^2}{\mu_F^2}\right)^{\epsilon}$$

  - partonic (physical) structure function at factorization scale μ_F

$$\hat{F}_{2,q} = e_q^2 \left(\delta(1-x) + \frac{\alpha_s}{4\pi} \left\{ c_{2,q}^{(1)}(x) - \ln\left(\frac{Q^2}{\mu_F^2}\right) \, P_{qq}^{(0)}(x) \right\} \right)$$

QCD evolution

- Evolution formulates dependence of cross sections for observable on momentum transfer
- Classic example: scaling violations of structure functions
  Gross, Wilczek ‘73; Politzer ‘73
- Physical cross section in factorization ansatz cannot depend on μ
  $Q^2\, \sigma_{\rm phys}\,(Q) = \hat{\sigma}_{\rm pt}\,(Q/\mu, \alpha_s(\mu))\, \otimes\, f\,(\mu)$
  - factorization scale μ arbitrary $\mu \dfrac{d\,\sigma_{\rm phys}}{d\mu} = 0$
- Immediate consequence **DGLAP**: Altarelli, Parisi ‘77

$$
\begin{aligned}
\mu \frac{d\, f(\mu)}{d\mu} &= P\,(\alpha_s(\mu))\, \otimes\, f(\mu) + \mathcal{O}\left(\frac{1}{Q^2}\right) \\
\mu \frac{d\, \hat{\sigma}_{\rm pt}\,(Q/\mu, \alpha_s(\mu))}{d\mu} &= -P\,(\alpha_s(\mu))\, \otimes\, \hat{\sigma}_{\rm pt}\,(Q/\mu, \alpha_s(\mu)) + \mathcal{O}\left(\frac{1}{Q^2}\right)
\end{aligned}
$$

- PDF evolution from renormalization group equation

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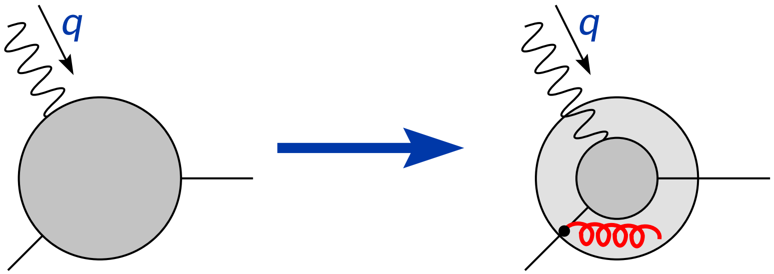
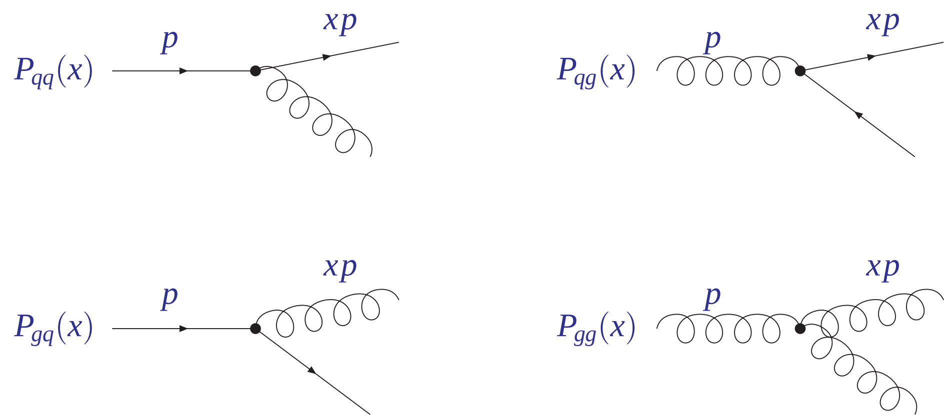
- PDF evolution from renormalization group equation
  - splitting functions calculable in QCD

Parton evolution



- Proton in resolution $1/Q \longrightarrow$ sensitive to lower momentum partons

- Feynman diagrams in leading order



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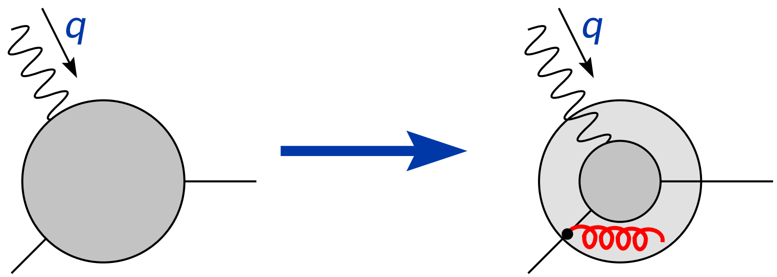

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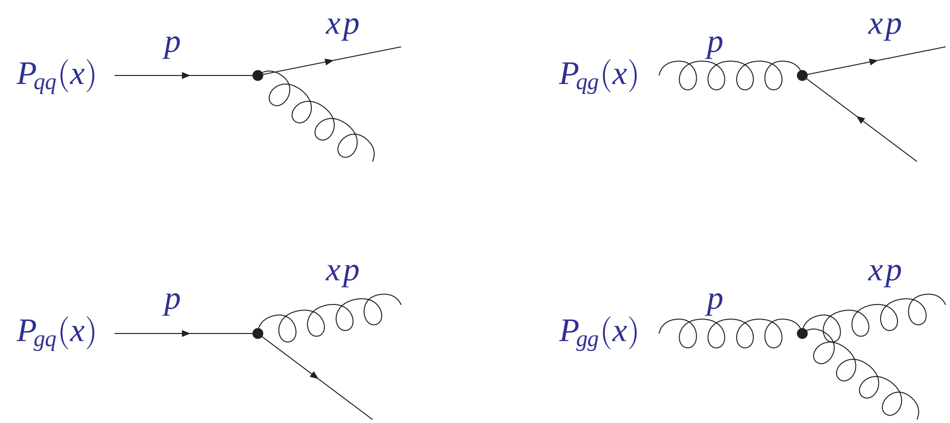

- Proton in resolution $1/Q \longrightarrow$ sensitive to lower momentum partons
- Evolution equations for parton distributions f_i
  - predictions from fits to reference processes (universality)

$$\frac{d}{d \ln \mu^2} f_i(x, \mu^2) = \sum_k \left[P_{ik}(\alpha_s(\mu^2)) \otimes f_k(\mu^2) \right](x)$$

- Splitting functions P

$$P = \underbrace{\alpha_s P^{(0)} + \alpha_s^2 P^{(1)}}_{} + \alpha_s^3 P^{(2)} + \dots$$

NLO: standard approximation (large uncertainties)

Complete set of splitting functions and PDFs

- Evolution equations
  - non-singlet ($2n_f - 1$ scalar) and singlet (2×2 matrix) equations

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- Non-singlet and singlet distributions $q^{\pm}$, $q^{\rm v}$ and $q_{\rm s}$, g

$$q^{\pm}_{{\rm ns},ik} = q_i \pm \bar{q}_i - (q_k \pm \bar{q}_k)$$ flavour asymmetries

$$q^{\rm v}_{\rm ns} = \sum_{r=1}^{n_f} (q_r - \bar{q}_r)$$ total valence distribution

$$q_{\rm s} = \sum_{r=1}^{n_f} (q_r + \bar{q}_r)$$ flavour singlet distribution, $f_i = \begin{pmatrix} q_{\rm s} \\ g \end{pmatrix}$

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$$q_{\rm s} = \sum_{r=1}^{n_f} (q_r + \bar{q}_r) \quad \text{flavour singlet distribution, } f_i = \begin{pmatrix} q_{\rm s} \\ g \end{pmatrix}$$

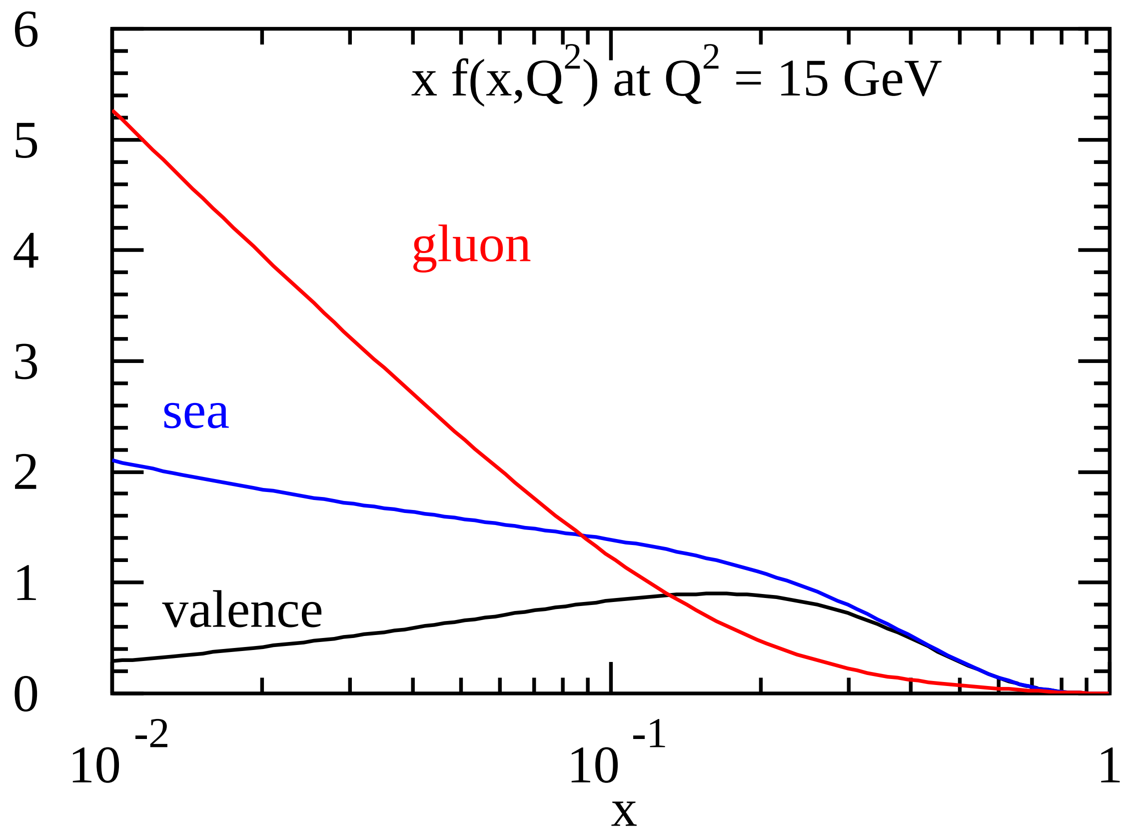
- Splitting function combinations

$$P^{\pm}_{\rm ns}\,, \qquad P^{\rm v}_{\rm ns} = P^{-}_{\rm ns} + P^{\rm s}_{\rm ns} \qquad \text{non-singlet}$$

$$P_{\rm s} = \begin{pmatrix} P_{\rm qq} & P_{\rm qg} \\ P_{\rm gq} & P_{\rm gg} \end{pmatrix}, \qquad P_{\rm qq} = P^{+}_{\rm ns} + P_{\rm ps} \qquad \text{singlet}$$

Parton distributions in proton

- Valence $q - \bar{q}$ (additive quantum numbers) sea (part with $q + \bar{q}$)



- Parameterization (bulk of data from deep-inelastic scattering)
  - structure function $F_2 \longrightarrow$ quark distribution
  - scale evolution (perturbative QCD) $\longrightarrow$ gluon distribution

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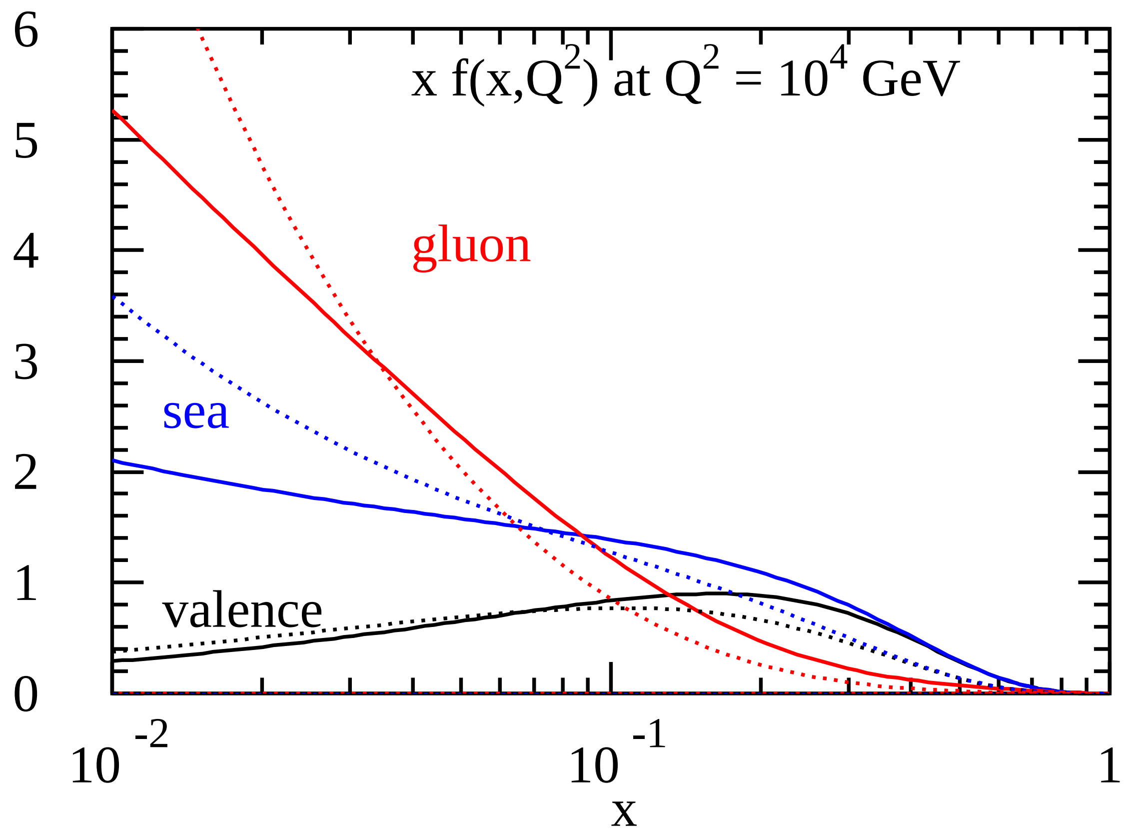

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Pocket partonometer

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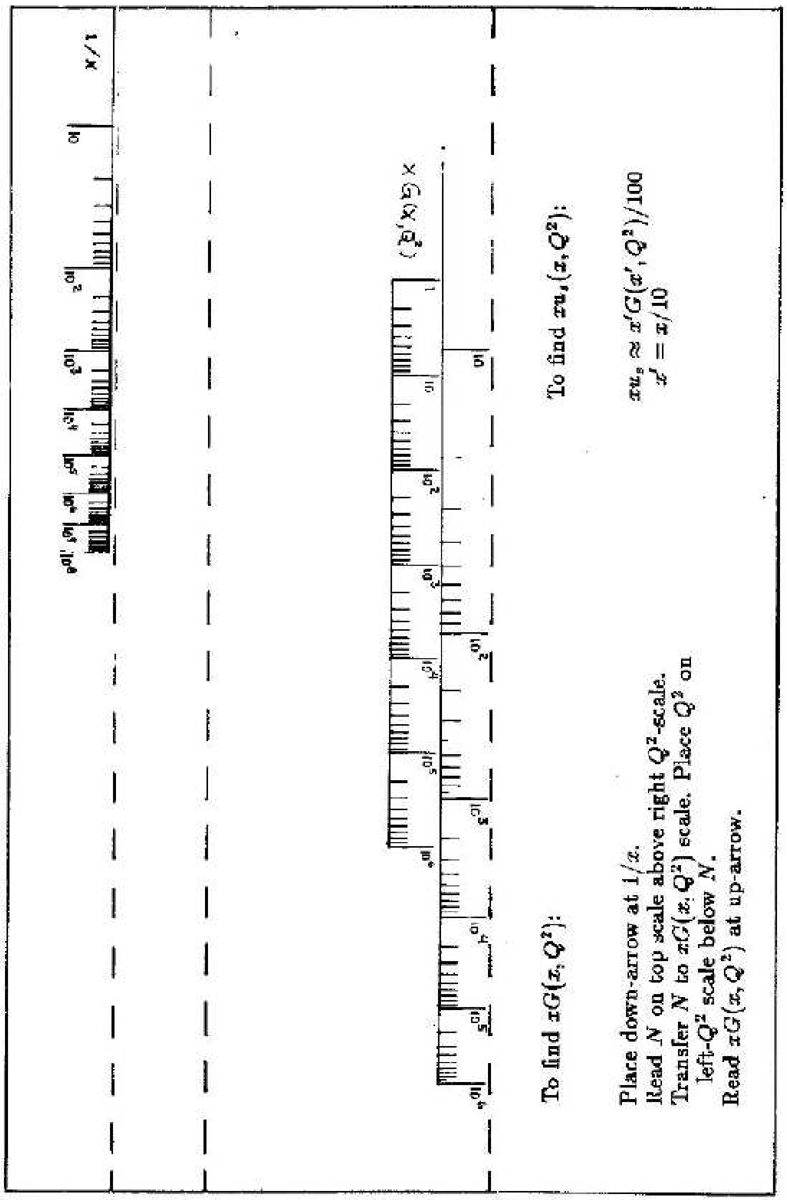

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for t- or heavier particle distributions one must model thresholds numerically such as done in ref. [4] [‡4]. However, departures from a symmetrically distributed sea, which complicate the boundary conditions, can be reproduced by the ratios $u_s \approx d_s \approx s_s \approx 2c_s \approx 2b_s$.

The analytic gluon solution (3), boundary conditions included, is calculated by the partonometer (fig. 2). The scales automate the logarithms of certain functions of $1/x$ and Q^2 left to the reader. In systematic testing the accuracy of the gizmo is at the 10–20% level depending on the operator's ability to read logarithmic scales. It is much better than interpolating between graphs such as fig. 1a. The speed is even faster than adding a new card [‡4] to an existing program that runs.

Gluon distributions are read off directly; see the example below. Quark sea distributions can be evaluated using the identity

$$xu_s(x, Q^2) = (2/b)\partial xG(x, Q^2)/\partial y, \tag{7}$$

and evaluating the derivative numerically. But wait! To minimize reading errors, one finds that the derivative above and the normalization change are roughly represented by

$$xu_s(x, Q^2) \approx x'G(x', Q^2)/100, \quad x' = x/10. \tag{8}$$

This estimate is actually quite close to the re-scaled $xu_s(x, Q^2)$ of ref. [5] and is not too bad a match to

[‡4] Private communication with well known phenomenologist.

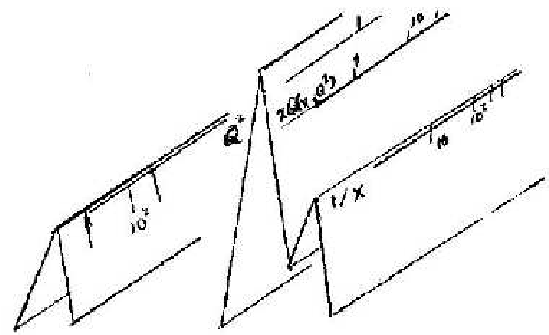
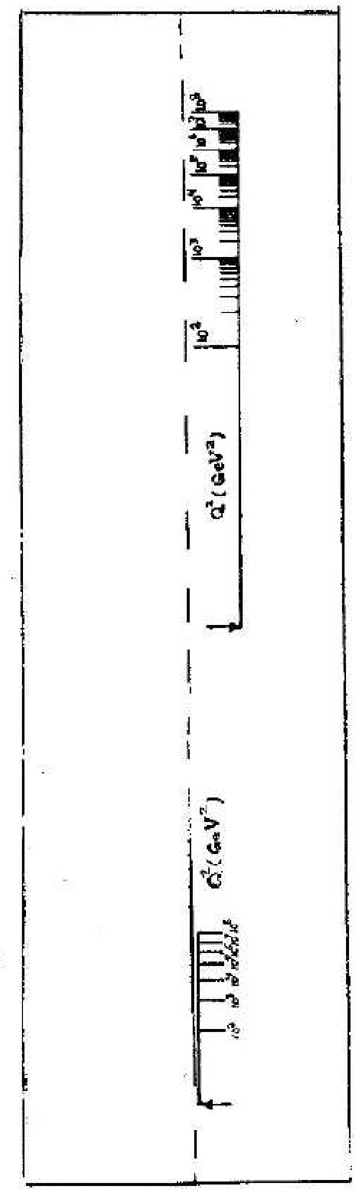
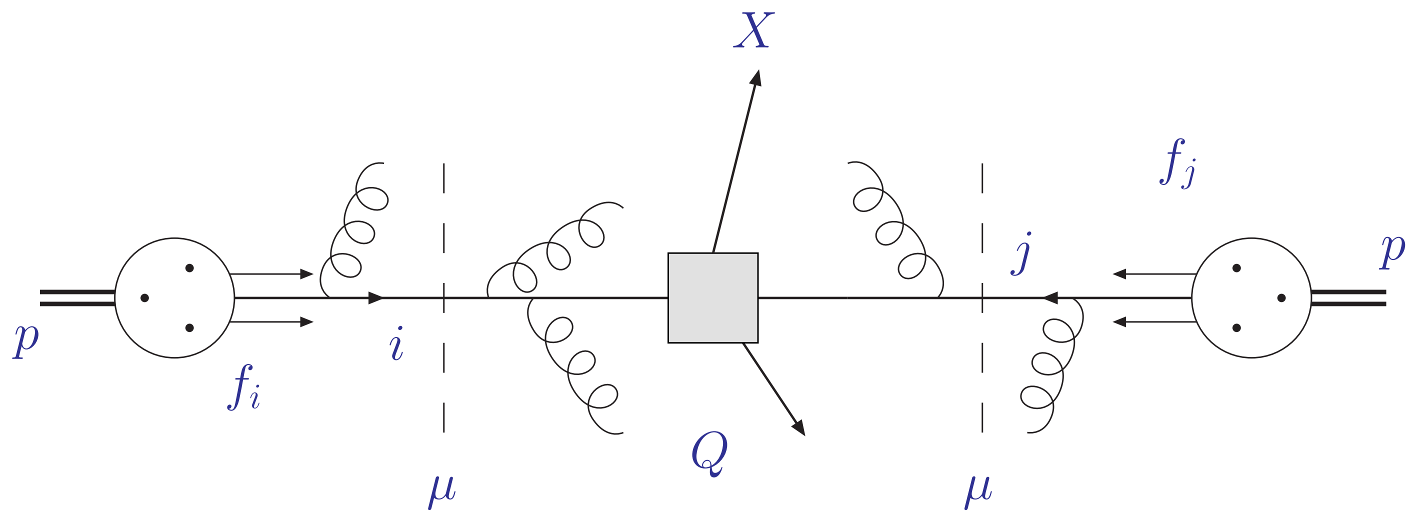


Fig. 2. The partonometer. To assemble: cut on solid lines, fold on dotted lines.



QCD factorization

- QCD factorization for hadron-hadron scattering

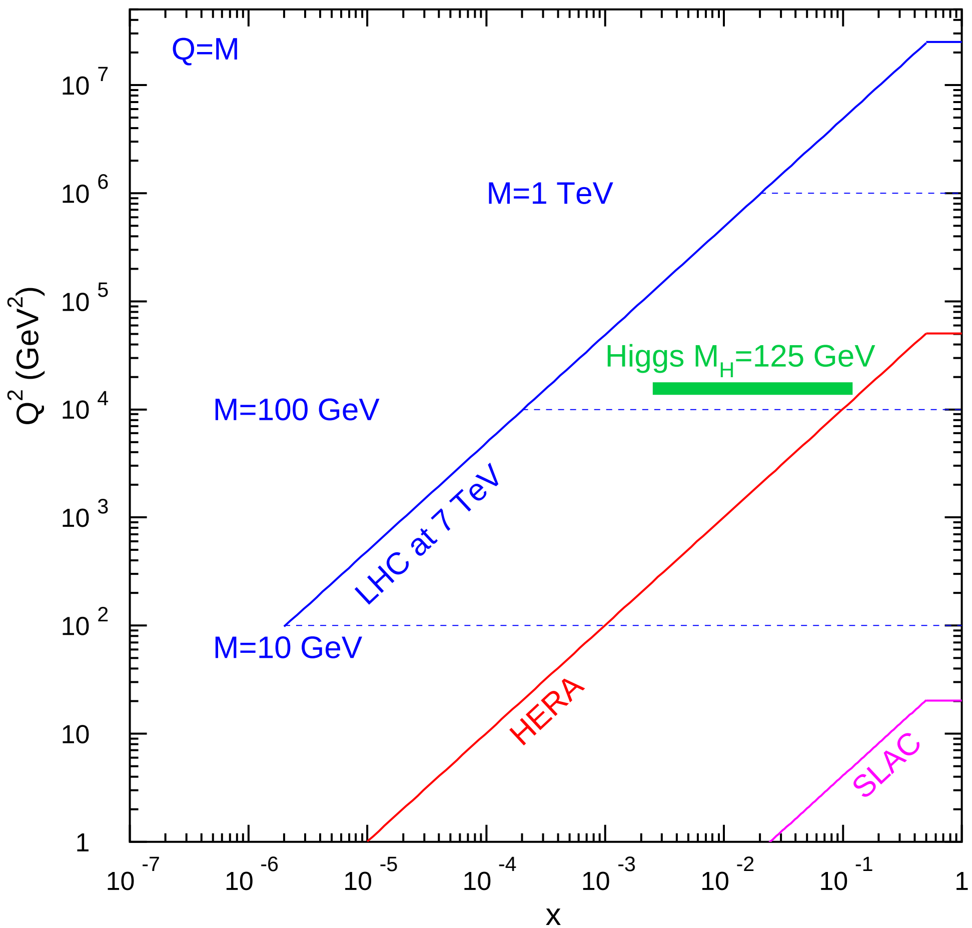


$$\sigma_{pp\to X} \;=\; \sum_{ij} f_i(\mu^2) \otimes \; f_j(\mu^2) \otimes \; \hat{\sigma}_{ij\to X}\left(\alpha_s(\mu^2), Q^2, \mu^2, m_X^2\right)$$

- Hard parton cross section $\hat{\sigma}_{ij\to X}$ calculable in perturbation theory
- Parton luminosity L_{ij} to be determined from comparison to data

$$L_{ij}(\mu^2) \;=\; f_i(\mu^2) \otimes \; f_j(\mu^2)$$

Parton luminosity at LHC



- LHC run at $\sqrt{s}$ = 7/8 TeV
  - parton kinematics well covered by HERA and fixed target experiments
- Parton kinematics at effective $\langle x \rangle = M/\sqrt{S}$
  - 100 GeV physics: small-x, sea partons
  - TeV scales: large-x

Wilson coefficients

- Recall QCD corrections to Wilson coefficients $c_{2,q}^{(1)}$ in $\hat{F}_{2,q}^{(1)}$ (cross section of hard parton scattering)

$$
\begin{aligned}
\hat{F}_{2,q}^{(1)} \quad = \quad & e_q^2\, C_F\, \frac{\alpha_s}{4\pi} \left(\frac{\mu^2}{Q^2}\right)^{\epsilon} \left\{ \frac{1}{\epsilon} \left(\frac{4}{1-x} - 2 - 2x + 3\delta(1-x) \right) \right. \\
& + 4\frac{\ln(1-x)}{1-x} - 3\frac{1}{1-x} - (9 + 4\zeta_2)\delta(1-x) \\
& \left. -2(1+x)(\ln(1-x) - \ln(x)) - 4\frac{1}{1-x}\ln(x) + 6 + 4x + \mathcal{O}(\epsilon) \right\}
\end{aligned}
$$

  - large radiative corrections as $x \to 1$
- Mellin transform with moments N (integral ransform $x \to N$)

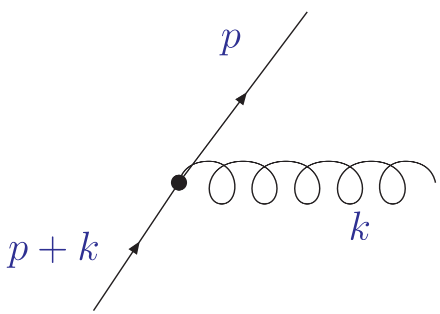
$$
f(N) \;=\; \int_0^1 dx\, x^{N-1} f(x)
$$

  - dictionary:

$$
\begin{aligned}
[\ln(1-x)/(1-x)]_+ \quad &\rightarrow \quad \ln^2 N \\
[1/(1-x)]_+ \quad &\rightarrow \quad \ln N \\
\delta(1-x) \quad &\rightarrow \quad \mathrm{const}_N
\end{aligned}
$$

Sudakov logarithms

- Recall perturbative QCD:
  - calculation of observables as series in $\alpha_s \ll 1$
  - but: large logarithmic corrections, $\ln(\dots) \gg 1$
    double logarithms (Sudakov)
- Soft and collinear regions of phase space
  - double logarithms from singular regions in Feynman diagrams
  - propagator vanishes for: $E_g = 0$, soft $\theta_{qg} = 0$ collinear



$$\frac{1}{(p+k)^2} = \frac{1}{2p\cdot k} = \frac{1}{2E_q E_g (1-\cos\theta_{qg})}$$

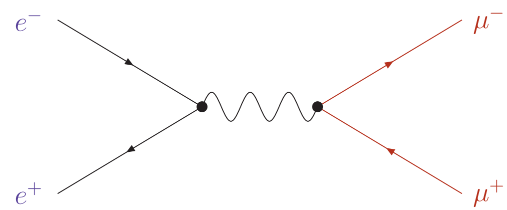
$$\alpha_s \int d^4k \frac{1}{(p+k)^2} \longrightarrow \alpha_s \int dE_g \, d\sin\theta_{qg} \, \frac{1}{2E_q E_g (1-\cos\theta_{qg})}$$

$$\longrightarrow \alpha_s \ln^2(\dots)$$

- Improved perturbation theory: resum logarithms to all orders
  - long history of resummation Sterman ‘87; Catani, Trentadue ‘88; ...

Sudakov logarithms in cross sections

- Intuitive aspects of higher order corrections
  - consider QED corrections for $e^+e^- \to \mu^+\mu^-$
  - lowest order, elastic



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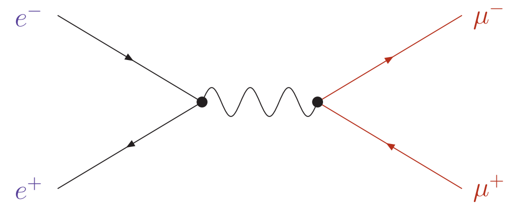
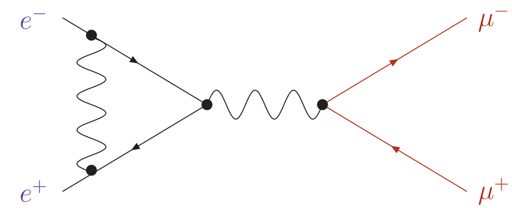
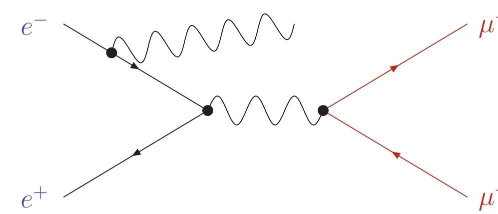


  - first order correction
    virtual < 0 (elastic)



  - first order correction
    Brems > 0 (inelastic)



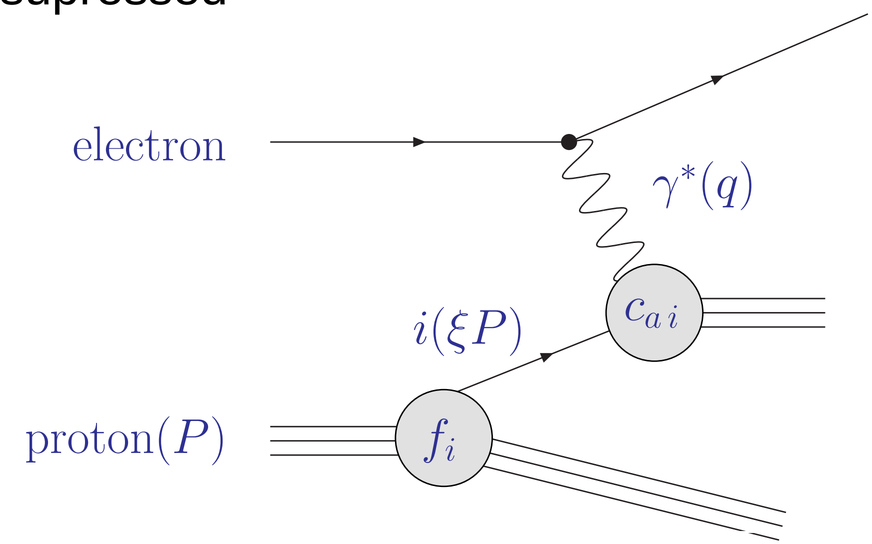
- at threshold for $\mu^+\mu^-$-pair creation
  - strong Sudakov-supression inelastic tendency

$$\sigma \sim \exp\left[-\alpha \ln^2(1 - 4m_\mu^2/s)\right]$$

  - universal factor for parton splittings (leading log accuracy)
    modelling of MC parton showers

Sudakov logarithms in QCD

- Hadronic reaction in ep DIS:
  - recall master equation $$x^{-1}F_2^p(x,Q^2) = \sum_i f_i^p \otimes c_{2,i}$$
  - initial partons: also Sudakov-supressed



- Wilson coefficients $c_{2,i}$
  - Sudakov-enhancement after mass factorization

$$c_{2,i} = \frac{x^{-1}F_2^p(x,Q^2)}{f_i} \simeq \frac{1}{\mathrm{e}^{-\alpha_s \ln^2(\dots)}} \simeq \mathrm{e}^{+\alpha_s \ln^2(\dots)}$$

  - large double logarithms

Resummation in a nut-shell

- Resummation
  - reorganize perturbative expansion $\longrightarrow$ stability
  - generating functional for higher orders of perturbation theory

$$
\begin{aligned}
\mathcal{O} &= 1 + \alpha \left(\ln^2 + \ln + 1\right) + \alpha^2 \left(\ln^4 + \ln^3 + \ln^2 + \ln + 1\right) + \ldots \\
&= \left(1 + \alpha\, 1 + \alpha^2\, 1 + \ldots\right) \exp\left(\alpha \, \ln^2 + \alpha \, \ln + \alpha^2 \, \ln + \ldots\right)
\end{aligned}
$$

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&= \left(1 + \alpha\, 1 + \alpha^2\, 1 + \ldots\right) \exp\left(\alpha \ln^2 + \alpha \ln + \alpha^2 \ln + \ldots\right)
\end{aligned}
$$

DIS Wilson coefficients

- Coefficient functions at large N/ large x have large logarithms at n$^{\mathrm{th}}$-order

$$
\alpha_s^n \left(\frac{\ln^{2n-1}(1-x)}{1-x} \right)_+ \longleftrightarrow \alpha_s^n \ln^{2n}(N)
$$

- Threshold resummation in Mellin space

$$
C^N = \left(1 + \alpha_s\, g_{01} + \alpha_s^2\, g_{02} + \ldots\right) \cdot \exp\left(G^N\right) + \mathcal{O}(N^{-1} \ln^n N)
$$

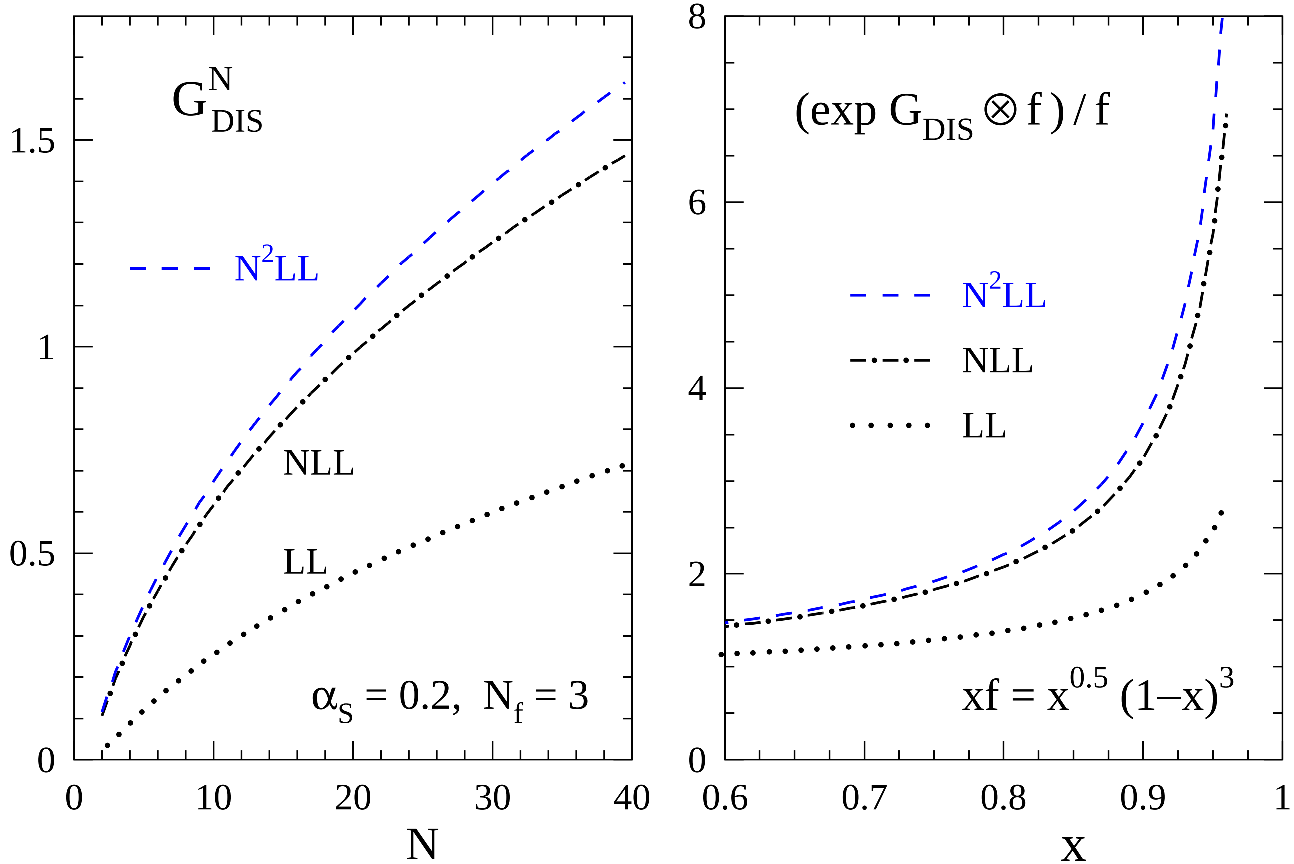
Accuracy under control

- Control over logarithms $\ln(N)$ with $\lambda = \beta_0 \alpha_s \ln(N)$ to N^{k}LL accuracy

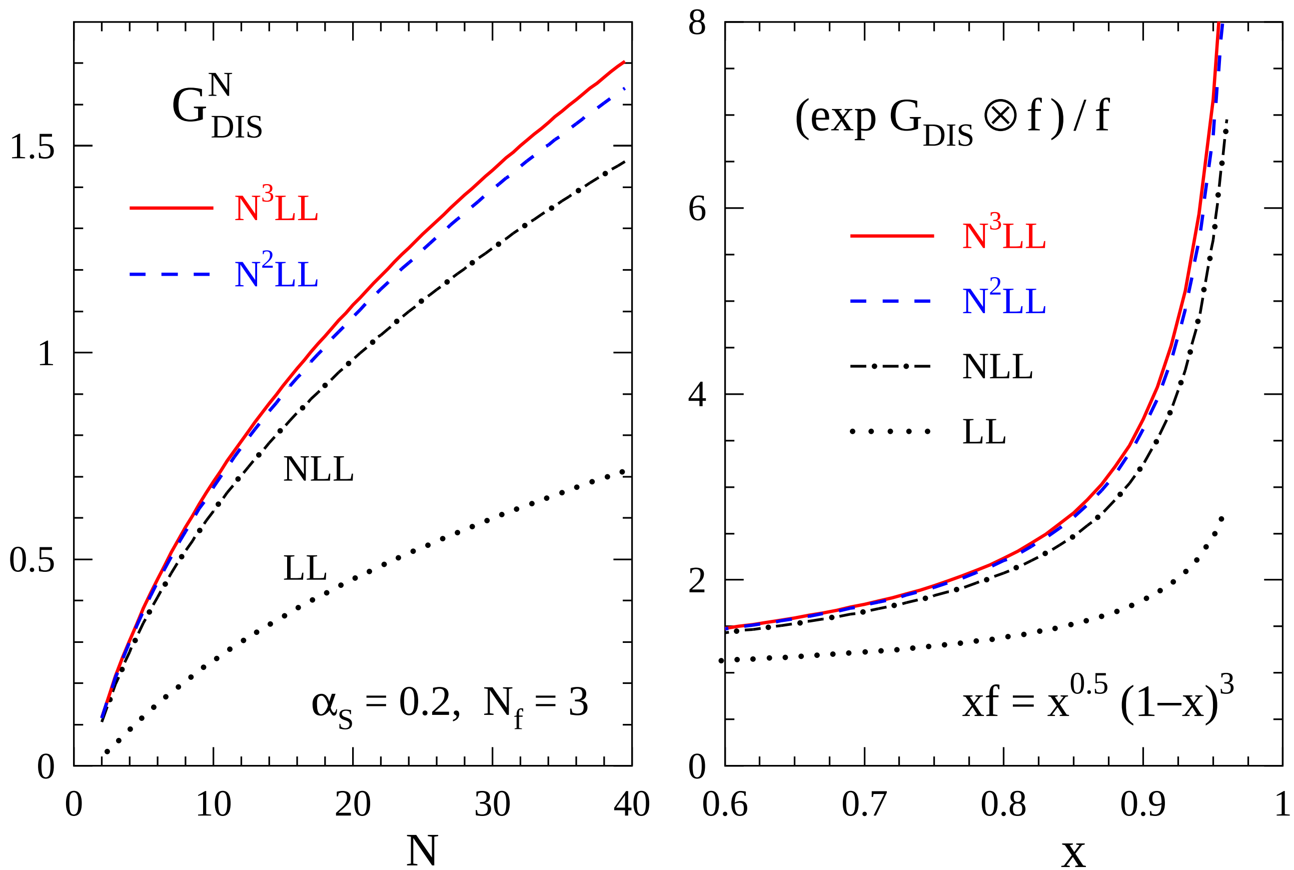
$$G^N = \ln(N) g_1(\lambda) + g_2(\lambda) + \alpha_s g_3(\lambda) + \alpha_s^2 g_4(\lambda) + \dots$$

  - $g_1(\lambda)$: LL Sterman '87; Appell, Mackenzie, Sterman '88
  - $g_2(\lambda)$: NLL Catani Trenatdue '89
  - $g_3(\lambda)$: NNLL or N^{2}LL Vogt '00; Catani, Grazzini, de Florian, Nason '03
  - $g_4(\lambda)$: N^{3}LL S.M., Vermaseren, Vogt '05
- Resummed G^N predicts fixed orders in perturbation theory
  - generating functional for towers of large logarithms

DIS resummation exponent



DIS resummation exponent



- Perturbative expansion very stable
- Resummation exponent generates perturbative expansion:
  - four-loop coeff. fct. $c_{2,q}^{(4)}$ known $\left(\frac{\ln^7(1-x)}{1-x}\right)_+, \ldots, \left(\frac{\ln(1-x)}{1-x}\right)_+$

Summary (part II)

Perturbative QCD at work

- Basics concepts of QCD
- Infrared safety
  - cancellation of soft and collinear singularities in inclusive observables
  - example $e^+e^- \to$ hadrons at NLO
- Factorization
  - scattering with initial state hadrons requires collinear factorization
  - separatin of long and short distance physics
  - parton distribution function
  - example $ep \to X$ in DIS at NLO
- Evolution
  - factorization induces evolution equations via renormalization group
- Resummation
  - large logarithms near threshold
  - radiative corrections (higher orders) important
  - essential to control theory uncertainties