Introduction to Lattice QCD II
A dedicated project

Karl Jansen

- **Leading order hadronic contribution to the muon anomalous magnetic moment**
  - motivation
  - describe the computation
  - new method for the lattice
  - results

A success of Quantum Field Theories: Electromagnetic Interaction

Quantum Electrodynamics (QED)

coupling of the electromagnetic interaction is small
$\Rightarrow$ perturbation theory 4-loop calculation

magnetic moment of the electron

$$\vec{\mu} = g_e \frac{e\hbar}{2m_e c} \vec{S}$$

$\vec{S}$ spin vector

deviation from $g_e = 2$: $a_e = (g_e - 2)/2$

$$\begin{aligned} a_e(\text{theory}) &= 1159652201.1(2.1)(27.1) \cdot 10^{-12} \\ a_e(\text{experiment}) &= 1159652188.4(4.3) \cdot 10^{-12} \end{aligned}$$

The question of the muon

magnetic moment of the muon

$$\vec{\mu} = g_\mu \frac{e\hbar}{2m_\mu c}\vec{S}$$

deviation from $g_\mu = 2$: $a_\mu = (g_\mu - 2)/2$

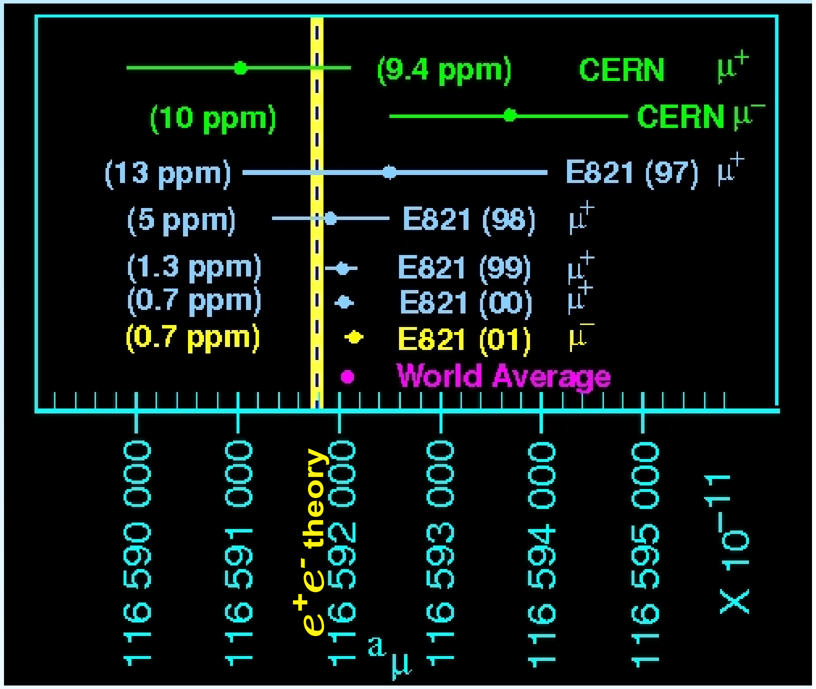
$$\begin{aligned} a_\mu(\text{theory}) &= 1.16591790(65) \cdot 10^{-3} \\ a_\mu(\text{experiment}) &= 1.16592080(63) \cdot 10^{-3} \end{aligned}$$

$\rightarrow$ there is a 3.2σ discrepancy

$$a_\mu(\text{experiment}) - a_\mu(\text{theory}) = 2.90(91) \cdot 10^{-9}$$

Motivation

E821 achieved ± 0.54 ppm. The e^+e^- based theory is at the ~0.4 ppm level. Difference is ~3.6 σ



$$a_\mu^{exp} = 116\,592\,089(63) \times 10^{-11} \ (0.54\,\text{ppm})$$

$$\Delta a_\mu \equiv a_\mu^{exp} - a_\mu^{SM} = (287 \pm 80) \times 10^{-11}$$

Theory: arXiv:1010.4180v1 [hep-ph] Davier, Hoecker, Malaescu, and Zhang, Tau2010

Why is this interesting?

$\rightarrow$ new proposed experiments at Fermilab (USA) and JPARC (Japan) ≈ 2015

brings down error

$\sigma_{\mathrm{ex}} = 6.3 \cdot 10^{-10} \rightarrow 1.6 \cdot 10^{-10}$

challenge: bring down theoretical error to same level

$\Rightarrow$ possibility

$$a_\mu(\mathrm{experiment}) - a_\mu(\mathrm{theory}) > 5\sigma$$

- possible breakdown of standard model?
- $a_\mu = a_\mu^{\mathrm{QED}} + a_\mu^{\mathrm{weak}} + a_\mu^{\mathrm{QCD}} + a_\mu^{\mathrm{NP}}$

$a_\mu^{NP} \propto m_{\mathrm{lepton}}^2 / m_{\mathrm{new}}^2$

$\rightarrow$ muon ideal to discover new physics

for τ experimental results too imprecise

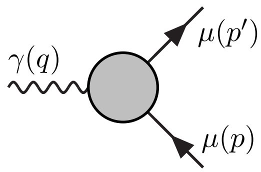
Is the lattice important?

$\rightarrow$ size of various contributions to a_μ

Contribution	σ^{th}
QCD-LO (α_s^2)	$5.3 \cdot 10^{-10}$
QCD-NLO (α_s^3)	$3.9 \cdot 10^{-10}$
QED/EW	$0.2 \cdot 10^{-10}$
Total	$6.6 \cdot 10^{-10}$

$\Rightarrow$ non-perturbative hadronic contributions are most important

The contributions to $g_\mu - 2$



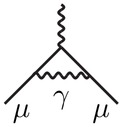
The vertex

- p, p' incoming and outgoing momenta
- $q = p - p'$ photon momentum

- put muon in magnetic field $\vec{B}$ interaction Hamiltonian $H = \vec{\mu}\vec{B}$
- measure interaction of muon with photon (magnetic field)

The simplest, QED contribution to $g_\mu - 2$

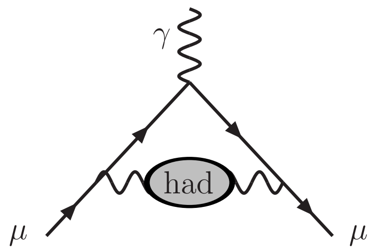
Schwinger, 1948



$$\Rightarrow \; a_\mu^{\mathrm{QED}(1)} = \frac{\alpha}{2\pi} \,,$$

α QED fine structure constant

The leading order hadronic contribution, $a_\mu^{\rm had}$



- $a_\mu^{\rm had}$ contributes almost 60% of theoretical error

- needs to be computed non-perturbatively

- computation of vacuum polarization tensor

$$\Pi_{\mu\nu}(q) = (q_\mu q_\nu - q^2 g_{\mu\nu})\Pi(q^2)$$

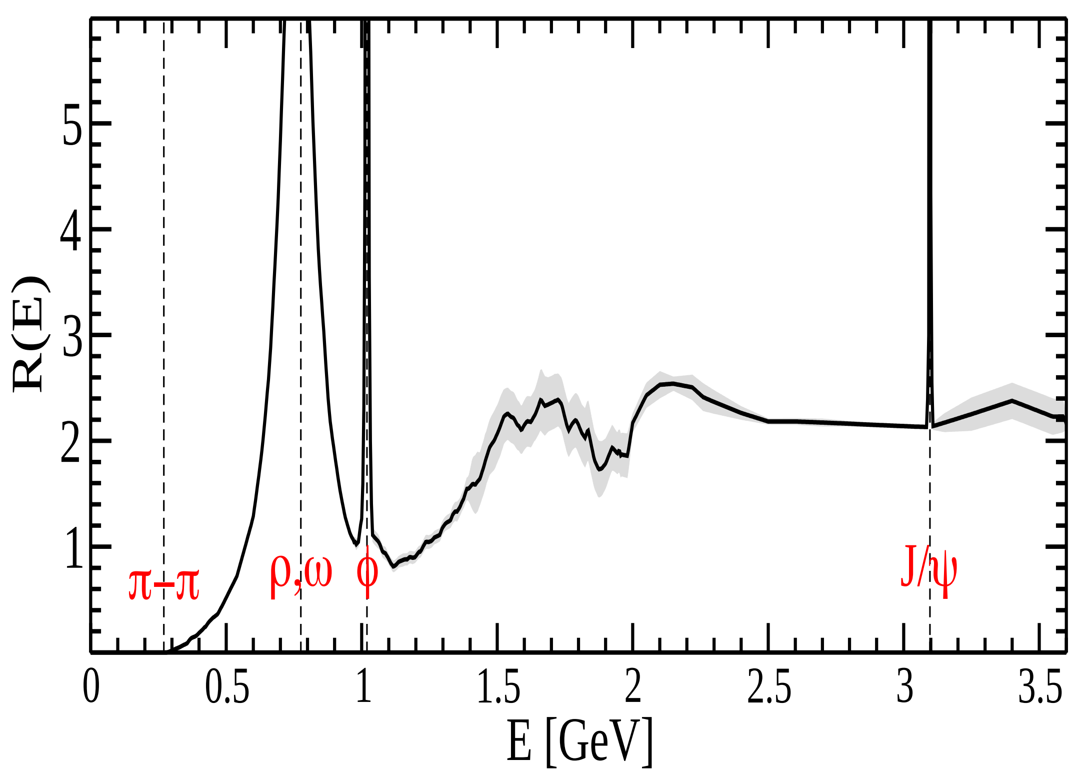
Relation to experimental extraction

connection between real and imaginary part of $\Pi(q^2)$

$$\Pi(q^2) - \Pi(0) = \frac{q^2}{\pi} \int_0^\infty ds \frac{\mathrm{Im}\Pi(s)}{s(s-q^2)}$$

$\mathrm{Im}\Pi(s)$ related to experimental data of total cross section in e^+e^- annihilation

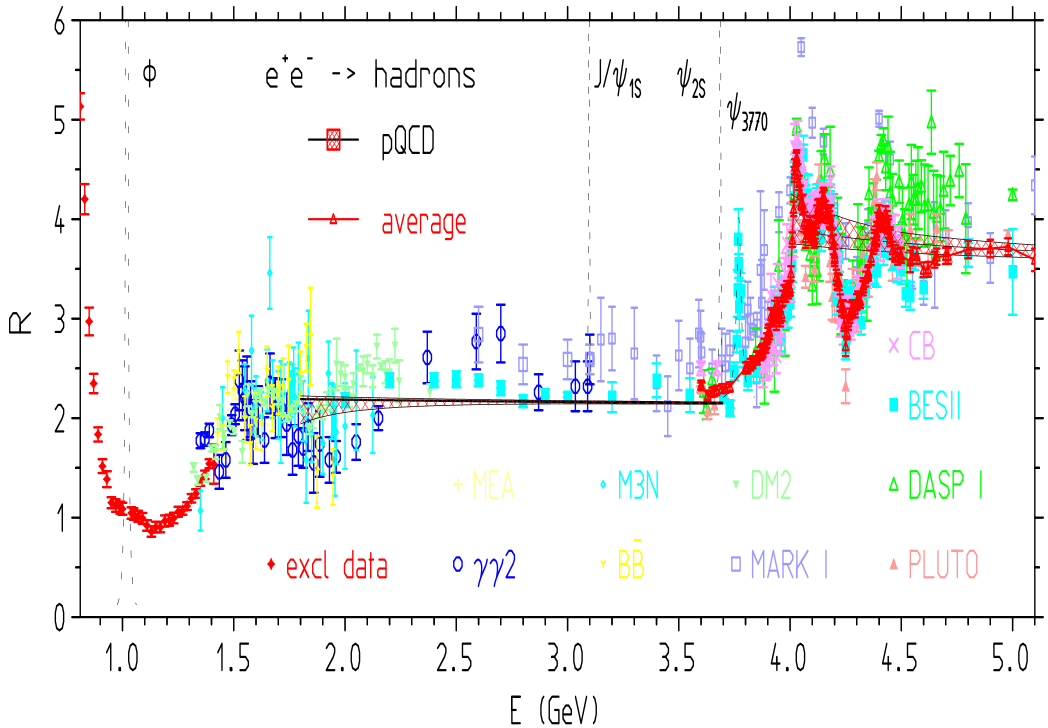
$$\mathrm{Im}\Pi(s) = \frac{\alpha}{3} R(s)$$



important contributions

- ρ, ω $(N_f = 2)$
- Φ $(N_f = 2+1)$
- J/Ψ $(N_f = 2+1+1)$

How the data really look like



- demanding analysis of $O(1000)$ channels
  Jegerlehner, Nyffeler, Phys.Rep.

Euclidean expression for hadronic contribution

$$a_\mu^{\text{had}} = \alpha^2 \int_0^\infty \frac{dQ^2}{Q^2} F\left(\frac{Q^2}{m_\mu^2}\right)\left(\Pi(Q^2) - \Pi(0)\right)$$

with $F\left(\frac{Q^2}{m_\mu^2}\right)$ a known function T. Blum

$\rightarrow$ need to compute vacuum polaization function

Continuum:

$$\Pi_{\mu\nu}(Q) = i\int d^4x e^{iQ\cdot(x-y)}\langle 0|TJ_\mu(x)J_\nu(y)|0\rangle$$

J_μ hadronic electromagnetic current

$$J_\mu(x) = \sum_f e_f \bar{\psi}_f(x)\gamma_\mu\psi_f(x) = \frac{2}{3}\bar{u}(x)\gamma_\mu u(x) - \frac{1}{3}\bar{d}(x)\gamma_\mu d(x) + \cdots$$

local vector current given is conserved

$$\partial_\mu J_\mu(x) = 0 \qquad \Rightarrow \qquad Q_\mu \Pi(Q)_{\mu\nu} = 0$$

eliminating the factor $Q_\mu Q_\nu - Q^2\delta_{\mu\nu}$

$\Rightarrow$ obtain $\Pi(Q^2)$

Going to the lattice

- work with $N_f = 2$ **twisted mass fermions**

$$\mathcal{S}_{tm} = \sum_x \bar{\chi}(x) \left[D_W + m_0 + i\mu\gamma_5\tau_3\right] \chi(x)$$

- use the conserved lattice current

Exercize: derive conserved current

use vector transformation

$$\delta_V \chi(x) = i\epsilon_V(x)\tau\chi(x) \;, \quad \delta_V \bar{\chi}(x) = -i\bar{\chi}(x)\tau\epsilon_V(x)$$

$$\tau = \begin{pmatrix} 2/3 & 0 \\ 0 & -1/3 \end{pmatrix} = \tfrac{1}{6}\mathbf{1} + \tfrac{1}{2}\tau^3$$

Can we also use the local current?

Conserved lattice current

$$J_\mu^{tm}(x) = \frac{1}{2}\left(\bar{\chi}(x)\tau(\gamma_\mu - r)U_\mu(x)\chi(x+\hat{\mu}) + \bar{\chi}(x+\hat{\mu})\tau(\gamma_\mu + r)U_\mu^\dagger(x)\chi(x)\right)$$

satisfying

$$\partial_\mu^* J_\mu^{tm}(x) = 0$$

with ∂_μ^* backward lattice derivative

Lattice vacuum polarization tensor

Fourier transformation of conserved vector current:

$$J_\mu^{tm}(\hat{Q}) = \sum_x e^{iQ\cdot(x+\hat{\mu}/2)} J_\mu^{tm}(x) \ , \quad \hat{Q}_\mu = 2\sin\left(\frac{Q_\mu}{2}\right)$$

leading to

$$\Pi_{\mu\nu}(\hat{Q}) = \frac{1}{V}\sum_{x,y} e^{iQ\cdot(x+\hat{\mu}/2-y-\hat{\nu}/2)} \langle J_\mu^{tm}(x) J_\nu^{tm}(y)\rangle$$

from which we extract the vacuum polarization function

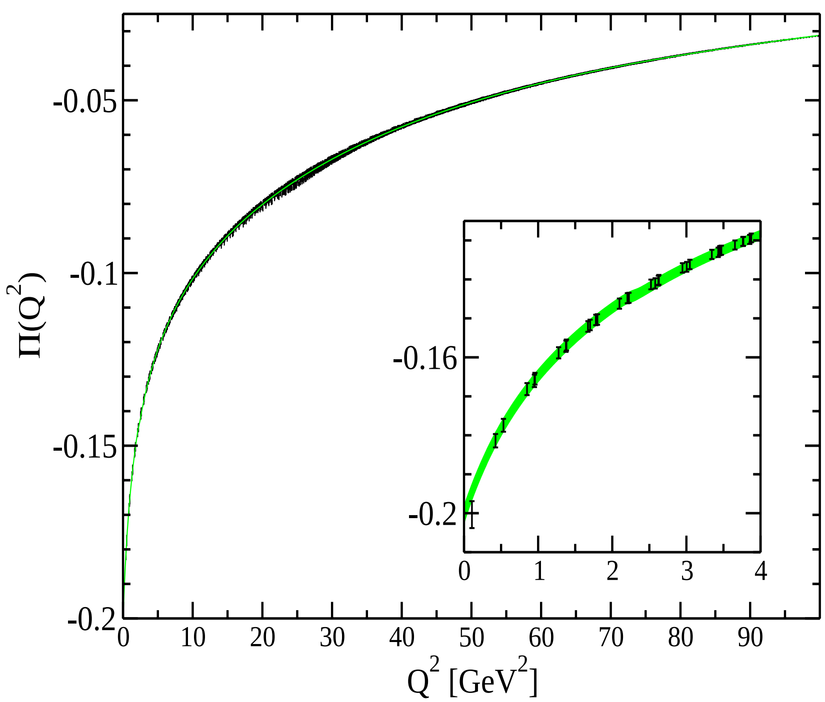
$$\Pi_{\mu\nu}(\hat{Q}) = (\hat{Q}_\mu \hat{Q}_\nu - \hat{Q}^2 \delta_{\mu\nu})\Pi(\hat{Q}^2)$$

Fit to vacuum polarization function

Fit function

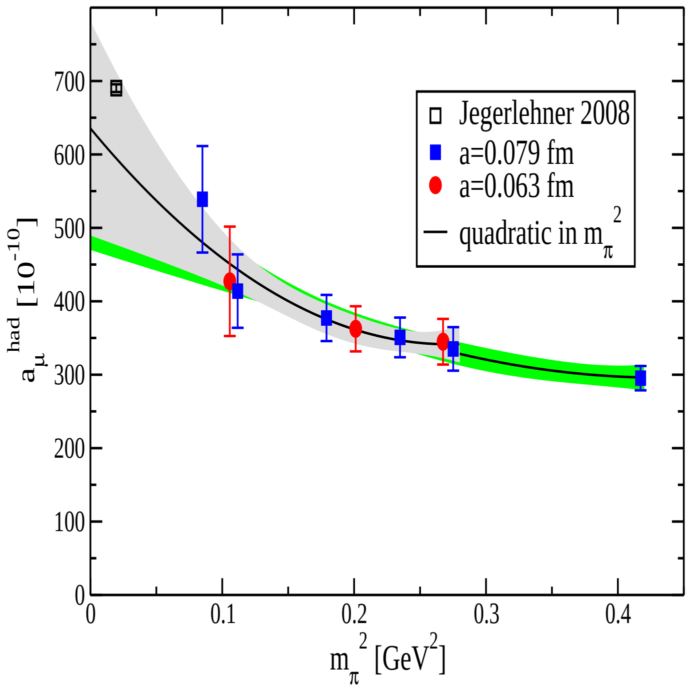
$$\Pi_{M,N}(Q^2) = -\frac{5}{9}\sum_{i=1}^{M}\frac{m_i^2}{Q^2+m_i^2} + \sum_{n=0}^{N} a_n (Q^2)^n$$

typically $i = 1, 2, 3$, $n = 0, 1, 2, 3$ (systematic error)



Do we control hadronic vacuum polarisation?

(Xu Feng, Dru Renner, Marcus Petschlies, K.J.; Lattice 2010)



- experiment: $a_{\mu,N_f=2}^{\text{hvp,exp}} = 5.66(05)10^{-8}$

- lattice: $a_{\mu,N_f=2}^{\text{hvp,old}} = 2.95(45)10^{-8}$

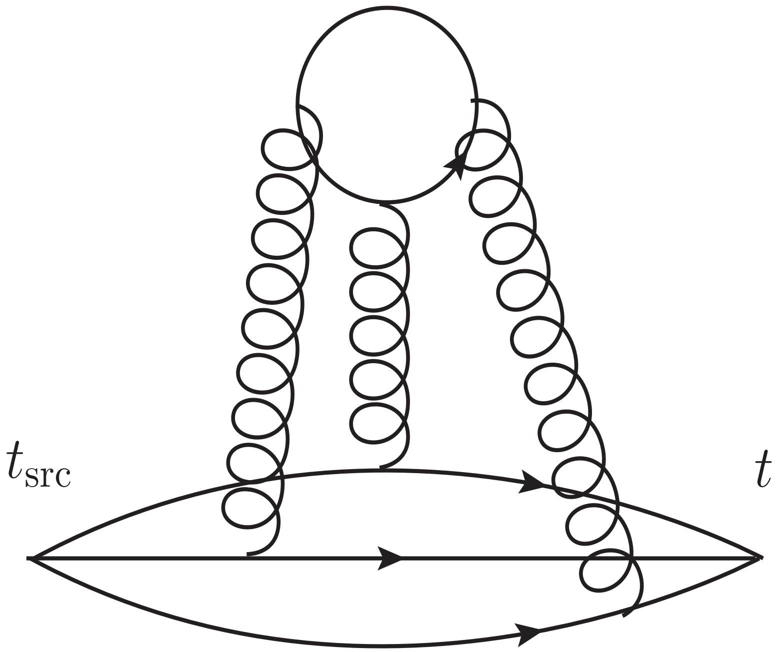
$\rightarrow$ misses the experimental value
$\rightarrow$ order of magnitude larger error

- have used different volumes

- have used different values of lattice spacing

Dis-connected contribution

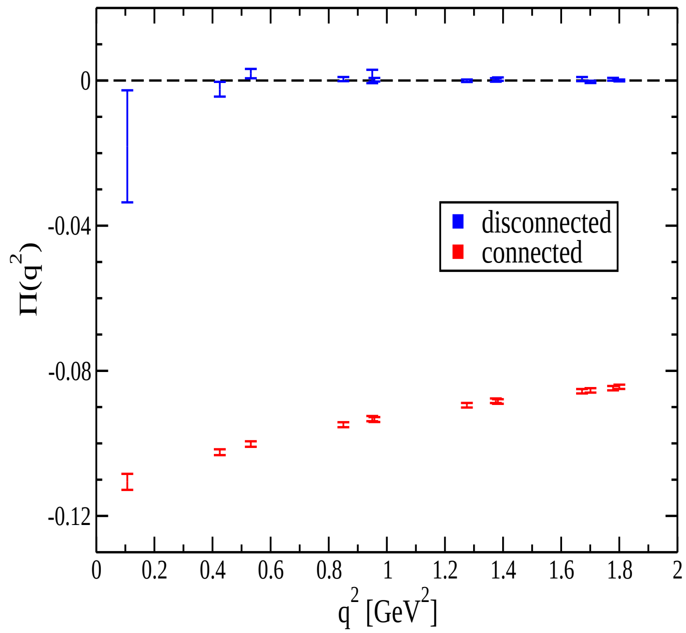
a graph representing dis-conected contributions



$\rightarrow$ has been basically always be neglected

Can it be the dis-connected contribution?

(Xu Feng, Dru Renner, Marcus Petschlies, K.J.)



- dedicated effort
- have included dis-connected contributions for first time
- smallness consistent with chiral perturbation theory (Della Morte, Jüttner)

Different extrapolation to the physical point

lattice: simulations at unphysical quark masses, demand only

$$\lim_{m_{\rm PS}\to m_\pi} a_l^{\rm hvp,latt} = a_l^{\rm hvp,phys}$$

$\Rightarrow$ flexibility to define $a_l^{\rm hvp,latt}$

standard definitions in the continuum

$$a_l^{\rm hvp} = \alpha^2 \int_0^\infty dQ^2 \frac{1}{Q^2}\omega(r)\Pi_R(Q^2)$$

$$\Pi_R(Q^2) = \Pi(Q^2) - \Pi(0)$$

$$\omega(r) = \frac{64}{r^2\left(1+\sqrt{1+4/r}\right)^4\sqrt{1+4/r}}$$

with $r = Q^2/m_l^2$

Redefinition of $a_l^{\mathrm{hvp,latt}}$

redefinition of r for lattice computations

$$r_{\mathrm{latt}} = Q^2 \cdot \frac{H^{\mathrm{phys}}}{H}$$

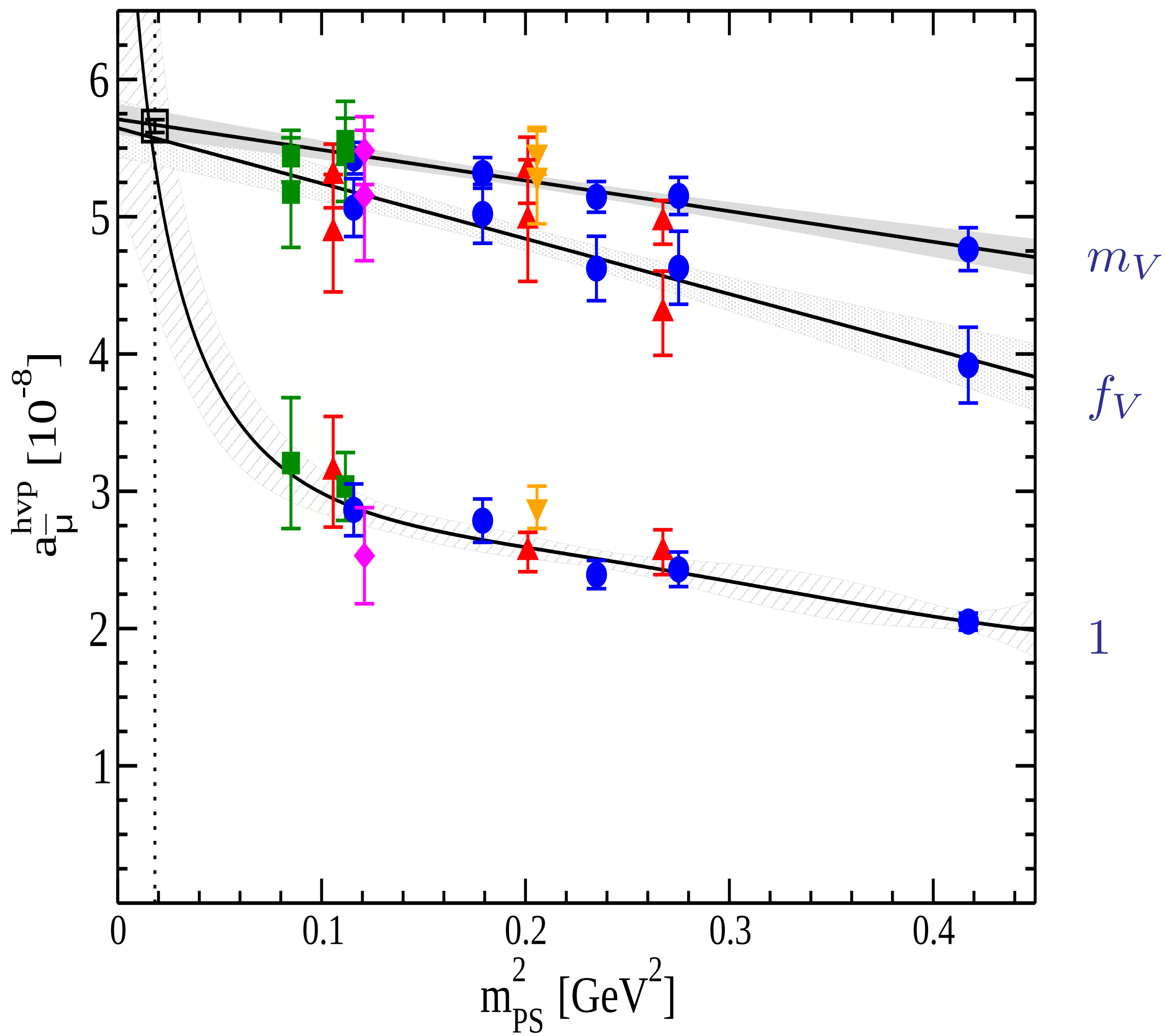
choices

- r_1: $H = 1$; $H^{\mathrm{phys}} = 1/m_l^2$
- r_2: $H = m_V^2(m_{\mathrm{PS}})$; $H^{\mathrm{phys}} = m_\rho^2/m_l^2$
- r_3: $H = f_V^2(m_{\mathrm{PS}})$; $H^{\mathrm{phys}} = f_\rho^2/m_l^2$

each definition of r will show a different dependence on m_{PS} but agree *by construction* at the physical point

remark: strategy often used in continuum limit extrapolations, e.g. charm quark mass determination

comparison using r_1, r_2, r_3



A new result from the lattice

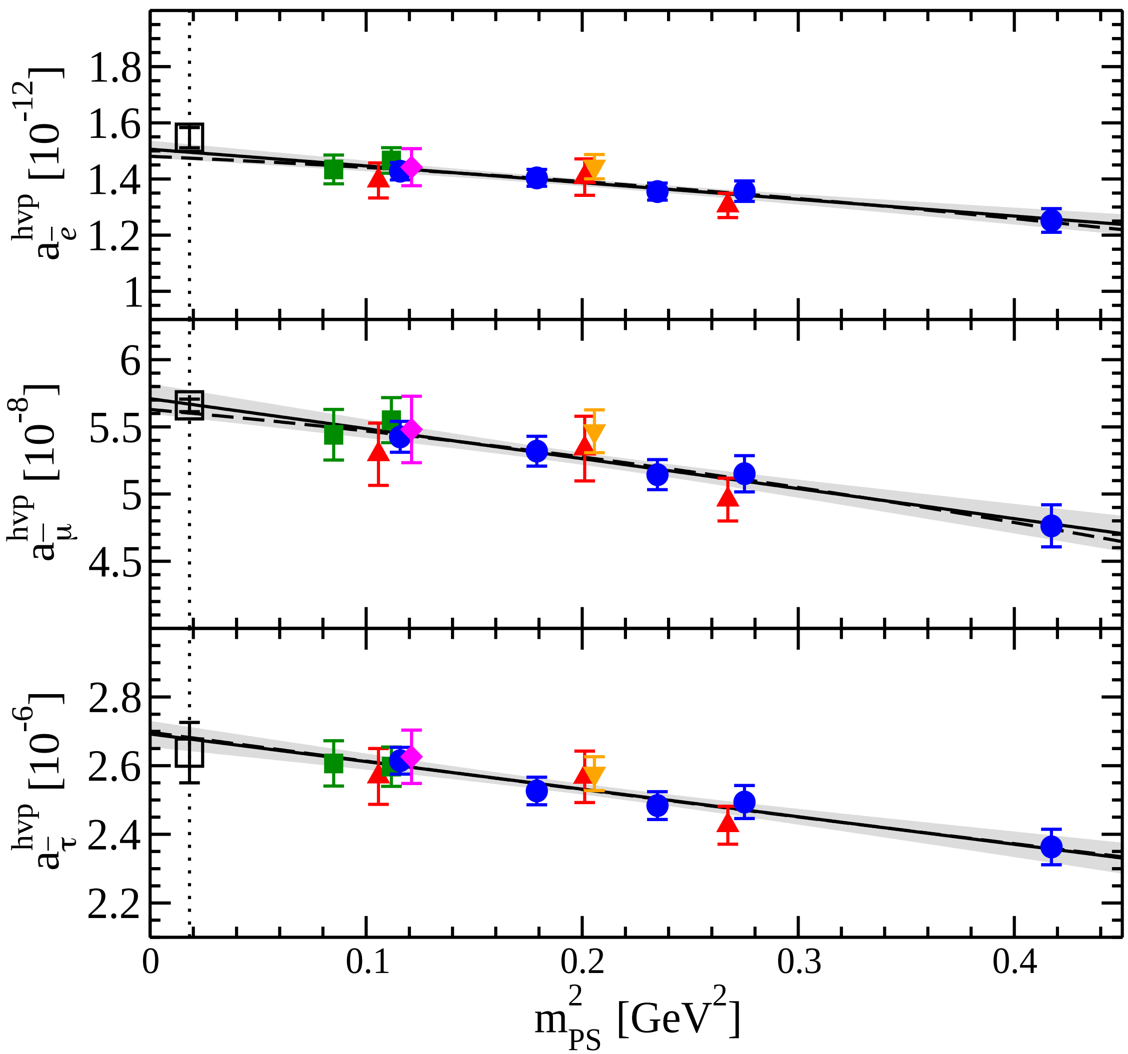
- experimental value: $a_{\mu,N_f=2}^{\mathrm{hvp,exp}} = 5.66(05)10^{-8}$
- from our old analysis: $a_{\mu,N_f=2}^{\mathrm{hvp,old}} = 2.95(45)10^{-8}$

$\longrightarrow$ misses the experimental value
$\longrightarrow$ order of magnitude larger error

- from our new analysis: $a_{\mu,N_f=2}^{\mathrm{hvp,new}} = 5.66(11)10^{-8}$

$\longrightarrow$ error (including systematics) almost matching experiment

Anomalous magnetic moments, a check

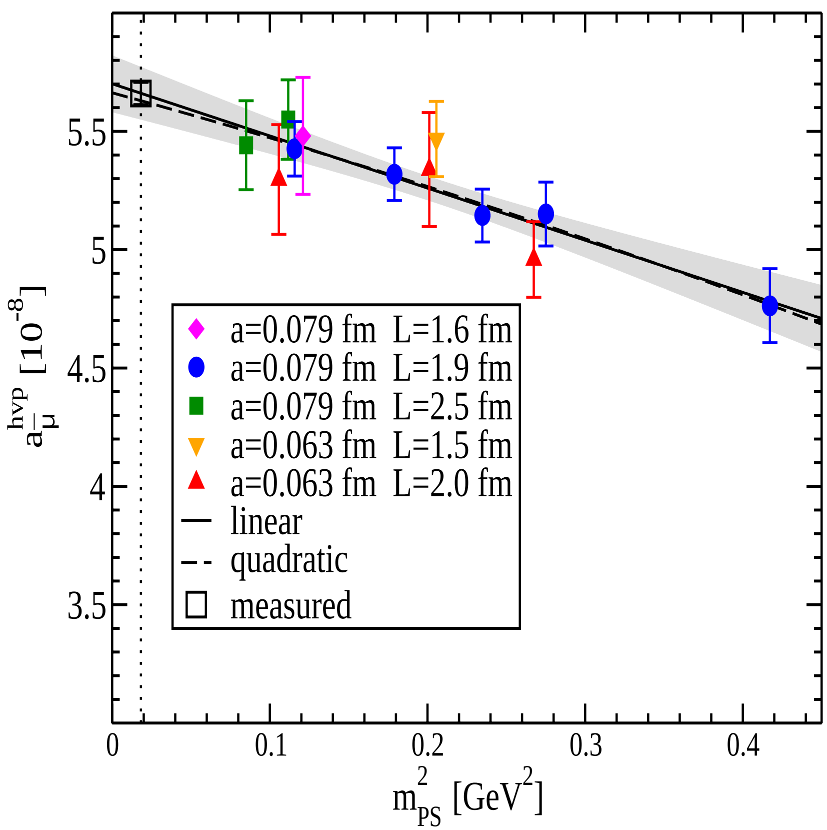


$a_e^{\rm hlo} = 1.513\,(43)\cdot 10^{-12}$ LQCD
$a_e^{\rm hlo} = 1.527\,(12)\cdot 10^{-12}$ PHENO

$a_\mu^{\rm hlo} = 5.72\,(16)\cdot 10^{-8}$ LQCD
$a_\mu^{\rm hlo} = 5.67\,(05)\cdot 10^{-8}$ PHENO

$a_\tau^{\rm hlo} = 2.650\,(54)\cdot 10^{-6}$ LQCD
$a_\tau^{\rm hlo} = 2.638\,(88)\cdot 10^{-12}$ PHENO

anomalous magnetic moment of muon



- have used different volumes
- have used different values of lattice spacing
- have included dis-connected contributions

⇒ can control systematic effects

Why it works: fitting the Q^2 dependence

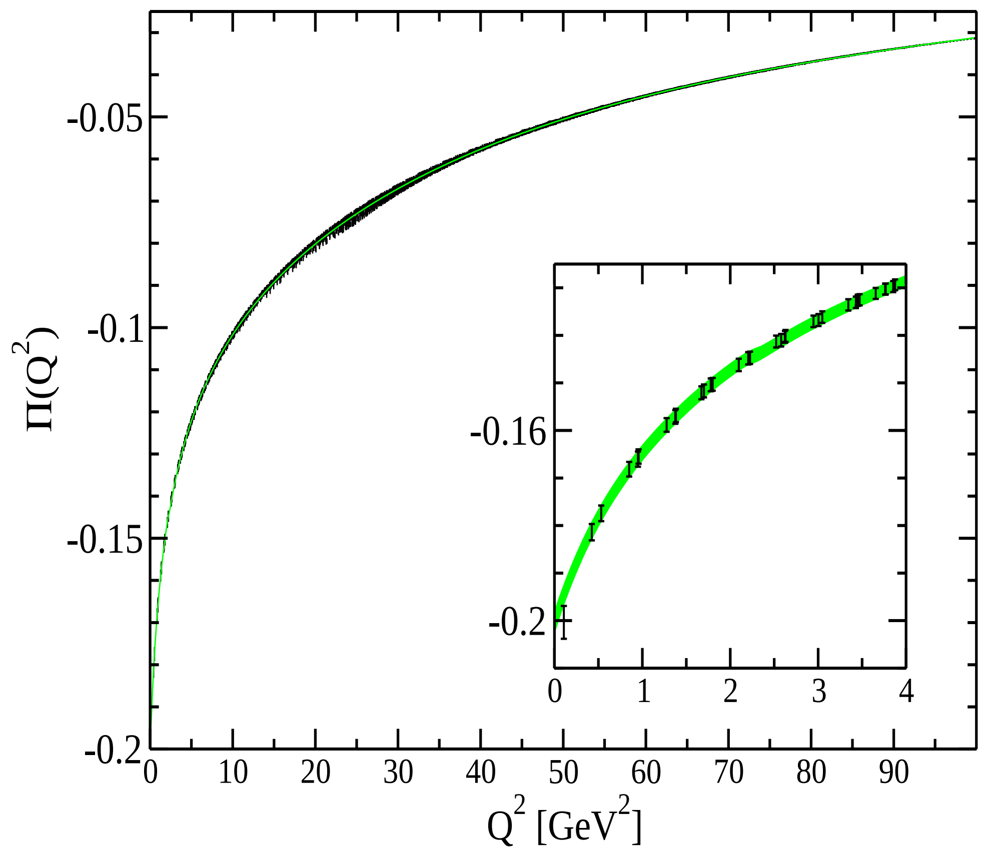
Fit function

$$\Pi_{M,N}(Q^2) = -\frac{5}{9}\sum_{i=1}^{M}\frac{m_i^2}{Q^2+m_i^2} + \sum_{n=0}^{N} a_n (Q^2)^n$$

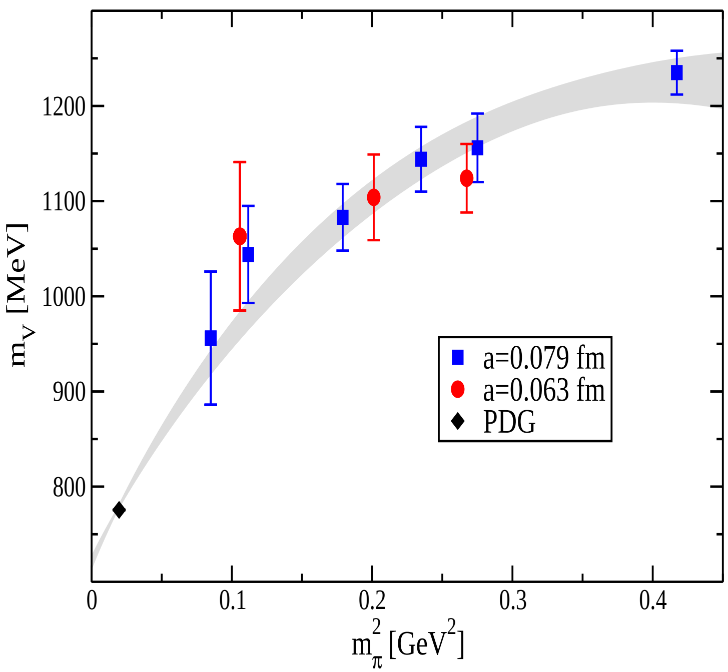
$i = 1$: ρ-meson $\rightarrow$ dominant contribution $\propto 5.010^{-8}$

$i = 2$: ω-meson $\propto 3.710^{-9}$

$i = 3$: ϕ-meson $\propto 3.410^{-9}$

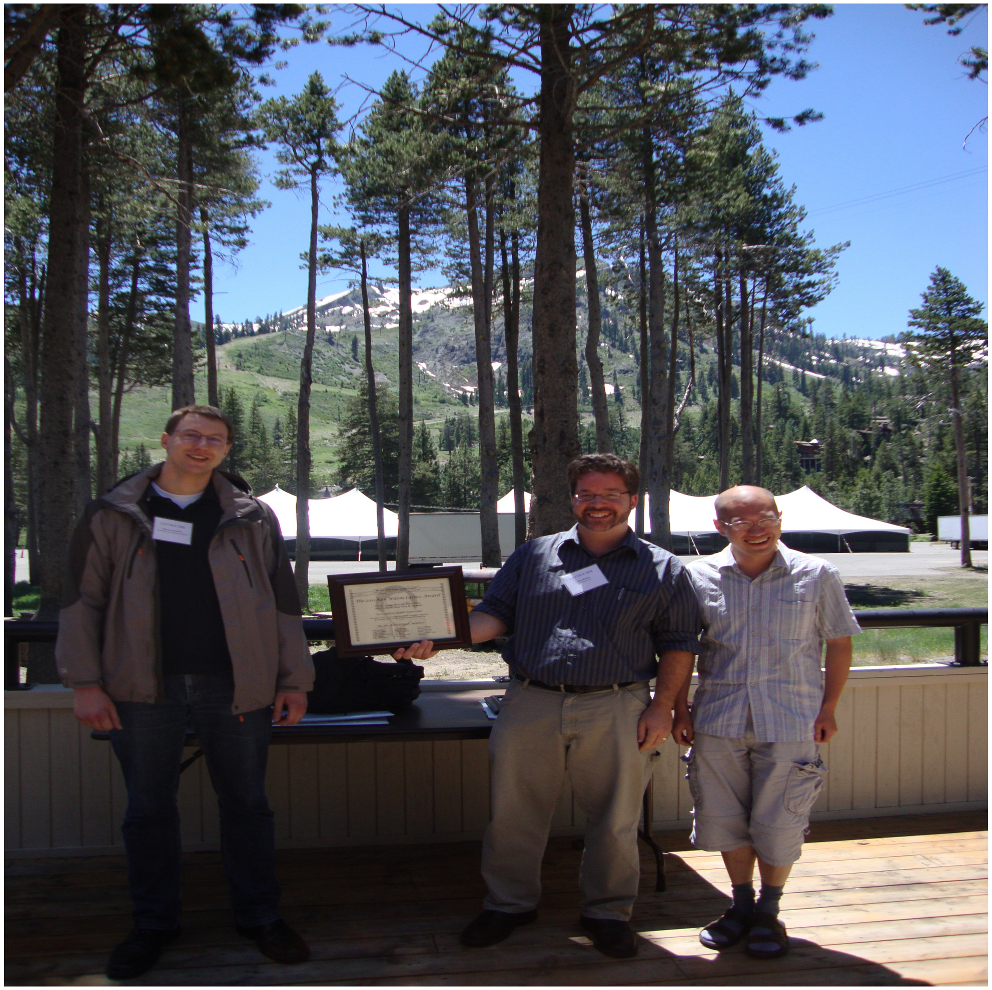


Why it works



- m_V consistent with resonance analysis
  (Feng, Renner, K.J.)
- strong dependence on $m_{\rm PS}$

Ken Wilson award



Next steps

Fit function

$$\Pi_{M,N}(Q^2) = -\frac{5}{9}\sum_{i=1}^{M}\frac{m_i^2}{Q^2+m_i^2} + \sum_{n=0}^{N} a_n (Q^2)^n$$

add $i = 4$: J/Ψ, $i = 5 \ldots$

- **ETMC** is performing simulations wit dynamical **up, down, strange and charm quarks**
  $\rightarrow$ unique opportunity

- avoids ambiguity with experiment comparison (what counts for $N_f = 2$?)

- generalized boundary conditions: $\Psi(L + a\hat{\mu}) = e^{i\theta}\Psi(x)$
  $\rightarrow$ θ *continuous* momentum
  $\rightarrow$ allows to realize arbitrary momenta on the lattice

INT Workshop on
The Hadronic Light-by-Light Contribution to the Muon Anomaly
February 28 - March 4, 2011

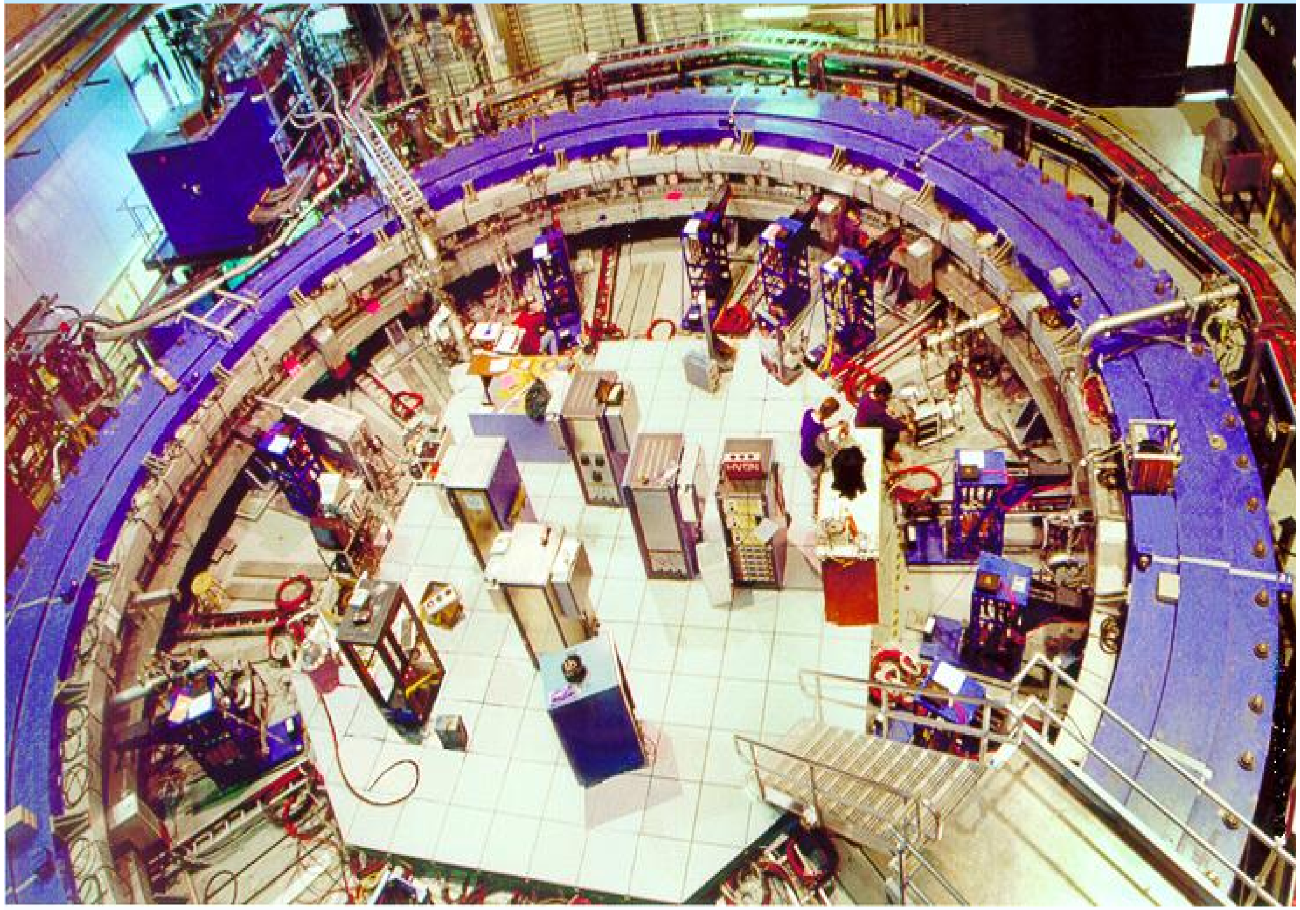
Some sildes from

Lee Roberts (g-2) collaboration

- new experiment **E989**
- move Brookhaven equipment to Fermilab

What is moved

muon (g - 2) storage ring



How and where is it moved

Sikorsky S64F 12.5 T hook weight (Outer coil/cryostat 8T)

- Transport coils to and from barge via Sikorsky aircrane
- Ship through St Lawrence -> Great Lakes -> Calumet SAG
- Subsystems can be transported overland, but probably more cost effective to ship steel on barge as well.

What is expected

Fermilab E989: Approved January 2011

- Re-locate the $(g - 2)$ storage ring to Fermilab
- Use the many proton storage rings to form the ideal proton beam
- Use one of the antiproton rings as a 900 m decay line to produce a pure muon beam
- Accumulate 21 times the statistics
- Improve the systematic errors
- Final goal: At least a factor of 4 more precise over E821

What can be achieved

E821

$$\left.\begin{array}{l}\sigma_{\rm stat} = \pm 0.46 \text{ ppm} \\ \sigma_{\rm syst} = \pm 0.28 \text{ ppm}\end{array}\right\} \sigma = \pm 0.54 \text{ ppm}$$

$$a_\mu^{exp} = 116\,592\,089(63) \times 10^{-11}$$

$$a_\mu^{SM} = 116\ 591\ 793 \pm 51$$

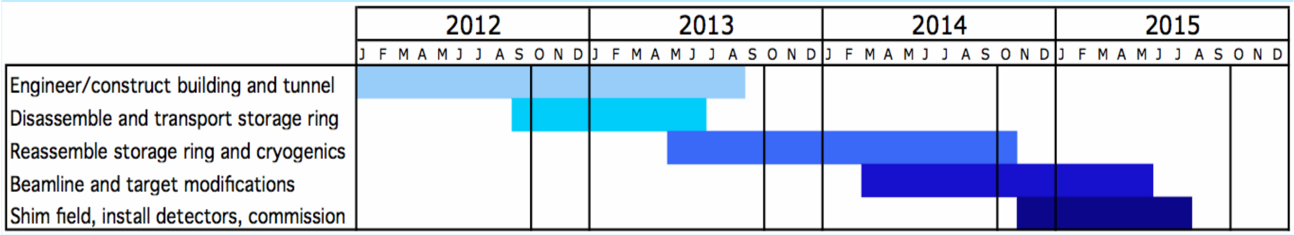
E989

$$\left.\begin{array}{l}\sigma_{\rm stat} = \pm 0.1 \text{ ppm} \\ \sigma_{\rm syst} = \pm 0.1 \text{ ppm}\end{array}\right\} \sigma = \pm 0.14 \text{ ppm}$$

$$a_\mu^{exp} = 116\,59x\,xxx(16) \times 10^{-11}$$

When will it happen

Timeline presented to DOE this week



The accuracy question

We need a precision $< 1\%$

- include explicit isospin breaking
- include electromagnetism
- need computation of light-by-light contribution
- reach small quark mass $\rightarrow$ physical point

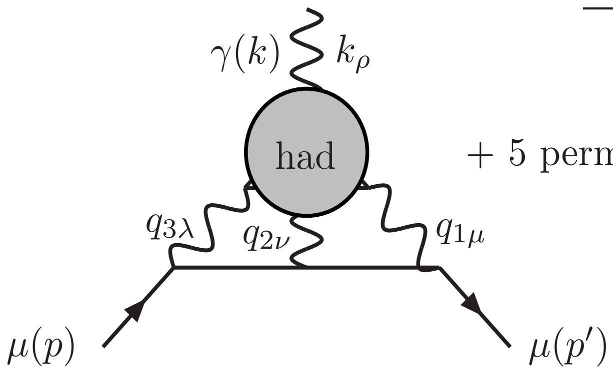
Much ado for young people

Simulation setup for $N_f = 2+1+1$
Configurations available through ILDG

β	a[fm]	L^3T/a^4	m_π[MeV]	status
1.9	≈ 0.085	$24^3 48$	300 – 500	ready
1.95	≈ 0.075	$32^3 64$	300 –500	ready
2.0	≈ 0.065	$32^3 64$	300	ready
2.1	≈ 0.055	$48^3 96$	300 – 500	running/ready
		$64^3 128$	230	thermalizing
		$64^3 128$	200	queued
		$96^3 192$	160	planned

- trajectory length always one
- 1000 trajectores for thermalization
- $\geq$ 5000 trajectores for measurements

Next, α_s^3, contribution



$+$ 5 permutations of the q_i

termed: *light-by-light scattering*

involves 4-point function

$$\Pi_{\mu\nu\alpha\beta}(q_1, q_2, q_3) = \int_{xyz} e^{iq_1\cdot x + iq_2\cdot y + iq_3\cdot z} \langle j_\mu(0) j_\nu(x) j_\alpha(y) j_\beta(z)\rangle$$

j_μ electromagnetic quark current

$$j_\mu = \tfrac{2}{3}\bar{u}\gamma_\mu u - \tfrac{1}{3}\bar{d}\gamma_\mu d - \tfrac{1}{3}\bar{s}\gamma_\mu s + \tfrac{2}{3}\bar{c}\gamma_\mu c$$

Momentum sources

(Alexandrou, Constantinou, Korzec, Panagopoulos, Stylianou)

$\leftarrow$ following Göckeler et.al.

for renormalization: need Green function in momentum space

$$G(p) = \frac{a^{12}}{V} \sum_{x,y,z,z'} e^{-ip(x-y)} \langle u(x)\overline{u}(z)\mathcal{J}(z,z')d(z')\overline{d}(y)\rangle$$

e.g. $\mathcal{J}(z,z')=\delta_{z,z'}\gamma_\mu$ corresponds to local vector current

sources:

$$b^a_\alpha(x) = e^{ipx}\delta_{\alpha\beta}\delta_{ab}$$

solve for

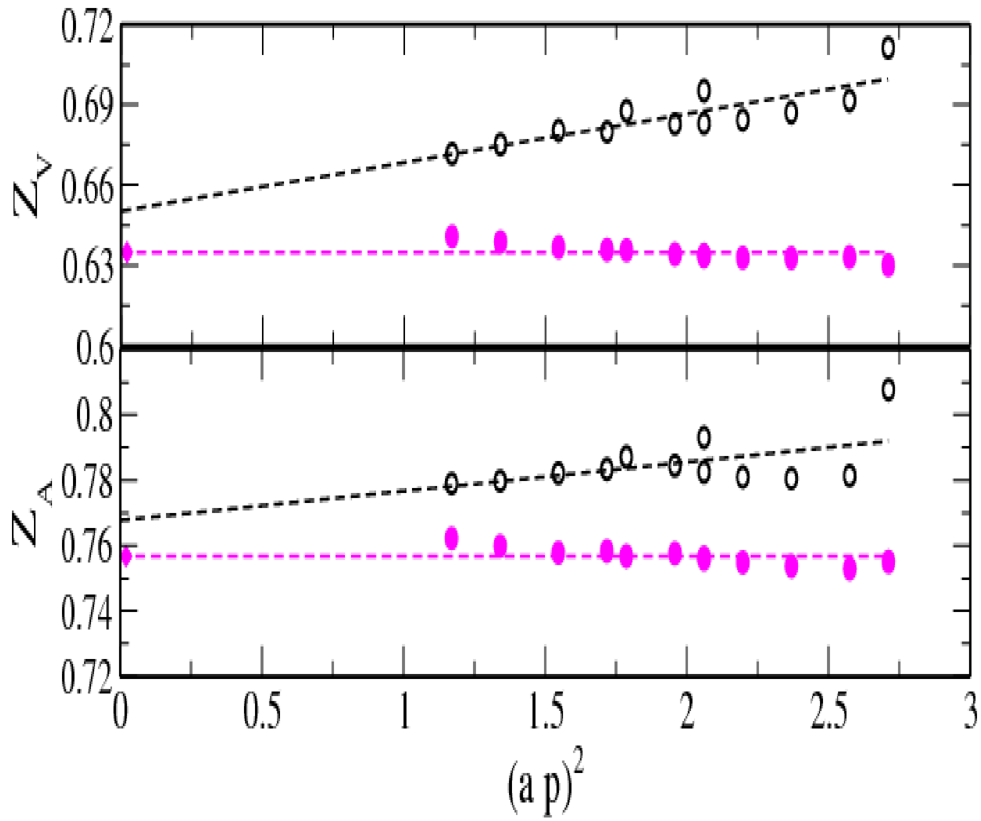
$$D_{\mathrm{latt}}G(p) = b$$

Advantage: very high, sub-percent precision data (only moderate statistics)

Disadvantage: need inversion for each momentum separately

$\rightarrow$ would need 3V inversions ...

Illustration of precision



with $O(a^2)$ effects

$O(a^2)$ effects pert. subtracted

with $O(a^2)$ effects

$O(a^2)$ effects pert. subtracted

use the momentum source method to attack the 4-point function as needed for light-by-light scattering (P. Rakow et.al., lattice'08)

Summary

- lattice calculation of muon anomalous magnetic moment
- looked hopeless first $\leftarrow$ order of magnitude larger error than experiment
- introduced new method $\rightarrow$ start to match experimental accuracy
- outlook
  - include first two quark generations
  - include isospin breaking and electromagnetism
  - attack light-by-light scattering