Analysis of the Schrödinger functional with chirally rotated boundary conditions

A. Shindler

University of Liverpool

Work done in collaboration with K. Jansen and J. Gonzalez Lopez

Glasgow University – September 29, 2008

Outline

- Motivation
- QCD and tmQCD actions
- Schrödinger functional boundary conditions
  - Standard (SF) and twisted SF I (tSF) boundary conditions
  - Twisted SF II (γ_5SF) boundary conditions
- Eigenvalue problem
  - SF and tSF boundary conditions
  - γ_5SF boundary conditions
- Quark propagator in the continuum at tree level of PT
  - SF and tSF boundary conditions
  - γ_5SF boundary conditions
- Conclusions and outlook

Motivating *twisted* Schrödinger functional b.c.

- Twisted mass lattice QCD simulations:
  - want mass independent non-perturbative renormalization for $N_f = 2+1+1$
  - need massless scheme - simulations close to or even at the chiral point
  - keep automatic O(a) improvement (R. Frezzotti and G.C. Rossi, hep-lat/0306014):

$$\Longrightarrow \langle \mathcal{O} \rangle_{\rm latt} = \langle \mathcal{O} \rangle_{\rm cont} + O(a^2)$$

- Why the Schrödinger functional scheme? (M. Lüscher et al., hep-lat/9207009)
  - Finite volume scheme $\Longrightarrow$ 'easy' simulations
  - Allows non-perturbative massless renormalization of QCD if suitable b.c. are chosen (non-zero bound)
  - Finite size techiques: non-perturbative running with renormalization scale $\frac{1}{L}$
- Why to twist the SF?
  - Wilson-type fermions with SF: bulk $O(a)$ effects
- Twisted SF I: tSF (S. Sint, hep-lat/0511034)
- Twisted SF II: γ_5SF (R. Frezzotti and G.C. Rossi, hep-lat/0507030)

Continuum fermion QCD and tmQCD actions

- Tree-level continuum QCD action with $N_f = 2$ mass degenerate quarks (Euclidean)

$$S_F[\bar{\psi}, \psi] = \int d^4x \, \bar{\psi}(x) \, D \, \psi(x) \quad ; \quad D := \gamma_\mu \partial_\mu + m$$

- Non-singlet axial rotation:

$$\psi(x) = e^{i\frac{\alpha}{2}\gamma_5\tau^3} \chi(x) \qquad \bar{\psi}(x) = \bar{\chi}(x) \, e^{i\frac{\alpha}{2}\gamma_5\tau^3}$$

- Tree-level continuum twisted mass QCD action

$$S_F^{tm}[\bar{\chi}, \chi] = \int d^4x \, \bar{\chi}(x) \, D_{tm} \, \chi(x) \quad ; \quad D_{tm} := \gamma_\mu \partial_\mu + m_q + i\mu_q\gamma_5\tau^3$$

QCD and tmQCD are equivalent in the continuum (same physics)

On the lattice at maximal twist ($\alpha = \frac{\pi}{2} \Leftrightarrow m_q = 0$)

automatic O(**a**) improvement

(R. Frezzotti and G.C. Rossi, hep-lat/0306014), (K. Cichy et al., arXiv:0802.3637)

SF boundaries: standard (SF) and twisted I (tSF)

- Dirichlet b.c. in the time direction (S. Sint, hep-lat/9312079, M. Lüscher, hep-lat/0603029)

$$P_+\psi(x)|_{x_0=0} = 0 \qquad P_-\psi(x)|_{x_0=T} = 0 \quad \text{via } \mathcal{T}$$

$$\bar{\psi}(x)P_-|_{x_0=0} = 0 \quad \text{via } \mathcal{C} \qquad \bar{\psi}(x)P_+|_{x_0=T} = 0 \quad \text{via } \mathcal{T} \text{ and } \mathcal{C}$$

$$P_\pm = \frac{1}{2}\left(1\!\!1 \pm \gamma_0\right)$$

- Maximally twisted SF (tSF) boundary conditions (S. Sint, hep-lat/0511034)

$$Q_+\chi(x)|_{x_0=0} = 0 \qquad Q_-\chi(x)|_{x_0=T} = 0 \quad \text{via } \mathcal{T}_{\frac{\pi}{2}}$$

$$\bar{\chi}(x)Q_+|_{x_0=0} = 0 \quad \text{via } \mathcal{C} \qquad \bar{\chi}(x)Q_-|_{x_0=T} = 0 \quad \text{via } \mathcal{T}_{\frac{\pi}{2}} \text{ and } \mathcal{C}$$

$$Q_\pm = \frac{1}{2}\left(1\!\!1 \pm i\gamma_0\gamma_5\tau^3\right)$$

SF↔tSF related by the same non-singlet axial rotation as QCD↔tmQCD

SF: obeys the discrete symmetries of QCD ($\mathcal{C}$, $\mathcal{P}$ and $\mathcal{T}$)

tSF: obeys the discrete symmetries of twisted mass QCD ($\mathcal{C}$, $\mathcal{P}_{\frac{\pi}{2}}$ and $\mathcal{T}_{\frac{\pi}{2}}$)

SF boundaries: twisted II (γ_5SF)

(R. Frezzotti and G.C. Rossi, hep-lat/0507030)

$\Pi_+\phi(x)|_{x_0=0} = 0$ $\Pi_-\phi(x)|_{x_0=T} = 0$ via $\mathcal{T}$

$\bar{\phi}(x)\Pi_-|_{x_0=0} = 0$ via $\mathcal{C}_F^{1,2}$ $\bar{\phi}(x)\Pi_+|_{x_0=T} = 0$ via $\mathcal{T}$ and $\mathcal{C}_F^{1,2}$

$$\Pi_\pm = \frac{1}{2}\left(\mathbb{1} \pm \gamma_5\tau^3\right)$$

$\mathcal{CPT}$ is still symmetry

No continuum transformation between SF $\leftrightarrow$ γ_5SF

Discrete symmetries different from both QCD and tmQCD

Check for the existence of a lower bound of the eigenvalue spectrum: SF and tSF

(S. Sint, hep-lat/9312079)

- Quadratic form of the action
- Well defined eigenvalue problem

Neumann b.c. for the complementary components

$$\text{SF}: \begin{cases} (\partial_0 - m)P_-\psi(x)|_{x_0=0} = 0, & (\partial_0 + m)P_+\psi(x)|_{x_0=T} = 0 \\ \bar{\psi}(x)P_+(\partial_0 + m)|_{x_0=0} = 0, & \bar{\psi}(x)P_-(\partial_0 - m)|_{x_0=T} = 0 \end{cases}$$

$$\text{tSF}: \left\{ (\partial_0 - \mu_q)Q_-\chi(x)|_{x_0=0} = 0, \quad (\partial_0 + \mu_q)Q_+\chi(x)|_{x_0=T} = 0 \right.$$

Form of the boundaries

$D^\dagger D$ has a discrete spectrum with a lower non-zero bound

$$\{\lambda_0^2(m=0)\}_{\text{SF}} = \left(\frac{\pi}{2\text{T}}\right)^2 \qquad \{\lambda_0^2(\mu_q=0)\}_{\text{tSF}} = \left(\frac{\pi}{2\text{T}}\right)^2 + m_q^2$$

Check for the existence of a lower bound of the eigenvalue spectrum: γ_5SF

- Quadratic form of the action for each flavour
- Well defined eigenvalue problem

$$(m_q + i\mu_q\gamma_5\tau_3)\,\Pi_-\phi(x)|_{x_0=0} = 0$$

$$(m_q + i\mu_q\gamma_5\tau_3)\,\Pi_+\phi(x)|_{x_0=T} = 0$$

Non-zero mass: Dirichlet b.c. for complementary components

only trivial solution

Zero mass: possible non trivial solution

possible zero eigenvalue

- Independent from the form of the mass term

Quark propagator: SF and tSF b.c.

$$\mathrm{SF}: \begin{cases} D(x)\; S^{\mathrm{SF}}(x,y) = \delta^4(x-y) & 0 < x_0, y_0 < T \\ P_+ S^{\mathrm{SF}}(x,y)|_{x_0=0} = 0 & P_- S^{\mathrm{SF}}(x,y)|_{x_0=T} = 0 \end{cases}$$

$$\mathrm{tSF}: \begin{cases} D_{\mathrm{mtm}}(x)\; S^{\mathrm{tSF}}(x,y) = \delta^4(x-y) & 0 < x_0, y_0 < T \\ Q_+ S^{\mathrm{tSF}}(x,y)|_{x_0=0} = 0 & Q_- S^{\mathrm{tSF}}(x,y)|_{x_0=T} = 0 \end{cases}$$

Boundary conditions on the right side satisfied

$$S^{\mathrm{SF}}(x,y)P_-|_{y_0=0} = 0 \qquad S^{\mathrm{SF}}(x,y)P_+|_{y_0=T} = 0$$

$$S^{\mathrm{tSF}}(x,y)Q_+|_{y_0=0} = 0 \qquad S^{\mathrm{tSF}}(x,y)Q_-|_{y_0=T} = 0$$

Existence of unique and non-trivial propagator in both cases

$S^{\mathrm{SF}}(x,y) \leftrightarrow S^{\mathrm{tSF}}(x,y)$ by the non-singlet axial rotation

$$S^{\mathrm{tSF}}(x,y) = e^{-i\frac{\pi}{4}\gamma_5\tau^3}\; S^{\mathrm{SF}}(x,y)\, e^{-i\frac{\pi}{4}\gamma_5\tau^3}$$

Quark propagator: γ_5SF b.c.

$$D(x)\ S(x,y) = \delta^4(x-y) \qquad 0 < x_0, y_0 < T$$

$$\Pi_+ S(x,y)|_{x_0=0} = 0 \qquad \Pi_- S(x,y)|_{x_0=T} = 0$$

Boundary conditions on the right side NOT satisfied

$$S(x,y)\Pi_-|_{y_0=0} \neq 0 \qquad S(x,y)\Pi_+|_{y_0=T} \neq 0$$

only trivial solution

It satisfies the boundaries obtained by charge conjugation

$$S(x,y)\Pi_+|_{y_0=0} = 0 \qquad S(x,y)\Pi_-|_{y_0=T} = 0$$

$\mathcal{P}$ and $\mathcal{CPT}$ violation

Conclusions and Outlook

- Studied properties of three different ways to implement SF b.c. by:
  - eigenvalue spectrum
  - quark propagator
- Conclusion:
  - SF and tSF: sound definition of QCD with SF b.c.
  - γ_5SF: open questions, need further investigations
- Ongoing work with tSF b.c.:
  - already obtained the lattice tree-level propagator with tSF
  - first numerical checks at tree-level of PT
- Near future:
  - lattice perturbation theory
  - QCD simulations with $N_f = 4$

Acknowledgments

- In collaboration with
  - DESY — K. Jansen
  - HU-DESY — J. Gonzalez Lopez
- Thanks to
  - HU zu Berlin — M. Müller Preussker
  - Berlin — SFB-TR9
    Theory group at DESY, Zeuthen
  - Dublin — S. Sint
  - Rome — R. Frezzotti and G. C. Rossi