Nucleon form factors and GPDs

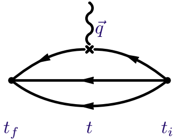
- calculation of (generalized) form factors
- G_E and G_M, G_A and G_p
- GPDs

ETMC collaboration meeting in Autrans, 18-20 March 2009

Matrix elements

We are interested in QCD matrix elements

$$\langle N(p_f, s_f)|\mathcal{O}|N(p_i, s_i)\rangle$$



where

- $|N(p_f, s_f)\rangle$, $|N(p_i, s_i)\rangle$ are nucleon states with final (initial) momentum $p_f(p_i)$ and spin $s_f(s_i)$
- $\mathcal{O}$ is an operator, e.g.
  - (conserved) vector current
  - axial current
  - operator emerging from OPE of a (non-local) light-cone operator

Form factors

QCD symmetries restrict the matrix element to a few possible terms. The q^2 dependent coefficients are the (generalized) form factors. In the vector-current case:

$$\begin{aligned}\langle N(p_f, s_f)|V_\mu(0)|N(p_i, s_i)\rangle &= \bar{u}(p_f, s_f)\mathcal{J}_\mu u(p_i, s_i)\\ \mathcal{J}_\mu &= \gamma_\mu F_1(q^2) + \frac{i\sigma_{\mu\nu}q^\nu}{2m_N}F_2(q^2)\end{aligned}$$

$q = p_f - p_i$ is the momentum transfer
F_1, F_2 are the Dirac and Pauli form factors.

$$\begin{aligned}G_E(q^2) &= F_1(q^2) - \frac{q^2}{(2m_N)^2}F_2(q^2)\\ G_M(q^2) &= F_1(q^2) + F_2(q^2)\end{aligned}$$

G_E, G_M are the electric and magnetic Sachs form factors.

(Generalized) form factors

Similarly for other operators

- $\bar{q}\gamma_\mu\gamma_5 q$
  $\Rightarrow$ Axial form factors G_A and G_p
- $\bar{q}\gamma_{\{\mu}\overleftrightarrow{D}_{\nu\}}q$
  $\Rightarrow$ Generalized form factors A_{20}, B_{20} and C_{20}
- $\bar{q}\gamma_5\gamma_{\{\mu}\overleftrightarrow{D}_{\nu\}}q$
  $\Rightarrow$ Generalized form factors $\tilde{A}_{20}$, $\tilde{B}_{20}$

In the limit of zero momentum transfer

- $G_E(q^2) \rightarrow 1$, charge
- $G_M(q^2) \rightarrow$ magnetic moment
- $G_A(q^2) \rightarrow g_A$, axial charge
- $A_{20}(q^2) \rightarrow \langle x \rangle_q$ 1st moment of density distribution
- $\tilde{A}_{20}(q^2) \rightarrow \langle x \rangle_{\Delta q}$ 1st moment of helicity distribution

Correlation functions

Measure two-point and three-point functions

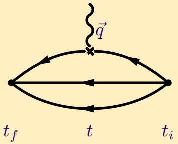
$$G(\vec{q},t) = \sum_{\vec{x}_f} e^{-i\vec{x}_f\cdot\vec{q}}\,\Gamma^0_{\beta\alpha}\,\langle J_\alpha(t_f,\vec{x}_f)\bar{J}_\beta(0)\rangle$$

$$G^\mu(\Gamma^\nu,\vec{q},t) = \sum_{\vec{x},\vec{x}_f} e^{i\vec{x}\cdot\vec{q}}\,\Gamma_{\beta\alpha}\,\langle J_\alpha(t_f,\vec{x}_f)\mathcal{O}^\mu(t,\vec{x})\bar{J}_\beta(0)\rangle$$

$\Gamma^0 = \frac{1}{4}(\mathbb{1}+\gamma_0)$ and Γ some 4×4 matrix

Kinematical setup

- Source at $t_i = 0$, $\vec{x} = 0$
- Sink at $t_f = 12$, $\vec{p}_f = 0$
- Operator $\mathcal{O}^\mu$ at t, $\vec{q} = -\vec{p}_i$



Ratios

Leading time dependence and overlap factors cancel in ratios:

$$R^\mu(t,\Gamma,\vec{q}) = \frac{G^\mu(\Gamma,\vec{q},t)}{G(\vec{0},t_f)} \sqrt{\frac{G(\vec{p}_i,t_f-t)G(\vec{0},t)G(\vec{0},t_f)}{G(\vec{0},t_f-t)G(\vec{p}_i,t)G(\vec{p}_i,t_f)}} \to \Pi^\mu(\Gamma,\vec{q})$$

If the separation between t_f and t_i sufficient: R^μ has a plateau
The plateau value is

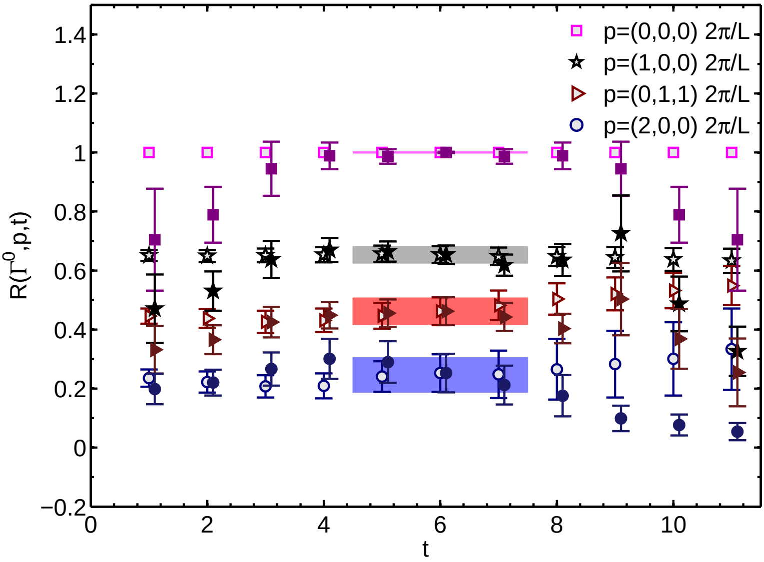
$$\Pi^\mu(\Gamma,\vec{q}) = \sqrt{\frac{2m^2}{E(E+m)}}\mathrm{tr}\left[\Gamma\frac{-i\not{p}_f+m}{2m}\mathcal{J}^\mu\frac{-i\not{p}_i+m}{2m}\right]$$

Types of sequential propagators

We do not want to make 16×24 inversions for all components of Γ instead chose a few useful combinations

- type 1: $\Gamma = \frac{1}{4}\left[\mathbb{1}+\gamma_0\right]$
- type 2: $\Gamma = \sum_{k=1}^{3}\frac{1}{4}\left[\mathbb{1}+\gamma_0\right]i\gamma_5\gamma_k$
- type 3: $\Gamma = \sum_{k=1}^{3}\left[\mathbb{1}+\gamma_k\right]$?

Plateaus



p=(0,0,0) 2π/L
p=(1,0,0) 2π/L
p=(0,1,1) 2π/L
p=(2,0,0) 2π/L
R(Γ^0,p,t)
t
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
0
2
4
6
8
10

Data analysis

- for a given q^2 calculate $\Pi^\mu(\Gamma, \vec{q})$ for all Γ, μ and $\vec{q}$. Call that vector b
- calculate the Jackknife errors of b: Δb
- get coefficients from trace algebra and summarize them in matrix A such that (example Axial FF)

$$\begin{array}{ccc} b & = & A & x \\ \begin{pmatrix} \uparrow \\ n \\ \downarrow \end{pmatrix} & = & \begin{pmatrix} & \uparrow & \\ \leftarrow & n \times 2 & \rightarrow \\ & \downarrow & \end{pmatrix} & \begin{pmatrix} G_A \\ G_p \end{pmatrix} \end{array}$$

- SVD: $A = U \Sigma V^\dagger$ therefore pseudo-inverse: $"A^{-1}" = V \Sigma^{-1} U^\dagger$
  $\Rightarrow$ solution in the least squares sense

$$\min_x \sum_i (A_{ij} x_j - b_i)^2$$

- or better: $b_i \rightarrow b_i / \Delta b_i$, $A_{ij} \rightarrow A_{ij} / \Delta b_i \Rightarrow$ minimize χ^2

Electric Sachs form factor

- iso-vector current
- $a = 0.089(1)\ fm$ from m_N
- $L = 2.13\ fm$ / $2.84 fm$
- $m_\pi = 447\ MeV$
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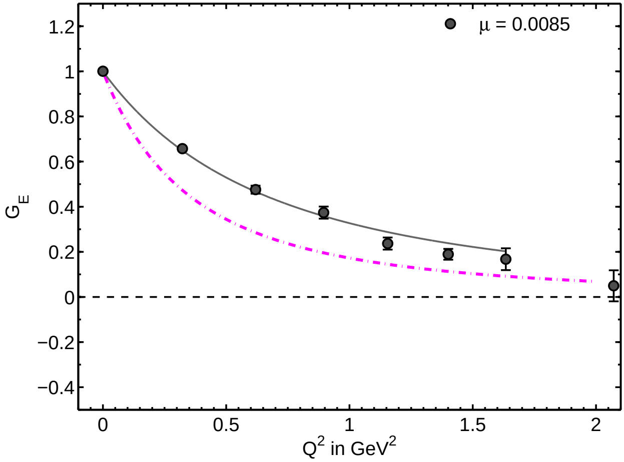

One parameter dipole fit: $G_E(Q^2) = 1/\left(1 + \frac{Q^2}{\Lambda_E^2}\right)^2$

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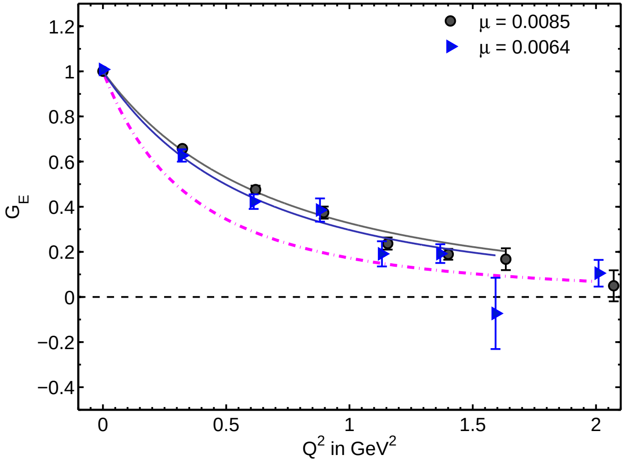

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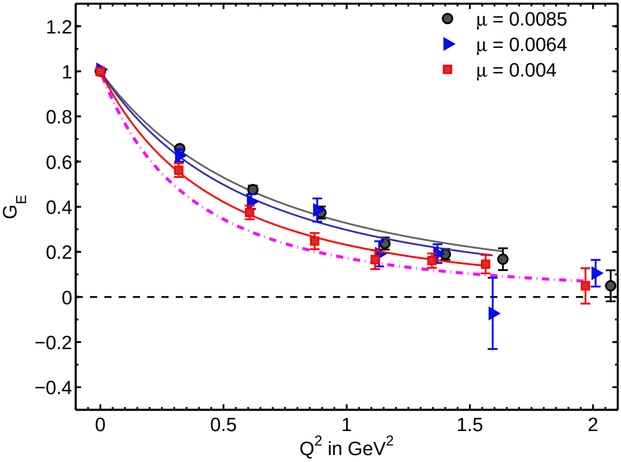

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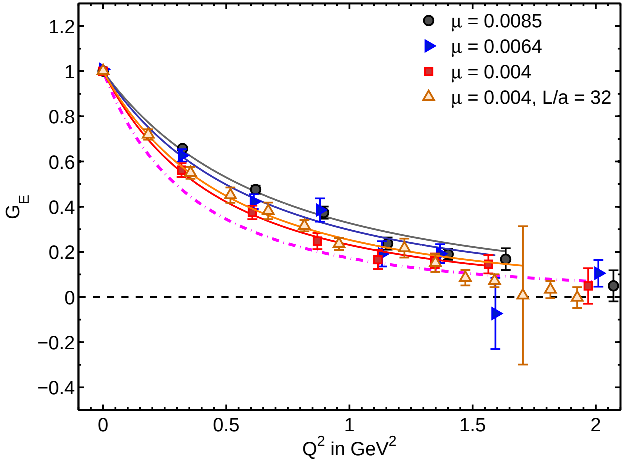

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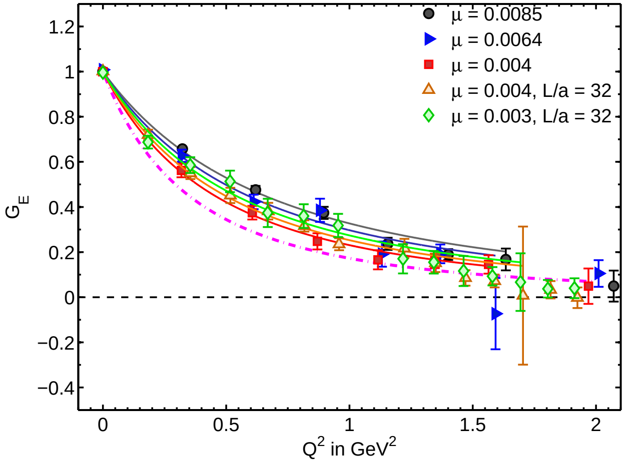

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Magnetic Sachs form factor

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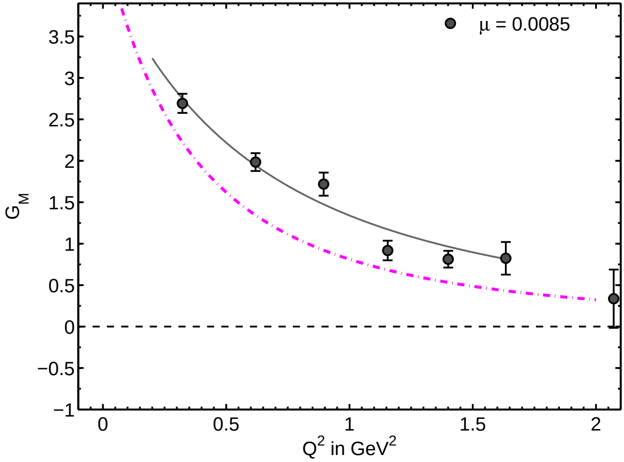

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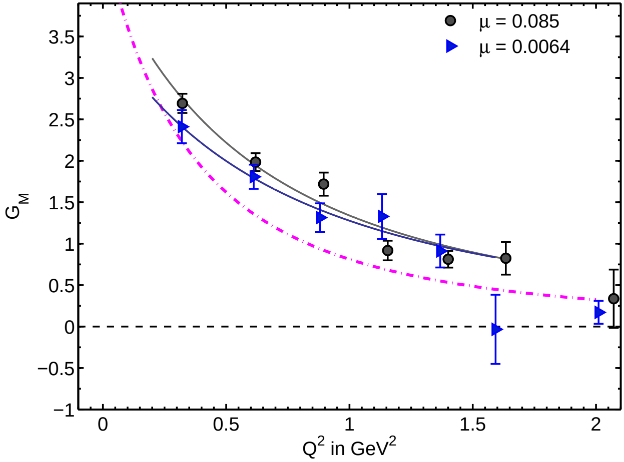

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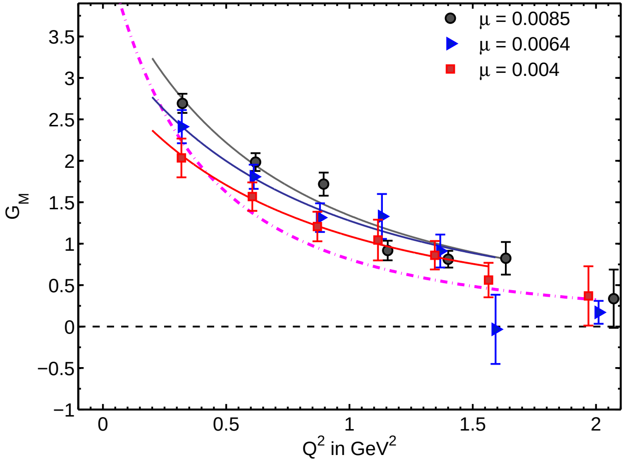

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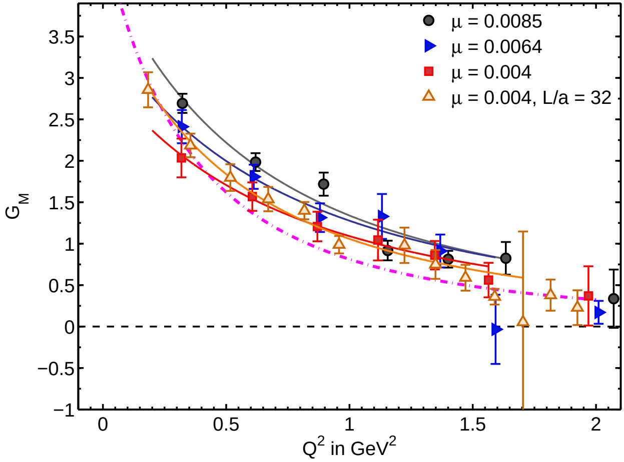

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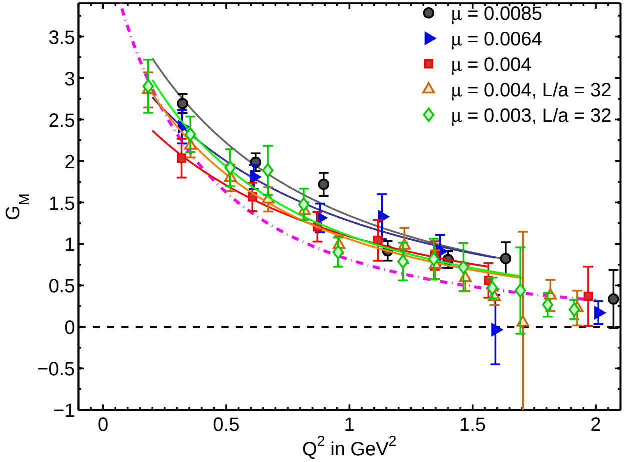

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Related observables

Magnetic moment

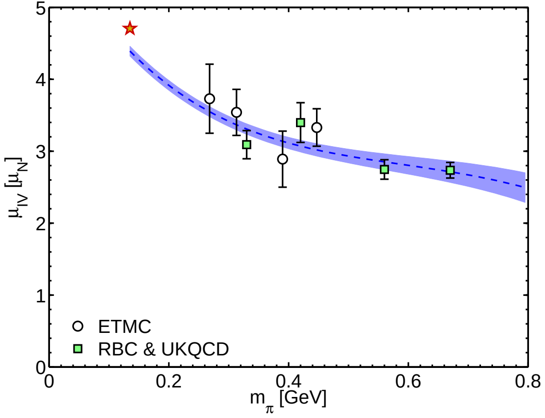
$$\mu_{IV} = G_M(0)\frac{e\hbar}{2m_N^{\mathrm{lat}}} = G_M(0)\frac{m_N^{\mathrm{exp}}}{m_N^{\mathrm{lat}}}\mu_N$$

Dirac and Pauli radius

The Dirac (Pauli) radius is given by

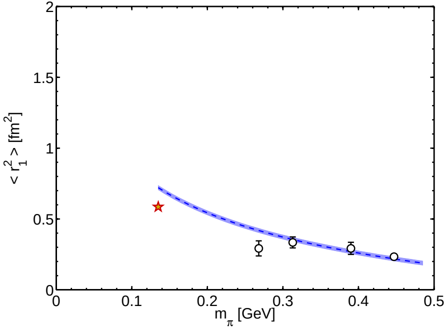
$$\begin{aligned} \langle r_i^2 \rangle &= -\frac{6}{F_i(Q^2)}\left.\frac{d\,F_i(Q^2)}{dQ^2}\right|_{Q^2=0} \\ \Rightarrow \langle r_1^2 \rangle &= \frac{12}{\Lambda_E^2} - \frac{3G_M(0)-3}{2m^2} \\ \langle r_2^2 \rangle &= \frac{6}{G_M(0)-1}\left(\frac{2G_M(0)}{\Lambda_M^2} - \frac{2}{\Lambda_E^2}\right) + \frac{3}{2m^2}\,. \end{aligned}$$

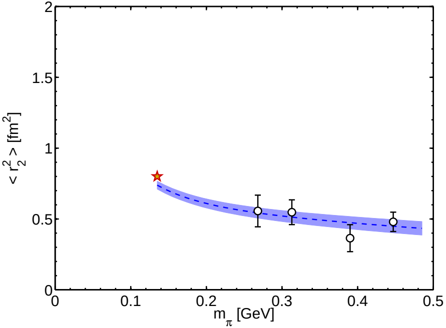
Magnetic moment



chiral extrapolation: [T.Hemmert, W. Weise, 2002], fit 3 parameters

$\langle r_1^2 \rangle$ and $\langle r_2^2 \rangle$





chiral extrapolation: [QCDSF, 2005], fit 1 parameter

Axial form factors: G_A

- local iso-vector axial current, $Z_A = 0.76$
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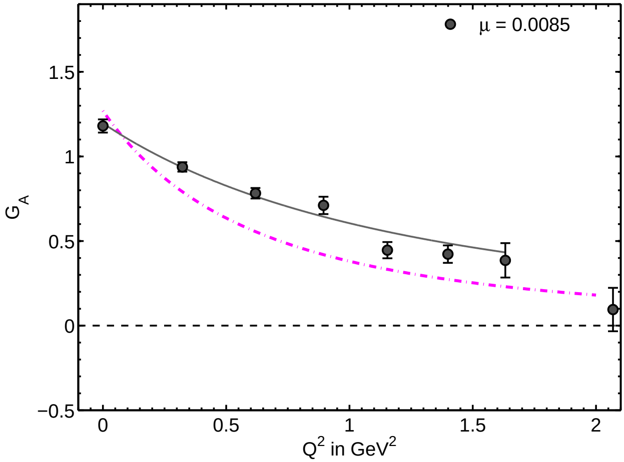

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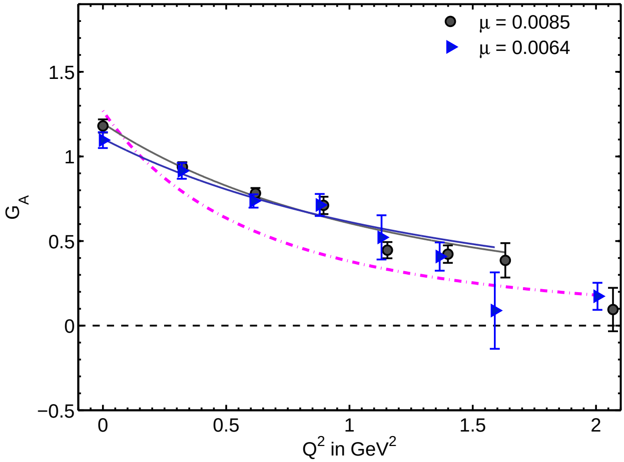

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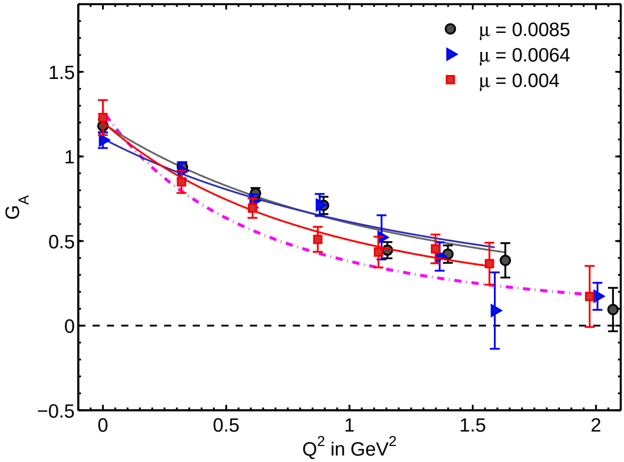

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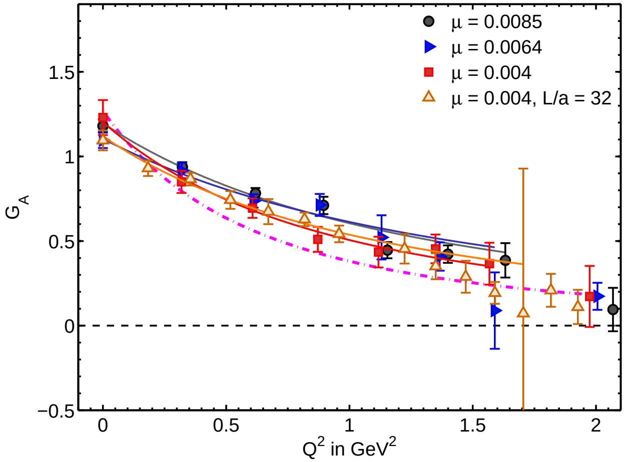

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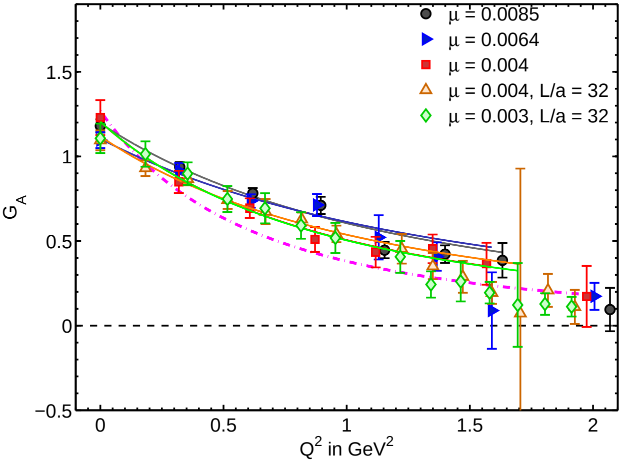

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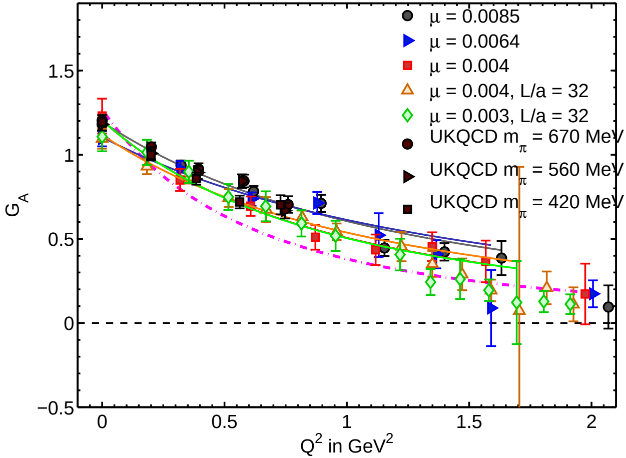

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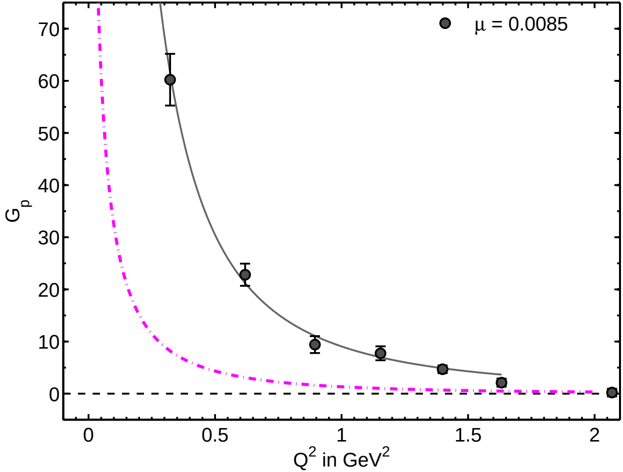

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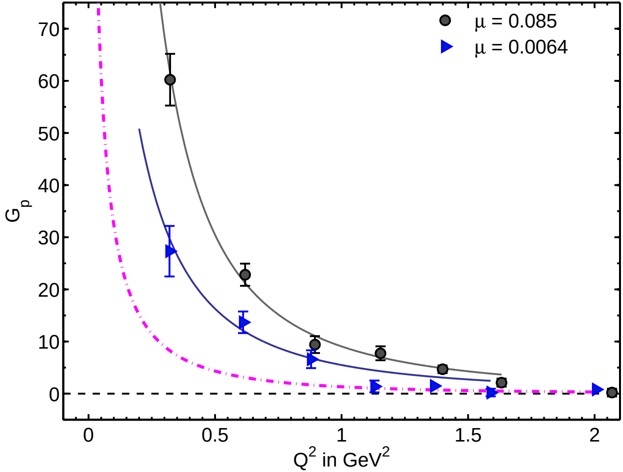

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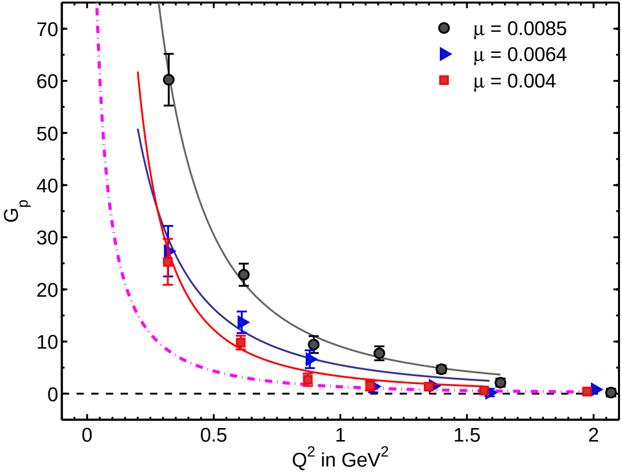

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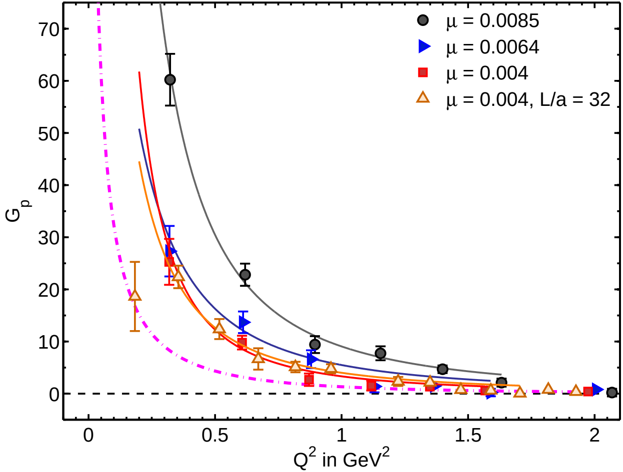

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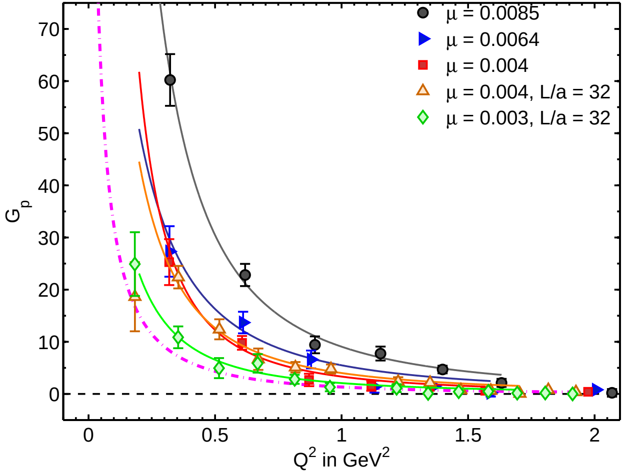
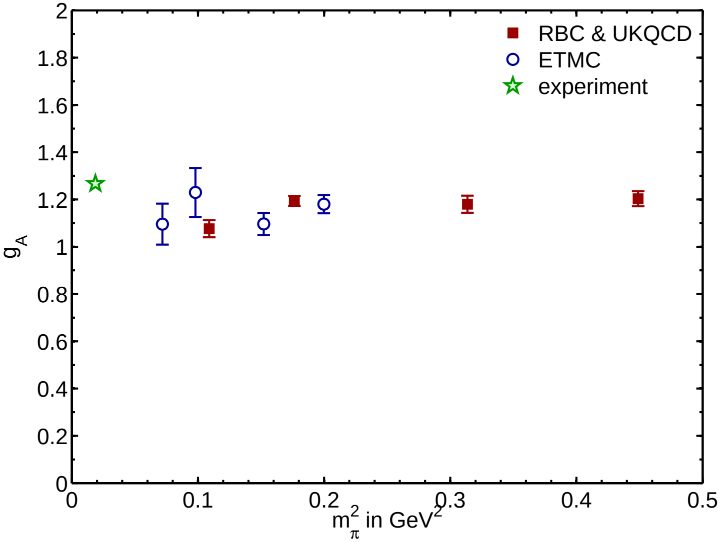
- local iso-vector axial current
- $a = 0.089(1)\ fm$ from m_N
- $L = 2.13\ fm$ / $2.84 fm$
- $m_\pi = 447\ MeV$
- $m_\pi = 390\ MeV$
- $m_\pi = 313\ MeV$
- $m_\pi = 297\ MeV$
- $m_\pi = 268\ MeV$



Two parameter dipole fit: $G_p(Q^2) = G_p(0)/\left(1 + \frac{Q^2}{\Lambda_p^2}\right)^2$

g_A


RBC & UKQCD
ETMC
experiment
g_A
m_π^2 in GeV^2
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
0.1
0.3
0.5

GPDs: trouble with $\mathcal{O}^{00}$

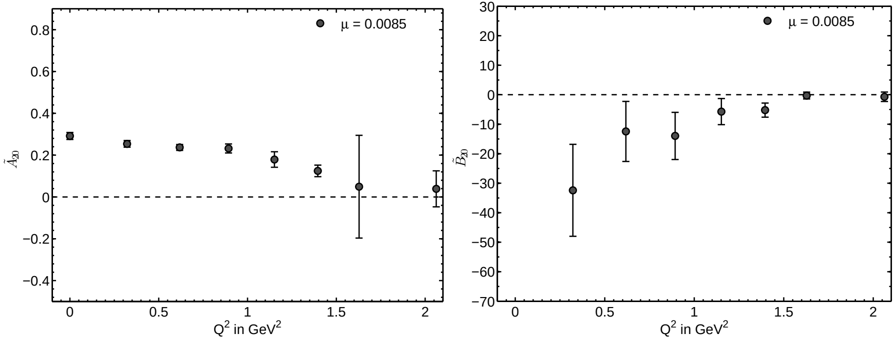
$\mathcal{O}_{\mu\nu} = \bar{q}\gamma_{\{\mu}\overleftrightarrow{D}_{\nu\}}q$ irreducible under $O(4)$ but not under the hyper-cubic group.

- $\mathcal{O}_{00} = \bar{q}\left[\gamma_0\overleftrightarrow{D}_0 - \frac{1}{3}(\gamma_1\overleftrightarrow{D}_1 + \gamma_2\overleftrightarrow{D}_2 + \gamma_3\overleftrightarrow{D}_3)\right]q$
  Coefficient matrix has rank 1 $\Rightarrow$ not possible to solve for A_{20}, B_{20} and C_{20} without additional sources (if any exist)
- $\mathcal{O}_{0k} = \bar{q}\left[\gamma_0\overleftrightarrow{D}_k + \gamma_k\overleftrightarrow{D}_0\right]q$
  Coefficient matrix has rank 2 $\Rightarrow$ not possible to solve for A_{20}, B_{20} and C_{20} without additional sources. In principle possible with type 1, type 2 and type 3.

If renormalization constants are known, a solution using both operators is possible with just type 1 and type 2 sequential propagators.

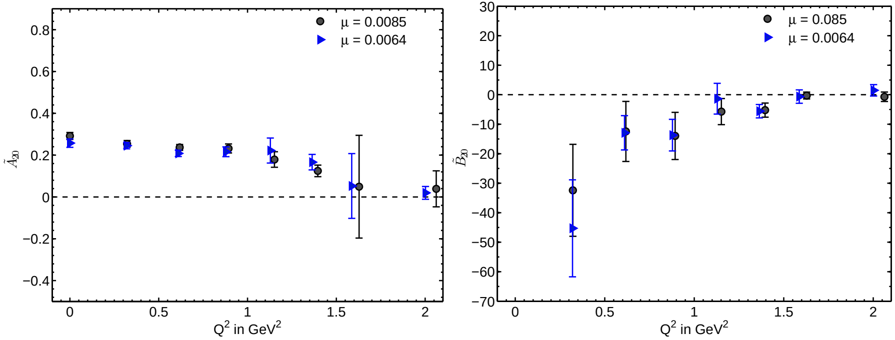
$\tilde{A}_{20}$ and $\tilde{B}_{20}$

Bare generalized form factors of operator $q\gamma_5\gamma_{\{0}\overleftrightarrow{D}_{\mu\}}q$



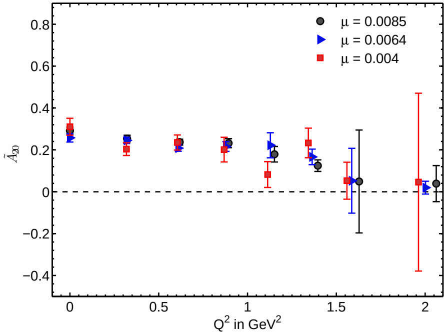
$\tilde{A}_{20}$ and $\tilde{B}_{20}$

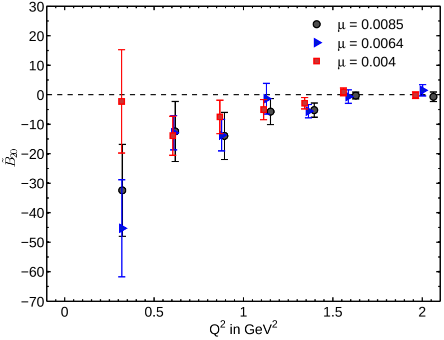
Bare generalized form factors of operator $q\gamma_5\gamma_{\{0}\overleftrightarrow{D}_{\mu\}}q$



$\tilde{A}_{20}$ and $\tilde{B}_{20}$

Bare generalized form factors of operator $q\gamma_5\gamma_{\{0}\overleftrightarrow{D}_{\mu\}}q$





Final remarks

- Vector form factors G_E and G_M
  - compatible with experiments
  - reasonable chiral behavior
  - no significant finite volume effects
- Axial form factors G_A and G_p
  - g_A reasonable
  - $G_A(q^2)$ in agreement with UKQCD/RBC
- GPDs
  - $q^2 = 0$ limit (structure functions) OK
  - need non-perturbative renormalization urgently
  - maybe worthwhile creating more types of sequential propagators
- to do
  - finalize form factor calculation and write up
  - independent analysis
  - look at disconnected contribution to iso-scalar form factors
  - finer lattices