The Drell–Levy–Yan Relation: ep vs e^+e^- Scattering to $O(\alpha_s^2)$

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DESY

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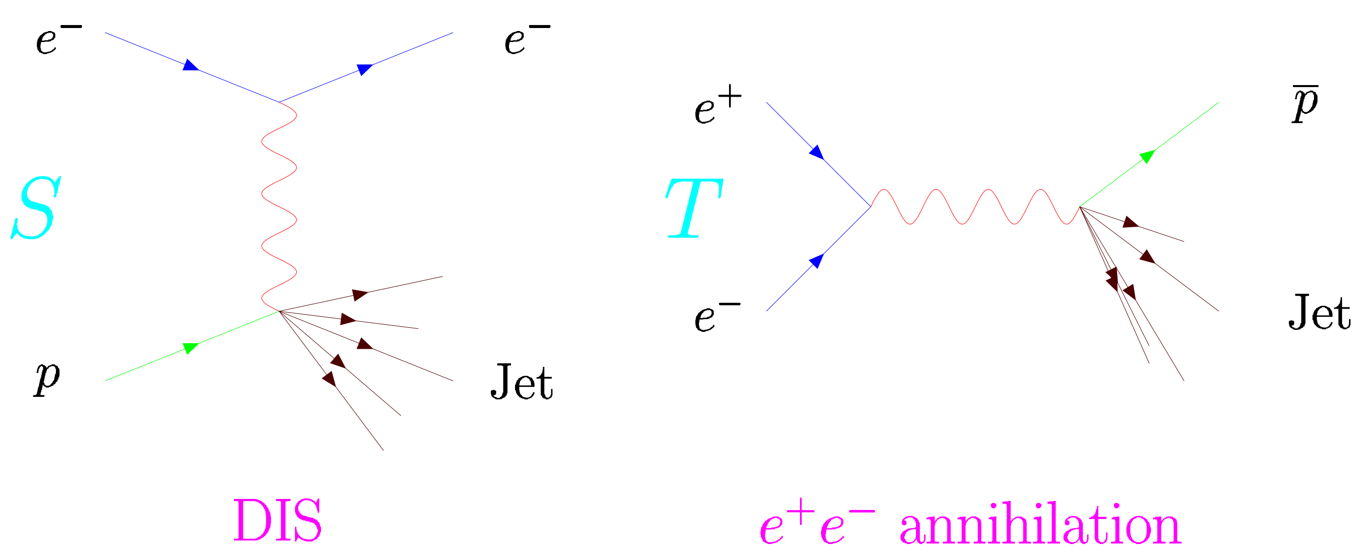
Based on : J. Blümlein, V. Ravindran, and W.L. van Neerven, hep-ph/0004182, Nucl. Phys. **B** in print

1. The DLY–Relation

Hard Processes :

Structure Functions: $e^- p \to e^- X$

Fragmentation Functions: $e^+ e^- \to \overline{p} X$



$$\frac{d^2\sigma}{dx dQ^2} \sim L_{\mu\nu} W^{\mu\nu}$$

Drell, Levy and Yan (1969) anticipated the crossing relation

$$W_{\mu\nu}^{(S)}(q, p) = -W_{\mu\nu}^{(T)}(q, -p)$$

for the two reactions.

- simple scalar-fermion ladder model with $\delta(1-z)$ source

What can we learn about the relation of these processes in QCD?

• Factorization of the Structure Functions ep and the Fragmentation Functions e^+e^- into:

- nonperturbative input at Q_0^2
- perturbative evolution kernel $Q_0^2 \rightarrow Q^2$

The Crossing Relations can be studied in perturbative QCD only for the Evolution Kernels.

Crossing for non-perturbative inputs? $\rightarrow$ LGT

Scaling Variables:

$$
\begin{aligned}
x_B &= \frac{Q^2}{2p.q}, \quad 0 \leq x_B \leq 1 \quad \text{DIS} \\
x_E &= \frac{2p.q}{Q^2}, \quad 0 \leq x_E \leq 1 \quad e^+e^- \text{ annihilation}
\end{aligned}
$$

Both domains are connected at $x = 1$, which is usually a singular point.

What are the Conditions for Continuation from one Domain to the other for the Evolution Kernels?

How is the Evolution of Structure Functions and Fragmentation Functions Related?

To which Order does this Relation exist?

History

• Early Investigations

Drell, Levy and Yan 1969,1970

Pestieau and Roy, 1969

Gribov and Lipatov 1971, 1972

Fishbane and Sullivan 1971

Suri 1971; Gatto and Preparata 1972, Dahmen and Steiner 1973

Gatto, Menotti, Vendramin 1971,1972

Landshoff, Polkinghorne, Short 1970-1973

Altarelli and Maiani 1972/73

Bukhvostov, Lipatov and Popov 1975

For a review see: P. Menotti, in: Proceedings of the Informal Meeting on Electromagnetic Interactions, Frascati, May 2–3, 1972, Pisa Preprint SNS 3/72.

• Later Developments

Curci, Furmanski and Petronzio 1980

Floratos, Lacaze and Kounnas 1981

Stratmann and Vogelsang 1997

Blümlein, Ravindran and van Neerven 1998/2000

2. Structure and Fragmentation Functions

$$F_i(x,Q^2) = \sum_{k=q,g} \left(\mathcal{F}_{ik}\left(\alpha_s(\mu_r^2), \frac{Q^2}{\mu^2}, \frac{\mu^2}{\mu_r^2}, \epsilon\right) \otimes \hat{f}_k\right)(x)\,,$$

$\hat{f}$ is the bare parton density and $\otimes$ denotes the Mellin–convolution,

$$(f \otimes g)(z) = \int_0^1 dz_1 \int_0^1 dz_2 f(z_1) g(z_2) \delta(z - z_1 z_2)\,.$$

with

$$\hat{\mathcal{F}}_{ik}(z, \alpha_s(\mu_r^2), \frac{Q^2}{\mu^2}, \frac{\mu^2}{\mu_r^2}, \epsilon) =$$
$$\sum_{l=q,g} \left(C_{i,l}\left(\alpha_s(\mu_r^2), \frac{Q^2}{\mu_f^2}, \frac{\mu_f^2}{\mu_r^2}\right) \otimes \Gamma_{lk}\left(\alpha_s(\mu_r^2), \frac{\mu_f^2}{\mu^2}, \frac{\mu_f^2}{\mu_r^2}, \epsilon\right)\right)(z)\,.$$

Γ_{lk} denotes the transition functions and $C_{i,l}$ the coefficient functions.

$$F_i(x,Q^2) = \sum_{l=q,g} \left(C_{i,l}\left(\alpha_s(\mu_r^2), \frac{Q^2}{\mu_f^2}, \frac{\mu_f^2}{\mu_r^2}\right) \otimes f_l\left(\alpha_s(\mu_r^2), \frac{\mu_f^2}{\mu^2}, \frac{\mu_f^2}{\mu_r^2}\right)\right)(x)\,,$$

The structure functions contain the renormalized parton densities f_l.

$$f_l\left(z, \alpha_s(\mu_r^2), \frac{\mu_f^2}{\mu^2}, \frac{\mu_f^2}{\mu_r^2}\right) = \sum_{k=q,g} \left(\Gamma_{lk}\left(\alpha_s(\mu_r^2), \frac{\mu_f^2}{\mu^2}, \frac{\mu_f^2}{\mu_r^2}, \epsilon\right) \otimes \hat{f}_k\right)(z)\,.$$

The Transition Functions obey the RGE :

$$\left(\left[\left\{\mu_f^2 \frac{\partial}{\partial \mu_f^2} + \beta(a_s(\mu_f^2)) \frac{\partial}{\partial a_s(\mu_f^2)}\right\} \mathbf{1} \delta_{lm} - \frac{1}{2} P_{lm}(a_s(\mu_f^2), \epsilon)\right]\right.$$
$$\left. \otimes \Gamma_{mk}\left(a_s(\mu_f^2), \frac{\mu_f^2}{\mu^2}, 1, \epsilon\right)\right)(z) = 0$$

where

$$a_s(\mu_f^2) \equiv \frac{\alpha_s(\mu_f^2)}{4\pi} \quad , \quad \mathbf{1} = \delta(1-z)\,,$$

The strong coupling constant evolves as

$$\mu_r^2 \frac{d\ a_s(\mu_r^2)}{d\ \mu_r^2} = -\beta_0 a_s^2(\mu_r^2) - \beta_1 a_s^3(\mu_r^2) \cdots ,$$

The Coefficient Functions obey the RGE :

$$\left(\left[\left\{ \mu_f^2 \frac{\partial}{\partial \mu_f^2} + \beta(a_s(\mu_f^2)) \frac{\partial}{\partial a_s(\mu_f^2)} \right\} \mathbf{1} \delta_{lm} + \frac{1}{2} P_{lm}(a_s(\mu_f^2), \epsilon) \right] \right.$$

$$\left. \otimes\, C_{i,m}\left(a_s(\mu_f^2), \frac{Q^2}{\mu_f^2}, 1 \right) \right)(z) = 0\,.$$

Factorization Scheme Transformations

$$\Gamma_{lk} \rightarrow \sum_{m=q,g} Z_{lm} \otimes \bar{\Gamma}_{mk} \quad , \quad C_{i,l} \rightarrow \sum_{m=q,g} \bar{C}_{i,m} \otimes Z_{ml}^{-1}$$

The splitting functions transform as:

$$P_{lk} = \sum_{\{m,n\}=q,g} Z_{lm} \otimes \bar{P}_{mn} \otimes (Z^{-1})_{nk} - 2\beta(a_s) \sum_{m=q,g} Z_{lm} \otimes \frac{d}{da_s}(Z^{-1})_{mk}$$

likewise the coefficient functions obey:

$$\begin{aligned} C_{i,q} &= \delta(1-z) + a_s\left(\bar{C}_{i,q}^{(1)} + Z_{qq}^{(1)}\right) + a_s^2\left(\bar{C}_{i,q}^{(2)} + Z_{qq}^{(2)} + (Z_{qq}^{(1)})^2\right. \\ &\quad \left. + Z_{qg}^{(1)} \otimes Z_{gq}^{(1)} + \bar{C}_{i,q}^{(1)} \otimes Z_{qq}^{(1)} + \bar{C}_{i,g}^{(1)} \otimes Z_{gq}^{(1)}\right) + \cdots , \\ C_{i,g} &= a_s\left(\bar{C}_{i,g}^{(1)} + Z_{qg}^{(1)}\right) + a_s^2\left(\bar{C}_{i,g}^{(2)} + Z_{qg}^{(2)} + Z_{qg}^{(1)} \otimes \left(Z_{gg}^{(1)} + Z_{qq}^{(1)}\right)\right. \\ &\quad \left. + \bar{C}_{i,q}^{(1)} \otimes Z_{qg}^{(1)} + \bar{C}_{i,g}^{(1)} \otimes Z_{gg}^{(1)}\right) + \cdots . \end{aligned}$$

The MELLIN Integrals above can be turned into a simple algebraic structure by the transform

$$f^{(N)} = \int_0^1 dz \; z^{N-1} \; f(z)$$

with

$$(f \otimes g)^N = \int_0^1 dz \; z^{N-1} \; (f \otimes g)(z) = f^N \cdot g^N \; .$$

The combination of the above RGE's leads thus to the RGE for the structure and fragmentation functions:

$$\left[\mu_f^2 \frac{\partial}{\partial \mu_f^2} + \beta(a_s(\mu_f^2)) \frac{\partial}{\partial a_s(\mu_f^2)} \right] F_i^N(x, Q^2) = 0,$$

which are scheme invariant.

Linearly Independent Pairs of Structure Functions can be expressed as combinations of the singlet quark and gluon densities $f_{q,g}$ and the respective coefficient functions.

$$\begin{aligned} F_I^N(Q^2) &= f_q^N \left(a_s(\mu_f^2), \frac{\mu_f^2}{Q_0^2} \right) C_{I,q}^N \left(a_s(\mu_f^2), \frac{Q^2}{\mu_f^2} \right) \\ &+ f_g^N \left(a_s(\mu_f^2), \frac{\mu_f^2}{Q_0^2} \right) C_{I,g}^N \left(a_s(\mu_f^2), \frac{Q^2}{\mu_f^2} \right), \qquad I = A, B. \end{aligned}$$

This decomposition is scheme and scale (μ_f) dependent.

$$\left(\begin{array}{c} F_A^N \\ F_B^N \end{array} \right) = \left(\begin{array}{cc} C_{Aq}^N & C_{Ag}^N \\ C_{Bq}^N & C_{Bg}^N \end{array} \right) \left(\begin{array}{c} f_q^N \\ f_g^N \end{array} \right) \; .$$

3. Scheme Invariant Combinations

Evolution Equations of Structure or Fragmentation Functions do normally exhibit FACTORIZATION AND RENORMALIZATION SCHEME DEPENDENCES. INSTEAD OF PROCESS-INDEPENDENT SCHEME-DEPENDENT EVOLUTION EQUATIONS FOR PARTONS ONE MAY THINK OF PROCESS-DEPENDENT SCHEME-INDEPENDENT EVOLUTION EQUATIONS FOR **Observables.**

Evolution Equations :

$$\frac{\partial}{\partial t}\begin{pmatrix} F_A^N \\ F_B^N \end{pmatrix} = -\frac{1}{4}\begin{pmatrix} K_{AA}^N & K_{AB}^N \\ K_{BA}^N & K_{BB}^N \end{pmatrix}\begin{pmatrix} F_A^N \\ F_B^N \end{pmatrix},$$

evolution variable

$$t = -\frac{2}{\beta_0}\ln\left(\frac{a_s(Q^2)}{a_s(Q_0^2)}\right),$$

physical evolution kernels

$$K_{IJ}^N = \left[-4\frac{\partial C_{I,m}^N(t)}{\partial t}\left(C^N\right)_{m,J}^{-1}(t) - \frac{\beta_0 a_s(Q^2)}{\beta(a_s(Q^2))}C_{I,m}^N(t)\gamma_{mn}^N(t)\left(C^N\right)_{n,J}^{-1}(t)\right]$$

with

$$K_{IJ}^N = \sum_{n=0}^{\infty} a_s^n(Q^2)\left(K^N\right)_{IJ}^{(n)}$$

Possible choices for F_A and F_B are F_2 and $\partial F_2/\partial t$ or F_2 and F_L. For these sets of physical observables we will examine the crossing-behaviour from S to T-Channel.

The dependence on the **renormalization scheme** is only removed if the perturbation series is summed to all orders.

System : $F_2(x,Q^2), \partial F_2/\partial t(x,Q^2)$

Leading Order :

$$
\begin{aligned}
K_{22}^{N(0)} &= 0 \\
K_{2d}^{N(0)} &= -4 \\
K_{d2}^{N(0)} &= \frac{1}{4}\left(\gamma_{qq}^{N(0)}\gamma_{gg}^{N(0)} - \gamma_{qg}^{N(0)}\gamma_{gq}^{N(0)}\right) \\
K_{dd}^{N(0)} &= \gamma_{qq}^{N(0)} + \gamma_{gg}^{N(0)}
\end{aligned}
$$

Next-to-Leading Order :

[Furmanski, Petronzio 1982]

$$
\begin{aligned}
K_{22}^{N(1)} &= K_{2d}^{N(1)} = 0 \\
K_{d2}^{N(1)} &= \frac{1}{4}\left[\gamma_{gg}^{N(0)}\gamma_{qq}^{N(1)} + \gamma_{gg}^{N(1)}\gamma_{qq}^{N(0)} - \gamma_{qg}^{N(1)}\gamma_{gq}^{N(0)} - \gamma_{qg}^{N(0)}\gamma_{gq}^{N(1)}\right] \\
&\quad -\frac{\beta_1}{2\beta_0}\left(\gamma_{qq}^{N(0)}\gamma_{gg}^{N(0)} - \gamma_{gq}^{N(0)}\gamma_{qg}^{N(0)}\right) \\
&\quad +\frac{\beta_0}{2}C_{2,q}^{N(1)}\left(\gamma_{qq}^{N(0)} + \gamma_{gg}^{N(0)} - 2\beta_0\right) \\
&\quad -\frac{\beta_0}{2}\frac{C_{2,g}^{N(1)}}{\gamma_{qg}^{N(0)}}\left[(\gamma_{qq}^{N(0)})^2 - \gamma_{qq}^{N(0)}\gamma_{gg}^{N(0)} + 2\gamma_{qg}^{N(0)}\gamma_{gq}^{N(0)} - 2\beta_0\gamma_{qq}^{N(0)}\right] \\
&\quad -\frac{\beta_0}{2}\left(\gamma_{qq}^{N(1)} - \frac{\gamma_{qq}^{N(0)}\gamma_{qg}^{N(1)}}{\gamma_{qg}^{N(0)}}\right)
\end{aligned}
\qquad (1)
$$

$$K_{dd}^{N(1)} = \gamma_{qq}^{N(1)} + \gamma_{gg}^{N(1)} - \frac{\beta_1}{\beta_0}\left(\gamma_{qq}^{N(0)} + \gamma_{gg}^{N(0)}\right) + 4\beta_0 C_{2,q}^{N(1)} - 2\beta_1$$

$$- \frac{2\beta_0}{\gamma_{qg}^{N(0)}}\left[C_{2,g}^{N(1)}\left(\gamma_{qq}^{N(0)} - \gamma_{gg}^{N(0)} - 2\beta_0\right) - \gamma_{qg}^{N(1)}\right]$$

System : $F_2(x,Q^2), F_L(x,Q^2)$

$(\widetilde{F}_L^N \equiv F_L^N/(a_s(Q^2)C_{L,g}^{N(1)}))$

Leading Order :

[Catani 1997]

$$K_{22}^{N(0)} = \gamma_{qq}^{N(0)} - \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}}\gamma_{qg}^{N(0)}$$

$$K_{2L}^{N(0)} = \gamma_{qg}^{N(0)}$$

$$K_{L2}^{N(0)} = \gamma_{gq}^{N(0)} - \left(\frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}}\right)^2 \gamma_{qg}^{N(0)}$$

$$K_{LL}^{N(0)} = \gamma_{gg}^{N(0)} + \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}}\gamma_{qg}^{N(0)} + \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}}\left(\gamma_{qq}^{N(0)} - \gamma_{gg}^{N(0)}\right)$$

Next-to-Leading Order :

[BRvN 2000]

$$K_{22}^{N(1)} = \gamma_{qq}^{N(1)} - \frac{\beta_1}{\beta_0}\gamma_{qq}^{N(0)} - \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}}\left(\gamma_{qg}^{N(1)} - \frac{\beta_1}{\beta_0}\gamma_{qg}^{N(0)}\right)$$

$$+ \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}}C_{2,g}^{N(1)}\gamma_{qq}^{N(0)} - \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}}C_{2,g}^{N(1)}\gamma_{gg}^{N(0)}$$

$$
\begin{aligned}
& - \left[\frac{C_{L,q}^{N(2)}}{C_{L,g}^{N(1)}} + \left(\frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} \right)^2 C_{2,g}^{N(1)} - \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} \frac{C_{L,g}^{N(2)}}{C_{L,g}^{N(1)}} \right] \gamma_{qg}^{N(0)} \\
& + C_{2,g}^{N(1)} \gamma_{gq}^{N(0)} + 2\beta_0 \left(C_{2,q}^{N(1)} - \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} C_{2,g}^{N(1)} \right) \\
K_{2L}^{N(1)} \quad = \quad & \gamma_{qg}^{N(1)} - \frac{\beta_1}{\beta_0} \gamma_{qg}^{N(0)} - C_{2,g}^{N(1)} (\gamma_{qq}^{N(0)} - \gamma_{gg}^{N(0)}) + 2\beta_0 C_{2,g}^{N(1)} \\
& + \left(C_{2,q}^{N(1)} + \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} C_{2,g}^{N(1)} - \frac{C_{L,g}^{N(2)}}{C_{L,g}^{N(1)}} \right) \gamma_{qg}^{N(0)} \\
K_{L2}^{N(1)} \quad = \quad & \gamma_{gq}^{N(1)} - \frac{\beta_1}{\beta_0} \gamma_{gq}^{N(0)} + \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} \left(\gamma_{qq}^{N(1)} - \frac{\beta_1}{\beta_0} \gamma_{qq}^{N(0)} \right) \\
& - \left(\frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} \right)^2 \left(\gamma_{qg}^{N(1)} - \frac{\beta_1}{\beta_0} \gamma_{qg}^{N(0)} \right) \\
& - \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} \left(\gamma_{gg}^{N(1)} - \frac{\beta_1}{\beta_0} \gamma_{gg}^{N(0)} \right) \\
& + \left[\frac{C_{L,q}^{N(2)}}{C_{L,g}^{N(1)}} - \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} C_{2,q}^{N(1)} + \left(\frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} \right)^2 C_{2,g}^{N(1)} \right] \gamma_{qq}^{N(0)} \\
& - \left[\left(\frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} \right)^3 C_{2,g}^{N(1)} + 2 \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} \frac{C_{L,q}^{N(2)}}{C_{L,g}^{N(1)}} - \left(\frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} \right)^2 \frac{C_{L,g}^{N(2)}}{C_{L,g}^{N(1)}} \right. \\
& \left. - \left(\frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} \right)^2 C_{2,q}^{N(1)} \right] \gamma_{qg}^{N(0)} \\
& + \left(\frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} C_{2,g}^{N(1)} - C_{2,q}^{N(1)} + \frac{C_{L,g}^{N(2)}}{C_{L,g}^{N(1)}} \right) \gamma_{gq}^{N(0)}
\end{aligned}
$$

$$
\begin{aligned}
& - \left[\frac{C_{L,q}^{N(2)}}{C_{L,g}^{N(1)}} + \left(\frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} \right)^2 C_{2,g}^{N(1)} - \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} C_{2,q}^{N(1)} \right] \gamma_{gg}^{N(0)} \\
& +2\beta_0 \left(\frac{C_{L,q}^{N(2)}}{C_{L,g}^{N(1)}} - \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} \frac{C_{L,g}^{N(2)}}{C_{L,g}^{N(1)}} \right)
\end{aligned}
$$

$$
\begin{aligned}
K_{LL}^{N(1)} = \; & \gamma_{gg}^{N(1)} - \frac{\beta_1}{\beta_0} \gamma_{gg}^{N(0)} + \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} \left(\gamma_{qg}^{N(1)} - \frac{\beta_1}{\beta_0} \gamma_{qg}^{N(0)} \right) \\
& - \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} C_{2,g}^{N(1)} \gamma_{qq}^{N(0)} + \left[\frac{C_{L,q}^{N(2)}}{C_{L,g}^{N(1)}} - \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} \frac{C_{L,g}^{N(2)}}{C_{L,g}^{N(1)}} \right. \\
& \left. + \left(\frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} \right)^2 C_{2,g}^{N(1)} \right] \gamma_{qg}^{N(0)} \\
& - C_{2,g}^{N(1)} \gamma_{gq}^{N(0)} + \frac{C_{L,q}^{N(1)}}{C_{L,g}^{N(1)}} C_{2,g}^{N(1)} \gamma_{gg}^{N(0)} + 2\beta_0 \frac{C_{L,g}^{N(2)}}{C_{L,g}^{N(1)}}
\end{aligned}
$$

4. DLY–Relations for Evolution Kernels

Original Crossing Relation:

$$W_{\mu\nu}^{T}(q,p) = -W_{\mu\nu}^{S}(q,-p)$$

[Drell et al 1969]

Modified for particles with different spin s_i in a simple ladder model

$$F_i^{(S)}(x_B) = -(-1)^{2(s_1+s_2)} x_E F_i^{(T)}\left(\frac{1}{x_E}\right) \quad , \quad i = 1, 2, L \, .$$

[Bukhvostov et al. 1975]

Similar relations are expected to hold for the QCD evolution kernels in LO.

In the evolution kernels singular contributions like

$$\delta(1-z), \qquad \left(\frac{\ln^i(1-z)}{1-z}\right)_+ = \delta(1-z)\frac{\ln^{i+1}\varepsilon}{(i+1)} + \theta(1-\varepsilon-z)\frac{\ln^i(1-z)}{(1-z)} \; ,$$

arise, which have to be continued analytically.

Continuation Rules: *[BRvN 2000]*

$$\begin{aligned}
P(z) &\rightarrow zP(1/z) \\
P_{ii} &\rightarrow -P_{ii} \\
P_{qg}, P_{gq} &\rightarrow \mathrm{cross\ color\ pre-factor} \\
\ln\left(Q^2/\mu_f^2\right)_{\mathrm{space-like}} &\rightarrow \ln\left(Q^2/\mu_f^2\right)_{\mathrm{time-like}} - i\pi \; . \\
\delta(1-z) &\rightarrow -\delta(1-z) \\
\ln(1-z) &\rightarrow \ln(1-z) - \ln(z) + i\pi \\
\ln(\varepsilon) &\rightarrow \ln(\varepsilon) + i\pi
\end{aligned}$$

LO unpolarized and polarized Splitting Functions:

$$
\begin{aligned}
P_{qq}^{(0)}(z) = \Delta P_{qq}^{(0)}(z) &= 4C_F\left[\frac{1+z^2}{(1-z)_+}+\frac{3}{2}\delta(1-z)\right] \\
P_{qG}^{(0)}(z) &= 8T_R N_f\left[z^2+(1-z)^2\right] \\
\Delta P_{qG}^{(0)}(z) &= 8T_R N_f\left[z^2-(1-z)^2\right] \\
P_{Gq}^{(0)}(z) &= 4C_F\frac{1+(1-z)^2}{z} \\
\Delta P_{Gq}^{(0)}(z) &= 4C_F\frac{1-(1-z)^2}{z} \\
P_{GG}^{(0)}(z) &= 8C_A\left[\frac{z}{(1-z)_+}+\frac{1-z}{z}+z(1-z)\right]+2\beta_0\delta(1-z) \\
\Delta P_{GG}^{(0)}(z) &= 8C_A\left[\frac{1}{(1-z)_+}+1-2z\right]+2\beta_0\delta(1-z)
\end{aligned}
$$

Crossing Relations:

$$
\bar{P}_{qq}^{(0)} = -zP_{qq}^{(0)}\left(\frac{1}{z}\right) \qquad \bar{P}_{qg}^{(0)} = \frac{C_F}{2N_fT_f}\,zP_{qg}^{(0)}\left(\frac{1}{z}\right)
$$

$$
\bar{P}_{gq}^{(0)} = \frac{2N_fT_f}{C_F}\,zP_{gq}^{(0)}\left(\frac{1}{z}\right) \qquad \bar{P}_{gg}^{(0)} = -zP_{gg}^{(0)}\left(\frac{1}{z}\right)\ ,
$$

Note the Color Factors!

Already here

$$
\delta(1-z) \rightarrow -\delta(1-z)\ .
$$

is required.

NLO unpolarized and polarized Splitting Functions:

$$\begin{aligned}
\bar{P}_{qq}^{(1)S} - P_{qq}^{(1)T} &= -2\beta_0 Z_{qq}^{T(1)} + Z_{qg}^{T(1)} \otimes \bar{P}_{gq}^{(0)} - Z_{gq}^{T(1)} \otimes \bar{P}_{qg}^{(0)} , \\
\bar{P}_{qg}^{(1)S} - P_{qg}^{(1)T} &= -2\beta_0 Z_{qg}^{T(1)} + Z_{qg}^{T(1)} \otimes (\bar{P}_{gg}^{(0)} - \bar{P}_{qq}^{(0)}) \\
&\quad + \bar{P}_{qg}^{(0)} \otimes (Z_{qq}^{T(1)} - Z_{gg}^{T(1)}) , \\
\bar{P}_{gq}^{(1)S} - P_{gq}^{(1)T} &= -2\beta_0 Z_{gq}^{T(1)} + Z_{gq}^{T(1)} \otimes (\bar{P}_{qq}^{(0)} - \bar{P}_{gg}^{(0)}) \\
&\quad + \bar{P}_{gq}^{(0)} \otimes (Z_{gg}^{T(1)} - Z_{qq}^{T(1)}) , \\
\bar{P}_{gg}^{(1)S} - P_{gg}^{(1)T} &= -2\beta_0 Z_{gg}^{T(1)} + Z_{gq}^{T(1)} \otimes \bar{P}_{qg}^{(0)} - Z_{qg}^{T(1)} \otimes \bar{P}_{gq}^{(0)} ,
\end{aligned}$$

where for unpolarized scattering

$$Z_{ij}^{T(1)} = P_{ji}^{(0)} \left(\ln(z) + a_{ji} \right) .$$

$$a_{qq} = a_{gg} = 0 \quad , \quad a_{qg} = -\frac{1}{2} \quad , \quad a_{gq} = \frac{1}{2} ,$$

and for polarized scattering

$$a_{ij} = 0 .$$

(c.f. also *[Stratmann, Vogelsang 1997]*)

NLO unpolarized and polarized Coefficient Functions:

$$
\begin{aligned}
C_{1,q}^{(T)(1)}(z) + \left\{ z\, C_{1,q}^{(S)(1)}\left(\frac{1}{z}\right)\right\} &= Z_{qq}^{(T)(1)} \\
\frac{1}{2}\left[C_{1,g}^{(T)(1)}(z) - \frac{C_F}{2N_f T_f}\left\{2z C_{1,g}^{(S)(1)}\left(\frac{1}{z}\right)\right\}\right] &= Z_{qg}^{(T)(1)}\ .
\end{aligned}
$$

NLO unpolarized Longitudinal Coefficient Functions:

$$
\begin{aligned}
C_{L,q}^{(T)(1)}(z) - \frac{z}{2} C_{L,q}^{(S)(1)}\left(\frac{1}{z}\right) &= 0\,, \\
\frac{1}{2}\left[C_{L,g}^{(T)(1)}(z) + \frac{C_F}{2N_f T_f}\left\{z\, C_{L,g}^{(S)(1)}\left(\frac{1}{z}\right)\right\}\right] &= 0\ .
\end{aligned}
$$

NNLO unpolarized Longitudinal Coefficient Functions:

Coefficient fcts. see

[Zijlstra, vN 1992,1994]

,

[Rijken, vN 1996,1997]

.

$$
\begin{aligned}
&C_{L,q}^{(T)(2)}(z) + \left\{-\frac{z}{2}\, C_{L,q}^{(S)(2)}\left(\frac{1}{z}\right)\right\} = \\
&\qquad Z_{qq}^{(T)(1)} \otimes \frac{z}{2} C_{Lq}^{(1)S}\left(\frac{1}{z}\right) + Z_{gq}^{(T)(1)} \otimes \frac{C_F}{2N_f T_f}\left\{-\frac{z}{2}\, C_{L,g}^{(S)(1)}\left(\frac{1}{z}\right)\right\}, \\
&\frac{1}{2}\left[C_{L,g}^{(T)(2)}(z) + \frac{C_F}{2N_f T_f}\left\{z C_{L,g}^{(S)(2)}\left(\frac{1}{z}\right)\right\}\right] = \\
&\qquad Z_{qg}^{(T)(1)} \otimes \frac{z}{2} C_{L,q}^{(S)(1)}\left(\frac{1}{z}\right) + Z_{gg}^{(T)(1)} \otimes \frac{C_F}{2N_f T_f}\left\{-\frac{z}{2}\, C_{L,g}^{(1)S}\left(\frac{1}{z}\right)\right\}\ .
\end{aligned}
$$

To derive these relations extensive use has to be made of convolution relations like

$$z\ln(z)\otimes\left(\frac{\ln(1-z)}{1-z}\right)_+ = z\Biggl[-\mathrm{S}_{1,2}(1-z)+\ln(1-z)-\frac{1}{z}\ln(1-z)-\ln(z)$$
$$+\frac{1}{2}\ln^2(1-z)\ln(z)-\ln(z)\ln(1-z)+\frac{1}{2}\ln^2(z)$$
$$-\frac{1}{2}\ln(1-z)\ln^2(z)-\mathrm{Li}_2(1-z)+\ln(1-z)$$
$$\times\mathrm{Li}_2(1-z)-\ln(z)\mathrm{Li}_2(1-z)-\mathrm{Li}_3(1-z)\Biggr]$$

$$\frac{\ln(z)}{(1-z)}\otimes z^2\ln(1-z) = \frac{1}{4}\Biggl\{-4-5z+9z^2-8\ \mathrm{S}_{1,2}(1-z)z^2+\ln(1-z)$$
$$+4z\ln(1-z)-5z^2\ln(1-z)-3\ln(z)-4z\ln(z)$$
$$+2\ln(1-z)\ln(z)+4z\ln(1-z)\ln(z)$$
$$-2z^2\ln(1-z)\ln^2(z)+\Biggl[2+4z-4z^2\ln(1-z)$$
$$-4z^2\ln(z)\Biggr]\mathrm{Li}_2(1-z)+4z^2\mathrm{Li}_3(1-z)\Biggr\}$$

and relations between NIELSEN integrals of various arguments (cf. *[JB, Kurth 1999]*)

$$\mathrm{Li}_2\left(1-\frac{1}{z}\right) = -\frac{1}{2}\ln^2(z)-\mathrm{Li}_2(1-z)\,,$$
$$\mathrm{S}_{1,2}\left(1-\frac{1}{z}\right) = -\frac{1}{6}\ln^3(z)+\mathrm{S}_{1,2}(1-z)\,,$$
$$\mathrm{Li}_3\left(1-\frac{1}{z}\right) = \frac{1}{6}\ln^3(z)+\mathrm{S}_{1,2}(1-z)-\mathrm{Li}_3(1-z)+\ln(z)\mathrm{Li}_2(1-z)\,,$$
$$\mathrm{S}_{1,2}\left(-\frac{1}{z}\right) = -\mathrm{S}_{1,2}(-z)+\mathrm{Li}_3(-z)-\ln(z)\mathrm{Li}_2(-z)-\frac{1}{6}\ln^3(z)+\zeta(3)\ .$$

Transformations for other NNLO coefficient functions, see *[BRvN 2000]*.

Transformation of the Physical Evolution Kernels

Define

$$\delta K_{IJ} := K_{IJ}^T - \overline{K}_{IJ}^S.$$

The transforms for the F_2-F_L system read:

$$\begin{aligned}
\delta K_{22}^{N(1)} &= \delta\gamma_{qq}^{N(1)} - \frac{\bar{C}_{L,q}^{N(1)}}{\bar{C}_{L,g}^{N(1)}}\delta\gamma_{gq}^{N(1)} + \frac{\bar{C}_{L,q}^{N(1)}}{\bar{C}_{L,g}^{N(1)}}\delta C_{2,g}^{N(1)}\bar{\gamma}_{qq}^{N(0)} \\
&\quad - \left[\frac{\delta C_{L,q}^{N(2)}}{\bar{C}_{L,g}^{N(1)}} + \left(\frac{\bar{C}_{L,q}^{N(1)}}{\bar{C}_{L,g}^{N(1)}}\right)^2 \delta C_{2,g}^{N(1)} - \frac{\bar{C}_{L,q}^{N(1)}}{\bar{C}_{L,g}^{N(1)}}\frac{\delta C_{L,g}^{N(2)}}{\bar{C}_{L,g}^{N(1)}}\right]\bar{\gamma}_{qg}^{N(0)} \\
&\quad + \bar{\gamma}_{gq}^{N(0)}\delta C_{2,g}^{N(1)} - \frac{\bar{C}_{L,q}^{N(1)}}{\bar{C}_{L,g}^{N(1)}}\bar{\gamma}_{gg}^{N(0)}\delta C_{2,g}^{N(1)} \\
&\quad + 2\beta_0\left(\delta C_{2,q}^{N(1)} - \frac{\bar{C}_{L,q}^{N(1)}}{\bar{C}_{L,g}^{N(1)}}\delta C_{2,g}^{N(1)}\right) \\
&= \delta\gamma_{qq}^{N(1)} - 2\beta_0 Z_{qq}^{(T)N(1)} - \bar{\gamma}_{gq}^{N(0)} Z_{qg}^{(T)N(1)} + \bar{\gamma}_{qg}^{N(0)} Z_{gq}^{(T)N(1)} \\
&\quad + \frac{\bar{C}_{L,q}^{N(1)}}{\bar{C}_{L,g}^{N(1)}}(-\delta\gamma_{gq}^{N(1)} + 2\beta_0 Z_{qg}^{(T)N(1)} - Z_{qg}^{(T)N(1)}\bar{\gamma}_{qq}^{N(0)} \\
&\quad + Z_{qq}^{(T)N(1)}\bar{\gamma}_{qg}^{N(0)} - Z_{gg}^{(T)N(1)}\bar{\gamma}_{qg}^{N(0)} + Z_{qg}^{(T)N(1)}\bar{\gamma}_{gg}^{N(0)})\,.
\end{aligned}$$

$$\delta K_{2L}^{N(1)} = \delta\gamma_{qg}^{N(1)} - 2\beta_0 Z_{qg}^{(T)N(1)} + Z_{qg}^{(T)N(1)}(\bar{\gamma}_{qq}^{N(0)} - \bar{\gamma}_{gg}^{N(0)}) - \bar{\gamma}_{qg}^{N(0)}(Z_{qq}^{(T)N(1)} - Z_{gg}^{(T)N(1)}),$$

$$\delta K_{LL}^{N(1)} = \delta\gamma_{gg}^{N(1)} - 2\beta_0\bar{\gamma}_{gg}^{N(0)} + Z_{qg}^{(T)N(1)}\bar{\gamma}_{gq}^{N(0)} - Z_{gq}^{(T)N(1)}\bar{\gamma}_{qg}^{N(0)}$$
$$+\frac{\bar{C}_{L,q}^{N(1)}}{\bar{C}_{L,g}^{N(1)}}\left[\delta\gamma_{gq}^{N(1)} - 2\beta_0 Z_{gq}^{(T)N(1)} + Z_{qq}^{(T)N(1)}\bar{\gamma}_{gq}^{N(0)}\right.$$
$$\left.-Z_{qg}^{(T)N(1)}\bar{\gamma}_{gg}^{N(0)} - Z_{qq}^{(T)N(1)}\bar{\gamma}_{qg}^{N(0)} + Z_{gg}^{(T)N(1)}\bar{\gamma}_{qg}^{N(0)}\right],$$

$$\delta K_{L2}^{N(1)} = \delta\gamma_{qg}^{N(1)} + \frac{\bar{C}_{L,q}^{N(1)}}{\bar{C}_{L,g}^{N(1)}}\delta\gamma_{qq}^{N(1)} - \left(\frac{\bar{C}_{L,q}^{N(1)}}{\bar{C}_{L,g}^{N(1)}}\right)^2\delta\gamma_{gq}^{N(1)}$$
$$-\frac{\bar{C}_{L,q}^{N(1)}}{\bar{C}_{L,g}^{N(1)}}\delta\gamma_{gg}^{N(1)} - 2\beta_0 Z_{qg}^{(T)N(1)} - Z_{gq}^{(T)N(1)}\bar{\gamma}_{qq}^{N(0)}$$
$$+Z_{qq}^{(T)N(1)}\bar{\gamma}_{gg}^{N(0)} - Z_{gg}^{(T)N(1)}\bar{\gamma}_{gq}^{N(0)} + Z_{gq}^{(T)N(1)}\bar{\gamma}_{qg}^{N(0)}$$
$$+\frac{\bar{C}_{L,q}^{N(1)}}{\bar{C}_{L,g}^{N(1)}}\left[-2\beta_0 Z_{qq}^{(T)N(1)} - Z_{qg}^{(T)N(1)}\bar{\gamma}_{gq}^{N(0)} + Z_{gq}^{(T)N(1)}\bar{\gamma}_{qg}^{N(0)}\right]$$
$$+\left(\frac{\bar{C}_{L,q}^{N(1)}}{\bar{C}_{L,g}^{N(1)}}\right)^2\left[2\beta_0 Z_{gq}^{(T)N(1)} - Z_{gq}^{(T)N(1)}\bar{\gamma}_{qq}^{N(0)} - Z_{gg}^{(T)N(1)}\bar{\gamma}_{qg}^{N(0)}\right.$$
$$\left.+Z_{qq}^{(T)N(1)}\bar{\gamma}_{qg}^{N(0)} + Z_{qg}^{(T)N(1)}\bar{\gamma}_{gg}^{N(0)}\right]$$
$$+\frac{\bar{C}_{L,q}^{N(1)}}{\bar{C}_{L,g}^{N(1)}}\left[2\beta_0 Z_{gg}^{(T)N(1)} - Z_{qg}^{(T)N(1)}\bar{\gamma}_{gq}^{N(0)} + Z_{gq}^{(T)N(1)}\bar{\gamma}_{qg}^{N(0)}\right]$$

The above operations are difficult to perform in x-space due to multiple direct & multiple inverse MELLIN convolutions.

The above substitutions yield:

$$\begin{aligned} \delta K_{22}^{N(1)} &= 0 \\ \delta K_{L2}^{N(1)} &= 0 \\ \delta K_{2L}^{N(1)} &= 0 \\ \delta K_{LL}^{N(1)} &= 0 \end{aligned}$$

$\Rightarrow$ DLY-Relation to $O(\alpha_s^2)$.

The transforms for the F_2-$\partial F_2/\partial t$ system read:

$$\begin{aligned} \delta K_{d2} &= \frac{\beta_0}{2}\left[\delta C_{2q}^{N(1)} - Z_{qq}^{(T)N(1)}\right]\left[\bar{\gamma}_{qq}^{N(0)} + \bar{\gamma}_{gg}^{N(0)} - 2\beta_0\right] \\ & -\frac{\beta_0}{2\bar{\gamma}_{gq}^{N(0)}}\left[\delta C_{2g}^{N(1)} - Z_{qg}^{(T)N(1)}\right] \\ & \left[(\bar{\gamma}_{qq}^{N(0)})^2 - \bar{\gamma}_{qq}^{N(0)}\bar{\gamma}_{gg}^{N(0)} + 2\bar{\gamma}_{qg}^{N(0)}\bar{\gamma}_{gq}^{N(0)} - 2\beta_0\bar{\gamma}_{qq}^{N(0)}\right] \end{aligned}$$

$$\begin{aligned} \delta K_{dd} &= -2\frac{\beta_0}{\bar{\gamma}_{gq}^{N(0)}}\left[\delta C_{2g}^{N(1)} - Z_{qg}^{(T)N(1)}\right]\left[\bar{\gamma}_{qq}^{N(0)} - \bar{\gamma}_{gg}^{N(0)} - 2\beta_0\right] \\ & +4\beta_0\left[\delta C_{1q}^{N(1)} - Z_{qq}^{(T)N(1)}\right] \end{aligned}$$

The above substitutions yield:

$$\delta K_{d2} = 0$$
$$\delta K_{dd} = 0$$

$\Rightarrow$ DLY-Relation to $O(\alpha_s^2)$.

$\Rightarrow$ The Evolution of Observables using physical evolution kernels is related by an analytic continuation from the space-like to the time-like domain up to $O(\alpha_s^2)$.

Gribov-Lipatov Relation (1972)

$$\overline{K}(x_E, Q^2) = K(x_B, Q^2)$$

This relation holds for the LO non-singlet contributions and some pieces in the NLO non-singlet contributions, but is generally violated beyond LO.

5. Conclusions

- The scale evolution of structure and fragmentation functions can be represented in terms of physical evolution kernels and observable non-perturbative input distributions.
- The physical evolution kernels of either choice of observables are related for the evolution of structure and fragmentation functions by an analytic continuation (DLY relation) from $0 \leq x < 1$ to $1 < x < \infty$ up to $O(\alpha_s^2)$.
  The Gribov–Lipatov relation is violated beyond LO.
- An extension of the present investigation to $O(\alpha_s^3)$ requires the knowledge of the hitherto unknown 3–loop singlet anomalous dimensions. The DLY relation for the evolution kernels is not necessarily expected to hold to arbitray high orders due to the emergence of new production thresholds for the s-channel process.
- An interesting test of QCD can be carried out in comparing the scaling violations of structure and fragmentation functions using factorization scheme–independent evolution equations.