Higher order QCD corrections for deep-inelastic scattering

Orsay, LHeC meeting

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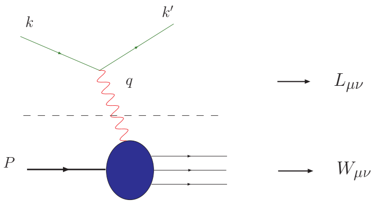
Outline

Higher order QCD corrections

- Massless corrections (unpolarized and polarized)
  - anomalous dimensions
  - massless Wilson coefficients
- Massive Wilson coefficients (unpolarized and polarized)
  - single mass corrections
  - two-mass corrections
- Scheme invariant evolution and highest precision for $\alpha_s(M_Z^2)$

- Goal: anomalous dimensions: 4 loops
- Goal: Wilson coefficients: 3 loops (massless and massive).
- Needed to match LHC, EIC, & LHeC measurement precision.
- EIC and LHeC shall run with deuterons.
- Use NS scheme–invariant evolution to perform an unprecedented $\alpha_s(M_Z^2)$ measurement form DIS.
- Already now one shall work towards a polarized LHC to perform *pp* polarized scattering and later polarized *ep* scattering.

Deep–inelastic scattering



- Kinematic invariants:

$$Q^2 = -q^2, \qquad x = \frac{Q^2}{2P.q}$$

- The cross section factorizes into leptonic and hadronic tensor:

$$\frac{d^2\sigma}{dQ^2 dx} \sim L_{\mu\nu} W^{\mu\nu}$$

- The hadronic tensor can be expressed through structure functions:

$$W_{\mu\nu} = \frac{1}{4\pi} \int d^4\xi \exp(iq\xi) \langle P, | \left[J_\mu^{\text{em}}(\xi), J_\nu^{\text{em}}(\xi)\right] | P\rangle$$

$$= \frac{1}{2x}\left(g_{\mu\nu} + \frac{q_\mu q_\nu}{Q^2}\right) F_L(x, Q^2) + \frac{2x}{Q^2}\left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu}\right) F_2(x, Q^2)$$

$$+ i\epsilon_{\mu\nu\rho\sigma} \frac{q^\rho S^\sigma}{q \cdot P} g_1(x, Q^2) + i\epsilon_{\mu\nu\rho\sigma} \frac{q^\rho \left(q \cdot P S^\sigma - q \cdot S P^\sigma\right)}{(q \cdot P)^2} g_2(x, Q^2)$$

- F_L, F_2, g_1 and g_2 contain contributions from both, charm and bottom quarks.

3–loop anomalous dimensions and massless Wilson coefficients

- All non-singlet anomalous dimensions were calculated in a fully automated way. S. Moch, J. Vermaseren, A. Vogt, Nucl.Phys.B 688 (2004) 101, 691 (2004) 129; JB, P. Marquard, C. Schneider, K. Schönwald, Nucl.Phys.B 971 (2021) 115542
- Also the polarized singlet anomalous dimensions were computed. S. Moch, J. Vermaseren, A. Vogt, Nucl.Phys.B 889 (2014) 351; JB, P. Marquard, C. Schneider, K. Schönwald, JHEP 01 (2022) 193
- The method used are off-shell gauge variant massless operator matrix elements, using our method of arbitrary high Mellin moments. JB, C. Schneider, Phys.Lett.B 771 (2017) 31.
- In the unpolarized case the complete singlet anomalous dimensions are obtained at 2-loop order, correcting errors in the foregoing literature. JB, P. Marquard, C. Schneider, K. Schönwald, Nucl.Phys.B 980 (2022) 115794
- The anomalous dimensions for transversity were calculated first.
- In earlier calculations S. Moch, J. Vermaseren, A. Vogt, Nucl.Phys.B 688 (2004) 101, 691 (2004) 129 various special assumptions have been made, which we all did not assume. Those were now verified for the case of general N for the first time by the present calculations.
- The present studies prepare for the 4–loop calculations.

3–loop anomalous dimensions and massless Wilson coefficients

The final goal is to calculate the anomalous dimensions to 4–loop order for general values of Mellin N. A series of moments and first results on the simplest color factors exist. S. Moch et al., JHEP 10 (2017) 041, Phys.Lett.B 825 (2022) 136853

- The upolarized 3–loop Wilson coefficients have been calculated in J. Vermaseren, A. Vogt, S. Moch, Nucl.Phys.B 724 (2005) 3; JB, P. Marquard, C. Schneider, K.Schönwald, 2208.14325 [hep-ph]
- The polarized 3–loop Wilson coefficients have been computed in JB, P. Marquard, C. Schneider, K.Schönwald, 2208.14325 [hep-ph]

Evolution equations

$$\Sigma(N) = \sum_{k=1}^{N_F} q_k(N) + \bar{q}_k(N)$$

$$\gamma_{ij} = \sum_{k=0}^{\infty} a_s^{k+1} \gamma_{ij}$$

$$\frac{dq^{\mathrm{NS},\pm,(\mathrm{s})}(N,a_s)}{d\ln(\mu^2)} = \sum_{k=0}^{\infty} a_s^{k+1} \gamma_{qq}^{(k),\mathrm{NS},\pm(\mathrm{s})}(N) \cdot q^{\mathrm{NS},\pm(\mathrm{s})}(N,a_s)$$

$$\frac{d}{d\ln(\mu^2)} \begin{pmatrix} \Sigma(N) \\ G(N) \end{pmatrix} = \sum_{k=0}^{\infty} a_s^{k+1} \begin{pmatrix} \gamma_{qq}^{(k)} & \gamma_{qg}^{(k)} \\ \gamma_{gq}^{(k)} & \gamma_{gg}^{(k)} \end{pmatrix} \cdot \begin{pmatrix} \Sigma(N) \\ G(N) \end{pmatrix}$$

The non-singlet anomalous dimension $\gamma_{qq}^{+,\mathrm{NS}}$

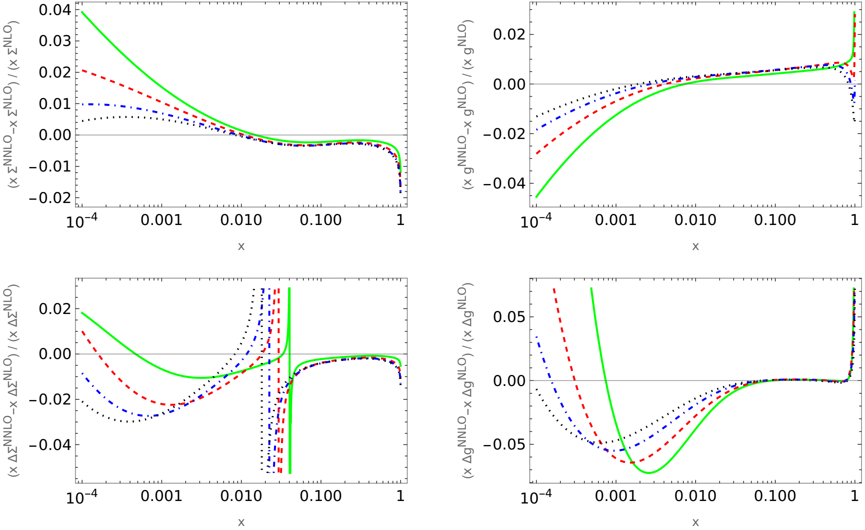
$$
\begin{aligned}
\gamma_{\rm NS}^{(2),+} = {} & \frac{1}{2}\left[1+(-1)^N\right] \\
& \times \Bigl\{ C_F^2 \Bigl\{ C_A \Bigl[\frac{72P_3}{N^2(1+N)^2}\zeta_3 + \frac{32P_{15}}{9N^2(1+N)^2}S_{-2,1} - \frac{16P_{17}}{9N^2(1+N)^2}S_3 + \frac{P_{33}}{18N^4(1+N)^4} \\
& + \Bigl(-\frac{16P_{29}}{9N^4(1+N)^4} - \frac{4288}{9}S_2 + \frac{64(-12+31N+31N^2)}{3N(1+N)}S_3 + 320S_4 - 1024S_{3,1} \\
& + \frac{64(-84+31N+31N^2)}{3N(1+N)}S_{-2,1} + 3712S_{-2,2} + 3840S_{-3,1} - 7168S_{-2,1,1} \Bigr) S_1 + \Bigl(256S_3 \\
& + 1792S_{-2,1} \Bigr) S_1^2 + \Bigl(\frac{4P_{19}}{9N^2(1+N)^2} - 832S_3 - 5248S_{-2,1} \Bigr) S_2 + \frac{352}{3}S_2^2 \\
& + \frac{16(-30+151N+151N^2)}{3N(1+N)}S_4 + \Bigl(-\frac{16P_{22}}{9N^2(1+N)^3} + (-\frac{64P_9}{9N^2(1+N)^2} - 256S_2 \Bigr) S_1 \\
& + \frac{32(12+31N+31N^2)S_2}{3N(1+N)} + 64S_3 + 5376S_{2,1} - 384S_{-2,1} + 576\zeta_3 \Bigr) S_{-2} \\
& + \Bigl(-\frac{32(8+3N+3N^2)}{N(1+N)} + 512S_1 \Bigr) S_{-2}^2 + \Bigl(\frac{32(108+31N+31N^2)}{3N(1+N)}S_1 - \frac{16P_{16}}{9N^2(1+N)^2} \\
& - 1152S_1^2 + 2624S_2 + 960S_{-2} \Bigr) S_{-3} + \Bigl(\frac{16(138+35N+35N^2)}{3N(1+N)} - 1472S_1 \Bigr) S_{-4} \\
& + 2304S_{-5} + 768S_{2,3} + 2688S_{2,-3} - \frac{64(-24+29N+29N^2)}{3N(1+N)}S_{3,1} - 768S_{4,1} \\
& + \frac{32(-174+31N+31N^2)}{3N(1+N)}S_{-2,2} - 3648S_{-2,3} - \frac{1920}{N(1+N)}S_{-3,1} + 1728S_{-4,1} \\
& - 5376S_{2,1,-2} + 1536S_{3,1,1} - \frac{128(-84+31N+31N^2)}{3N(1+N)}S_{-2,1,1} - 1536S_{-2,1,-2} \\
& - 5376S_{-2,2,1} - 5376S_{-3,1,1} + 10752S_{-2,1,1,1} \Bigr] + T_F N_F \Bigl[-\frac{16P_5}{9N^2(1+N)^2}S_2 + \frac{4P_{34}}{9N^4(1+N)^4} \\
& + \Bigl(-\frac{8P_{13}}{9N^2(1+N)^2} + \frac{1280}{9}S_2 - \frac{512}{3}S_3 - \frac{512}{3}S_{-2,1} + 128\zeta_3 \Bigr) S_1 - \frac{128}{3}S_2^2 \\
& + \frac{64(12+29N+29N^2)}{9N(1+N)}S_3 - \frac{512}{3}S_4 + \Bigl(-\frac{128(-3+10N+16N^2)}{9N^2(1+N)^2} + \frac{2560}{9}S_1 - \frac{256}{3}S_2 \Bigr) \\
& \times S_{-2} + \Bigl(\frac{128(3+10N+10N^2)}{9N(1+N)} - \frac{256}{3}S_1 \Bigr) S_{-3} - \frac{256}{3}S_{-4} - \frac{256(-3+10N+10N^2)}{9N(1+N)}S_{-2,1} \\
& + \frac{256}{3}S_{3,1} - \frac{256}{3}S_{-2,2} + \frac{1024}{3}S_{-2,1,1} - \frac{32(2+3N+3N^2)}{N(1+N)}\zeta_3 \Bigr] \Bigr\} \\
& + C_F \Bigl\{ T_F^2 N_F^2 \Bigl[\frac{8P_{28}}{27N^3(1+N)^3} - \frac{128}{27}S_1 - \frac{640}{27}S_2 + \frac{128}{9}S_3 \Bigr] + C_A^2 \Bigl[-\frac{24P_3}{N^2(1+N)^2}\zeta_3 \\
& \frac{32P_{11}}{9N^2(1+N)^2}S_{-2,1} + \frac{8P_{18}}{9N^2(1+N)^2}S_3 + \frac{P_{32}}{54N^3(1+N)^3} + \Bigl(\frac{4P_{35}}{3N^4(1+N)^4} \\
& - \frac{16(-8+11N+11N^2)}{N(1+N)}S_3 - 256S_4 + 512S_{3,1} - \frac{64(-24+11N+11N^2)}{3N(1+N)}S_{-2,1} \\
& - 1024S_{-2,2} - 1024S_{-3,1} + 2048S_{-2,1,1} \Bigr) S_1 + \Bigl(-128S_3 - 512S_{-2,1} \Bigr) S_1^2 + \Bigl(-\frac{8344}{27} \\
& + 384S_3 + 1536S_{-2,1} \Bigr) S_2 - \frac{16(-24+55N+55N^2)}{3N(1+N)}S_4 + 64S_5 + \Bigl(\frac{32P_{10}}{9N^2(1+N)^2}S_1 \\
& + \frac{16P_{27}}{9N^3(1+N)^3} - \frac{352}{3}S_2 - 64S_3 - 1536S_{2,1} + 128S_{-2,1} - 192\zeta_3 \Bigr) S_{-2} \\
& + \Bigl(\frac{48(2+N+N^2)}{N(1+N)} - 192S_1 \Bigr) S_{-2}^2 + \Bigl(\frac{16P_{12}}{9N^2(1+N)^2} - \frac{32(24+11N+11N^2)}{3N(1+N)}S_1 \\
& + 256S_1^2 - 768S_2 - 320S_{-2} \Bigr) S_{-3} + \Bigl(-\frac{16(30+13N+13N^2)}{3N(1+N)} + 320S_1 \Bigr) S_{-4} \\
& - 704S_{-5} - 384S_{2,3} - 768S_{2,-3} + \frac{64(-12+11N+11N^2)}{3N(1+N)}S_{3,1} + 384S_{4,1} \\
& - \frac{32(-48+11N+11N^2)}{3N(1+N)}S_{-2,2} + 1088S_{-2,3} + \frac{512}{N(1+N)}S_{-3,1} - 448S_{-4,1} \\
& + 1536S_{2,1,-2} - 768S_{3,1,1} + \frac{128(-24+11N+11N^2)}{3N(1+N)}S_{-2,1,1} + 512S_{-2,1,-2} + 1536[S_{-2,2,1} \\
& + S_{-3,1,1}] - 3072S_{-2,1,1,1} \Bigr] + C_A T_F N_F \Bigl[-\frac{8P_{31}}{27N^3(1+N)^3} + \Bigl(-\frac{16P_{30}}{27N^3(1+N)^3} + 64S_3 \\
& + \frac{256}{3}S_{-2,1} - 128\zeta_3 \Bigr) S_1 + \frac{5344}{27}S_2 - \frac{32(3+14N+14N^2)}{3N(1+N)}S_3 + \frac{320}{3}S_4 + \Bigl(-\frac{1280}{9}S_1 \\
& + \frac{64(-3+10N+16N^2)}{9N^2(1+N)^2} + \frac{128}{3}S_2 \Bigr) S_{-2} + \Bigl(-\frac{64(3+10N+10N^2)}{9N(1+N)} + \frac{128}{3}S_1 \Bigr) S_{-3} \\
& + \frac{128}{3}S_{-4} - \frac{256}{3}S_{3,1} + \frac{128(-3+10N+10N^2)}{9N(1+N)}S_{-2,1} + \frac{128}{3}S_{-2,2} - \frac{512}{3}S_{-2,1,1} \\
& + \frac{32(2+3N+3N^2)}{N(1+N)}\zeta_3 \Bigr] \Bigr\} + C_F^3 \Bigl\{ -\frac{48P_3}{N^2(1+N)^2}\zeta_3 + \frac{8P_4}{N^2(1+N)^2}S_3 + \frac{P_{36}}{N^5(1+N)^5} \\
& + \Bigl(\frac{8P_{26}}{N^4(1+N)^4} - \frac{128(1+2N)}{N^2(1+N)^2}S_2 + 128S_2^2 - 384S_3 + 128S_4 + 512S_{3,1} - 3328S_{-2,2} \\
& - \frac{384(-4+N+N^2)}{N(1+N)}S_{-2,1} - 3584S_{-3,1} + 6144S_{-2,1,1} \Bigr) S_1 + \Bigl(-\frac{64(1+3N+3N^2)}{N^3(1+N)^3} \\
& - 1536S_{-2,1} \Bigr) S_1^2 + \Bigl(\frac{4P_{25}}{N^3(1+N)^3} + 512S_3 + 4352S_{-2,1} \Bigr) S_2 - \frac{32(2+3N+3N^2)}{N(1+N)}S_2^2 \\
& \frac{32(2+15N+15N^2)}{N(1+N)}S_4 + (\frac{32P_{24}}{N^3(1+N)^3} + \Bigl(-\frac{128(5+7N+3N^2)}{N^2(1+N)^2} + 512S_2 \Bigr) S_1 \\
& - \frac{64(4+3N+3N^2)}{N(1+N)}S_2 + 128S_3 - 4608S_{2,1} + 256S_{-2,1} - 384\zeta_3 \Bigr) S_{-2} + \Bigl(\frac{128}{N(1+N)} \\
& - 256S_1 \Bigr) S_{-2}^2 + \Bigl(\frac{32(8+5N+9N^2)}{N^2(1+N)^2} - \frac{64(20+3N+3N^2)}{N(1+N)}S_1 + 1280S_1^2 - 2176S_2 \\
& - 640S_{-2} \Bigr) S_{-3} + \Bigl(-\frac{32(26+3N+3N^2)}{N(1+N)} + 1664S_1 \Bigr) S_{-4} - 1792S_{-5} - 384S_{2,3} \\
& - 2304S_{2,-3} + \frac{128(-2+3N+3N^2)}{N(1+N)}S_{3,1} + 384S_{4,1} - \frac{64(4-N+3N^2)}{N^2(1+N)^2}S_{-2,1} \\
& - \frac{64(-26+3N+3N^2)}{N(1+N)}S_{-2,2} + 2944S_{-2,3} + \frac{1792}{N(1+N)}S_{-3,1} - 1664S_{-4,1} \\
& + 4608S_{2,1,-2} - 768S_{3,1,1} + \frac{768(-4+N+N^2)}{N(1+N)}S_{-2,1,1} + 1024S_{-2,1,-2} \\
& + 4608[S_{-2,2,1} + S_{-3,1,1}] - 9216S_{-2,1,1,1} \Bigr\} \Bigr\},
\end{aligned}
$$

The unpolarized and polarized Wilson coefficients fill around 160 pages in its most compact representations.

The polarized singlet anomalous dimension: $\Delta\gamma_{gg}^{(2)}$

$$
\begin{aligned}
\Delta\gamma_{gg}^{(2)} = \; & C_A T_F^2 N_F^2 \Biggl[-\frac{16P_8}{27N^2(1+N)^2}S_1 - \frac{4P_{48}}{27N^3(1+N)^3}\Biggr] + C_F\Biggl[T_F^2 N_F^2\Biggl[-\frac{8P_{59}}{27N^4(1+N)^4} \\
& + \frac{64(N-1)(2+N)\bigl(-6-8N+N^2\bigr)}{9N^3(1+N)^3}S_1 + \frac{32(N-1)(2+N)}{3N^2(1+N)^2}S_1^2 \\
& - \frac{32(N-1)(2+N)}{N^2(1+N)^2}S_2\Biggr] + C_A T_F N_F\Biggl[\frac{8P_6}{N^3(1+N)^3}S_2 - \frac{8P_9}{3N^3(1+N)^3}S_1^2 \\
& + \frac{2P_{77}}{27(N-1)N^5(1+N)^5(2+N)} + \Biggl(-\frac{8P_{67}}{9(-1+N)N^4(1+N)^4(2+N)} \\
& - \frac{32(N-1)(2+N)}{N^2(1+N)^2}S_2 + 128\zeta_3\Biggr)S_1 + \frac{32(N-1)(2+N)}{3N^2(1+N)^2}S_1^3 - \frac{32\bigl(34+N+N^2\bigr)}{3N^2(1+N)^2} \\
& \times S_3 + \Biggl(\frac{128P_2}{(N-1)N^2(1+N)^2(2+N)}S_1 - \frac{32P_{23}}{(N-1)N^2(1+N)^3(2+N)}\Biggr)S_{-2} \\
& - \frac{192\bigl(4-N-N^2\bigr)}{N^2(1+N)^2}S_{-3} + \frac{64(N-1)(2+N)}{N^2(1+N)^2}S_{2,1} - \frac{128\bigl(-8+N+N^2\bigr)}{N^2(1+N)^2}S_{-2,1} \\
& - \frac{64(-3+N)(4+N)}{N^2(1+N)^2}\zeta_3\Biggr]\Biggr] + C_A^3\Biggl[\frac{64P_{16}}{9N^2(1+N)^2}S_{-2,1} - \frac{32P_{18}}{9N^2(1+N)^2}S_3 \\
& + \frac{P_{74}}{27(N-1)N^5(1+N)^5(2+N)} + \Biggl(\frac{4P_{69}}{9(N-1)N^4(1+N)^4(2+N)} \\
& - \frac{64P_{17}}{9N^2(1+N)^2}S_2 + 128S_2^2 + \frac{16\bigl(-96+11N+11N^2\bigr)}{3N(1+N)}S_3 + 192S_4 \\
& + \frac{1024}{N(1+N)}S_{-2,1} - 640S_{-2,2} - 768S_{-3,1} + 1024S_{-2,1,1}\Biggr)S_1 \\
& + \Biggl(-\frac{256\bigl(1+3N+3N^2\bigr)}{N^3(1+N)^3} + 128S_3 - 256S_{-2,1}\Biggr)S_1^2 + \Biggl(-\frac{16P_{41}}{9N^3(1+N)^3} \\
& + 64S_3 + 640S_{-2,1}\Biggr)S_2 - \frac{256}{N(1+N)}S_2^2 - \frac{384}{N(1+N)}S_4 + 64S_5 \\
& + \Biggl(\frac{32P_{52}}{9(N-1)N^3(1+N)^3(2+N)} + \Bigl(-\frac{64P_{32}}{9(-1+N)N(1+N)^2(2+N)} + 256S_2\Biggr) \\
& \times S_1 - \frac{512}{N(1+N)}S_2 + 128S_3 - 768S_{2,1}\Biggr)S_{-2} + \Biggl(-\frac{16\bigl(24+11N+11N^2\bigr)}{3N(1+N)} \\
& + 64S_1\Biggr)S_{-2}^2 + \Biggl(-\frac{32P_{15}}{9N^2(1+N)^2} - \frac{1536}{N(1+N)}S_1 + 384S_1^2 - 320S_2\Biggr)S_{-3} \\
& + \Biggl(-\frac{1024}{N(1+N)} + 512S_1\Biggr)S_{-4} - 192S_{-5} - 384S_{2,-3} + \frac{1280}{N(1+N)}S_{-2,2} \\
& + 384S_{-2,3} + \frac{1536}{N(1+N)}S_{-3,1} - 384S_{-4,1} + 768S_{2,1,-2} - \frac{2048}{N(1+N)}S_{-2,1,1} \\
& + 768[S_{-2,2,1} + S_{-3,1,1}] - 1536S_{-2,1,1,1}\Biggr] \\
& + C_F^2 T_F N_F\Biggl[-\frac{4P_{75}}{(N-1)N^5(1+N)^5(2+N)} + \Biggl(\frac{32(N-1)(2+N)S_2}{N^2(1+N)^2} \\
& - \frac{16P_{42}}{N^4(1+N)^4}\Biggr)S_1 + \frac{8(N-1)(2+N)\bigl(2+3N+3N^2\bigr)}{N^3(1+N)^3}S_1^2 - \frac{32(N-1)(2+N)}{3N^2(1+N)^2} \\
& \times S_1^3 - \frac{8(2+N)\bigl(2-11N-16N^2+9N^3\bigr)}{N^3(1+N)^3}S_2 + \frac{32\bigl(10+7N+7N^2\bigr)}{3N^2(1+N)^2}S_3 \\
& + \Biggl(-\frac{64\bigl(10+N+N^2\bigr)}{(N-1)N(1+N)(2+N)} + \frac{512}{N^2(1+N)^2}S_1\Biggr)S_{-2} + \frac{256}{N^2(1+N)^2}S_{-3} \\
& - \frac{64(N-1)(2+N)}{N^2(1+N)^2}S_{2,1} - \frac{512}{N^2(1+N)^2}S_{-2,1} + \frac{192\bigl(-2-N-N^2\bigr)}{N^2(1+N)^2}\zeta_3\Biggr] \\
& + C_A^2 T_F N_F\Biggl[\frac{32P_4}{9N^2(1+N)^2}S_2 + \frac{32P_{11}}{9N^2(1+N)^2}S_{-3} - \frac{64P_{11}}{9N^2(1+N)^2}S_{-2,1} \\
& + \frac{16P_{13}}{9N^2(1+N)^2}S_3 + \frac{2P_{76}}{27(N-1)N^5(1+N)^5(2+N)} + \Biggl(\frac{1280}{9}S_2 - \frac{64}{3}S_3 \\
& - \frac{8P_{68}}{27(-1+N)N^4(1+N)^4(2+N)} - 128\zeta_3\Biggr)S_1 + \frac{64}{3}S_{-2}^2 \\
& + \Biggl(\frac{64P_{45}}{9(N-1)N^2(1+N)^2(2+N)}S_1 - \frac{32P_{50}}{9(N-1)N^3(1+N)^3(2+N)}\Biggr)S_{-2} \\
& + \frac{128\bigl(-3+2N+2N^2\bigr)}{N^2(1+N)^2}\zeta_3\Biggr]
\end{aligned}
$$

The unpolarized and polarized NNLO evolution



$Q^2 = 10, 10^2, 10^3, 10^4$ GeV^2 dotted, dash-dotted, dashed, full lines.

Heavy Quark Corrections to Deep-Inelastic Scattering

- The scaling violations of the heavy quark corrections are quite different from those of massless quarks.
- Work in the region $Q^2 \gg m_Q^2$, also to avoid higher twist corrections.
- Under these conditions the heavy flavor corrections are given by the massive operator matrix elements (OMEs) A_{ij} and the massless process-dependent Wilson coefficients.
- Analytic calculations are possible to the 3–loop level for single and two mass corrections.
- The corrections are needed in the unpolarized and the polarized case.
- The massive OMEs also form the transition matrix element in the Variable Flavor Number Scheme describing the behaviour of massive and massless partons in the high energy range at colliders.
- Measurement goals: improve the current values at the theoretical side.
    - $\alpha_s(M_Z^2) = 0.1147 \pm 0.0008$
    - $m_c(m_c) = 1.252 \pm 0.018(exp) \begin{array}{c} +0.03 \\ -0.02 \end{array} (scale), \begin{array}{c} +0.00 \\ -0.07 \end{array} (thy)\mathrm{GeV}$ ($\overline{\mathrm{MS}}$-scheme)

The massive Wilson coefficients at large Q^2

$$L_{q,(2,L)}^{\rm NS}(N_F+1) = a_s^2\left[A_{qq,Q}^{(2),\rm NS}(N_F+1)\delta_2 + \hat{C}_{q,(2,L)}^{(2),\rm NS}(N_F)\right] + a_s^3\left[A_{qq,Q}^{(3),\rm NS}(N_F+1)\delta_2 + A_{qq,Q}^{(2),\rm NS}(N_F+1)C_{q,(2,L)}^{(1),\rm NS}(N_F+1) + \hat{C}_{q,(2,L)}^{(3),\rm NS}(N_F)\right]$$

$$L_{q,(2,L)}^{\rm PS}(N_F+1) = a_s^3\left[A_{qq,Q}^{(3),\rm PS}(N_F+1)\delta_2 + N_F A_{gq,Q}^{(2),\rm NS}(N_F)\tilde{C}_{g,(2,L)}^{(1),\rm NS}(N_F+1) + N_F\hat{\tilde{C}}_{q,(2,L)}^{(3),\rm PS}(N_F)\right]$$

$$L_{g,(2,L)}^{\rm S}(N_F+1) = a_s^2 A_{gg,Q}^{(1)}(N_F+1)N_F\tilde{C}_{g,(2,L)}^{(2)}(N_F+1) + a_s^3\left[A_{qg,Q}^{(3)}(N_F+1)\delta_2 + A_{gg,Q}^{(1)}(N_F+1)N_F\tilde{C}_{g,(2,L)}^{(2)}(N_F+1)\right.$$
$$\left. + A_{gg,Q}^{(2)}(N_F+1)N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + A_{Qg}^{(1)}(N_F+1)N_F\tilde{C}_{q,(2,L)}^{(2),\rm PS}(N_F+1) + N_F\hat{\tilde{C}}_{g,(2,L)}^{(3)}(N_F)\right]$$

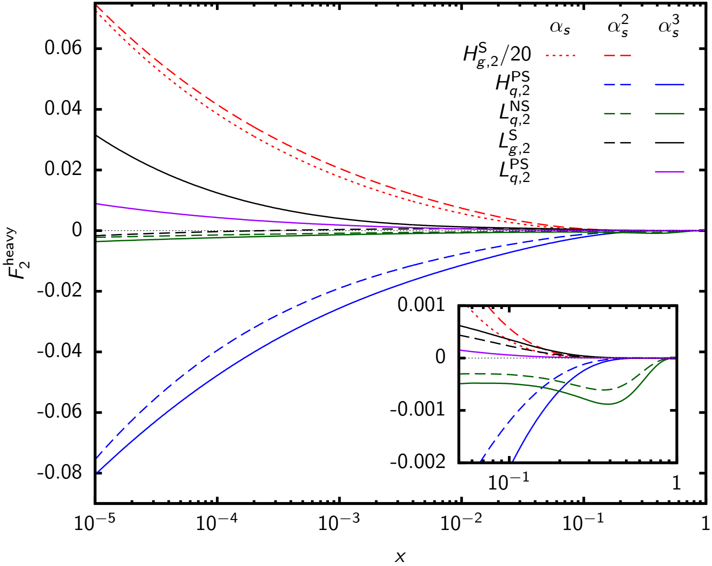
$$H_{q,(2,L)}^{\rm PS}(N_F+1) = a_s^2\left[A_{Qq}^{(2),\rm PS}(N_F+1)\delta_2 + \tilde{C}_{q,(2,L)}^{(2),\rm PS}(N_F+1)\right]$$
$$+ a_s^3\left[A_{Qq}^{(3),\rm PS}(N_F+1)\delta_2 + A_{gq,Q}^{(2)}(N_F+1)\tilde{C}_{g,(1,L)}^{(2)}(N_F+1) + A_{Qq}^{(2),\rm PS}(N_F+1)\tilde{C}_{q,(2,L)}^{(1),\rm NS}(N_F+1) + \tilde{C}_{q,(2,L)}^{(3),\rm PS}(N_F+1)\right]$$

$$H_{g,(2,L)}^{\rm S}(N_F+1) = a_s\left[A_{Qg}^{(1)}(N_F+1)\delta_2 + \tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\right]$$
$$+ a_s^2\left[A_{Qg}^{(2)}(N_F+1)\delta_2 + A_{Qg}^{(1)}(N_F+1)\tilde{C}_{q,(2,L)}^{(1)}(N_F+1) + A_{gg,Q}^{(1)}(N_F+1)\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + \tilde{C}_{g,(2,L)}^{(2)}(N_F+1)\right]$$
$$+ a_s^3\left[A_{Qg}^{(3)}(N_F+1)\delta_2 + A_{Qg}^{(2)}(N_F+1)\tilde{C}_{q,(2,L)}^{(1)}(N_F+1) + A_{gg,Q}^{(2)}(N_F+1)\tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\right.$$
$$\left. + A_{Qg}^{(1)}(N_F+1)\tilde{C}_{q,(2,L)}^{(2),\rm S}(N_F+1) + A_{gg,Q}^{(1)}(N_F+1)\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + \tilde{C}_{g,(2,L)}^{(3)}(N_F+1)\right]$$

- All first order factorizable contributions and $O(1000)$ fixed moments of $A_{Qg}^{(3)}$ are known.
- The case for two different masses obeys an analogous representation.

I. Bierenbaum, JB, S. Klein, Nucl.Phys.B 820 (2009) 417

Heavy Flavor contribution to F_2



JB et al. PoS QCDEV2017 (2017) 031

The variable flavor number scheme

- Matching conditions for parton distribution functions:

$$f_k(N_F+2) + f_{\bar{k}}(N_F+2) = A_{qq,Q}^{\rm NS}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) \cdot \left[f_k(N_F) + f_{\bar{k}}(N_F)\right] + \frac{1}{N_F} A_{qq,Q}^{\rm PS}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) \cdot \Sigma(N_F)$$
$$+ \frac{1}{N_F} A_{qg,Q}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) \cdot G(N_F)\,,$$

$$f_Q(N_F+2) + f_{\bar{Q}}(N_F+2) = A_{Qq}^{\rm PS}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2},\right) \cdot \Sigma(N_F) + A_{Qg}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) \cdot G(N_F)\,,$$

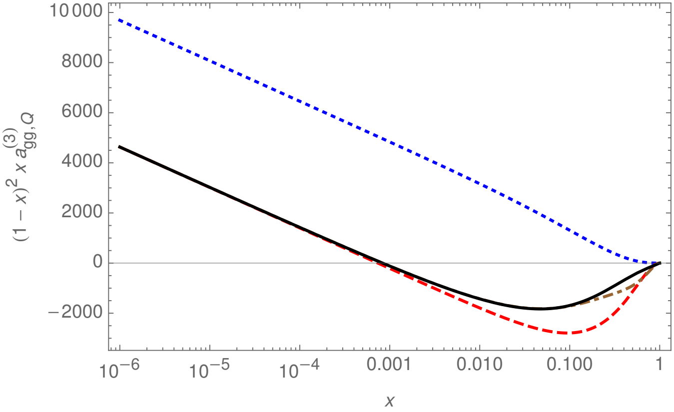
$$\Sigma(N_F+2) = \left[A_{qq,Q}^{\rm NS}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) + A_{qq,Q}^{\rm PS}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) + A_{Qq}^{\rm PS}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right)\right] \cdot \Sigma(N_F)$$
$$+ \left[A_{qg,Q}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) + A_{Qg}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right)\right] \cdot G(N_F)\,,$$

$$G(N_F+2) = A_{gq,Q}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) \cdot \Sigma(N_F) + A_{gg,Q}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) \cdot G(N_F)\,.$$

J. Ablinger et al., Nucl.Phys.B 921 (2017) 585

Results: $A_{gg,Q}^{(3)}$

Single mass contributions: just finished.

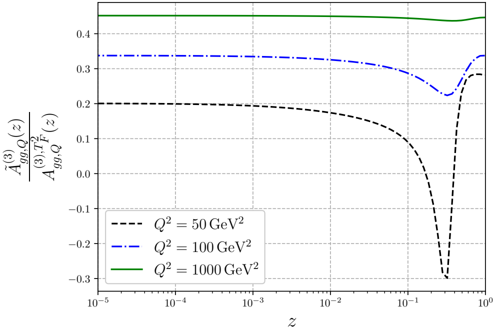


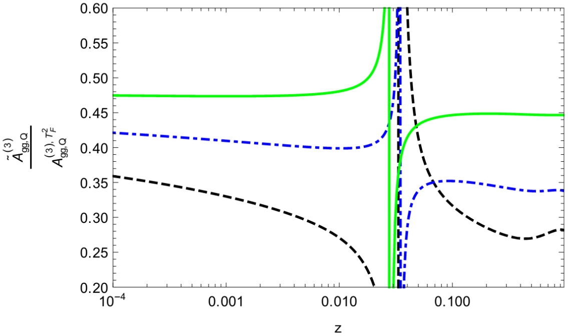
blue dotted: small x "leading term": $\propto \ln(x)/x$ completely off.

full line: complete result

Results: $A_{gg,Q}^{(3)}$

The two mass contributions over the whole T_F^2- contributions to the OME $A_{gg,Q}^{(3)}$:



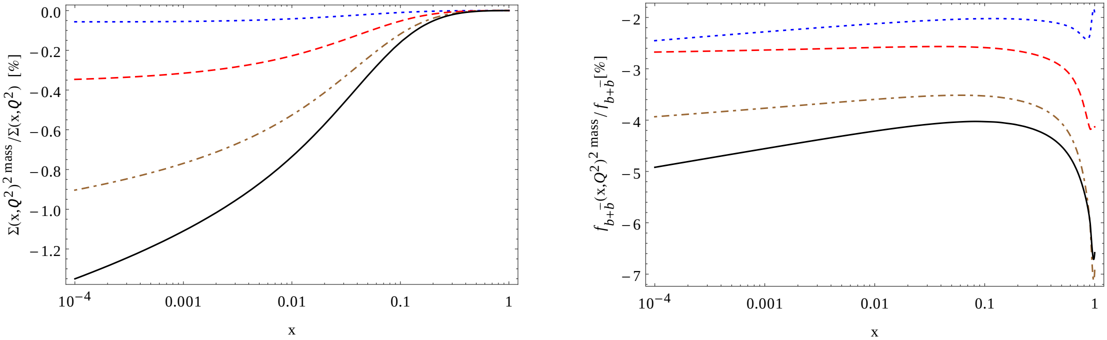


J. Ablinger, JB, A. De Freitas, C. Schneider , and K. Schönwald, Nucl.Phys.B 932 (2018) 129

J. Ablinger, JB, A. De Freitas, A. Goedicke, M. Saragnese, C. Schneider, K. Schönwald, Nucl.Phys.B 955 (2020) 115059

The variable flavor number scheme at NLO

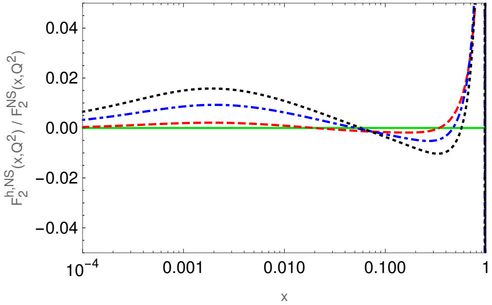
$\Sigma(x, Q^2)^{\text{2-mass}} / \Sigma(x, Q^2)$ $\quad$ $f_{b+\bar{b}}(x, Q^2)^{\text{2-mass}} / f_{b+\bar{b}}(x, Q^2)$

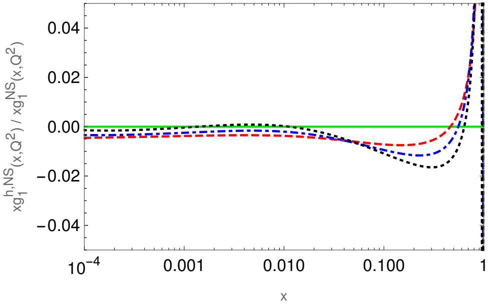


- The ratio of the 2-mass contributions to the singlet parton distribution $\Sigma(x, Q^2)$ (left) and the heavy flavor parton distribution $f_{b+\bar{b}}(x, Q^2)$ (right) over their full form in percent for $m_c = 1.59$ GeV, $m_b = 4.78$ GeV in the on-shell scheme. Dash-dotted line: $Q^2 = 30$ GeV^2; Dotted line: $Q^2 = 30$ GeV^2; Dashed line: $Q^2 = 100$ GeV^2; Dash-dotted line: $Q^2 = 1000$ GeV^2; Full line: $Q^2 = 10000$ GeV^2.
- For the PDFs the `NNLO` variant of ABMP16 with $N_f = 3$ flavors was used. Alekhin et al., Phys. Rev. D 96 (2017) 1

JB, A. De Freitas, C. Schneider, K. Schönwald, Phys.Lett.B 782 (2018) 362

Scheme–invariant evolution





Left: The relative contribution of the heavy flavor contributions due to *c* and *b* quarks to the structure function $F_2^{\rm NS}$ at N^{3}LO; dashed lines: 100 GeV^2; dashed-dotted lines: 1000 GeV^2; dotted lines: 10000 GeV^2. Right: The same for the structure function $xg_1^{\rm NS}$ at N^{3}LO.

Measured input: $v_3^+(x, Q_0^2) = x[(u - \bar{u}) - (d - \bar{d})]$ No Gluons.

JB, M.Saragnese, Phys.Lett.B 820 (2021) 136589

Conclusions and Outlook

- Significant progress has been made in the field of analytic massless and massive 3-loop corrections in QCD and QED (up to two scales).
- Technologies are available to solve problems footing on complex alphabets (also with more than one variable).
- Calculation of QCD power corrections at 2-loop order.
- 3-loop massless results: anomalous dimensions, Wilson coefficients: unpolarized & polarized.
- Just finished: $A_{gg,Q}^{(3)}$ unpolarized and polarized.
- Work towards finishing $A_{Qg}^{(3)}$ in analytic form.
- Non–singlet scheme invariant evolution, including heavy flavor corrections to N^3LO.
- On the way to: the 4–loop anomalous dimensions.