

3-Loop Corrections to Heavy Flavor Wilson Coefficients in Deep-Inelastic Scattering

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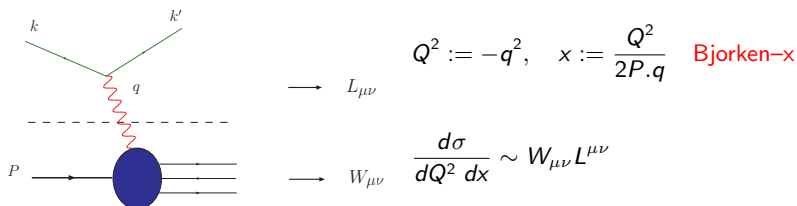
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Introduction

Unpolarized Deep-Inelastic Scattering (DIS):



$$W_{\mu\nu}(q, P, s) = \frac{1}{4\pi} \int d^4\xi \exp(iq\xi) \langle P, s | [J_\mu^{em}(\xi), J_\nu^{em}(0)] | P, s \rangle =$$

$$\frac{1}{2x} \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) F_L(x, Q^2) + \frac{2x}{Q^2} \left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu} \right) F_2(x, Q^2) .$$

Structure Functions: $F_{2,L}$

contain light and heavy quark contributions.

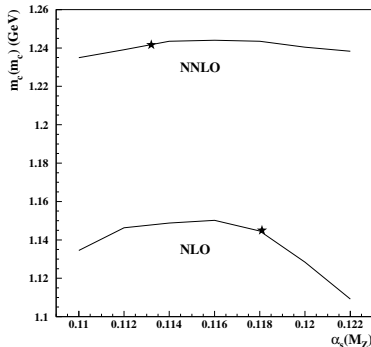
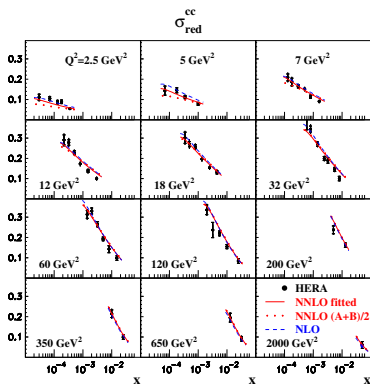
$\alpha_s(M_Z^2)$ from NNLO DIS(+) analyses

	$\alpha_s(M_Z^2)$	
BBG	$0.1134^{+0.0019}_{-0.0021}$	valence analysis, NNLO
GRS	0.112	valence analysis, NNLO
ABKM	0.1135 ± 0.0014	HQ: FFNS $N_f = 3$
JR	0.1128 ± 0.0010	dynamical approach
JR	0.1162 ± 0.0006	including NLO-jets
MSTW	0.1171 ± 0.0014	
Thorne	0.1136	[DIS+DY+HT*] (2014)
ABM11 _J	$0.1134 - 0.1149 \pm 0.0012$	Tevatron jets (NLO) incl.
ABM13	0.1133 ± 0.0011	
ABM13	0.1132 ± 0.0011	(without jets)
ABM16	0.1149 ± 0.0009	+ new HERA, + $t\bar{t}$
CTEQ	0.1159..0.1162	
CTEQ	0.1140	(without jets)
NN21	$0.1174 \pm 0.0006 \pm 0.0001$	
Gehrmann et al.	$0.1131^{+0.0028}_{-0.0022}$	e^+e^- thrust
Abbate et al.	0.1140 ± 0.0015	e^+e^- thrust
BBG	$0.1141^{+0.0020}_{-0.0022}$	valence analysis, $N^3\text{LO}$

$$\Delta_{\text{TH}}\alpha_s = \alpha_s(N^3\text{LO}) - \alpha_s(\text{NNLO}) + \Delta_{\text{HQ}} = +0.0009 \pm 0.0006_{\text{HQ}}$$

NNLO accuracy is needed to analyze the world data. \implies NNLO HQ corrections needed.

Deep-Inelastic Scattering (DIS):



NNLO:

S. Alekhin, J. Blümlein, K. Daum, K. Lipka, Phys.Lett. B720 (2013) 172 [1212.2355]

$$m_c(m_c) = 1.252 \pm 0.02(\text{exp}) \begin{matrix} +0.03 \\ -0.02 \end{matrix} (\text{scale}) \begin{matrix} +0.00 \\ -0.07 \end{matrix} (\text{thy})\text{GeV},$$

$$m_b(m_b) = 3.83 \pm 0.12\text{GeV}$$

$$m_t(m_t) = 160.9 \pm 1.1\text{GeV}$$

Yet approximate NNLO treatment [Kawamura et al. [1205.5227]].

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Publications: Mathematics

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Factorization of the Structure Functions

At leading twist the structure functions factorize in terms of a Mellin convolution

$$F_{(2,L)}(x, Q^2) = \sum_j \underbrace{C_{j,(2,L)} \left(x, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right)}_{\text{perturbative}} \otimes \underbrace{f_j(x, \mu^2)}_{\text{nonpert.}}$$

into (pert.) **Wilson coefficients** and (nonpert.) **parton distribution functions (PDFs)**.

\otimes denotes the Mellin convolution

$$f(x) \otimes g(x) \equiv \int_0^1 dy \int_0^1 dz \delta(x - yz) f(y) g(z) .$$

The subsequent calculations are performed in Mellin space, where \otimes reduces to a multiplication, due to the Mellin transformation

$$\hat{f}(N) = \int_0^1 dx x^{N-1} f(x) .$$

Wilson coefficients:

$$C_{j,(2,L)} \left(N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right) = C_{j,(2,L)} \left(N, \frac{Q^2}{\mu^2} \right) + H_{j,(2,L)} \left(N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right) .$$

At $Q^2 \gg m^2$ the heavy flavor part

$$H_{j,(2,L)} \left(N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right) = \sum_i C_{i,(2,L)} \left(N, \frac{Q^2}{\mu^2} \right) A_{ij} \left(\frac{m^2}{\mu^2}, N \right)$$

[Buza, Matiounine, Smith, van Neerven 1996 Nucl.Phys.B]

factorizes into the light flavor Wilson coefficients C and the massive operator matrix elements (OMEs) of local operators O_i between partonic states j

$$A_{ij} \left(\frac{m^2}{\mu^2}, N \right) = \langle j | O_i | j \rangle .$$

→ additional Feynman rules with local operator insertions for partonic matrix elements.

The unpolarized light flavor Wilson coefficients are known up to NNLO

[Moch, Vermaseren, Vogt, 2005 Nucl.Phys.B].

For $F_2(x, Q^2)$: at $Q^2 \gtrsim 10m^2$ the asymptotic representation holds at the 1% level.

Status of OME calculations

Leading Order: [Witten 1976, Babcock, Sivers, Wolfram 1978, Shifman, Vainshtein, Zakharov 1978, Leveille, Weiler 1979, Glück, Reya 1979, Glück, Hoffmann, Reya 1982]

Next-to-Leading Order:

[Laenen, van Neerven, Riemersma, Smith 1993]

$Q^2 \gg m^2$: via IBP [Buza, Matiounine, Smith, Migneron, van Neerven 1996]

Compact results via ${}_pF_q$'s [Bierenbaum, Blümlein, Klein, 2007]

$O(\alpha_s^2 \varepsilon)$ (for general N) [Bierenbaum, Blümlein, Klein 2008, 2009]

Next-to-Next-to-Leading Order: $Q^2 \gg m^2$

- ▶ Moments for F_2 : $N = 2 \dots 10(14)$ [Bierenbaum, Blümlein, Klein 2009]
mapping large expressions to [MATAD, Steinhauser 2000]
- ▶ Contributions to transversity: $N = 1 \dots 13$ [Blümlein, Klein, Tödtli 2009]
- ▶ Two masses $m_1 \neq m_2 \rightarrow$ Moments $N = 2, 4, 6$ [JB, Wißbrock 2011]

At 3-loop order for general values of N :

- ▶ All OMEs: terms $O(n_f T_F^2 C_{A/F})$ to F_2 [Ablinger et al. 2011, 2012]
- ▶ First contributions to $O(T_F^2 C_{A/F})$ $A_{gg,Q}$ [Ablinger et al. 2014]

The Wilson Coefficients at large Q^2

$$\begin{aligned}
 2014 \quad L_{q,(2,L)}^{\text{NS}}(N_F + 1) &= a_s^2 \left[A_{qq,Q}^{(2),\text{NS}}(N_F + 1) \delta_2 + \hat{C}_{q,(2,L)}^{(2),\text{NS}}(N_F) \right] \\
 &+ a_s^3 \left[A_{qq,Q}^{(3),\text{NS}}(N_F + 1) \delta_2 + A_{qq,Q}^{(2),\text{NS}}(N_F + 1) C_{q,(2,L)}^{(1),\text{NS}}(N_F + 1) + \hat{C}_{q,(2,L)}^{(3),\text{NS}}(N_F) \right] \\
 2010 \quad L_{q,(2,L)}^{\text{PS}}(N_F + 1) &= a_s^3 \left[A_{qq,Q}^{(3),\text{PS}}(N_F + 1) \delta_2 + A_{qq,Q}^{(2)}(N_F) N_F \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) + N_F \hat{C}_{q,(2,L)}^{\hat{(3),\text{PS}}}(N_F) \right. \\
 2010 \quad L_{g,(2,L)}^{\text{S}}(N_F + 1) &= a_s^2 A_{gg,Q}^{(1)}(N_F + 1) N_F \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) + a_s^3 \left[A_{qq,Q}^{(3)}(N_F + 1) \delta_2 \right. \\
 &+ A_{gg,Q}^{(1)}(N_F + 1) N_F \tilde{C}_{g,(2,L)}^{(2)}(N_F + 1) + A_{gg,Q}^{(2)}(N_F + 1) N_F \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) \\
 &+ A_{Qg}^{(1)}(N_F + 1) N_F \tilde{C}_{q,(2,L)}^{(2),\text{PS}}(N_F + 1) + N_F \hat{C}_{g,(2,L)}^{\hat{(3)}}(N_F) \left. \right], \\
 2014 \quad H_{q,(2,L)}^{\text{PS}}(N_F + 1) &= a_s^2 \left[A_{Qq}^{(2),\text{PS}}(N_F + 1) \delta_2 + \tilde{C}_{q,(2,L)}^{(2),\text{PS}}(N_F + 1) \right] + a_s^3 \left[A_{Qq}^{(3),\text{PS}}(N_F + 1) \delta_2 \right. \\
 &+ \tilde{C}_{q,(2,L)}^{(3),\text{PS}}(N_F + 1) + A_{qq,Q}^{(2)}(N_F + 1) \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) \\
 &+ A_{Qq}^{(2),\text{PS}}(N_F + 1) C_{q,(2,L)}^{(1),\text{NS}}(N_F + 1) \left. \right], \\
 H_{g,(2,L)}^{\text{S}}(N_F + 1) &= a_s \left[A_{Qg}^{(1)}(N_F + 1) \delta_2 + \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) \right] + a_s^2 \left[A_{Qg}^{(2)}(N_F + 1) \delta_2 \right. \\
 &+ A_{Qg}^{(1)}(N_F + 1) C_{q,(2,L)}^{(1),\text{NS}}(N_F + 1) + A_{gg,Q}^{(1)}(N_F + 1) \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) \\
 &+ \tilde{C}_{g,(2,L)}^{(2)}(N_F + 1) \left. \right] + a_s^3 \left[A_{Qg}^{(3)}(N_F + 1) \delta_2 + A_{Qg}^{(2)}(N_F + 1) C_{q,(2,L)}^{(1),\text{NS}}(N_F + 1) \right. \\
 &+ A_{gg,Q}^{(2)}(N_F + 1) \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) + A_{Qg}^{(1)}(N_F + 1) \left\{ C_{q,(2,L)}^{(2),\text{NS}}(N_F + 1) \right. \\
 &+ \left. \left. \tilde{C}_{q,(2,L)}^{(2),\text{PS}}(N_F + 1) \right\} + A_{gg,Q}^{(1)}(N_F + 1) \tilde{C}_{g,(2,L)}^{(2)}(N_F + 1) + \tilde{C}_{g,(2,L)}^{(3)}(N_F + 1) \right]
 \end{aligned}$$

[Ablinger et al. 2010, Ablinger et al., 2014a, Ablinger et al., 2014b]

Variable Flavor Number Scheme

$$f_k(n_f + 1, \mu^2) + f_{\bar{k}}(n_f + 1, \mu^2) = A_{qq,Q}^{\text{NS}}\left(n_f, \frac{\mu^2}{m^2}\right) \otimes \left[f_k(n_f, \mu^2) + f_{\bar{k}}(n_f, \mu^2) \right] \\ + \tilde{A}_{qq,Q}^{\text{PS}}\left(n_f, \frac{\mu^2}{m^2}\right) \otimes \Sigma(n_f, \mu^2) + \tilde{A}_{qg,Q}^{\text{S}}\left(n_f, \frac{\mu^2}{m^2}\right) \otimes G(n_f, \mu^2)$$

$$f_{Q+\bar{Q}}(n_f + 1, \mu^2) = \tilde{A}_{Qq}^{\text{PS}}\left(n_f, \frac{\mu^2}{m^2}\right) \otimes \Sigma(n_f, \mu^2) + \tilde{A}_{Qg}^{\text{S}}\left(n_f, \frac{\mu^2}{m^2}\right) \otimes G(n_f, \mu^2).$$

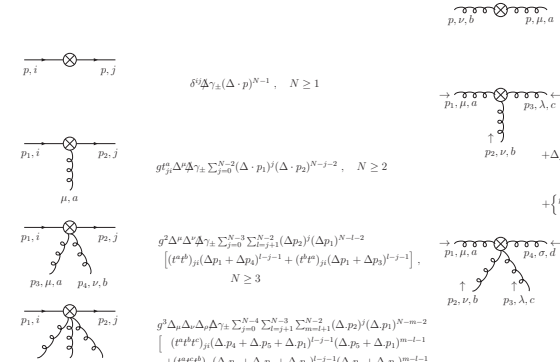
$$G(n_f + 1, \mu^2) = A_{gq,Q}^{\text{S}}\left(n_f, \frac{\mu^2}{m^2}\right) \otimes \Sigma(n_f, \mu^2) + A_{gg,Q}^{\text{S}}\left(n_f, \frac{\mu^2}{m^2}\right) \otimes G(n_f, \mu^2).$$

$$\Sigma(n_f + 1, \mu^2) = \sum_{k=1}^{n_f+1} \left[f_k(n_f + 1, \mu^2) + f_{\bar{k}}(n_f + 1, \mu^2) \right] \\ = \left[A_{qq,Q}^{\text{NS}}\left(n_f, \frac{\mu^2}{m^2}\right) + n_f \tilde{A}_{qq,Q}^{\text{PS}}\left(n_f, \frac{\mu^2}{m^2}\right) + \tilde{A}_{Qq}^{\text{PS}}\left(n_f, \frac{\mu^2}{m^2}\right) \right] \\ \otimes \Sigma(n_f, \mu^2) \\ + \left[n_f \tilde{A}_{qg,Q}^{\text{S}}\left(n_f, \frac{\mu^2}{m^2}\right) + \tilde{A}_{Qg}^{\text{S}}\left(n_f, \frac{\mu^2}{m^2}\right) \right] \otimes G(n_f, \mu^2)$$

All master integrals for $A_{gg}^{(3)}$ have been completed (June 2015).

Calculation of the 3-loop operator matrix elements

The OMEs are calculated using the QCD Feynman rules together with the following operator insertion Feynman rules:



$$\overline{p, \nu, b} \otimes p, \mu, a$$

$$\frac{1+(-1)^N \delta^{ab} (\Delta \cdot p)^{N-2}}{2} \left[g_{\mu\nu} (\Delta \cdot p)^2 - (\Delta_\mu p_\nu + \Delta_\nu p_\mu) \Delta \cdot p + p^2 \Delta_\mu \Delta_\nu \right], \quad N \geq 2$$

$$\delta^{ij} \Delta_\mu \gamma_\pm (\Delta \cdot p)^{N-1}, \quad N \geq 1$$

$$g t_{ji}^a \Delta_\mu \Delta_\nu \Delta_\rho \gamma_\pm \sum_{j=0}^{N-2} \sum_{l=j+1}^{N-2} (\Delta \cdot p_1)^j (\Delta \cdot p_2)^{N-j-2}, \quad N \geq 2$$

$$g^2 \Delta^\mu \Delta^\nu \Delta^\rho \Delta_\sigma \gamma_\pm \sum_{j=0}^{N-3} \sum_{l=j+1}^{N-2} (\Delta p_2)^j (\Delta p_1)^{N-l-2} \left[(t^a t^b)_{ji} (\Delta p_1 + \Delta p_4)^{l-j-1} + (t^b t^a)_{ji} (\Delta p_1 + \Delta p_3)^{l-j-1} \right], \quad N \geq 3$$

$$g^3 \Delta_\mu \Delta_\nu \Delta_\rho \Delta_\sigma \Delta_\tau \gamma_\pm \sum_{j=0}^{N-4} \sum_{l=j+1}^{N-3} \sum_{m=l+1}^{N-2} (\Delta p_2)^j (\Delta p_1)^{N-m-2} \left[(t^a t^b t^c)_{jil} (\Delta p_4 + \Delta p_5 + \Delta p_1)^{l-j-1} (\Delta p_5 + \Delta p_1)^{m-l-1} + (t^a t^c t^b)_{jil} (\Delta p_4 + \Delta p_5 + \Delta p_1)^{l-j-1} (\Delta p_4 + \Delta p_1)^{m-l-1} + (t^b t^a t^c)_{jil} (\Delta p_3 + \Delta p_5 + \Delta p_1)^{l-j-1} (\Delta p_5 + \Delta p_1)^{m-l-1} + (t^b t^c t^a)_{jil} (\Delta p_3 + \Delta p_5 + \Delta p_1)^{l-j-1} (\Delta p_3 + \Delta p_1)^{m-l-1} + (t^c t^a t^b)_{jil} (\Delta p_3 + \Delta p_4 + \Delta p_1)^{l-j-1} (\Delta p_4 + \Delta p_1)^{m-l-1} + (t^c t^b t^a)_{jil} (\Delta p_3 + \Delta p_4 + \Delta p_1)^{l-j-1} (\Delta p_3 + \Delta p_1)^{m-l-1} \right], \quad N \geq 4$$

$$\gamma_+ = 1, \quad \gamma_- = \gamma_5.$$

$$-ig \frac{1+(-1)^N}{2} f^{abc} \left(\left[(\Delta_\nu g_{\lambda\mu} - \Delta_\lambda g_{\mu\nu}) \Delta \cdot p_1 + \Delta_\mu (p_{1,\nu} \Delta_\lambda - p_{1,\lambda} \Delta_\nu) \right] (\Delta \cdot p_1)^{N-2} + \Delta_\lambda \left[\Delta \cdot p_1 p_{2,\mu} \Delta_\nu + \Delta \cdot p_2 p_{1,\nu} \Delta_\mu - \Delta \cdot p_1 \Delta \cdot p_2 g_{\mu\nu} - p_1 \cdot p_2 \Delta_\mu \Delta_\nu \right] \times \sum_{j=0}^{N-3} (-\Delta \cdot p_1)^j (\Delta \cdot p_2)^{N-3-j} + \left\{ \begin{matrix} p_1 \rightarrow p_2 \rightarrow p_3 \rightarrow p_1 \\ \mu \rightarrow \nu \rightarrow \lambda \rightarrow \mu \end{matrix} \right\} + \left\{ \begin{matrix} p_1 \rightarrow p_3 \rightarrow p_2 \rightarrow p_1 \\ \mu \rightarrow \lambda \rightarrow \nu \rightarrow \mu \end{matrix} \right\} \right), \quad N \geq 2$$

$$g^2 \frac{1+(-1)^N}{2} \left(f^{abc} f^{cde} O_{\mu\nu\lambda\sigma} (p_1, p_2, p_3, p_4) + f^{a\alpha\epsilon} f^{b\delta\epsilon} O_{\mu\lambda\nu\sigma} (p_1, p_3, p_2, p_4) + f^{a\delta\epsilon} f^{b\alpha\epsilon} O_{\mu\nu\sigma\lambda} (p_1, p_4, p_2, p_3) \right),$$

$$O_{\mu\nu\lambda\sigma} (p_1, p_2, p_3, p_4) = \Delta_\nu \Delta_\lambda \left\{ -g_{\mu\sigma} (\Delta \cdot p_3 + \Delta \cdot p_4)^{N-2} + [p_{4,\mu} \Delta_\sigma - \Delta \cdot p_4 g_{\mu\sigma}] \sum_{i=0}^{N-3} (\Delta \cdot p_3 + \Delta \cdot p_4)^i (\Delta \cdot p_4)^{N-3-i} - [p_{1,\sigma} \Delta_\mu - \Delta \cdot p_1 g_{\mu\sigma}] \sum_{i=0}^{N-3} (-\Delta \cdot p_1)^i (\Delta \cdot p_3 + \Delta \cdot p_4)^{N-3-i} + [\Delta \cdot p_1 \Delta \cdot p_4 g_{\mu\sigma} + p_1 \cdot p_4 \Delta_\mu \Delta_\sigma - \Delta \cdot p_4 p_{1,\sigma} \Delta_\mu - \Delta \cdot p_1 p_{4,\mu} \Delta_\sigma] \times \sum_{i=0}^{N-4} \sum_{j=0}^i (-\Delta \cdot p_1)^{N-4-i} (\Delta \cdot p_3 + \Delta \cdot p_4)^{i-j} (\Delta \cdot p_4)^j \right\} - \left\{ \begin{matrix} p_1 \leftrightarrow p_2 \\ \mu \leftrightarrow \nu \end{matrix} \right\} - \left\{ \begin{matrix} p_3 \leftrightarrow p_4 \\ \lambda \leftrightarrow \sigma \end{matrix} \right\} + \left\{ \begin{matrix} p_1 \leftrightarrow p_2, p_3 \leftrightarrow p_4 \\ \mu \leftrightarrow \nu, \lambda \leftrightarrow \sigma \end{matrix} \right\}, \quad N \geq 2$$

The diagrams are generated using **QGRAF** [Nogueira 1993 J. Comput. Phys].

	$A_{qq,Q}^{(3),NS}$	$A_{gq,Q}^{(3)}$	$A_{Qq}^{(3),PS}$	$A_{gg,Q}^{(3)}$	$A_{Qg}^{(3)}$
No. diagrams	110	86	125	642	1233

A **FORM** [Vermaseren 2000] program was written in order to perform the γ -matrix algebra in the numerator of all diagrams, which are then expressed as a linear combination of scalar integrals.

$$A_{qq,Q}^{(3),NS} \rightarrow 7426 \text{ scalar integrals.}$$

$$A_{gq,Q}^{(3)} \rightarrow 12529 \text{ scalar integrals.}$$

$$A_{Qq}^{(3),PS} \rightarrow 5470 \text{ scalar integrals.}$$

\Rightarrow Need to use integration by parts identities.

\Rightarrow The reduction for all OMEs has been completed.

\Rightarrow Use special computers: 12 units with overall 3.2 TB RAM, 97 TB fast disc, hundreds of mathematica lic. ; IBP: several TB of final relations.

Integration by parts

We use **Reduze** [A. von Manteuffel, C. Studerus, 2012] to express all scalar integrals required in the calculation in terms of a small(er) set of master integrals.

Reduze is a **C++** program based on **Laporta's algorithm**.

$$(\Delta \cdot k)^N \rightarrow \sum_{N=0}^{\infty} x^N (\Delta \cdot k)^N = \frac{1}{1 - x\Delta \cdot k}$$

⇒ additional propagator.

Number of master integrals:

$$A_{qq,Q}^{(3),NS} \rightarrow 35 \text{ master integrals } \checkmark.$$

$$A_{gq,Q}^{(3)} \rightarrow 41 \text{ master integrals } \checkmark.$$

$$A_{Qq}^{(3),PS} \rightarrow 66 \text{ master integrals } \checkmark.$$

$$A_{gg,Q}^{(3)} \rightarrow 205 \text{ master integrals } \checkmark.$$

$$A_{Qg}^{(3)} \rightarrow 340 \text{ master integrals. (224 done by June 2015.)}$$

116 master integrals **to be done** ⇒ **CIS-type**
24 integral families are required and implemented in Reduze.

Calculation of the master integrals

For the calculation of the master integrals we use a wide variety of tools:

- ▶ Hypergeometric functions.
- ▶ Summation methods based on difference fields, implemented in the Mathematica program **Sigma** [C. Schneider, 2005–].
 - ▶ Reduction of the sums to a small number of key sums.
 - ▶ Expansion the summands in ε .
 - ▶ Simplification by symbolic summation algorithms based on $\Pi\Sigma$ -fields [Karr 1981 J. ACM, Schneider 2005–].
 - ▶ Harmonic sums, polylogarithms and their various generalizations are algebraically reduced using the package HarmonicSums [Ablinger 2010, 2013, Ablinger, Blümlein, Schneider 2011,2013].
- ▶ Mellin-Barnes representations.
- ▶ In the case of **convergent** massive 3-loop Feynman integrals, they can be performed in terms of **Hyperlogarithms** [Generalization of a method by F. Brown, 2008, to non-vanishing masses and local operators].
- ▶ Systems of Differential Equations.
- ▶ Almkvist-Zeilberger Theorem as Integration Method.

Emergence of new nested sums :

$$\begin{aligned}
 & \sum_{i=1}^N \binom{2i}{i} (-2)^i \sum_{j=1}^i \frac{1}{j \binom{2j}{j}} S_{1,2} \left(\frac{1}{2}, -1; j \right) \\
 &= \int_0^1 dx \frac{x^N - 1}{x - 1} \sqrt{\frac{x}{8+x}} [H_{w_{17}, -1, 0}^*(x) - 2H_{w_{18}, -1, 0}^*(x)] \\
 &+ \frac{\zeta_2}{2} \int_0^1 dx \frac{(-x)^N - 1}{x + 1} \sqrt{\frac{x}{8+x}} [H_{12}^*(x) - 2H_{13}^*(x)] \\
 &+ c_3 \int_0^1 dx \frac{(-8x)^N - 1}{x + \frac{1}{8}} \sqrt{\frac{x}{1-x}},
 \end{aligned}$$

$$w_{12} = \frac{1}{\sqrt{x(8-x)}},$$

$$w_{13} = \frac{1}{(2-x)\sqrt{x(8-x)}},$$

$$w_{17} = \frac{1}{\sqrt{x(8+x)}},$$

$$w_{18} = \frac{1}{(2+x)\sqrt{x(8+x)}}.$$

Non-iterative Iterative Integrals

The live after iterative integrals and/or differential equations factorizing completely to 1st order:

- Iterative integrals/nested sums in QFT have been very well understood during the last 18 years since 1998. [J. Vermaseren, E. Remiddi, JB];
- Now even general alphabets (including up to root valued letters).
- Even single-scale Feynman integrals lead beyond that [Sabry's kite, 1962]
- Currently worked out by the community. [Ablinger, Adams, Ananthanarayan, Behring, Bijnnes, JB,

Bloch, Bogner, Brown, De Freitas, Gangl, Ghosh, Hebbar, Hoeij, Imamoglu, Laporta, Levin, Müller-Stach, Remiddi, Schneider, Schweitzer, Tancredi, Vidunnas, Weinzierl, Zagier, Zayadeh, ...]

$$\mathbb{H}_{a_1, \dots, a_{m-1}; \{a_m; F_m(r(y_m))\}, a_{m+1}, \dots, a_q(x)} = \int_0^x dy_1 f_{a_1}(y_1) \int_0^{y_1} dy_2 \dots \int_0^{y_{m-1}} dy_m f_{a_m}(y_m) F_m[r(y_m)] \times H_{a_{m+1}, \dots, a_q}(y_{m+1}),$$

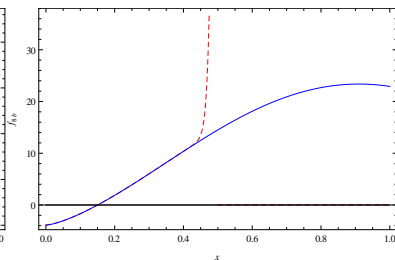
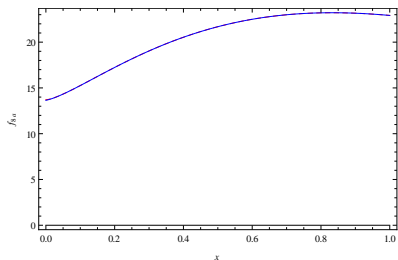
$$F[r(y)] = \int_0^1 dz g(z, r(y)), \quad r(y) \in \mathbb{Q}[y],$$

$$\psi_{1a}^{(0)}(x) = \sqrt{2\sqrt{3}\pi} \frac{x^2(x^2 - 1)^2(x^2 - 9)^2}{(x^2 + 3)^4} {}_2F_1\left[\frac{4}{3}, \frac{5}{3}; z\right]$$

$$z = \frac{x^2(x^2 - 9)^2}{(x^2 + 3)^3}.$$

CIS-series; In some cases: complete elliptic integrals at very special rational arguments. Highly precise numerical representations already available. On the structural side: Relations to elliptic polylogarithms [in the elliptic case].

Non-iterative Iterative Integrals



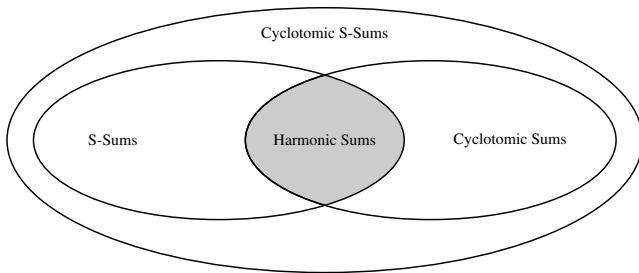
- Have to handle branch-points in case.
 - Relations due to shuffle algebras.
 - Further relations due to triangle group; Important relations between different solutions of the homogeneous equations.
- Most of the master integrals infected by the new CIS solutions are iterated integrals over a few of the former ones.
- We have identified the whole respective tree in case of our project.
 - It would be interesting to view the corresponding situation in case of $\sigma(pp \rightarrow t\bar{t})(\hat{s})$.

Spill-Off:

New Mathematical Function Classes and Algebras

- ▶ **1998:** Harmonic Sums [Vermaseren; JB]
- ▶ **1999:** Harmonic Polylogarithms [Remiddi, Vermaseren]
- ▶ **2001:** Generalized Harmonic Sums [Moch, Uwer, Weinzierl]
- ▶ **2004:** Infinite harmonic (inverse) binomial sums [Davydychev, Kalmykov; Weinzierl]
- ▶ **2011:** (generalized) Cyclotomic Harmonic Sums, polylogarithms and numbers [Ablinger, JB, Schneider]
- ▶ **2013:** Systematic Theory of Generalized Harmonic Sums, polylogarithms and numbers [Ablinger, JB, Schneider]
- ▶ **2014:** Finite nested Generalized Cyclotomic Harmonic Sums with (inverse) Binomial Weights [Ablinger, JB, Raab, Schneider]
- ▶ **2016:** Elliptic integrals with (involved) rational arguments appear in part of the functions of our project already as base cases. They stem from Heun equations. [since April 2016.] [Ablinger, Behring, JB, De Freitas, van Hoeij, Raab, Schneider, DESY16-147].

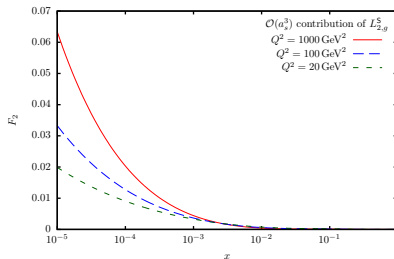
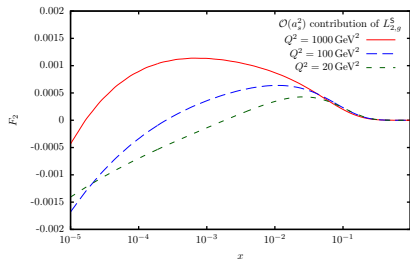
Particle Physics Generates **NEW** Mathematics.



Nested (inverse) binomial sums; CIS functions [Elliptic Integrals and iterations on them]

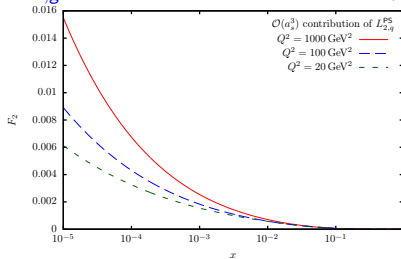
More and more onion skins to be added during these calculations.

Numerical Results : $L_{g,2}^S$ and $L_{q,2}^{PS}$

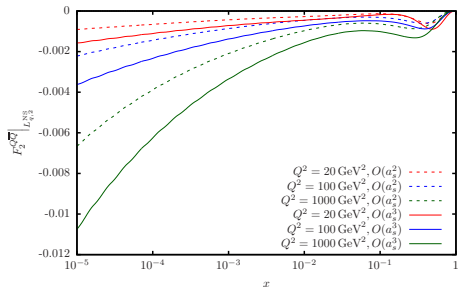


$\mathcal{O}(a_s^2)$ $L_{2,g}^S$

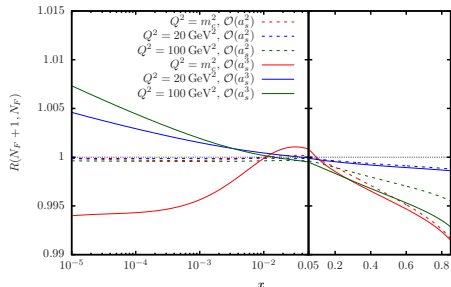
$\mathcal{O}(a_s^3)$ $L_{2,g}^S$



$L_{q,2}^{PS}$

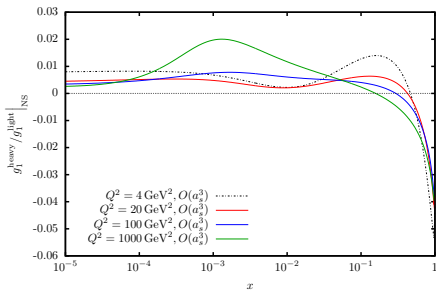


Contribution to $F_2(x, Q^2)$

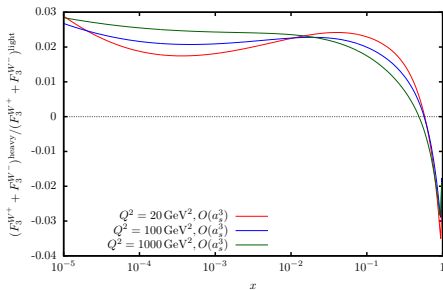


VFNS matching

NS corrections to $g_{1(2)}(x, Q^2)$ and $x F_3^{W^+ + W^-}$



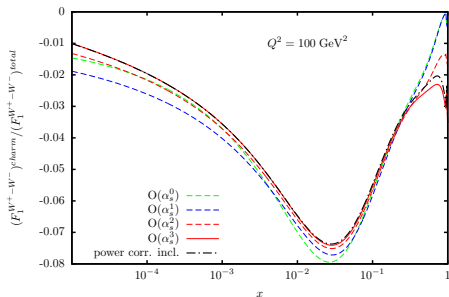
$$g_1(x, Q^2)$$



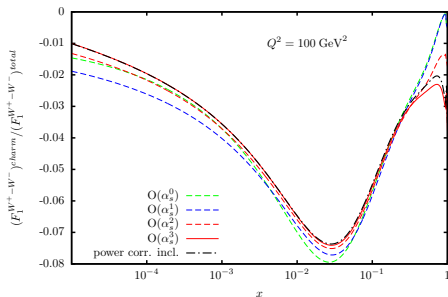
$$x F_3^{W^+ + W^-}(x, Q^2)$$

The corrections to $g_2(x, Q^2)$ are obtained using the Wandzura-Wilczek relation.

NS corrections to $F_1^{W^+ - W^-}$ and $F_2^{W^+ - W^-}$



$$F_1^{W^+ - W^-}(x, Q^2)$$

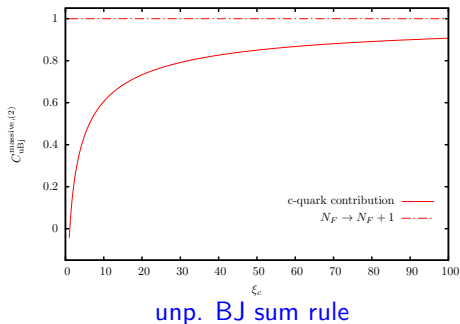
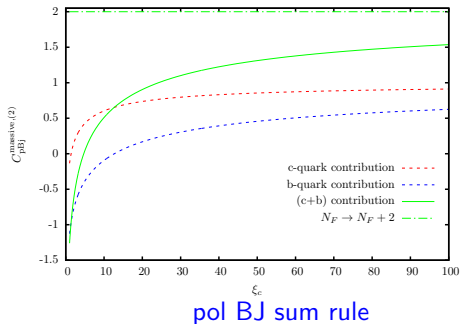


$$F_2^{W^+ - W^-}(x, Q^2)$$

The massless corrections are due to Davies, Vogt, Moch, Vermaseren, LT-1084.

from: A. Behring et al., DESY 16-148

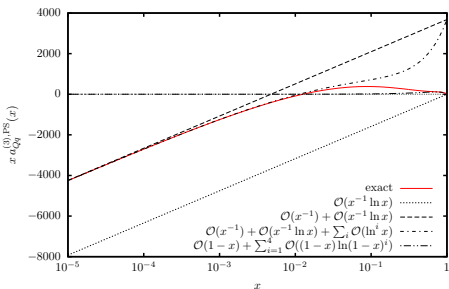
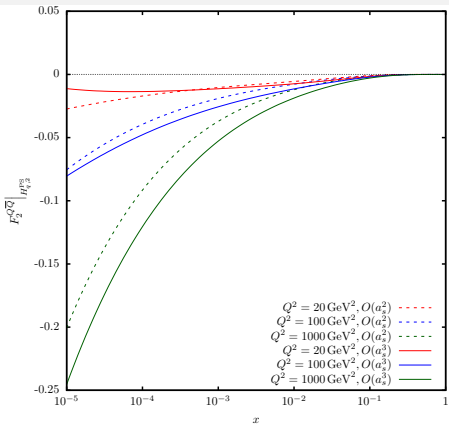
$O(\alpha_s^2)$ Complete NS corrections



Note the negative corrections at low Q^2 !

Already for charm it takes quite a while to become massless.

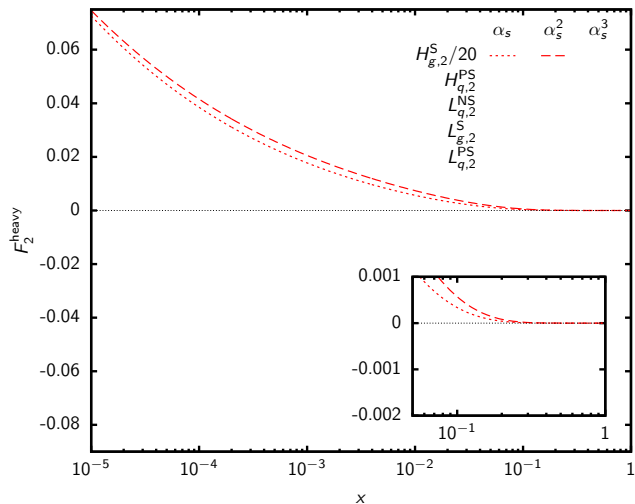
JB, G. Falcioni, A. De Freitas, Nucl. Phys. B910 (2016) 568.


 $a_{Qq}^{(3),\text{PS}}$


Contribution to $F_2(x, Q^2)$

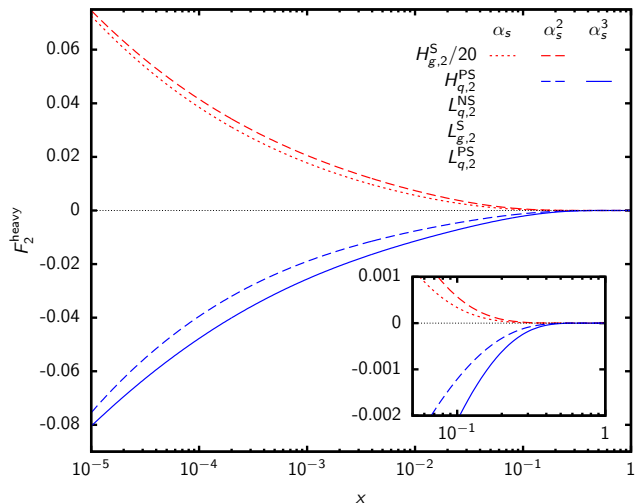
The leading small x approximation corresponding to CCH, 1991, departs from the physical result everywhere except for $x = 1$.

The present NC corrections to $F_2(x, Q^2)$



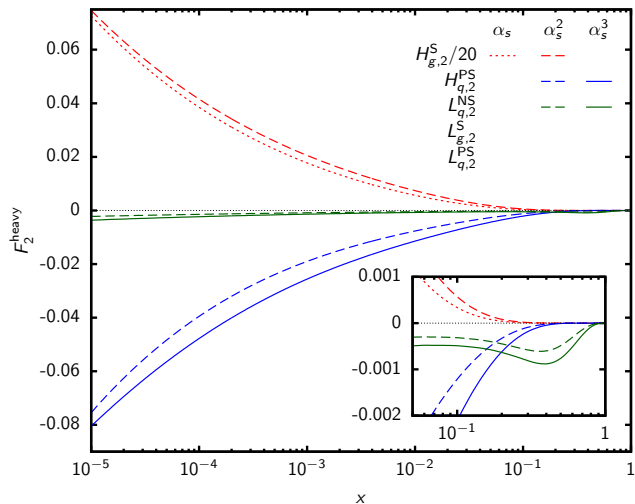
$Q^2 = 100\text{GeV}^2$ [$H_{g,2}^S$ scaled down by a factor 20.]

The present NC corrections to $F_2(x, Q^2)$



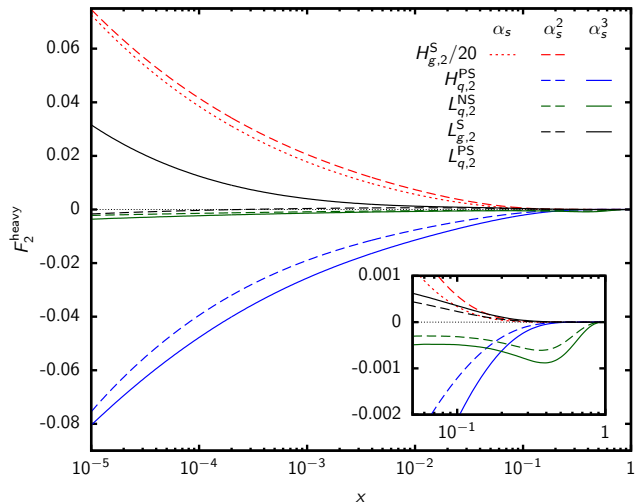
$Q^2 = 100\text{GeV}^2$ [$H_{g,2}^S$ scaled down by a factor 20.]

The present NC corrections to $F_2(x, Q^2)$



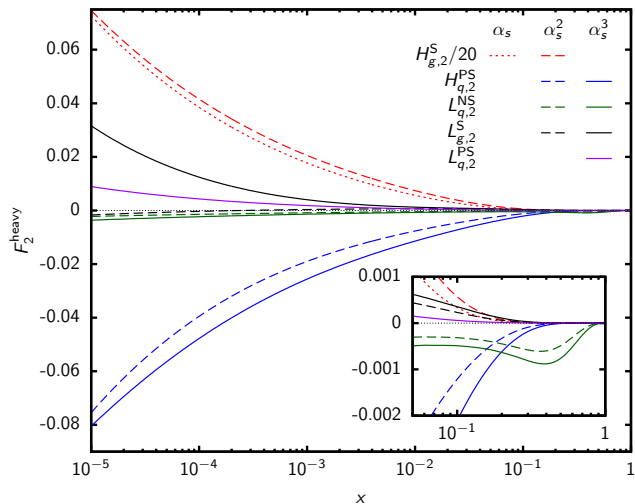
$Q^2 = 100\text{GeV}^2$ [$H_{g,2}^S$ scaled down by a factor 20.]

The present NC corrections to $F_2(x, Q^2)$



$Q^2 = 100\text{GeV}^2$ [$H_{g,2}^S$ scaled down by a factor 20.]

The present NC corrections to $F_2(x, Q^2)$



$Q^2 = 100\text{GeV}^2$ [$H_{g,2}^S$ scaled down by a factor 20.]

3-Loop OME: $A_{gg,Q}$

$$\begin{aligned}
 a_{gg,Q}^{(3)} = & \frac{1 + (-1)^N}{2} \left\{ C_F^2 T_F \left[\frac{16(N^2 + N + 2)}{N^2(N + 1)^2} \sum_{i=1}^N \frac{\binom{2i}{i} \left(\sum_{j=1}^i \frac{4^j S_1(j-1)}{\binom{2j}{j} j^2} - 7\zeta_3 \right)}{4^i (i + 1)^2} - \frac{4P_{69} S_1^2}{3(N - 1)N^4(N + 1)^4(N + 2)} \right. \right. \\
 & \left. \left. + \tilde{\gamma}_{gq}^{(0)} \left(\frac{128(S_{-4} - S_{-3} S_1 + S_{-3,1} + 2S_{-2,2})}{3N(N + 1)(N + 2)} + \frac{4(5N^2 + 5N - 22) S_1^2 S_2}{3N(N + 1)(N + 2)} + \dots \right) + \dots \right] \right. \\
 & + C_A C_F T_F \left[\frac{16P_{42}}{3(N - 1)N^2(N + 1)^2(N + 2)} \sum_{i=1}^N \frac{\binom{2i}{i} \left(\sum_{j=1}^i \frac{4^j S_1(j-1)}{\binom{2j}{j} j^2} - 7\zeta_3 \right)}{4^i (i + 1)^2} + \frac{32P_2 S_{-2,2}}{(N - 1)N^2(N + 1)^2(N + 2)} \right. \\
 & \left. - \frac{64P_{14} S_{-2,1,1}}{3(N - 1)N^2(N + 1)^2(N + 2)} - \frac{16P_{23} S_{-4}}{3(N - 1)N^2(N + 1)^2(N + 2)} + \frac{4P_{63} S_4}{3(N - 2)(N - 1)N^2(N + 1)^2(N + 2)} + \dots \right] \\
 & + C_A^2 T_F \left[-\frac{4P_{46}}{3(N - 1)N^2(N + 1)^2(N + 2)} \sum_{i=1}^N \frac{\binom{2i}{i} \left(\sum_{j=1}^i \frac{4^j S_1(j-1)}{\binom{2j}{j} j^2} - 7\zeta_3 \right)}{4^i (i + 1)^2} + \frac{256P_5 S_{-2,2}}{9(N - 1)N^2(N + 1)^2(N + 2)} \right. \\
 & \left. + \frac{32P_{30} S_{-2,1,1} + 16P_{35} S_{-3,1} + 16P_{44} S_{-4}}{9(N - 1)N^2(N + 1)^2(N + 2)} + \frac{16P_{52} S_{-2}^2}{27(N - 1)N^2(N + 1)^2(N + 2)} + \frac{8P_{36} S_2^2}{9(N - 1)N^2(N + 1)^2} + \dots \right] \\
 & + C_F T_F^2 \left[-\frac{16P_{48} \binom{2N}{N} 4^{-N} \left(\sum_{i=1}^N \frac{4^i S_1(i-1)}{\binom{2i}{i} j^2} - 7\zeta_3 \right)}{3(N - 1)N(N + 1)^2(N + 2)(2N - 3)(2N - 1)} - \frac{32P_{86} S_1}{81(N - 1)N^4(N + 1)^4(N + 2)(2N - 3)(2N - 1)} \right. \\
 & \left. + \frac{16P_{45} S_1^2}{27(N - 1)N^3(N + 1)^3(N + 2)} - \frac{16P_{45} S_2}{9(N - 1)N^3(N + 1)^3(N + 2)} + \dots \right] + \dots \left. \right\} \quad (1)
 \end{aligned}$$

Also, with this calculation we were able to re-derive the three loop anomalous dimension $\gamma_{gg}^{(3)}$ for the terms $\propto T_F$, and obtained agreement with the literature.

Moments for graphs with two massive lines ($m_1 \neq m_2$)

$$\begin{aligned}
 a_{Q_q}^{(3)}(N=6) = & \frac{1}{2} \left\{ T_F^2 C_A \left[\frac{69882273800453}{367569090000} - \frac{395296}{19845} \zeta_3 + \frac{1316809}{39690} \zeta_2 + \frac{832369820129}{14586075000} x + \frac{1511074426112}{624023544375} x^2 - \frac{84840004938801319}{690973782403905000} x^3 \right. \right. \\
 & + \ln\left(\frac{m_2^2}{\mu^2}\right) \left[\frac{117716442229}{194481000} + \frac{78496}{2205} \zeta_2 - \frac{1406143531}{69457500} x - \frac{105157957}{180093375} x^2 + \frac{2287164970759}{7669816654500} x^3 \right] \\
 & + \ln^2\left(\frac{m_2^2}{\mu^2}\right) \left[\frac{2668087}{79380} + \frac{112669}{661500} x - \frac{49373}{51975} x^2 - \frac{31340489}{34054020} x^3 \right] + \ln^3\left(\frac{m_2^2}{\mu^2}\right) \frac{324148}{19845} + \ln^2\left(\frac{m_2^2}{\mu^2}\right) \ln\left(\frac{m_1^2}{\mu^2}\right) \frac{156992}{6615} \\
 & + \ln\left(\frac{m_2^2}{\mu^2}\right) \ln\left(\frac{m_1^2}{\mu^2}\right) \left[\frac{128234}{3969} - \frac{112669}{330750} x + \frac{98746}{51975} x^2 + \frac{31340489}{17027010} x^3 \right] + \ln\left(\frac{m_2^2}{\mu^2}\right) \ln^2\left(\frac{m_1^2}{\mu^2}\right) \frac{68332}{6615} \\
 & + \ln\left(\frac{m_1^2}{\mu^2}\right) \left[\frac{83755534727}{583443000} + \frac{78496}{2205} \zeta_2 + \frac{1406143531}{69457500} x + \frac{105157957}{180093375} x^2 - \frac{2287164970759}{7669816654500} x^3 \right] \\
 & + \ln^2\left(\frac{m_1^2}{\mu^2}\right) \left. \left[\frac{2668087}{79380} + \frac{112669}{661500} x - \frac{49373}{51975} x^2 - \frac{31340489}{34054020} x^3 \right] + \ln^3\left(\frac{m_1^2}{\mu^2}\right) \frac{412808}{19845} \right\} \\
 & + T_F^2 C_F \left\{ -\frac{3161811182177}{71471767500} + \frac{447392}{19845} \zeta_3 + \frac{9568018}{4862025} \zeta_2 - \frac{64855635472}{2552563125} x + \frac{1048702178522}{97070329125} x^2 + \frac{1980566069882672}{2467763508585375} x^3 \right. \\
 & + \ln\left(\frac{m_2^2}{\mu^2}\right) \left[\frac{1786067629}{204205050} - \frac{111848}{15435} \zeta_2 - \frac{128543024}{24310125} x - \frac{22957168}{3361743} x^2 - \frac{2511536080}{2191376187} x^3 \right] \\
 & + \ln^2\left(\frac{m_2^2}{\mu^2}\right) \left[\frac{3232799}{4862025} + \frac{752432}{231525} x + \frac{177944}{40425} x^2 + \frac{127858928}{42567525} x^3 \right] - \ln^3\left(\frac{m_2^2}{\mu^2}\right) \frac{111848}{19845} - \ln^2\left(\frac{m_2^2}{\mu^2}\right) \ln\left(\frac{m_1^2}{\mu^2}\right) \frac{223696}{46305} \\
 & + \ln\left(\frac{m_2^2}{\mu^2}\right) \ln\left(\frac{m_1^2}{\mu^2}\right) \left[\frac{22238456}{4862025} - \frac{1504864}{231525} x - \frac{355888}{40425} x^2 - \frac{255717856}{42567525} x^3 \right] + \ln\left(\frac{m_2^2}{\mu^2}\right) \ln^2\left(\frac{m_1^2}{\mu^2}\right) \frac{223696}{46305} \\
 & + \ln\left(\frac{m_1^2}{\mu^2}\right) \left[-\frac{24797875607}{1021025250} - \frac{111848}{15435} \zeta_2 + \frac{128543024}{24310125} x + \frac{22957168}{3361743} x^2 + \frac{2511536080}{2191376187} x^3 \right] \\
 & \left. + \ln^2\left(\frac{m_1^2}{\mu^2}\right) \left[\frac{3232799}{4862025} + \frac{752432}{231525} x + \frac{177944}{40425} x^2 + \frac{127858928}{42567525} x^3 \right] - \ln^3\left(\frac{m_1^2}{\mu^2}\right) \frac{1230328}{138915} \right\} + O(x^4 \ln^3(x))
 \end{aligned}$$

→ $q_2 e / \exp$ [Harlander, Seidensticker, Steinhauser 1999] $\times = m_1^2 / m_2^2$

Analytic general N results are available for $A_{qq,Q}^{\text{NS}}$, $A_{qq,Q}^{\text{PS}}$ and the scalar integrals of $A_{gg,Q}$.

Conclusions

- ▶ 2009: 10-14 Mellin Moments for all massive 3-loop OMEs, WC.
2010: Wilson Coefficients $L_q^{(3),PS}(N)$, $L_g^{(3),S}(N)$.
- ▶ 2013: Ladder, V-Graph and Benz-topologies for graphs, with no singularities in ε can be systematically calculated for **general N** .
- ▶ Here **new functions** occur (including a larger number of root-letters in iterated integrals).
- ▶ 2014 $L_q^{NS,(3)}$, $A_{gq,Q}^{S,(3)}$, $A_{qq,Q}^{NS,TR(3)}$, $H_{2,q}^{PS(3)}$ and $A_{Qq}^{PS(3)}$ were completed.
- ▶ A method for the calculation of **graphs with two massive lines** of equal masses and operator insertions has been developed and applied $A_{gg,Q}^{(3)}$.
- ▶ The method can be generalized to the case of unequal masses. Here the moments for $N = 2, 4, 6$ for all graphs with two quark lines of unequal masses are now known [\rightarrow **extended renormalization**]; for some OMEs the complete 2-mass structure has been computed.
- ▶ The $O(\alpha_s^2)$ charged current Wilson coefficients have been completed.

Conclusions

- ▶ The corresponding 3-loop anomalous dimensions were computed, those for **transversity** for the first time ab initio; those for the **PS-case** independently for the first time.
- ▶ In all NS-cases [NC and CC] we also computed **all power corrections at $O(a_s^2)$** and the associated sum rules in the inclusive case improving an earlier result by JB & W. van Neerven.
- ▶ All master integrals based on iterative integrals over **whatsoever alphabet** for $A_{gg,Q}^{(3)}$ and $A_{Qg}^{(3)}$ have been computed and $A_{gg,Q}^{(3)}$ is known for any even integer moment $N \geq 2$. Here all the topologies, including the ladder- and V-topologies have been solved.
- ▶ We have all the principal means to reconstruct $A_{Qg}^{(3)}$ systematically at very high accuracy. The full analytic solution will request more mathematical efforts.
- ▶ Different new computer-algebra and mathematical technologies were developed. These efforts will continue. The technologies are certainly useful for various present and upcoming calculations for the LHC and ILC.