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The Evolution of Singlet Structure Functions at Small x

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DESY

J. Blümlein and A.Vogt Phys. Rev. D57 (1998) 1.
J. Blümlein and A.Vogt hep-ph/9712546, Phys. Rev. D58 (1998) 014020.
J. Blümlein, V. Ravindran and W.L. van Neerven, DESY 98-036, hep-ph/9806357
J. Blümlein, V. Ravindran, W.L. van Neerven, and A. Vogt DESY 98-067, hep-ph/9806368

1. Introduction

- SCALING VIOLATIONS OF STRUCTURE FUNCTIONS ARE DESCRIBED BY RENORMALIZATION GROUP EQUATIONS.

  TWIST 2 – MASS FACTORIZATION

  TWO LOOP LEVEL : COMPLETELY KNOWN

- LARGE EFFECTS : SMALL x

  Lx , NLx RESUMMATION

  ⋮

  P_{ij} (gg MOST RECENT)

LIPATOV,
FADIN et al. > 1975

CIAFALONI, CATANI, HAUTMANN 1990, 1994 (CH)

CIAFALONI, CAMICI 1996/98

- ADVANTAGE OF THE Q_0-SCHEME (CONF. INVARIANCE)
  CIAFALONI 1995.

- IMPORTANCE OF MEDIUM & LARGE x TERMS

  JB 1993
  W. v NEERVEN 1993
  ⋮

- CONSEQUENCES FOR F_2, F_L.

2. LO and NLO Small x Resummation and Evolution Equations

$$F_i(x,Q^2) = \sum_{r=1}^{2N_f} a_{ir}\, c_{i,r}(x,Q^2) \otimes q_r(x,Q^2) + a_{ig}\, c_{i,g}(x,Q^2) \otimes g(x,Q^2)\,,$$

EVOLUTION EQUATIONS:

$$\frac{\partial\, q_{\rm NS}^{\pm}(x,Q^2)}{\partial \ln Q^2} = P_{\rm NS}^{\pm}(x,\alpha_s) \otimes q_{\rm NS}^{\pm}(x,Q^2)\,,$$

$$\frac{\partial\, \boldsymbol{q}_{\rm S}(x,Q^2)}{\partial \ln Q^2} = \boldsymbol{P}_{\rm S}(x,\alpha_s) \otimes \boldsymbol{q}_{\rm S}(x,Q^2)\,.$$

$$\frac{da_s}{d\ln Q^2} = -\sum_{k=0}^{\infty} a_s^{k+2}\beta_k\,.$$

ALL ORDER RESUMMATION:

$$P^{\pm}(x,a_s) = \sum_{l=0}^{\infty} a_s^{l+1} P_l^{\pm}(x)\,,$$

$$\boldsymbol{P}(x,a_s) \equiv \begin{pmatrix} P_{qq}(x,a_s) & P_{qg}(x,a_s) \\ P_{gq}(x,a_s) & P_{gg}(x,a_s) \end{pmatrix} = \sum_{l=0}^{\infty} a_s^{l+1} \boldsymbol{P}_l(x)\,,$$

$$c_{i,j}(x,Q^2) = \delta(1-x)\delta_{jq} + \sum_{l=1}^{\infty} a_s^l c_{ij,l}(x)\,.$$

- LO + NLO exact
- BEYOND: Lx, NLx RESUMMATION
- LO, NLO MOTIVATED: MODEL STUDIES FOR LESS SINGULAR TERMS

NLX: $O(\alpha\,(\alpha/(N-1))^{k})$

$$\gamma(N, a_s) = -2\int_0^1 dx\, x^{N-1}\boldsymbol{P}(x, a_s)\,.$$

LX:
$$\gamma_L(N, a_S) = -2\begin{pmatrix} 0 & 0 \\ C_F/C_A & 1 \end{pmatrix}\gamma_L(N, \alpha_s)\,.$$
JAROSEWICZ

$$\rho \equiv \frac{N}{\overline{\alpha}_s} = 2\psi(1) - \psi(\gamma_L) - \psi(1-\gamma_L) \equiv \chi[\gamma_L],$$

$$\gamma_L(N, a_s) = \sum_{k=1}^{\infty} g_k^{(0)}\left(\frac{\overline{\alpha}_s}{N}\right)^k\,.$$

NLX:
$$\gamma_{\mathrm{NL}}(N, \alpha_s) = -2\begin{pmatrix} \frac{C_F}{C_A}[\gamma_{\mathrm{NL}} - \frac{8}{3}a_s T_F] & \gamma_{\mathrm{NL}} \\ \gamma_{gq,\mathrm{NL}} & \gamma_{gg,\mathrm{NL}} \end{pmatrix},$$

$\uparrow$

STILL UNKNOWN.

$$
\begin{aligned}
{}^{\mathrm{DIS}}_{\mathrm{NL}}(N, a_s) &= \gamma_{\mathrm{NL}}^{Q_0}(N, a_s)R(\gamma_L) = T_F\frac{\alpha_s}{3\pi}\frac{2+3\gamma_L-3\gamma_L^2}{3-2\gamma_L}\frac{[B(1-\gamma_L, 1+\gamma_L)]^3}{B(2+2\gamma_L, 2-2\gamma_L)}R(\gamma_l \\
&= 2\frac{\alpha_s}{3\pi}T_F\sum_{k=1}^{\infty} g_k^{qg,(1)}\left(\frac{\overline{\alpha}_s}{N}\right)^k,
\end{aligned}
$$

CATANI, HAUTMANN 94

$$R(\gamma) = \left[\frac{\Gamma(1-\gamma)\chi(\gamma)}{\Gamma(1+\gamma)\{-\gamma\chi'(\gamma)\}}\right]^{1/2}\exp\left[\gamma\psi(1) + \int_0^{\gamma} d\zeta\,\frac{\psi'(1)-\psi'(1-\zeta)}{\chi(\zeta)}\right]$$

$$\gamma_{\mathrm{NL}} = \hat{\gamma}_{\mathrm{NL}}(\gamma_L)\cdot\alpha_s\,.$$

SINGULARITIES OF THE PROBLEM:

$N \in \mathbb{C}$

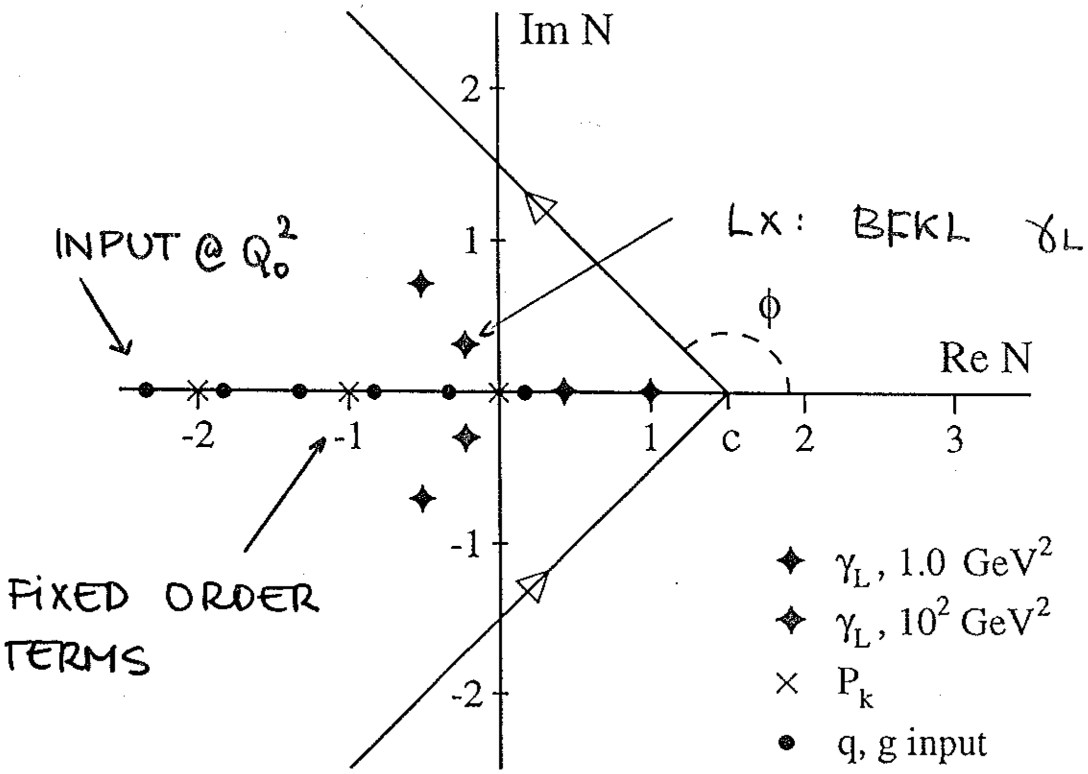


Fig. 4

- SOLUTION OF THE RGE'S IN MELLIN SPACE
- EXACT ACCOUNT FOR ALL COMMUTATION RELATIONS

$[P_{ij}^{\ell}, P_{ij}^{m}] \neq 0$ FOR $m \neq \ell$.

3. Lx and NLx Anomalous Dimensions in the Conformal Limit and Fixed Order Results

The Bethe-Salpeter Equation
BFKL

$$(N-1)G_N(q_1,q_2) = \delta^{D-2}(q_1-q_2)+\int d^{D-2}q_3\; K(q_1,q_3)\; G_N(q_3,q_2)$$

with

$$K(q_1,q_2) = \delta^{D-2}(q_1-q_2)2\omega(q_1)+K_{\mathrm{real}}(q_1,q_2)+K_{\mathrm{virt}}(q_1,q_2)$$

This equation is infrared finite.

ALSO IN NLO.

The Kernel and its Eigenvalue ($\gamma_+ \to \gamma_{gg}$)

HOW TO EXTRACT THE ANOM. DIMENSION ?

DIS : $q_1^2 \gg q_2^2$

CONF. INV. NO CONF. INV.

$$\int d^{D-2}q_2\; K(q_1,q_2)\; \left(q_2^2\right)^{\gamma-1} = \overline{\alpha}_s\left[\chi_0(\gamma)-\frac{\overline{\alpha}_s}{4}\delta(\gamma,q_1^2,\mu^2)\right]\; \left(q_1^2\right)^{\gamma-1}$$

with

$$\overline{\alpha}_s = \frac{N_c}{\pi}\alpha_s(\mu^2)$$

and

$$\chi_0(\gamma) = 2\,\psi(1)-\psi(\gamma)-\psi(1-\gamma)$$

LX: $\gamma \leftrightarrow 1-\gamma$ SYMMETRY

$$\delta(\gamma,q_1^2,\mu^2) = \frac{\beta_0}{3}\left\{\underbrace{\chi_0(\gamma)\log\left(\frac{q_1^2}{\mu^2}\right)}_{\text{SCALE DEP.}}+\frac{1}{2}\left[\chi_0^2(\gamma)+\chi_0'(\gamma)\right]\right\}+\hat{\chi}_1^{\mathrm{symm}}(\gamma)$$

ASYM.

Conformal Limit and the Anomalous Dimension

$$[M_{\mu\nu}, D] \quad = \quad 0$$

Asymptotic scale and conformal invariance :
K. Symanzik, 1971
G. Parisi, 1972

$$\left[\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g} + \gamma_m\, m\frac{\partial}{\partial m} + \gamma_{O_k} - n\gamma_\Phi\right] E_k^n = 0$$

$$m \;=\; 0, \qquad \beta \;=\; 0$$

$$\Rightarrow \qquad \left[\mu\frac{\partial}{\partial\mu} + \gamma_{O_k} - n\gamma_\Phi\right] E_k^n = 0$$

$$\boxed{E_k^n(\mu^2) \quad = \quad E_k^n(\mu_0^2)\left(\frac{\mu^2}{\mu_0^2}\right)^{\frac{1}{2}(\gamma_{O_k} - n\gamma_\Phi)}}$$

$$\gamma_{O_k} - n\gamma_\Phi \quad \equiv \quad \Gamma_k^n \quad = \quad \sum_{l=1}^{\infty} a^l \gamma_l^{k,n}$$

ALL ORDERS.

$a =$ fixed coupling constant.

THE CONFORMAL PART EXPONENTIATES TO ALL ORDERS.

V. Fadin, L. Lipatov, 1998;
G. Camici, M. Ciafaloni 1997,1998 :

$$\begin{aligned}
\chi_1(\gamma_L) &= \frac{\beta_0}{6}\left[\chi_0^2(\gamma_L) + \chi_0'(\gamma_L)\right] - \left(\frac{67}{9} - 2\zeta(2) - \frac{10}{27}N_f\right)\chi_0(\gamma_L) \\
&\quad - 6\zeta(3) + \left[\chi_0^2(\gamma_L) + \chi_0'(\gamma_L)\right]' + 4\Phi(\gamma_L) - \frac{\pi^3}{\sin^2(\pi\gamma_L)} \\
&\quad + \frac{\pi^2}{\sin^2(\pi\gamma_L)}\frac{\cos(\pi\gamma_L)}{1-2\gamma_L} \\
&\qquad\qquad \times\left[(22-\beta_0) + \frac{\gamma_L(1-\gamma_L)}{(1+2\gamma_L)(3-2\gamma_L)}\left(1+\frac{N_f}{3}\right)\right]
\end{aligned}$$

with

$$\begin{aligned}
\Phi(\gamma) &= \int_0^1 dz \frac{1}{1+z}\left[z^{\gamma-1} + z^{\gamma}\right]\left[\mathrm{Li}_2(1) - \mathrm{Li}_2(z)\right] \\
&= \frac{1}{\gamma^2}\left[\psi(\gamma+1) - \psi(1)\right] \\
&+ \sum_{n=1}^{\infty}(-1)^n\left[\frac{\psi(n+1+\gamma)-\psi(1)}{(n+\gamma)^2} - \frac{\psi(n+1-\gamma)-\psi(1)}{(n-\gamma)^2}\right] \\
&= \frac{1}{\gamma}\sum_{l=2}^{\infty}(-1)^l\zeta(l)\gamma^{l-2} + \sum_{k=0}^{\infty}\left[\frac{2\pi^2}{3}\eta(2k+2) + \underline{c_{2k+1}}\right]\gamma^{2k+1}
\end{aligned}$$

$$\eta(k) = \zeta(k)\left[1 - 2^{1-k}\right]$$

The coefficients c_{2k+1} belong to a <u>NEW CLASS of transcendentals</u> since their corresponding Mellin-sum for $k \in \mathbf{N}$ is not reducible, e.g. J. Blümlein, S. Kurth, 1997.

$$c_k = -\frac{2}{k!}\int_0^1 dz \log^k\left(\frac{1}{z}\right)\frac{\mathrm{Li}_2(\mathrm{z})}{1+z}$$

The structure of $\chi_1(\gamma)$

$$\gamma_L \quad \longleftrightarrow \quad 1-\gamma_L$$

$$\chi_1(\gamma_L) = \left[\frac{\beta_0}{6} + \frac{d}{d\gamma_L}\right]\left[\chi_0^2(\gamma_L) + \chi_0'(\gamma_L)\right] + \hat{\chi}_1^{\rm symm}(\gamma_L)$$

FROM: RUNNING α

universal terms

$$\chi_1(\gamma_L) = \frac{\beta_0}{6}\left[\chi_0^2(\gamma_L) + \chi_0'(\gamma_L)\right] - \left(\frac{67}{9} - 2\zeta(2) - \frac{10}{27}N_f\right)\chi_0(\gamma_L)$$

3LOOP !

$$- 6\zeta(3) + \left[\chi_0^2(\gamma_L) + \chi_0'(\gamma_L)\right]' + 4\Phi(\gamma_L) - \frac{\pi^3}{\sin^2(\pi\gamma_L)}$$

$$+ \frac{\pi^2}{\sin^2(\pi\gamma_L)}\frac{\cos(\pi\gamma_L)}{1-2\gamma_L}$$
$$\times\left[(22-\beta_0) + \frac{\gamma_L(1-\gamma_L)}{(1+2\gamma_L)(3-2\gamma_L)}\left(1+\frac{N_f}{3}\right)\right]$$

no "running" β – function ! **!** (G- SELFENERGY)

- KORCHEMSKY: q-Regge-trajectory: above term $6\zeta(3)$
  g-Regge-trajectory: Different result.
  3-Loop Term affected ! - Important to clarify.

FIRST AT:

----	1 LOOP	$1/\gamma_L^2$
----	2 LOOP	$1/\gamma_L$
----	3 LOOP	1

COMPARISON WITH FIXED ORDER RESULTS:

LX: $\dfrac{\bar{\alpha}_s}{N-1} + 0 \cdot \left(\dfrac{\bar{\alpha}_s}{N-1}\right)^2$ CONF. INVARIANCE

$\uparrow$

$P_{gg}^{(0)}$ $\quad$ $P_{gg}^{(1)}$: NO C_A^2 TERM. $\propto 1/(N-1)^2$.

NLX: CATANI, HAUTMANN: '94

Q_0 - SCHEME:

- $$\gamma_{qg}^{NLX}(N, a_s) = \frac{\alpha_s}{6\pi} T_F \frac{2+3\gamma - 3\gamma^2}{3-2\gamma} \frac{[B(1-\gamma, 1+\gamma)]^3}{B(2-2\gamma, 2+2\gamma)}$$

$$= \frac{2}{3} T_F \frac{\alpha}{\pi} + T_F \frac{13}{3} \left(\frac{\alpha}{\pi}\right)^2$$

CONF. INVARIANCE

(q_1^2, q_2^2 SPACE)

F+L, C+C '98: FROM: $(22-\beta_0)\frac{1}{\gamma}$ TERM.

- $$\gamma_{gg}^{NLX}(N, a_S) = \bar{\alpha}\left[-\frac{1}{12}\left(11 + \frac{2}{3} N_f\right)\right]$$

$$- \bar{\alpha}^2 \frac{1}{N-1} \frac{1}{4}\left(\frac{23}{27}\right) N_f$$

NOTE: $Q_0 \rightarrow$ DIS SCHEME! $\quad$ BOTH ARE DUE TO CONF. INVARIANCE.

$$1 = \frac{\bar{\alpha}}{N-1}\left[\chi_0(\gamma_+) + \alpha\left[\chi_1(\gamma_+) - 2[\chi_0\chi_0'](\gamma_+)\right]\right]$$

$$\Delta\gamma_+ = -\alpha\,\frac{\chi_1(\gamma) - 2\chi_0\chi_0'}{\chi_0'}$$

$$\gamma_\pm \simeq \begin{cases} \gamma_{gg} + \frac{C_F}{C_A}\gamma_{qg} & + \ldots \\ \gamma_{qq} - \frac{C_F}{C_A}\gamma_{qg} & \text{JUST FIXED ORDER} + \ldots \end{cases}$$

DIS SCHEME, RUNNING COUPLING:

$$\gamma_{gg}^{DIS} = \hat{\gamma}_{gg}^{Q_0} + \frac{\beta_0}{4\pi}\alpha^2\frac{d\log R(\gamma)}{d\alpha_s} + \frac{C_F}{C_A}(1 - R(\gamma))\,\gamma_{qg}^{Q_0}$$
$$+ \frac{\beta_0}{4\pi}\alpha^2\frac{d\log[\gamma\sqrt{-\chi_0'(\gamma)}]}{d\alpha_s}.$$

3 LOOP:

$$\gamma^{gg}_{NLX,3} = \left(\frac{3\alpha}{\pi}\right)^3 \frac{1}{(N-1)^2} \frac{1}{4}\left[\frac{395}{27} + \frac{71}{81} N_f - \frac{\pi^2}{18}\left(11 + \frac{2}{3} N_f\right) - 2\zeta(3)\right]$$

$\uparrow$

NO: $C_A^3 \left(\frac{\alpha}{\pi}\right)^3 \frac{1}{(N-1)^3}$ TERM.

HERE FOR THE 1ST TIME

RUNNING α - EFFECTS CONTRIBUTE !

STARTING WITH 4-LOOP:

- $\bar{\alpha}\left(\frac{\bar{\alpha}}{N-1}\right)^3 \left[3.21\left(\frac{\beta_0}{4} - \frac{2}{9}\frac{C_F}{C_A} T_F\right) + \dots \right]$ $\quad R(\gamma)$ $\quad Q_0 \to$ DIS SCHEME
- $G^g_+(t) = \exp \int_0^t dt \left[\gamma_+(t) - \frac{d}{dt}\log\left(\gamma_+ \sqrt{-\chi_0'(\gamma_+)}\right)\right]$

$$= \gamma_+ - \frac{\beta_0}{12}\frac{\bar{\alpha}^4}{(N-1)^3}\left[6\zeta_3 - 2\zeta_3 \frac{\bar{\alpha}}{N-1} + 20\, g_5\left(\frac{\bar{\alpha}}{N\ldots}\right.\right. + \dots \Big]$$

$\uparrow$

VERY BIG TERM !

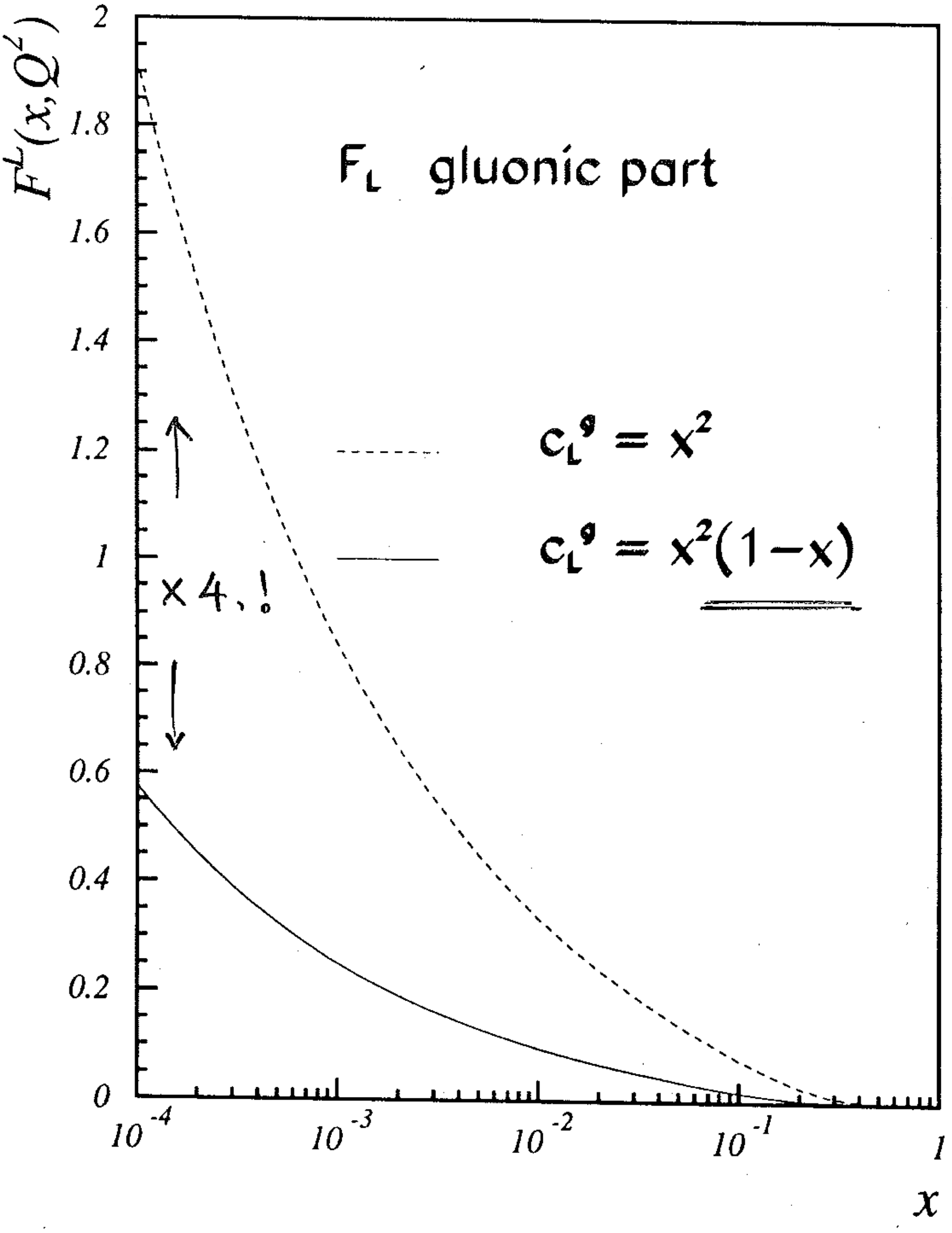
k	$g^{(0)}_{k,qg}$	$g^{(1)}_{k,qg}$ (Q_0)	$g^{(1)}_{k,qg}$ (DIS)	r_k	c^L_k
0	1.00000 E+0	1.00000 E+0	1.00000 E+0	1.00000 E+0	1.00000 E+0
1	0.00000 E+0	2.16667 E+0	2.16667 E+0	0.00000 E+0	−3.33333 E−1
2	0.00000 E+0	2.29951 E+0	2.29951 E+0	0.00000 E+0	2.13284 E+0
3	2.40411 E+1	5.06561 E+0	8.27109 E+0	3.20549 E+0	2.27231 E+0
4	0.00000 E+0	8.79145 E+0	1.49249 E+1	−8.11742 E−1	4.34344 E−1
5	2.07386 E+1	1.90521 E+1	2.92268 E+1	4.56248 E+1	2.02643 E+1
6	1.73393 E+1	4.58482 E+1	1.02812 E+2	3.27070 E+1	2.30315 E+1
7	2.01670 E+0	9.24159 E+1	1.94887 E+2	−2.95476 E+1	3.46449 E+1
8	3.98863 E+1	2.31063 E+2	4.85100 E+2	1.08183 E+2	2.65004 E+2
9	1.68747 E+2	5.59958 E+2	1.52444 E+3	3.99588 E+2	3.30038 E+2
10	6.99881 E+1	1.24822 E+3	3.11451 E+3	1.33228 E+2	8.50371 E+2
11	6.61253 E+2	3.25381 E+3	8.58375 E+3	2.10243 E+3	3.90849 E+3
12	1.94531 E+3	7.93653 E+3	2.47571 E+4	5.51142 E+3	5.67433 E+3
13	1.71768 E+3	1.89275 E+4	5.47435 E+4	5.30316 E+3	1.77680 E+4
14	1.06433 E+4	4.98520 E+4	1.56195 E+5	3.85296 E+4	6.21982 E+4
15	2.55668 E+4	1.23011 E+5	4.26980 E+5	8.49086 E+4	1.07028 E+5
16	3.67813 E+4	3.06504 E+5	1.01111 E+6	1.40384 E+5	3.51475 E+5
17	1.71685 E+5	8.07771 E+5	2.89398 E+6	6.94998 E+5	1.05058 E+6
18	3.75379 E+5	2.02210 E+6	7.69042 E+6	1.44307 E+6	2.10341 E+6
19	7.36025 E+5	5.17873 E+6	1.91919 E+7	3.22738 E+6	6.80747 E+6

k	$g^{q\bar{q}\,(a)}_{k,gg}$ (Q_0)	$g^{q\bar{q}\,(b)}_{k,gg}$ (Q_0)	$g^{q\bar{q}\,(a)}_{k,gg}$ (DIS)	$g^{q\bar{q}\,(b)}_{k,gg}$ (DIS)	$\Delta g^{gg}_{k,gg}$
0	−1.00000 E+0	0.00000 E+0	−1.00000 E+0	0.00000 E+0	−1.65000 E+1
1	−3.83333 E+0	0.00000 E+0	−3.83333 E+0	0.00000 E+0	0.00000 E+0
2	−2.29951 E+0	0.00000 E+0	−2.29951 E+0	0.00000 E+0	−2.78734 E+1
3	6.42072 E+0	−1.19004 E+2	−6.04506 E+0	3.96679 E+1	−2.25279 E+2
4	−2.59764 E+1	0.00000 E+0	−2.81814 E+1	−5.35750 E+1	−1.65583 E+2
5	5.75787 E+0	−3.42186 E+2	−2.60988 E+1	3.42186 E+1	−7.24788 E+2
6	1.21690 E+2	−2.28879 E+3	−9.43607 E+1	4.40583 E+2	−3.14501 E+3
7	−2.66365 E+2	−6.98786 E+2	−3.54981 E+2	−7.39527 E+2	−3.49585 E+3
8	5.43807 E+2	−1.11881 E+4	−4.27828 E+2	1.11801 E+3	−1.51028 E+4
9	1.96852 E+3	−4.10835 E+4	−1.67366 E+3	4.86665 E+3	−4.91970 E+4
10	−2.04998 E+3	−3.39345 E+4	−5.21390 E+3	−9.10195 E+3	−7.46877 E+4
11	1.49302 E+4	−2.75933 E+5	−7.99079 E+3	2.40902 E+4	−2.99245 E+5
12	3.33837 E+4	−7.55104 E+5	−3.05607 E+4	5.32758 E+4	−8.31843 E+5
13	9.19579 E+3	−1.10387 E+6	−8.37332 E+4	−9.58437 E+4	−1.59528 E+6
14	3.35804 E+5	−6.12763 E+6	−1.57171 E+5	4.46747 E+5	−5.82155 E+6
15	6.26484 E+5	−1.45966 E+7	−5.64262 E+5	5.92510 E+5	−1.49497 E+7
16	9.72892 E+5	−3.01102 E+7	−1.43675 E+6	−6.85258 E+5	−3.37088 E+7
17	7.05626 E+6	−1.30018 E+8	−3.14592 E+6	7.71985 E+6	−1.12828 E+8
18	1.29507 E+7	−2.96814 E+8	−1.05144 E+7	7.22515 E+6	−2.81522 E+8
19	3.18568 E+7	−7.45406 E+8	−2.59548 E+7	2.95797 E+6	−7.03719 E+8

Table 1: The numerical expansion coefficients for the anomalous dimensions and coefficient functions

THE RÔLE OF 'MEDIUM' x [illegible]... :

EXAMPLE :



– JB, MAR '93 (DURHAM)

– VAN NEERVEN '93

$$F \propto K(x) \otimes G(x)$$

peak @ small x
small @ large x

MEDIUM x TERMS ARE IMPORTANT.

FIXED ORDER REVISITED: SUBLEADING TERMS

$$
\begin{aligned}
\gamma_{qq,\mathrm{LO}} &= +10.8793\,N - 6.82222\,N^2 + O(N^3)\,, \\
\gamma_{qg,\mathrm{LO}} &= -10.6667 + 11.5556\,N - 13.1852\,N^2 + O(N^3)\,, \\
\gamma_{gq,\mathrm{LO}} &= -\frac{10.6667}{N} + 8.00000 - 9.3333\,N + 10.0000\,N^2 + O(N^3)\,, \\
\gamma_{gg,\mathrm{LO}} &= -\frac{24.0000}{N} + 27.3333 - 5.1883\,N + 17.0395\,N^2 + O(N^3)\,.
\end{aligned}
$$

$$
\begin{aligned}
\gamma^{\mathrm{DIS}}_{qq,\mathrm{NLO}} &= -\frac{123.259}{N} + 405.863 - 684.836\,N + 1197.52N^2 + O(N^3)\,, \\
\gamma^{\mathrm{DIS}}_{qg,\mathrm{NLO}} &= -\frac{277.333}{N} + 846.222 - 1706.18\,N + 2622.76\,N^2 + O(N^3)\,, \\
\gamma^{\mathrm{DIS}}_{gq,\mathrm{NLO}} &= +\frac{91.2593}{N} - 453.512 + 809.030\,N - 1344.89\,N^2 + O(N^3)\,, \\
\gamma^{\mathrm{DIS}}_{gg,\mathrm{NLO}} &= +\frac{245.333}{N} - 988.210 + 2093.25\,N - 3109.08\,N^2 + O(N^3)\,.
\end{aligned}
$$

$$
\begin{aligned}
\gamma^{\overline{\mathrm{MS}}}_{qq,\mathrm{NLO}} &= -\frac{94.8148}{N} + 253.026 - 337.185\,N + 623.259\,N^2 + O(N^3)\,, \\
\gamma^{\overline{\mathrm{MS}}}_{qg,\mathrm{NLO}} &= -\frac{213.333}{N} + 461.449 - 889.687\,N + 1501.16\,N^2 + O(N^3)\,, \\
\gamma^{\overline{\mathrm{MS}}}_{gq,\mathrm{NLO}} &= +\frac{62.8148}{N} - 361.805 + 658.108\,N - 1048.43\,N^2 + O(N^3)\,, \\
\gamma^{\overline{\mathrm{MS}}}_{gq,\mathrm{NLO}} &= +\frac{216.889}{N} - 790.928 + 1616.55\,N - 2423.77\,N^2 + O(N^3)\,.
\end{aligned}
$$

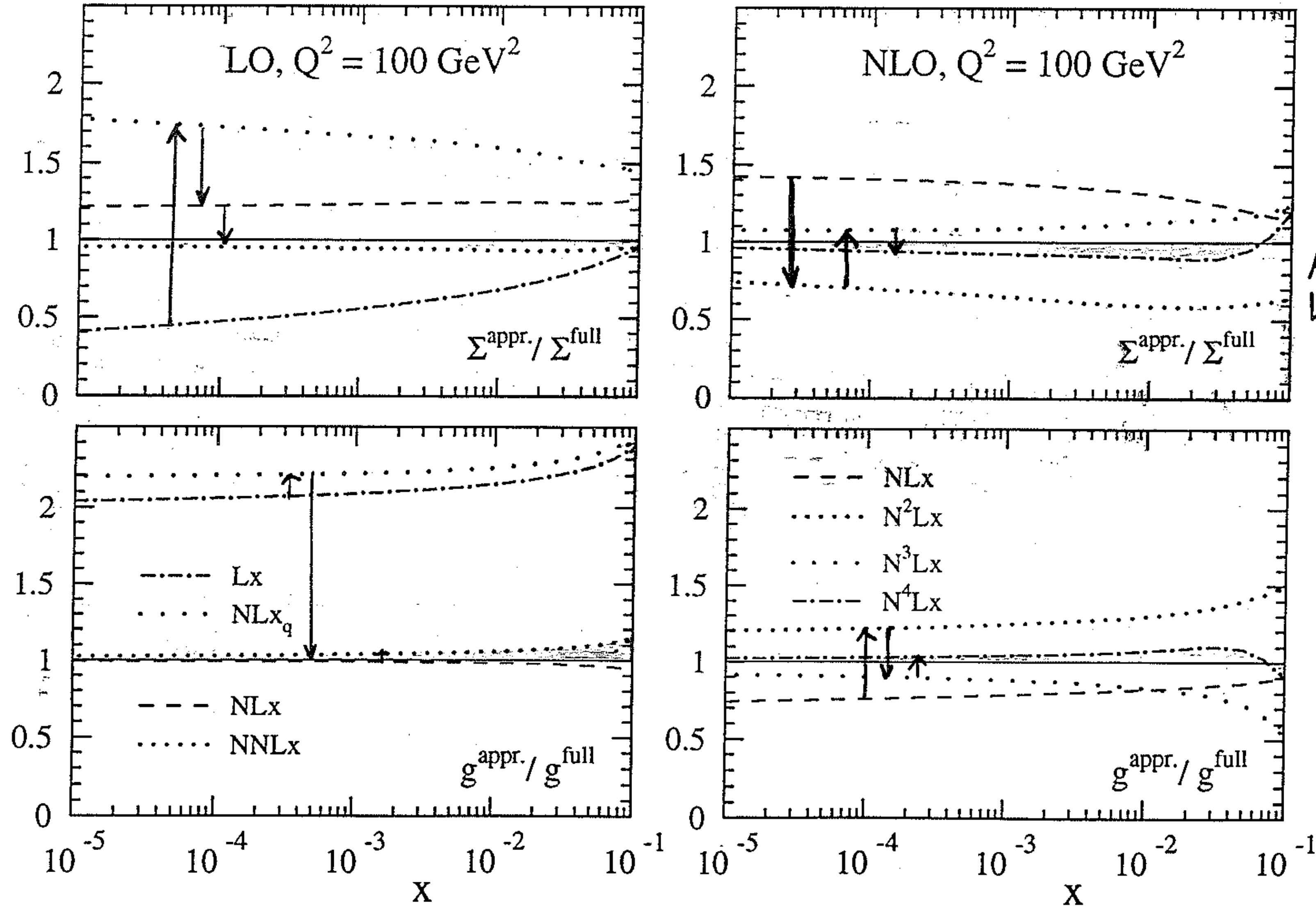
SUBL TERMS: qq, qg, gq 'MODEL'

$$
\gamma_{ij} \to \gamma_{ij}\,(1 - 2N + N^{\alpha}) \quad \begin{cases} C: & \alpha = 2 \\ D: & \alpha = 3 \end{cases}
$$

(CONSERVATIVE).

gg: NLx, ONLY N^{α} TERM ADDED.

HOW MANY $1/(N-1)$ TERMS ARE NEEDED TO GET FIXED ORDER RESULTS ?



AT LEAST 4.

Fig. 6

4. Numerical results:
i) Resummed Splitting Functions

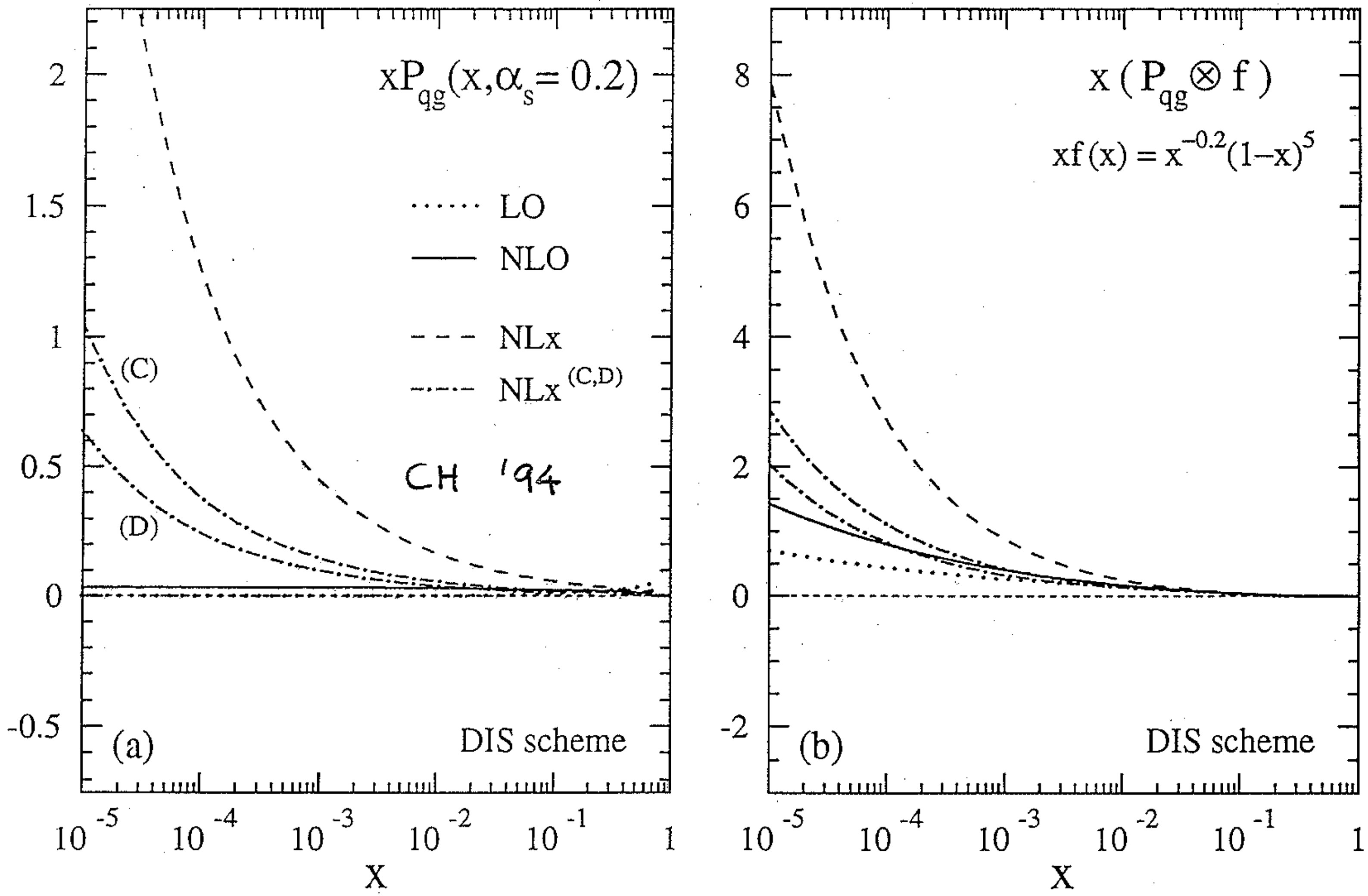
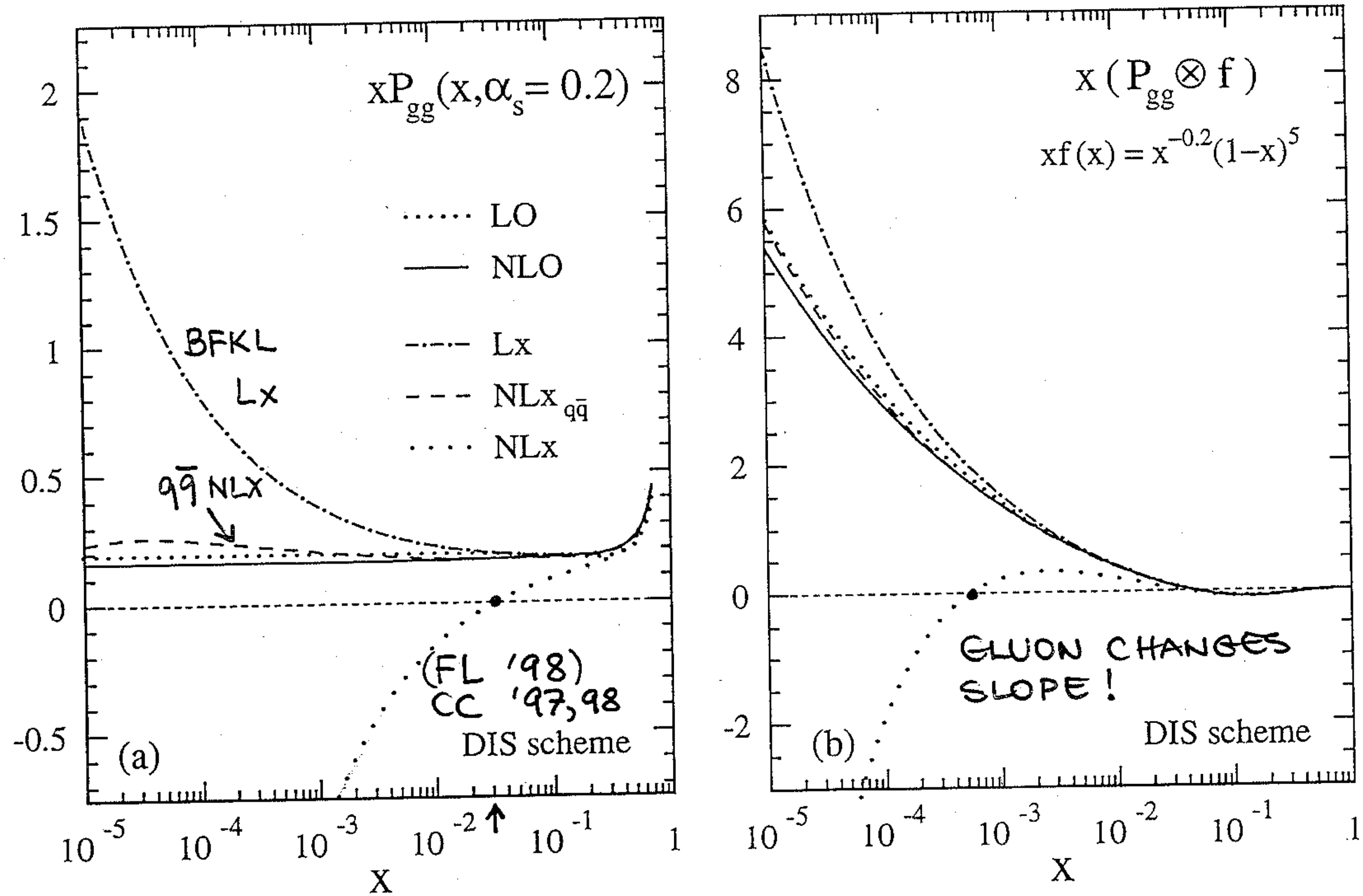


Fig. 2

Resummed splitting function

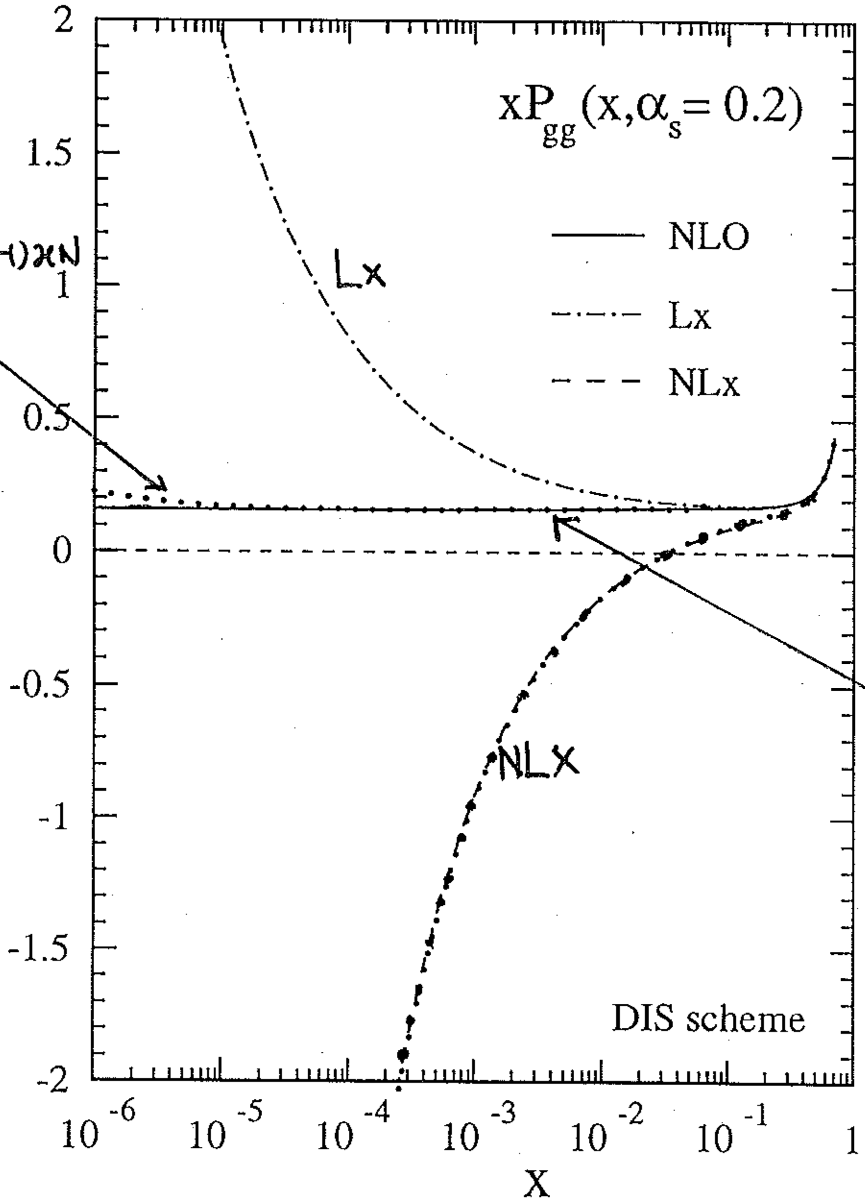
Blümlein, A.V.



- LOSS OF PROBABILITY INTERPRETATION
- WHOLE! $P_{gg} < 0$

Blümlein, A.V.

$\sum C_{\ell L} \left(\frac{\bar{\alpha}}{N}\right)^{\ell} e^{-(\ell-1)\varkappa N}$



NLO !

WHAT IFF THIS IS THE TOP OF AN ICEBERG ?

$$Lx = \sum_{\ell=1}^{\infty} C_L^{\ell} \left(\frac{\bar{\alpha}}{N}\right)^{\ell}$$

$$NLx \simeq \sum_{\ell=1}^{\infty} C_L^{\ell} \left(\frac{\bar{\alpha}}{N}\right)^{\ell} (1 - (\ell-1)\varkappa N) \quad \cdots\cdots \text{ LINE.}$$

$\varkappa(\alpha = 0.2) \simeq 3$

LINEAR TERM OUT OF AN EXPONENTIAL ?

4. Numerical results: ii) F_2 and F_L

EFFECT OF POSSIBLE SUB-LEADING TERMS.

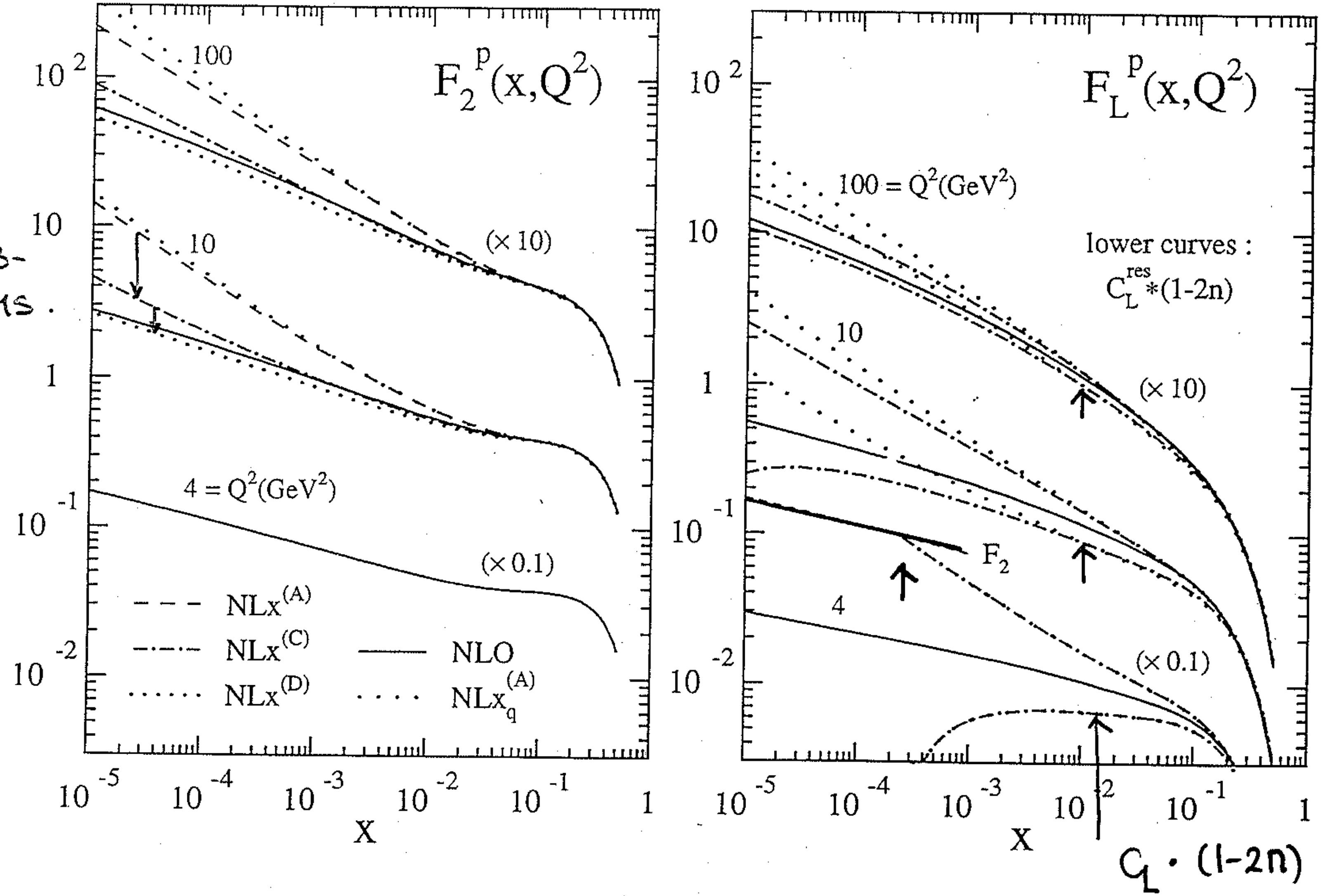


Fig. 10

AN ASPECT IN $O(\alpha_s^3)$: THE $\zeta(3)$ TERM.

TRAJECTORY:

$$\omega = \omega_q + \omega_g = -\bar{g}^2\left[\frac{2}{\varepsilon} + 2L\right]$$
$$-\bar{g}^4\left\{c_0\left[\frac{1}{\varepsilon^2} - L^2\right] + c_1\left[\frac{1}{\varepsilon} + 2L\right] + c_2\right\}$$

$$c_0 = \frac{\beta_0}{3} \equiv \frac{11}{3} - \frac{2}{3}\frac{N_f}{N_c}$$

$$c_1 = \frac{67}{9} - 2\zeta(2) - \frac{10}{9}\frac{N_f}{N_c}$$

$$c_2 = -\frac{440}{27} + \frac{56}{27}\frac{N_f}{N_c} + \left\{\begin{array}{ll} 2\zeta(3) & \text{FADIN, FIORE, KOTSKY} \\ 14\zeta(3) & \text{KORCHEMSKAJA, I} \end{array}\right.$$

$\uparrow$ IN ω_g.

$\longrightarrow$ SOME MORE EASIY INTEGRALS,

1 NON-TRIVIAL ONE $\rightarrow$ $\zeta(3)$ IN ALL OF THEM!

$$I_2 = -\frac{\partial}{\partial\nu}\hat{I}_2(\nu)\big|_{\nu=0}$$

$$\hat{I}_2(\nu) = \bar{g}^4 \frac{1}{\pi^{2(1+\varepsilon)}}\frac{1}{\Gamma^4(1-\varepsilon)}\int \frac{d^{2(1+\varepsilon)}q_1\, d^{2(1+\varepsilon)}q_2\, (q^2)^{2(1+\varepsilon)+\nu}}{q_1^2 q_2^2 (q-q_1)^2 (q-q_2)^2 \left[(q_1-q_2)^2\right]^{\nu}}$$

JB, V. RAVINDRAN, W. van NEERVEN, DESY 98-067, hep-ph 9806357.

$$\hat{I}_2(\nu) = \bar{g}^4\, C(\nu)\, [\nu\, S_2(\nu)]$$

$$= \bar{g}^4 \left[C_0 + \nu C_1\right]\left[(S_{10}+S_{20}) + \nu(S_{11}+S_{21})\right] + O(\nu^2)$$

$$C_0 = -\frac{6}{\varepsilon^2} + \frac{6}{\varepsilon} + 12\left[1+\zeta(2)\right] + 12\left[2-\zeta(2)-5\,\underline{\zeta(3)}\right]\varepsilon + \dots$$

$$C_1 = -\frac{4}{\varepsilon^3} - \frac{2}{\varepsilon^2} - \frac{4}{\varepsilon}\left[1-2\zeta(2)\right] - 4\left[2-\zeta(2)-8\,\underline{\zeta(3)}\right] + \dots$$

$$S_{10} = \frac{1}{3} + \frac{1}{3}\varepsilon + \left[1-2\zeta(2)\right]\varepsilon^2 + \left[3-2\zeta(2)-\underline{2\zeta(3)}\right]\varepsilon^3 + \dots$$

$$S_{11} = \frac{5}{18}\frac{1}{\varepsilon} + \frac{17}{18} + \frac{3}{2}\varepsilon + \left[\frac{3}{2}+2\zeta(2)+\frac{4}{3}\,\underline{\zeta(3)}\right]\varepsilon^2 + \dots$$

$$S_{20} = -1-\varepsilon-\left[3-2\zeta(2)\right]\varepsilon^2 + \left[9-2\zeta(2)-6\,\underline{\zeta(3)}\right]\varepsilon^3 + \dots$$

$$S_{21} = \frac{5}{2}\varepsilon + \Big[12-\underbrace{2\zeta(2)-6\,\underline{\zeta(3)}}_{}\Big]\varepsilon^2$$

↑ requires to evaluate 8 transcendental integrals.

$$I_2 = -\bar{g}^4\left[\frac{1}{\varepsilon^3} + \frac{0}{\varepsilon^2} - \frac{2}{\varepsilon}\zeta(2) - 26\,\zeta(3)\right]$$

↑ FFK ✓

KR no.

{ may have diff. reason!

IMPORTANCE:

- $\gamma_+ = \gamma_{gg} + \frac{4}{9}\gamma_{qq}$: $N_f = 4$ → 2.123 OFF! (3LOOP).

- "RISING POWER" $\omega = 2.64762\,\alpha_s\left[1-6.36402\,\alpha_s\right]$

↓ • 0.6210

$\omega_{conf} = 2.65\,\alpha_s\left[1-2.55 \cdot \underline{\underline{0.62}}\,\alpha_s\right]$ (CONFORMAL PART!)

5. THE s^{ω} - BEHAVIOR

$$\omega(\alpha_s) = \frac{12}{\pi} \log 2 \; \alpha_s \left[1 - c(N_f)\,\alpha_s\right]$$ F, L '98

$$N_f = 4$$

- $\omega = 2.65\,\alpha_s\,[1 - 6.36\,\alpha_s]$

  $\omega_{max} = 0.10 \simeq 0.0808$ (DL) @ $Q^2 = 8.7 \cdot 10^6\ \mathrm{GeV}^2$

  $\omega < 0$ FOR $Q^2 < 600\ \mathrm{GeV}^2$

  e.g. $\omega = -0.35$ $Q^2 = 20\ \mathrm{GeV}^2$!

- $\omega_{conf} = 2.65\,\alpha_s\,[1 - 2.55\,\alpha_s]$

  $\omega_{max} = 0.26$, $Q^2_{max} \simeq 90\ \mathrm{GeV}^2$

  $\omega_{conf} > 0$ FOR $Q^2 > 2\ \mathrm{GeV}^2$.

  · NOTE: DIFFERENCE TO MUELLER et al.

EFFECT OF LESS SINGULAR TERMS ?

→ DOES ω_{conf} STABILIZE ? i.e. $O\left(\alpha_s^2 \left(\frac{\alpha_s}{N-1}\right)^k\right)$

$\ll O\left(\alpha_s \left(\frac{\alpha_s}{N-1}\right)^k\right)$

?

6. CONCLUSIONS

1) THE AGREEMENT OF THE EXPANDED γ_{ij}^{Lx} TO $O(\alpha_s^2)$ (ALL KNOWN TERMS) IS A CONSEQUENCE OF CONFORMAL INVARIANCE.

2) NO $\zeta(3)$ PROBLEM – IMPORTANT FOR FUTURE COMPARISONS AT $O(\alpha_s^3)$

3) MOST CRUCIAL POINT γ_{ij}^{Lx} @ $O(\alpha_s^4)$ AND FUTURE FIXED ORDER RESULTS ; RUNNING COUPL. EFFECTS & SCHEME.

4) THE CONFORMAL PART OF K_{NLx} HAS STABLE IMPACT ON γ_{ij} & s^{ω}

5) HOWEVER : RUNNING α_s (AND OTHER SCALE) EFFECTS YIELD DIVERGING RESULTS : γ_{ij} @ NLx ORDER

$\longrightarrow$ CURED BY LESS SINGULAR TERMS.
($\rightarrow$ TO BE EVALUATED)

6) COMPARISONS WITH HO CALCULATIONS ARE BADLY NEEDED $\leftrightarrow$ MORE DETAILED UNDERSTANDING.