3-Loop Heavy Flavor Corrections to Deep-Inelastic Scattering and the VFNS

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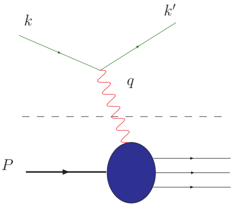
Introduction

Unpolarized Deep–Inelastic Scattering (DIS):



$\longrightarrow \quad L_{\mu\nu}$

$\longrightarrow \quad W_{\mu\nu}$

$$Q^2 := -q^2, \quad x := \frac{Q^2}{2P.q} \quad \text{Bjorken–x}$$

$$\frac{d\sigma}{dQ^2\, dx} \sim W_{\mu\nu} L^{\mu\nu}$$

$$W_{\mu\nu}(q,P,s) = \frac{1}{4\pi} \int d^4\xi \exp(iq\xi) \langle P,s \mid [J_\mu^{em}(\xi), J_\nu^{em}(0)] \mid P,s \rangle =$$

$$\frac{1}{2x}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) F_L(x,Q^2) + \frac{2x}{Q^2}\left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu}\right) F_2(x,Q^2)\ .$$

Structure Functions: $F_{2,L}$ contain light and heavy quark contributions.
At 3-Loop order also graphs with two heavy quarks of different mass contribute.
$\Longrightarrow$ Single and 2-mass contributions: c and b quarks in one graph.

Introduction

Why are Heavy Flavor Contributions important ?

- They form a significant contribution to F_2 and F_L particularly at small x and high Q^2
- concise 3-loop corrections are needed to determine $\alpha_s(M_Z)$, m_c and perhaps m_b
- The accuracy of measurements at the LHC reaches a level of precision requiring 3-loop VFNS matching

NNLO: S. Alekhin, J. Blümlein, S. Moch and R. Placakyte, Phys. Rev. D **96** (2017) no.1, 014011 [arXiv:1701.05838 [hep-ph]].

$\alpha_s(M_Z^2) = 0.1147 \pm 0.0008$

$m_c(m_c) = 1.252 \pm 0.018(exp) \begin{array}{c} +0.03 \\ -0.02 \end{array} (scale) \begin{array}{c} +0.00 \\ -0.07 \end{array} (thy)\mathrm{GeV}$,

$m_b(m_b) = 3.84 \pm 0.12\mathrm{GeV}$

$m_t(m_t) = 160.9 \pm 1.1\mathrm{GeV}$ [all in $\overline{\mathrm{MS}}$ scheme.]

Yet approximate NNLO treatment H. Kawamura et al. Nucl. Phys. B **864** (2012) 399 [arXiv:1205.5727].

NS & PS corrections are exact J. Ablinger et al. Nucl. Phys. B **886** (2014) 733 [arXiv:1406.4654 [hep-ph]]; Nucl. Phys. B **890** (2014) 48 [arXiv:1409.1135 [hep-ph]].

Factorization of the Structure Functions

At leading twist the structure functions factorize in terms of a Mellin convolution

$$F_{(2,L)}(x,Q^2) = \sum_j \underbrace{\mathbb{C}_{j,(2,L)}\left(x, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right)}_{perturbative} \otimes \underbrace{f_j(x,\mu^2)}_{nonpert.}$$

into (pert.) Wilson coefficients and (nonpert.) parton distribution functions (PDFs).
$\otimes$ denotes the Mellin convolution

$$f(x) \otimes g(x) \equiv \int_0^1 dy \int_0^1 dz\, \delta(x - yz) f(y) g(z) \; .$$

The subsequent calculations are performed in Mellin space, where $\otimes$ reduces to a multiplication, due to the Mellin transformation

$$\hat{f}(N) = \int_0^1 dx\, x^{N-1} f(x) \; .$$

Wilson coefficients:

$$\mathbb{C}_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right) = C_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2}\right) + H_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right) \ .$$

At $Q^2 \gg m^2$ the heavy flavor part

$$H_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right) = \sum_i C_{i,(2,L)}\left(N,\frac{Q^2}{\mu^2}\right) A_{ij}\left(\frac{m^2}{\mu^2},N\right)$$

[Buza, Matiounine, Smith, van Neerven 1996 Nucl.Phys.B]

factorizes into the light flavor Wilson coefficients C and the massive operator matrix elements (OMEs) of local operators O_i between partonic states j

$$A_{ij}\left(\frac{m^2}{\mu^2},N\right) = \langle j \mid O_i \mid j\rangle \ .$$

$\rightarrow$ additional Feynman rules with local operator insertions for partonic matrix elements.

The unpolarized light flavor Wilson coefficients are known up to NNLO

[Moch, Vermaseren, Vogt, 2005 Nucl.Phys.B].

For $F_2(x,Q^2)$: at $Q^2 \gtrsim 10m^2$ the asymptotic representation holds at the 1% level.

Status of OME calculations

Leading Order: [Witten 1976, Babcock, Sivers, Wolfram 1978, Shifman, Vainshtein, Zakharov 1978, Leveille, Weiler 1979, Glück, Reya 1979, Glück, Hoffmann, Reya 1982]

Next-to-Leading Order:

[Laenen, van Neerven, Riemersma, Smith 1993]

$Q^2 \gg m^2$: via IBP [Buza, Matiounine, Smith, Migneron, van Neerven 1996]

Compact results via ${}_pF_q$'s [Bierenbaum, Blümlein, Klein, 2007]

$O(\alpha_s^2 \varepsilon)$ (for general N) [Bierenbaum, Blümlein, Klein 2008, 2009]

Next-to-Next-to-Leading Order: $Q^2 \gg m^2$

- Moments for F_2: $N = 2...10(14)$ [Bierenbaum, Blümlein, Klein 2009] mapping large expressions to [MATAD, Steinhauser 2000]
- Contributions to transversity: $N = 1...13$ [Blümlein, Klein, Tödtli 2009]
- Two masses $m_1 \neq m_2 \rightarrow$ Moments $\mathrm{N} = 2,4,6$ [JB, Wißbrock 2011]

At 3-loop order for general values of N: Topic of this talk.

The Wilson Coefficients at large Q^2

$$
\begin{aligned}
\text{2014} \quad L_{q,(2,L)}^{\mathsf{NS}}(N_F+1) &= a_s^2\Big[A_{qq,Q}^{(2),\mathsf{NS}}(N_F+1)\ \delta_2 + \hat{C}_{q,(2,L)}^{(2),\mathsf{NS}}(N_F)\Big] \\
&+ a_s^3\Big[A_{qq,Q}^{(3),\mathsf{NS}}(N_F+1)\ \delta_2 + A_{qq,Q}^{(2),\mathsf{NS}}(N_F+1)C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1) + \hat{C}_{q,(2,L)}^{(3),\mathsf{NS}}(N_F)\Big] \\
\text{2010} \quad L_{q,(2,L)}^{\mathsf{PS}}(N_F+1) &= a_s^3\Big[\ A_{qq,Q}^{(3),\mathsf{PS}}(N_F+1)\ \delta_2 + A_{gq,Q}^{(2)}(N_F)\ \ N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + N_F\hat{\tilde{C}}_{q,(2,L)}^{(3),\mathsf{PS}}(N_F \\
\text{2010} \quad L_{g,(2,L)}^{\mathsf{S}}(N_F+1) &= a_s^2 A_{gg,Q}^{(1)}(N_F+1)N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + a_s^3\Big[\ A_{gg,Q}^{(3)}(N_F+1)\ \delta_2 \\
&+A_{gg,Q}^{(1)}(N_F+1)\ \ N_F\tilde{C}_{g,(2,L)}^{(2)}(N_F+1) + A_{gg,Q}^{(2)}(N_F+1)\ \ N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+ \\
&+ A_{Qg}^{(1)}(N_F+1)\ \ N_F\tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+1) + N_F\hat{\tilde{C}}_{g,(2,L)}^{(3)}(N_F)\Big]\ , \\
\text{2014} \quad H_{q,(2,L)}^{\mathsf{PS}}(N_F+1) &= a_s^2\Big[\ A_{Qq}^{(2),\mathsf{PS}}(N_F+1)\ \delta_2 +\ \tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+1)\Big] + a_s^3\Big[\ A_{Qq}^{(3),\mathsf{PS}}(N_F+1)\ \delta_2 \\
&+ \tilde{C}_{q,(2,L)}^{(3),\mathsf{PS}}(N_F+1) + A_{gq,Q}^{(2)}(N_F+1)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+1) \\
&+A_{Qq}^{(2),\mathsf{PS}}(N_F+1)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1)\Big]\ , \\
H_{g,(2,L)}^{\mathsf{S}}(N_F+1) &= a_s\Big[\ A_{Qg}^{(1)}(N_F+1)\ \delta_2 +\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\Big] + a_s^2\Big[\ A_{Qg}^{(2)}(N_F+1)\ \delta_2 \\
&+ A_{Qg}^{(1)}(N_F+1)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1) +\ A_{gg,Q}^{(1)}(N_F+1)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+1) \\
&+ \tilde{C}_{g,(2,L)}^{(2)}(N_F+1)\Big] +\ a_s^3\Big[\ A_{Qg}^{(3)}(N_F+1)\ \delta_2 +\ A_{Qg}^{(2)}(N_F+1)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+ \\
&+ A_{gg,Q}^{(2)}(N_F+1)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+1) +\ A_{Qg}^{(1)}(N_F+1)\Big\{C_{q,(2,L)}^{(2),\mathsf{NS}}(N_F+1) \\
&+ \tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+1)\Big\} +\ A_{gg,Q}^{(1)}(N_F+1)\ \tilde{C}_{g,(2,L)}^{(2)}(N_F+1) +\ \tilde{C}_{g,(2,L)}^{(3)}(N_F+1
\end{aligned}
$$

All first order factorizable contributions to $H_{Qg}^{(3)}$ are known since 2017.
All logarithmic corrections are known since 2010.

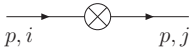
Variable Flavor Number Scheme

$$
\begin{aligned}
f_k(n_f+1,\mu^2) + f_{\bar{k}}(n_f+1,\mu^2) &= A_{qq,Q}^{\rm NS}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes \left[f_k(n_f,\mu^2) + f_{\bar{k}}(n_f,\mu^2)\right] \\
&\quad + \tilde{A}_{qq,Q}^{\rm PS}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes \Sigma(n_f,\mu^2) + \tilde{A}_{qg,Q}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes G(n_f,\mu^2) \\
f_{Q+\bar{Q}}(n_f+1,\mu^2) &= \tilde{A}_{Qq}^{\rm PS}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes \Sigma(n_f,\mu^2) + \tilde{A}_{Qg}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes G(n_f,\mu^2)\,. \\
G(n_f+1,\mu^2) &= A_{gq,Q}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes \Sigma(n_f,\mu^2) + A_{gg,Q}^{S}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes G(n_f,\mu^2)\,. \\
\Sigma(n_f+1,\mu^2) &= \sum_{k=1}^{n_f+1}\left[f_k(n_f+1,\mu^2) + f_{\bar{k}}(n_f+1,\mu^2)\right] \\
&= \left[A_{qq,Q}^{\rm NS}\left(n_f,\frac{\mu^2}{m^2}\right) + n_f\tilde{A}_{qq,Q}^{\rm PS}\left(n_f,\frac{\mu^2}{m^2}\right) + \tilde{A}_{Qq}^{\rm PS}\left(n_f,\frac{\mu^2}{m^2}\right)\right] \\
&\quad \otimes \Sigma(n_f,\mu^2) \\
&\quad + \left[n_f\tilde{A}_{qg,Q}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right) + \tilde{A}_{Qg}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right)\right] \otimes G(n_f,\mu^2)
\end{aligned}
$$

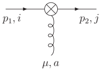
There are generalizations necessary in the 2-mass case.

J. Ablinger et al., Nucl.Phys. B921 (2017) 585 [arXiv:1705.07030]

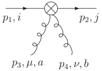
The OMEs are calculated using the QCD Feynman rules together with the following operator insertion Feynman rules:



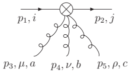
$$\delta^{ij} \not{\Delta} \gamma_\pm (\Delta \cdot p)^{N-1} , \quad N \geq 1$$



$$g t^a_{ji} \Delta^\mu \not{\Delta} \gamma_\pm \sum_{j=0}^{N-2} (\Delta \cdot p_1)^j (\Delta \cdot p_2)^{N-j-2} , \quad N \geq 2$$



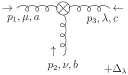
$$g^2 \Delta^\mu \Delta^\nu \not{\Delta} \gamma_\pm \sum_{j=0}^{N-3} \sum_{l=j+1}^{N-2} (\Delta p_2)^j (\Delta p_1)^{N-l-2} \Big[(t^a t^b)_{ji} (\Delta p_1 + \Delta p_4)^{l-j-1} + (t^b t^a)_{ji} (\Delta p_1 + \Delta p_3)^{l-j-1} \Big] , \quad N \geq 3$$



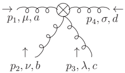
$$\begin{aligned} & g^3 \Delta_\mu \Delta_\nu \Delta_\rho \not{\Delta} \gamma_\pm \sum_{j=0}^{N-4} \sum_{l=j+1}^{N-3} \sum_{m=l+1}^{N-2} (\Delta.p_2)^j (\Delta.p_1)^{N-m-2} \\ & \Big[\ (t^a t^b t^c)_{ji} (\Delta.p_4 + \Delta.p_5 + \Delta.p_1)^{l-j-1} (\Delta.p_5 + \Delta.p_1)^{m-l-1} \\ & + (t^a t^c t^b)_{ji} (\Delta.p_4 + \Delta.p_5 + \Delta.p_1)^{l-j-1} (\Delta.p_4 + \Delta.p_1)^{m-l-1} \\ & + (t^b t^a t^c)_{ji} (\Delta.p_3 + \Delta.p_5 + \Delta.p_1)^{l-j-1} (\Delta.p_5 + \Delta.p_1)^{m-l-1} \\ & + (t^b t^c t^a)_{ji} (\Delta.p_3 + \Delta.p_5 + \Delta.p_1)^{l-j-1} (\Delta.p_3 + \Delta.p_1)^{m-l-1} \\ & + (t^c t^a t^b)_{ji} (\Delta.p_3 + \Delta.p_4 + \Delta.p_1)^{l-j-1} (\Delta.p_4 + \Delta.p_1)^{m-l-1} \\ & + (t^c t^b t^a)_{ji} (\Delta.p_3 + \Delta.p_4 + \Delta.p_1)^{l-j-1} (\Delta.p_3 + \Delta.p_1)^{m-l-1} \Big] , \\ & N \geq 4 \end{aligned}$$

$$\gamma_+ = 1 , \quad \gamma_- = \gamma_5 .$$

$$\frac{1+(-1)^N}{2} \delta^{ab} (\Delta \cdot p)^{N-2} \Big[g_{\mu\nu} (\Delta \cdot p)^2 - (\Delta_\mu p_\nu + \Delta_\nu p_\mu) \Delta \cdot p + p^2 \Delta_\mu \Delta_\nu \Big] , \quad N \geq 2$$



$$\begin{aligned} & -ig \frac{1+(-1)^N}{2} f^{abc} \Bigg(\\ & \Big[(\Delta_\nu g_{\lambda\mu} - \Delta_\lambda g_{\mu\nu}) \Delta \cdot p_1 + \Delta_\mu (p_{1,\nu} \Delta_\lambda - p_{1,\lambda} \Delta_\nu) \Big] (\Delta \cdot p_1)^{N-2} \\ & + \Delta_\lambda \Big[\Delta \cdot p_1 p_{2,\mu} \Delta_\nu + \Delta \cdot p_2 p_{1,\nu} \Delta_\mu - \Delta \cdot p_1 \Delta \cdot p_2 g_{\mu\nu} - p_1 \cdot p_2 \Delta_\mu \Delta_\nu \Big] \\ & \times \sum_{j=0}^{N-3} (-\Delta \cdot p_1)^j (\Delta \cdot p_2)^{N-3-j} \\ & + \left\{ \begin{smallmatrix} p_1 \to p_2 \to p_3 \to p_1 \\ \mu \to \nu \to \lambda \to \mu \end{smallmatrix} \right\} + \left\{ \begin{smallmatrix} p_1 \to p_3 \to p_2 \to p_1 \\ \mu \to \lambda \to \nu \to \mu \end{smallmatrix} \right\} \Bigg) , \quad N \geq 2 \end{aligned}$$



$$\begin{aligned} & g^2 \frac{1+(-1)^N}{2} \Bigg(f^{abe} f^{cde} O_{\mu\nu\lambda\sigma}(p_1, p_2, p_3, p_4) \\ & + f^{ace} f^{bde} O_{\mu\lambda\nu\sigma}(p_1, p_3, p_2, p_4) + f^{ade} f^{bce} O_{\mu\sigma\nu\lambda}(p_1, p_4, p_2, p_3) \Bigg) , \\ & O_{\mu\nu\lambda\sigma}(p_1, p_2, p_3, p_4) = \Delta_\nu \Delta_\lambda \Bigg\{ -g_{\mu\sigma} (\Delta \cdot p_3 + \Delta \cdot p_4)^{N-2} \\ & + [p_{4,\mu} \Delta_\sigma - \Delta \cdot p_4 g_{\mu\sigma}] \sum_{i=0}^{N-3} (\Delta \cdot p_3 + \Delta \cdot p_4)^i (\Delta \cdot p_4)^{N-3-i} \\ & - [p_{1,\sigma} \Delta_\mu - \Delta \cdot p_1 g_{\mu\sigma}] \sum_{i=0}^{N-3} (-\Delta \cdot p_1)^i (\Delta \cdot p_3 + \Delta \cdot p_4)^{N-3-i} \\ & + [\Delta \cdot p_1 \Delta \cdot p_4 g_{\mu\sigma} + p_1 \cdot p_4 \Delta_\mu \Delta_\sigma - \Delta \cdot p_4 p_{1,\sigma} \Delta_\mu - \Delta \cdot p_1 p_{4,\mu} \Delta_\sigma] \\ & \times \sum_{i=0}^{N-4} \sum_{j=0}^{i} (-\Delta \cdot p_1)^{N-4-i} (\Delta \cdot p_3 + \Delta \cdot p_4)^{i-j} (\Delta \cdot p_4)^j \Bigg\} \\ & - \left\{ \begin{smallmatrix} p_1 \leftrightarrow p_2 \\ \mu \leftrightarrow \nu \end{smallmatrix} \right\} - \left\{ \begin{smallmatrix} p_3 \leftrightarrow p_4 \\ \lambda \leftrightarrow \sigma \end{smallmatrix} \right\} + \left\{ \begin{smallmatrix} p_1 \leftrightarrow p_2, \ p_3 \leftrightarrow p_4 \\ \mu \leftrightarrow \nu, \ \lambda \leftrightarrow \sigma \end{smallmatrix} \right\} , \quad N \geq 2 \end{aligned}$$

The diagrams are generated using QGRAF [Nogueira 1993 J. Comput. Phys].

	$A_{qq,Q}^{(3),\mathrm{NS}}$	$A_{gq,Q}^{(3)}$	$A_{Qq}^{(3),\mathrm{PS}}$	$A_{gg,Q}^{(3)}$	$A_{Qg}^{(3)}$
No. diagrams	110	86	125	642	1358

A FORM [Vermaseren 2000] program was written in order to perform the γ-matrix algebra in the numerator of all diagrams, which are then expressed as a linear combination of scalar integrals.

$$
\begin{aligned}
A_{qq,Q}^{(3),\mathrm{NS}} &\rightarrow 7426 \text{ scalar integrals.} \\
A_{gq,Q}^{(3)} &\rightarrow 12529 \text{ scalar integrals.} \\
A_{Qq}^{(3),\mathrm{PS}} &\rightarrow 5470 \text{ scalar integrals.}
\end{aligned}
$$

$\Longrightarrow$ Need to use integration by parts identities.

$\Longrightarrow$ The reduction for all OMEs has been completed.

$\Longrightarrow$ Use special computers: 14 units with overall 8.5 TB RAM, $>$ 250 TB fast disc, hundreds of mathematica lic. ; IBP: several TB of final relations.

The Method of Arbitrary High Moments

JB, C. Schneider, Phys. Lett. B 771 (2017) 31.

- Exploit the vast amount of IBP relations to obtain and to solve the master integrals for fixed values of N recursively to larger and larger values of N.
- Project onto each color/ζ factor to obtain a large set of rational moments.
- Use guessing methods JB, M. Kauers, S. Klein, C. Schneider, Comput. Phys. Commun. 180 (2009) 2143 to obtain a difference equation for each of these terms (usually large in both degree and order).
- Solve these difference equations using `Sigma`. It will find the solution in case of 1st order factorization or split all first order terms of the solution and returns the remaining recurrence which is not 1st order factorizing.
- One may solve all 1st order factorizable problems this way, over whatsoever alphabet (not requiring a particular choice of basis) : HPLs, Kummer iterated integrals, cyclotomic HPLs, root-iterated integrals, ...

Examples:

- 3-loop anomalous dimensions [possible also at higher loop]
- 3 loop massive OMEs, as for their 1st order factorizable parts.

The 3-loop anomalous dimensions

$$
\begin{aligned}
\gamma_{qg}^{(2)} = {} & C_A N_F^2 T_F^2 \Biggl\{ -\frac{5N^2+8N+10}{N(N+1)(N+2)} \frac{128}{9} S_{-2} - \frac{64 P_8}{9N(N+1)^2(N+2)^2} S_1^2 \\
& - \frac{64 P_9}{9N(N+1)^2(N+2)^2} S_2 + \frac{64 P_{25}}{27N(N+1)^3(N+2)^3} S_1 + \frac{16 P_{34}}{27(N-1)N^4(N+1)^4(N+2)^4} \\
& + p_{qg}^{(0)}(N) \left(\frac{32}{9} S_1^3 - \frac{32}{3} S_1 S_2 + \frac{64}{9} S_3 + \frac{128}{3} S_{-3} + \frac{128}{3} S_{2,1} \right) \Biggr\} \\
& + C_F N_F^2 T_F^2 \Biggl\{ \frac{5N^2+3N+2}{N^2(N+1)(N+2)} \frac{32}{3} S_2 + \frac{10N^3+13N^2+29N+6}{N^2(N+1)(N+2)} \frac{32}{9} S_1^2 \\
& - \frac{32 P_{12}}{27N^2(N+1)^2(N+2)} S_1 + \frac{4 P_{38}}{27(N-1)N^5(N+1)^5(N+2)^4} \\
& + p_{qg}^{(0)}(N) \left(-\frac{32}{9} S_1^3 - \frac{32}{3} S_1 S_2 + \frac{320}{9} S_3 \right) \Biggr\} \\
& + C_A C_F N_F T_F \Biggl\{ -128 \frac{N^3-7N^2-6N+4}{N^2(N+1)^2(N+2)} S_{-2,1} + \frac{32 P_5}{N^2(N+1)^2(N+2)} S_{-3} \\
& + \frac{16 P_{18}}{9(N-1)N^2(N+1)^2(N+2)^2} S_1^3 - \frac{16 P_{24}}{9(N-1)N^2(N+1)^2(N+2)^2} S_3 \\
& - \frac{8 P_{27}}{9(N-1)N^3(N+1)^3(N+2)^2} S_1^2 + \frac{8 P_{29}}{3(N-1)N^3(N+1)^3(N+2)^3} S_2 \\
& + \frac{P_{37}}{27(N-1)N^5(N+1)^5(N+2)^4} + p_{qg}^{(0)}(N) \Biggl[\left(\frac{640}{3} S_3 - 384 S_{2,1} \right) S_1 + \frac{32}{3} S_1^4 \\
& + 160 S_1^2 S_2 - 64 S_2^2 + \left(192 S_1^2 + 64 S_2 \right) S_{-2} + 96 S_{-2}^2 + 224 S_{-4} - 64 S_{2,-2} + 64 S_{3,1} \\
& + 192 S_{2,1,1} - 256 S_{-2,1,1} - 192 S_1 \zeta_3 \Biggr] - \frac{192 P_{17}}{(N-1)N^2(N+1)^2(N+2)^2} \zeta_3 \\
& + \left(\frac{16 P_{16}}{3(N-1)N^2(N+1)^2(N+2)^2} S_2 + \frac{16 P_{35}}{27(N-1)N^4(N+1)^4(N+2)^4} \right) S_1 \\
& + \left[-\frac{32 P_{15}}{N^3(N+1)^3(N+2)} + \frac{128\left(N^3-13N^2-14N-2\right)}{N^2(N+1)^2(N+2)} S_1 \right] S_{-2} \\
& + \frac{96N(N+1) p_{qg}^{(0)}(N)^2}{N-1} S_{2,1} \Biggr\} \\
& + C_A^2 N_F T_F \Biggl\{ -\frac{64 P_{11}}{(N-1)N^2(N+1)^2(N+2)^2} S_{-2,1} - \frac{16 P_{20}}{9(N-1)N^2(N+1)^2(N+2)^2} S_3 \\
& - \frac{32 P_{21}}{3(N-1)N^2(N+1)^2(N+2)^2} S_{-3} - \frac{8 P_{22}}{9(N-1)N^2(N+1)^2(N+2)^2} S_1^3 \\
& + \frac{16 P_{32}}{9(N-1)^2N^3(N+1)^3(N+2)^3} S_1^2 + \frac{16 P_{33}}{9(N-1)^2N^3(N+1)^3(N+2)^3} S_2 \\
& - \frac{8 P_{39}}{27(N-1)^2N^5(N+1)^5(N+2)^5} + p_{qg}^{(0)}(N) \Biggl[-\frac{32 P_{10}}{3(N-1)N(N+1)(N+2)} S_{2,1} \\
& + \left(-\frac{704}{3} S_3 + 128 S_{2,1} + 512 S_{-2,1} \right) S_1 - 512 S_{-3} S_1 - \frac{16}{3} S_1^4 - 160 S_1^2 S_2 - 16 S_2^2 - 32 S_4 \\
& + \left(-192 S_1^2 + 320 S_2 \right) S_{-2} - 96 S_{-2}^2 + 96 S_{-4} - 448 S_{2,-2} - 128 S_{3,1} + 512 S_{-3,1} \\
& - 768 S_{-2,1,1} + 192 S_1 \zeta_3 \Biggr] + \frac{96(N-2)(N+3) P_4}{(N-1)N^2(N+1)^2(N+2)^2} \zeta_3 \\
& + \left(\frac{8 P_{19}}{3(N-1)N^2(N+1)^2(N+2)^2} S_2 - \frac{8 P_{36}}{27(N-1)^2N^4(N+1)^4(N+2)^4} \right) S_1 \\
& + \left(-\frac{64 P_{13}}{(N-1)N^2(N+1)^2(N+2)^2} S_1 + \frac{32 P_{30}}{9(N-1)N^3(N+1)^3(N+2)^3} \right) S_{-2} \Biggr\} \\
& + C_F^2 N_F T_F \Biggl\{ \frac{P_{31}}{N^5(N+1)^5(N+2)} - \frac{8 P_3}{3N^2(N+1)^2(N+2)} S_1^3 - \frac{16 P_6}{3N^2(N+1)^2(N+2)} S_3 \\
& + \frac{64 P_{14}}{N^3(N+1)^2(N+2)} S_{-2} - \frac{8 P_{23}}{N^3(N+1)^3(N+2)} S_1^2 + \frac{8 P_{26}}{N^3(N+1)^3(N+2)} S_2 \\
& + p_{qg}^{(0)}(N) \Biggl[\left(-\frac{704}{3} S_3 + 256 S_{2,1} \right) S_1 - 256 S_{-3} S_1 - \frac{16}{3} S_1^4 - 48 S_2^2 - 160 S_4 - 64 S_{-2}^2 \\
& - 192 S_{-4} - \frac{128}{N(N+1)} S_{2,1} - 128 S_{2,-2} + 64 S_{3,1} + 256 S_{-3,1} - 192 S_{2,1,1} \Biggr] \\
& + \frac{96(N-1)\left(3N^2+3N-2\right)}{N^2(N+1)^2} \zeta_3 - 256 \frac{2-N+N^2}{N^2(N+1)(N+2)} \left[S_{-2} S_1 - S_{-2,1} \right] \\
& + \left(-\frac{8 P_{28}}{N^4(N+1)^4(N+2)} - \frac{8 P_7}{N^2(N+1)^2(N+2)} S_2 \right) S_1 - \frac{128(N-1)}{(N+1)^2(N+2)} S_{-3} \Biggr\},
\end{aligned}
$$

Completed the calculation of all contributing 3-loop anomalous dimensions. Here, $\gamma_{qq}^{(2),\mathrm{PS}}$ and $\gamma_{qg}^{(2)}$ are complete; the others are $\propto T_F$;
Also the corresponding polarized 3-loop anomalous dimensions have been calculated very recently.

Confirmed the results of Moch, Vermaseren, Vogt, 2004 & 2014

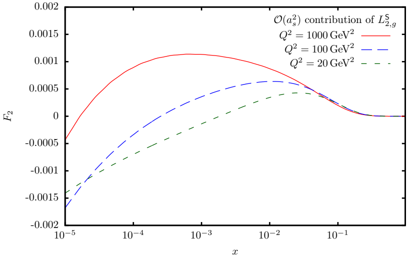
The 1st order factorizable contributions to $A_{Qg}^{(3)}$

$$
\begin{aligned}
a_{Qg}^{(3)}(N) = {} & C_A^2 T_F \Big\{ t_{C_A^2 T_F}(N) + t_{C_A^2 T_F \zeta_3}(N)\zeta_3 \\
& + \frac{72(-2+N)(3+N)P_2}{(N-1)N^2(1+N)^2(2+N)^2}\zeta_4 - \frac{4P_{17}}{(N-1)N^2(1+N)^2(2+N)^2}\mathsf{B}_4 \\
& + p_{qg}^{(0)} \Big[-16\mathsf{B}_4 S_1 + 144\zeta_4 S_1 + \Big(-16S_1^3 - \frac{4P_7}{3(N-1)N(1+N)(2+N)}S_2 - 32S_1S_2 \\
& - 8S_3 + \Big(-\frac{8P_8}{3(N-1)N(1+N)(2+N)} - 48S_1 \Big) S_{-2} - 8S_{-3} + 16S_{-2,1} \Big)\zeta_2 \Big] \\
& + \Big[\frac{2P_{35}}{9(N-1)^2N^4(1+N)^4(2+N)^4} + \frac{4P_{31}}{9(N-1)^2N^3(1+N)^3(2+N)^3}S_1 \\
& - \frac{4P_{18}}{3(N-1)N^2(1+N)^2(2+N)^2}S_1^2 \Big]\zeta_2 \Big\} + C_A T_F^2 \Big\{ t_{C_A T_F^2}(N) + t_{C_A T_F^2 \zeta_3}(N)\zeta_3 \\
& + N_F \Big\{ -\frac{8P_{36}}{243(N-1)N^5(1+N)^5(2+N)^5} + p_{qg}^{(0)} \Big[\Big(\frac{1888}{27}S_3 + \frac{224}{9}S_{2,1} \Big) S_1 \\
& + \frac{32}{27}S_1^4 + \frac{176}{9}S_1^2S_2 + \frac{80}{9}S_2^2 + \frac{640}{9}S_4 + \Big(-\frac{64(2N-1)}{(N-1)N}S_1 + \frac{128}{3}S_2 \Big) S_{-2} + \frac{64}{9}S_{-4} \\
& - \frac{32}{3}S_{1,-3} - \frac{64}{3}S_{2,-2} - \frac{32}{9}S_{3,1} + \frac{64(2N-1)}{(N-1)N}S_{-2,1} + 64S_{1,1,-2} - \frac{416}{9}S_{2,1,1} \\
& + \Big(\frac{16}{3}S_1^2 + \frac{16}{3}S_2 + \frac{32}{3}S_{-2} \Big)\zeta_2 + \Big(\frac{448\big(1+N+N^2\big)}{9(N-1)N(1+N)(2+N)} - \frac{224}{9}S_1 \Big)\zeta_3 \Big] \\
& + \Big(\frac{16P_{32}}{243(N-1)N^2(1+N)^4(2+N)^4} - \frac{16P_{10}}{27N(1+N)^2(2+N)^2}S_2 \Big) S_1 \\
& + \frac{8P_{20}}{81N(1+N)^3(2+N)^3}S_1^2 - \frac{16P_9}{81N(1+N)^2(2+N)^2}S_1^3 \\
& + \frac{8P_{26}}{81(N-1)N^3(1+N)^3(2+N)^3}S_2 - \frac{32P_{22}}{81(N-1)N^2(1+N)^2(2+N)^2}S_3 \\
& + \frac{32P_{21}}{81N(1+N)^3(2+N)^3}S_{-2} + \frac{32P_{15}}{27(N-1)N^2(1+N)^2(2+N)^2}S_{-3} \\
& - \frac{64P_{16}}{9(N-1)N^2(1+N)^2(2+N)^2}S_{1,-2} - \frac{64P_4}{27N(1+N)^2(2+N)^2}S_{2,1} \\
& + \Big(-\frac{4P_{27}}{9(N-1)N^3(1+N)^3(2+N)^3} - \frac{16P_5}{9N(1+N)^2(2+N)^2}S_1 \Big)\zeta_2 \Big\} \\
& + \Big(-\frac{4P_{28}}{9(N-1)N^3(1+N)^3(2+N)^3} + \frac{160\big(4-N+N^2+4N^3+N^4\big)}{9N(1+N)^2(2+N)^2}S_1 \Big)\zeta_2 \\
& + p_{qg}^{(0)} \Big(\frac{40}{3}S_1^2 + \frac{40}{3}S_2 + \frac{80}{3}S_{-2} \Big)\zeta_2 \Big\} + C_F^2 T_F \Big\{ t_{C_F^2 T_F}(N) + t_{C_F^2 T_F \zeta_3}(N)\zeta_3 \\
& - \frac{16(N-1)\big(-2+3N+3N^2\big)}{N^2(1+N)^2}\mathsf{B}_4 + \frac{72(N-1)\big(-2+3N+3N^2\big)}{N^2(1+N)^2}\zeta_4 \\
& + \Big[\frac{P_{23}}{2N^4(1+N)^4(2+N)} + \frac{8P_{19}}{N^3(1+N)^3(2+N)}S_1 + \frac{4P_3}{N^2(1+N)^2(2+N)}S_1^2 \Big]\zeta_2 \\
& + p_{qg}^{(0)} \Big[-16S_1^3 - \frac{8\big(2+3N+3N^2\big)}{N(1+N)}S_2 + 32S_1S_2 + 16S_3 \\
& + \Big(-\frac{16}{N(1+N)} + 32S_1 \Big) S_{-2} + 16S_{-3} - 32S_{-2,1} \Big]\zeta_2 \Big\} + C_F \Big\{ C_A T_F \Big\{ t_{C_F C_A T_F}(N) + \\
& t_{C_F C_A T_F \zeta_3}(N)\zeta_3 + \frac{32P_{13}}{(N-1)N^2(1+N)^2(2+N)^2} \Big(\mathsf{B}_4 - \frac{9}{2}\zeta_4 \Big) \\
& + p_{qg}^{(0)} \Big[32\mathsf{B}_4 S_1 - 144\zeta_4 S_1 + \Big(32S_1^3 - \frac{12P_1}{(N-1)N(1+N)(2+N)}S_2 - 8S_3 \\
& + \Big(-\frac{8\big(1+3N+3N^2\big)}{N(1+N)} + 16S_1 \Big) S_{-2} - 8S_{-3} + 16S_{-2,1} \Big)\zeta_2 \Big] \\
& + \Big(\frac{P_{25}}{18(N-1)N^3(1+N)^3(2+N)^3} - \frac{4P_{29}}{9(N-1)N^3(1+N)^3(2+N)^3}S_1 \\
& + \frac{8P_{12}}{3(N-1)N^2(1+N)^2(2+N)^2}S_1^2 \Big)\zeta_2 \Big\} + T_F^2 \Big\{ t_{C_F T_F^2}(N) + t_{C_F T_F^2 \zeta_3}(N)\zeta_3 \\
& + N_F \Big\{ \frac{P_{37}}{243(-1+N)N^6(1+N)^6(2+N)^5} + p_{qg}^{(0)} \Big[\Big(-\frac{256}{27}S_3 - \frac{128}{3}S_{2,1} \Big) S_1 \\
& - \frac{32}{27}S_1^4 - \frac{64}{9}S_1^2S_2 - \frac{128}{9}S_2^2 + \frac{256}{9}S_4 - \frac{128}{3}S_{3,1} + \frac{256}{3}S_{2,1,1} - \frac{16}{3}S_1^2\zeta_2 \\
& + \Big(-\frac{56P_{14}}{9(-1+N)N^2(1+N)^2(2+N)} + \frac{224}{9}S_1 \Big)\zeta_3 \Big] + \Big(-\frac{16P_{11}}{243N^2(1+N)^3(2+N)} \\
& + \frac{32\big(24+83N+49N^2+10N^3\big)}{27N^2(1+N)(2+N)}S_2 \Big) S_1 - \frac{32P_6}{81N^2(1+N)^2(2+N)}S_1^2 \\
& + \frac{32\big(24+83N+49N^2+10N^3\big)}{81N^2(1+N)(2+N)}S_1^3 + \frac{8P_{33}}{27(-1+N)N^4(1+N)^4(2+N)^3}S_2 \\
& - \frac{16P_{24}}{81(-1+N)N^3(1+N)^3(2+N)^2}S_3 - \frac{128\big(-2-3N+N^2\big)}{3N^2(1+N)(2+N)}S_{2,1} \\
& + \Big(\frac{2(-2+N)P_{30}}{9(-1+N)N^4(1+N)^4(2+N)^3} + \frac{16\big(12+28N+11N^2+5N^3\big)}{9N^2(1+N)(2+N)}S_1 \Big)\zeta_2 \Big\} \\
& + \Big(\frac{2P_{34}}{9(-1+N)N^4(1+N)^4(2+N)^3} + \frac{80\big(6+11N+4N^2+N^3\big)}{9N^2(1+N)(2+N)}S_1 \Big)\zeta_2 \\
& + p_{qg}^{(0)} \Big(-\frac{40}{3}S_1^2 + 8S_2 \Big)\zeta_2 \Big\}\Big\} - \frac{64}{9}p_{qg}^{(0)} T_F^3 \zeta_3,
\end{aligned}
$$

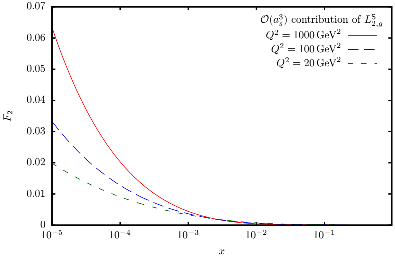
18 out of 28 color/ζ factors were completed. (2000 moments)

From 8000 moments we got the T_F^2 difference equations also; 4 of them are no longer first order factorizable. For them and other remaining color/ζ factors we have the 1st order terms too, i.e. we obtained the analytic results for 1122 of 1358 diagrams.

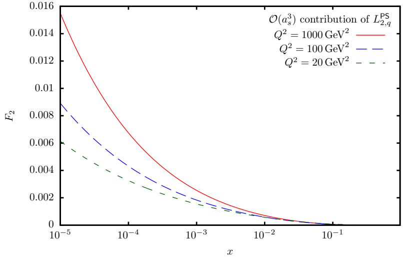
Numerical Results : $L_{g,2}^{\mathrm{S}}$ and $L_{q,2}^{\mathrm{PS}}$



$O(a_s^2)$ $L_{2,g}^{\mathrm{S}}$

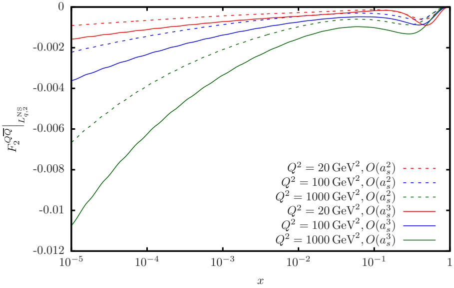


$O(a_s^3)$ $L_{2,g}^{\mathrm{S}}$

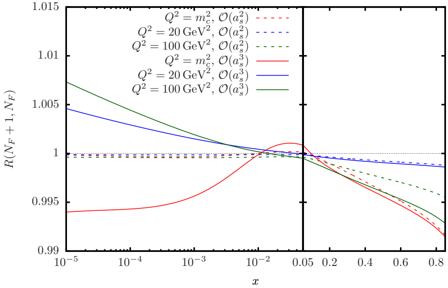


$L_{q,2}^{\mathrm{PS}}$

$L_{q,2}^{\mathrm{NS}}$

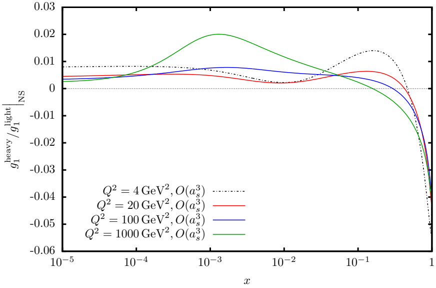


Contribution to $F_2(x, Q^2)$

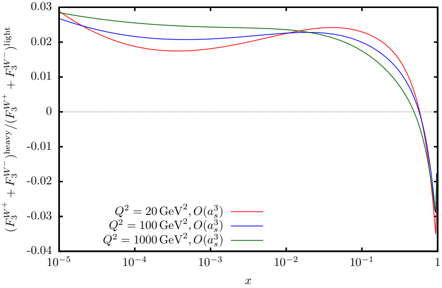


VFNS matching

NS corrections to $g_{1(2)}(x, Q^2)$ and $xF_3^{W^+ + W^-}$



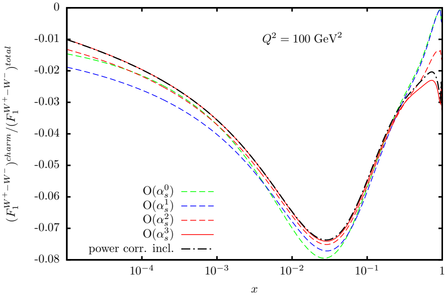
$g_1(x, Q^2)$



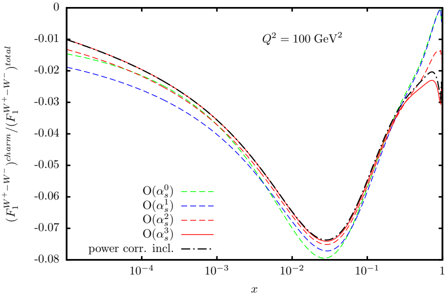
$xF_3^{W^+ + W^-}(x, Q^2)$

The corrections to $g_2(x, Q^2)$ are obtained using the Wandzura-Wilczek relation.

NS corrections to $F_1^{W^+-W^-}$ and $F_2^{W^+-W^-}$



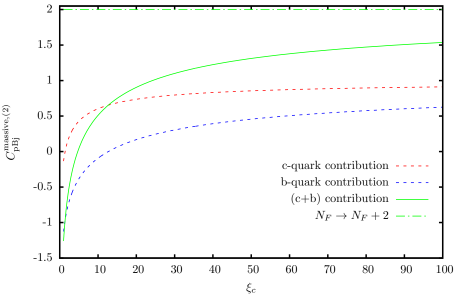
$F_1^{W^+-W^-}(x, Q^2)$



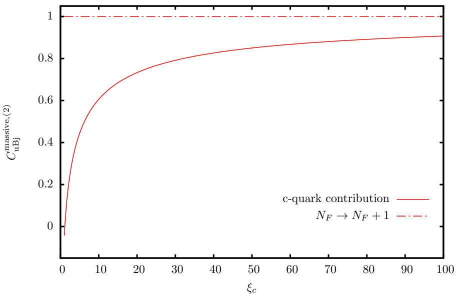
$F_2^{W^+-W^-}(x, Q^2)$

The massless corrections are due to [A. Vogt et al. arXiv:1606.08907 [hep-ph].]
from [A. Behring et al. Phys. Rev. D **94** (2016) no.11, 114006 [arXiv:1609.06255 [hep-ph]]].

$O(\alpha_s^2)$ Complete NS corrections



pol BJ sum rule

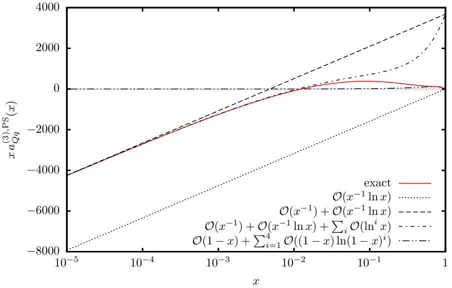


unp. BJ sum rule

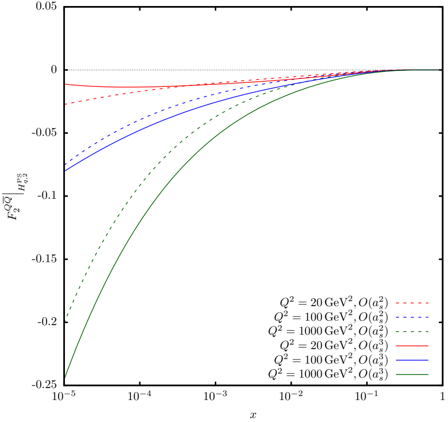
Note the negative corrections at low Q^2!
Already for charm it takes quite a while to become massless.

JB, G. Falcioni, A. De Freitas, Nucl. Phys. B910 (2016) 568.

$H_{q,2}^{\mathrm{PS}}$



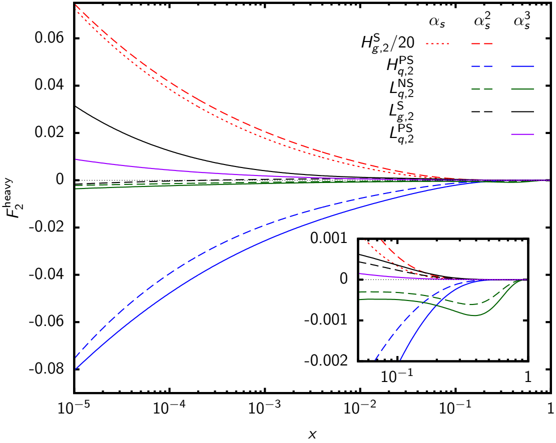
$a_{Qq}^{(3),\mathrm{PS}}$



Contribution to $F_2(x, Q^2)$

The leading small x approximation corresponding to High-energy factorization and small x heavy flavor production S. Catani, M. Ciafaloni, F. Hautmann, Nucl.Phys. B366 (1991) 135 departs from the physical result everywhere except for $x = 1$ (dotted line).

The present NC corrections to $F_2(x, Q^2)$



$Q^2 = 100 \mathrm{GeV}^2$ [$H^S_{g,2}$ scaled down by a factor 20.]

We have calculated 18 of 28 color and ζ-factors of $A_{Qg}^{(3)}$, as well as 2000 moments analytically. (MATAD, 2009: $N \leq 10$).

Here the method of arbitrary high moments proved to be crucial.

3-Loop OME: $A_{gg,Q}$

$$
\begin{aligned}
a_{gg,Q}^{(3)} = \; & \frac{1+(-1)^N}{2}\Bigg\{ C_F^2 T_F \Bigg[\frac{16(N^2+N+2)}{N^2(N+1)^2} \sum_{i=1}^{N} \frac{\binom{2i}{i}\left(\sum_{j=1}^{i} \frac{4^j S_1(j-1)}{\binom{2j}{j} j^2} - 7\zeta_3\right)}{4^i (i+1)^2} - \frac{4P_{69} S_1^2}{3(N-1)N^4(N+1)^4(N+2)} \\
& + \tilde{\gamma}_{gq}^{(0)} \left(\frac{128\left(S_{-4} - S_{-3}S_1 + S_{-3,1} + 2S_{-2,2}\right)}{3N(N+1)(N+2)} + \frac{4(5N^2+5N-22)S_1^2 S_2}{3N(N+1)(N+2)} + \cdots \right) + \cdots \Bigg] \\
& + C_A C_F T_F \Bigg[\frac{16P_{42}}{3(N-1)N^2(N+1)^2(N+2)} \sum_{i=1}^{N} \frac{\binom{2i}{i}\left(\sum_{j=1}^{i} \frac{4^j S_1(j-1)}{\binom{2j}{j} j^2} - 7\zeta_3\right)}{4^i (i+1)^2} + \frac{32P_2 S_{-2,2}}{(N-1)N^2(N+1)^2(N+2)} \\
& - \frac{64P_{14} S_{-2,1,1}}{3(N-1)N^2(N+1)^2(N+2)} - \frac{16P_{23} S_{-4}}{3(N-1)N^2(N+1)^2(N+2)} + \frac{4P_{63} S_4}{3(N-2)(N-1)N^2(N+1)^2(N+2)} + \cdots \Bigg] \\
& + C_A^2 T_F \Bigg[-\frac{4P_{46}}{3(N-1)N^2(N+1)^2(N+2)} \sum_{i=1}^{N} \frac{\binom{2i}{i}\left(\sum_{j=1}^{i} \frac{4^j S_1(j-1)}{\binom{2j}{j} j^2} - 7\zeta_3\right)}{4^i (i+1)^2} + \frac{256P_5 S_{-2,2}}{9(N-1)N^2(N+1)^2(N+2)} \\
& + \frac{32P_{30} S_{-2,1,1} + 16P_{35} S_{-3,1} + 16P_{44} S_{-4}}{9(N-1)N^2(N+1)^2(N+2)} + \frac{16P_{52} S_{-2}^2}{27(N-1)N^2(N+1)^2(N+2)} + \frac{8P_{36} S_2^2}{9(N-1)N^2(N+1)^2} + \cdots \Bigg] \\
& + C_F T_F^2 \Bigg[-\frac{16P_{48}\binom{2N}{N}4^{-N}\left(\sum_{i=1}^{N} \frac{4^i S_1(i-1)}{\binom{2i}{i} i^2} - 7\zeta_3\right)}{3(N-1)N(N+1)^2(N+2)(2N-3)(2N-1)} - \frac{32P_{86} S_1}{81(N-1)N^4(N+1)^4(N+2)(2N-3)(2N-1)} \\
& + \frac{16P_{45} S_1^2}{27(N-1)N^3(N+1)^3(N+2)} - \frac{16P_{45} S_2}{9(N-1)N^3(N+1)^3(N+2)} + \cdots \Bigg] + \cdots \Bigg\} \qquad (1)
\end{aligned}
$$

Also, with this calculation we were able to re-derive the three loop anomalous dimension $\gamma_{gg}^{(3)}$ for the terms $\propto T_F$, and obtained agreement with the literature. The x-space representation is underway.

The NC PS contributions to $F_2(x, Q^2)$ and $F_L(x, Q^2)$

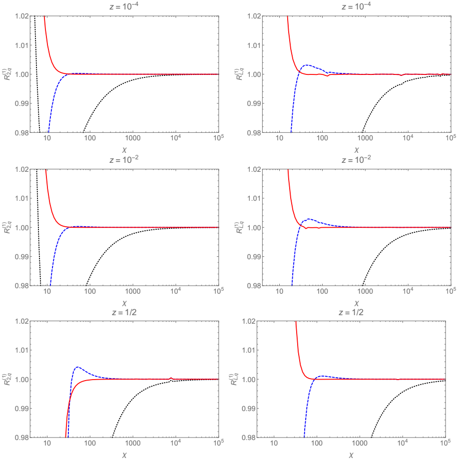
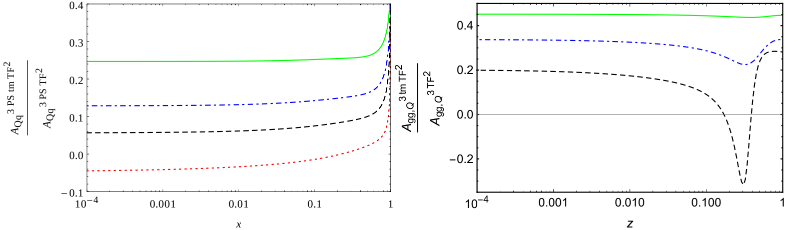


Figure 1: The ratios $R^{(1)}_{2,q}$ (left) and $R^{(1)}_{L,q}$ (right) as a function of $\chi = Q^2/m^2$ for different values of z gradually improved with κ suppressed terms. Dotted lines: asymptotic result; dashed lines: $O(m^2/Q^2)$ improved; solid lines : $O((m^2/Q^2)^2)$ improved.

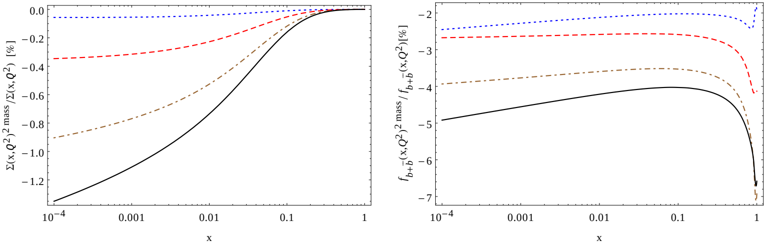
Different convergence range for F_2 and F_L w.r.t Q^2 at $O(\alpha_s^2)$.

3-Loop 2 Mass contributions: PS and gg



The 2-mass contributions are a significant part of the the T_F^2 terms (also in the other channels).

2-Loop 2 Mass VFNS: Singlet and b-quark distributions



Visible effects at high luminosity @ NLO already.

Charm and bottom decouple basically at the same scale.

Conclusions

- 2009: 10-14 Mellin Moments for all massive 3-loop OMEs, coefficient functions.
  2010: Coefficient functions $L_q^{(3),\mathrm{PS}}(N)$, $L_g^{(3),\mathrm{S}}(N)$.
- 2013: Ladder, V-Graph and Benz-topologies for graphs, with no singularities in ε can be systematically calculated for general N.
- Here new functions occur (including a larger number of root-letters in iterated integrals).
- 2014 $L_q^{\mathrm{NS},(3)}$, $A_{gq,Q}^{\mathrm{S},(3)}$, $A_{qq,Q}^{\mathrm{NS,TR}(3)}$, $H_{2,q}^{\mathrm{PS}(3)}$ and $A_{Qq}^{\mathrm{PS}(3)}$ were completed.
- The $O(\alpha_s^2)$ charged current Wilson coefficients have been completed.
- All corresponding 3-loop anomalous dimensions were computed, those for transversity for the first time ab initio; those for the PS- and the qg-case independently for the first time.
- In all NS-cases [NC and CC] we also computed all power corrections at $O(a_s^2)$ and the associated sum rules in the inclusive case improving an earlier result by JB & W. van Neerven.

Conclusions

- All master integrals based on iterative integrals over whatsoever alphabet for $A_{gg,Q}^{(3)}$ and $A_{Qg}^{(3)}$ have been computed and $A_{gg,Q}^{(3)}$ is known for any even integer moment $N \geq 2$. Here all the topologies, including the ladder- and V-topologies have been solved.
- The method of high moments allowed to efficiently derive a series of important general N expressions. We are able to generate analytically moments up to $\mathrm{N} = 2000$ to 8000 at 3-loop order.
- We have all the principal means to reconstruct $A_{Qg}^{(3)}$ systematically at very high accuracy. The full analytic solution will request more mathematical efforts.
- Different new computer-algebra and mathematical technologies were developed. These efforts will continue. The technologies are certainly useful for various present and upcoming calculations for the LHC and ILC.
- Quite a series of 2-mass three loop corrections have been also obtained.

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