Mathematical Structure of QCD Wilson Coefficients and Anomalous Dimensions

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DESY

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Parton Distributions in the 21st Century, Seattle, WA, November 2004

1. Introduction

Consider hard scattering processes in massless field theories:

QCD, QED, $m_i \to 0$

Factorization Theorem Leading Twist:

The cross section σ factorizes as

$$\sigma = \sum_k \sigma_{k,W} \otimes f_k$$

σ_W perturbative Wilson Coefficient

f non-perturbative Parton Density

$\otimes$ Mellin convolution

$$\begin{aligned} [A \otimes B](x) &= \int_0^1 dx_1 \int_0^1 dx_2 \delta(x - x_1 x_2) A(x_1) B(x_2) \\ \mathbf{M}[A \otimes B](N) &= \mathbf{M}[A](N) \cdot \mathbf{M}[B](N) \end{aligned}$$

with the Mellin transform :

$$\mathbf{M}[f(x)](N) = \int_0^1 dx x^{N-1} f(x), \quad Re[N] > c$$

Observation :

Feynman Amplitudes seem to obey the **Mellin Symmetry**

i.e. to significantly simplify in **Mellin Space**

2. x Space Results

Usual Starting Point of Higher Order Calculations :

$\Longrightarrow$ Nielsen type Integrals and their Generalization

$$S_{n,p,q}(x) = \frac{(-1)^{n+p+q-1}}{(n-1)!p!q!} \int_0^1 \frac{dz}{z} \ln^{(n-1)}(z) \ln^p(1-zx) \ln^q(1+zx)$$

Special Cases:

$$\begin{aligned} \mathrm{Li}_n(x) &= S_{n-1,1}(x) & w = n \\ \frac{d\mathrm{Li}_2(\pm x)}{d\ln(x)} &= -\ln(1 \mp x) & w = 1 \\ \mathrm{Li}_0(x) &= \frac{x}{1-x} & w = 0 \end{aligned}$$

van Neerven, Zijlstra 1992

$$
\begin{aligned}
c_{2,-}^{(2)}(x) &= C_F\,(C_F - C_A/2)\,\times \\
&\left\{ \frac{1+x^2}{1-x} \left[\left[4\ln^2(x) - 16\ln(x)\ln(1+x) - 16\mathrm{Li}_2(-x) - 8\zeta_2 \right] \ln(1-x) \right.\right. \\
&+ \left[-2\ln^2(x) + 20\ln(x)\ln(1+x) - 8\ln^2(1+x) + 8\mathrm{Li}_2(1-x) + 16\mathrm{Li}_2(-x) - 8 \right] \ln(x) \\
&- 16\ln(1+x)\mathrm{Li}_2(-x) - 8\zeta_2\ln(1+x) - 16\left[\mathrm{Li}_3\left(-\frac{1-x}{1+x}\right) - \mathrm{Li}_3\left(\frac{1-x}{1+x}\right) \right] \\
&\left. - 16\mathrm{Li}_2(1-x) + 8S_{1,2}(1-x) + 8\mathrm{Li}_3(-x) - 16S_{1,2}(-x) + 8\zeta_3 \right] \\
&+ (4+20x)\left[\ln^2(x)\ln(1+x) - 2\ln(x)\ln^2(1+x) - 2\zeta_2\ln(1+x) - 4\ln(1+x)\mathrm{Li}_2(-x) \right. \\
&\left. + 2\mathrm{Li}_3(-x) - 4S_{1,2}(-x) + 2\zeta_3 \right] + \left(32 + 32x + 48x^2 - \frac{72}{5}x^3 + \frac{8}{5x^2} \right) \\
&\times \left[\mathrm{Li}_2(-x) + ln(x)\ln(1+x) \right] + 8(1+x)\left[\mathrm{Li}_s(1-x) + \ln(x)\ln(1-x) \right] + 16(1-x)\ln(1-x) \\
&+ \left(-4 - 16x - 24x^2 + \frac{36}{5}x^3 \right)\ln^2(x) + \frac{1}{5}\left(-26 - 106x + 72x^2 - \frac{8}{x} \right)\ln(x) \\
&\left. + \left(-4 + 20x + 48x^2 - \frac{72}{5}x^3 \right)\zeta_2 + \frac{1}{5}\left(-162 + 82x + 72x^2 + \frac{8}{x} \right) \right\}
\end{aligned}
$$

.... several other pages for $c_2^{(+)}(x), c_2^G(x), c_L^{(q,G)}(x)$

$\Longrightarrow$ 77 Functions @ 2 Loops

- $\Longrightarrow$ partly rather complicated arguments
- $\Longrightarrow$ relations are not directly visible ...

The 77 functions do roughly correspond in number to the number of all possible harmonic sums up to weight w=4: 80.

x Space Results

No.	$f(z)$	$M[f](N) = \int_0^1 dz z^{N-1} f(z)$
1	$\delta(1-z)$	1
2	z^r	$\frac{1}{N+r}$
3	$\left(\frac{1}{1-z}\right)_+$	$-S_1(N-1)$
4	$\frac{1}{1+z}$	$(-1)^{N-1}[\log(2) - S_1(N-1)]$ $+\frac{1+(-1)^{N-1}}{2} S_1\left(\frac{N-1}{2}\right) - \frac{1-(-1)^{N-1}}{2} S_1\left(\frac{N-2}{2}\right)$
5	$z^r \log^n(z)$	$\frac{(-1)^n}{(N+r)^{n+1}}\Gamma(n+1)$
6	$z^r \log(1-z)$	$-\frac{S_1(N+r)}{N+r}$
7	$z^r \log^2(1-z)$	$\frac{S_1^2(N+r) + S_2(N+r)}{N+r}$
8	$z^r \log^3(1-z)$	$-\frac{S_1^3(N+r) + 3S_1(N+r)S_2(N+r) + 2S_3(N+r)}{N+r}$
9	$\left[\frac{\log(1-z)}{1-z}\right]_+$	$\frac{1}{2}S_1^2(N-1) + \frac{1}{2}S_2(N-1)$
10	$\left[\frac{\log^2(1-z)}{1-z}\right]_+$	$-\left[\frac{1}{3}S_1^3(N-1) + S_1(N-1)S_2(N-1) + \frac{2}{3}S_3(N-1)\right]$
11	$\left[\frac{\log^3(1-z)}{1-z}\right]_+$	$\frac{1}{4}S_1^4(N-1) + \frac{3}{2}S_1^2(N-1)S_2(N-1)$ $+\frac{3}{4}S_2^2(N-1) + 2S_1(N-1)S_3(N-1)$ $+\frac{3}{2}S_4(N-1)$
12	$\frac{\log^n(z)}{1-z}$	$(-1)^{n+1}\Gamma(n+1)\left[S_{n+1}(N-1) - \zeta(n+1)\right]$

Only single sums.

No.	$f(z)$	$M[f](N)$
64	$\dfrac{\mathrm{Li}_3(-z)}{1+z}$	$(-1)^{N-1}\Big\{S_{3,-1}(N-1)+\big[S_3(N-1)-S_{-3}(N-1)\big]\log 2$ $+\frac{1}{2}\zeta(2)S_{-2}(N-1)-\frac{3}{4}\zeta(3)S_{-1}(N-1)$ $+\frac{1}{8}\zeta^2(2)-\frac{3}{4}\zeta(3)\log 2\Big\}$
65	$\mathrm{Li}_3(1-z)$	$\frac{1}{N}[S_1(N)S_2(N)-\zeta(2)S_1(N)+S_3(N)$ $-S_{2,1}(N)+\zeta(3)]$
66	$\dfrac{\mathrm{Li}_3(1-z)}{1-z}$	$-S_{1,1,2}(N-1)+\frac{1}{2}\zeta(2)S_1^2(N-1)+\frac{1}{2}\zeta(2)S_2(N-1)$ $-\zeta(3)S_1(N-1)+\frac{2}{5}\zeta^2(2)$
67	$\dfrac{\mathrm{Li}_3(1-z)}{1+z}$	$(-1)^{N-1}\Big[S_{-1,1,2}(N-1)-\zeta(2)S_{-1,1}(N-1)$ $+\zeta(3)S_{-1}(N-1)+\mathrm{Li}_4\left(\frac{1}{2}\right)-\frac{9}{20}\zeta^2(2)$ $+\frac{7}{8}\zeta(3)\log 2+\frac{1}{2}\zeta(2)\log^2 2+\frac{1}{24}\log^4 2\Big]$
68	$\mathrm{Li}_3\left(\frac{1-z}{1+z}\right)$ $-\mathrm{Li}_3\left(-\frac{1-z}{1+z}\right)$	$\frac{(-1)^N}{N}\Big[-S_{-1,2}(N)-S_{-2,1}(N)+S_1(N)S_{-2}(N)$ $+S_{-3}(N)$ $+\zeta(2)S_{-1}(N)+\frac{1}{2}\zeta(2)S_1(N)-\frac{7}{8}\zeta(3)+\frac{3}{2}\zeta(2)\log 2\Big]$ $+\frac{1}{N}\Big[-S_{-1,-2}(N)-S_{2,1}(N)+S_1(N)S_2(N)+S_3(N)$ $-\frac{1}{2}\zeta(2)S_{-1}(N)-\zeta(2)S_1(N)+\frac{21}{8}\zeta(3)-\frac{3}{2}\zeta(2)\log 2\Big]$
69	$\frac{1}{1+z}\Big[\mathrm{Li}_3\left(\frac{1-z}{1+z}\right)$ $-\mathrm{Li}_3\left(-\frac{1-z}{1+z}\right)\Big]$	$(-1)^{N-1}\Big\{S_{1,1,-2}(N-1)-S_{1,-1,2}(N-1)$ $+S_{-1,1,2}(N-1)-S_{-1,-1,-2}(N-1)$ $+2\zeta(2)S_{1,-1}(N-1)+\frac{1}{4}\zeta(2)S_1^2(N-1)-\frac{1}{4}\zeta(2)S_{-1}^2(N-1)$ $-\zeta(2)S_1(N-1)S_{-1}(N-1)-\zeta(2)S_{-2}(N-1)$ $-\left[\frac{7}{8}\zeta(3)-\frac{3}{2}\zeta(2)\log 2\right]S_1(N-1)$ $+\left[\frac{21}{8}\zeta(3)-\frac{3}{2}\zeta(2)\log 2\right]S_{-1}(N-1)$ $-2\mathrm{Li}_4\left(\frac{1}{2}\right)+\frac{19}{40}\zeta^2(2)+\frac{1}{2}\zeta(2)\log^2 2-\frac{1}{12}\log^4 2\Big\}$

2 loop coefficient functions $\Longrightarrow$ Nested Harmonic Sums of Weight $w=4$

x Space Results

$$
\begin{aligned}
&S_{-1,-1,-2}(N) = \\
&(-1)^{N+1}\mathbf{M}\left\{\frac{1}{1+x}\left[F_1(x)+\log(1-x)\mathrm{Li}_2(-x)\right]\right\}(N) \\
&+(-1)^{N+1}\mathbf{M}\left\{\frac{1}{1+x}\left[\frac{1}{2}\mathrm{S}_{1,2}(x^2)-\mathrm{S}_{1,2}(x)-\mathrm{S}_{1,2}(-x)\right]\right\}(N) \\
&+\frac{1}{2}\zeta(2)\left[S_{-1,1}(N)-S_{-1,-1}(N)\right] \\
&+\left[\frac{9}{8}\zeta(3)-\frac{3}{2}\zeta(2)\log(2)-\frac{1}{6}\log^3(2)\right]S_{-1}(N) \\
&-\frac{1}{10}\zeta(2)^2+\frac{17}{8}\zeta(3)\log(2)-\frac{7}{4}\zeta(2)\log^2(2)-\frac{1}{6}\log^4(2)
\end{aligned}
$$

with

$$
\begin{aligned}
F_1(x) \quad = \quad & S_{1,2}\left(\frac{1-x}{2}\right)+S_{1,2}(1-x)-S_{1,2}\left(\frac{1-x}{1+x}\right) \\
&+S_{1,2}\left(\frac{1}{1+x}\right)-\ln(2)\left(\frac{1-x}{2}\right) \\
&+\frac{1}{2}\ln^2(2)\ln\left(\frac{1+x}{2}\right)-\ln(2)\mathrm{Li}_2\left(\frac{1-x}{1+x}\right)
\end{aligned}
$$

$F_1(x)$, although of complicated structure, it reduces completely via algebraic relations

$\Rightarrow$ Mellin polynomial of simpler objects

These objects can be very complicated integrals. J.B., van Neerven, Ravindran, Kawamura 2000, 2003

3. Multiple Harmonic Sums to Level 6

The simplest example :

$$P_{qq}(x) = \left(\frac{1+x^2}{1-x}\right)_+ = \frac{2}{(1-x)_+} + \ldots$$

$$\int_0^1 dx \frac{x^{N-1}}{(1-x)_+} = -\sum_{k=0}^{N-2} \int_0^1 dx x^k = -\sum_{k=1}^{N-1} \frac{1}{k} = -S_1(N-1)$$

Alternating sums :

$$S_{-1}(N-1) = (-1)^{N-1} \mathbf{M}\left[\frac{1}{1+x}\right](N) - \ln(2) = \int_0^1 dx \frac{x^{N-1}}{(1-x)_+} = \sum_{k=1}^{N-1} \frac{(-1)^k}{k}$$

(Finite for $N \rightarrow \infty$.)

General case :

$$S_{a_1,\ldots,a_l}(N) = \sum_{k_1=1}^{N} \frac{(\mathrm{sign}(a_1))^{k_1}}{k_1^{|a_1|}} \sum_{k_2=1}^{k_1} \frac{(\mathrm{sign}(a_2))^{k_2}}{k_2^{|a_2|}} \ldots$$

Vermaseren, 1997

All Mellin transforms occurring in massless Field Theories for 1-Parameter Quantities can be represented by Harmonic Sums

(at least to 3–loop order).

Algebraic Relations

First relation:

L. Euler, 1775

$$S_{m,n} + S_{n,m} = S_m \cdot S_n + S_{m+n}, \qquad m, n > 0$$

Generalized to alternating sums by

$$\begin{aligned} S_{m,n} + S_{n,m} &= S_m \cdot S_n + S_{m \wedge n}, \\ m \wedge n &= [|m| + |n|] \, \mathrm{sign}(m) \mathrm{sign}(n) \end{aligned}$$

Ternary relations: Sita Ramachandra Rao, 1984,
4-ary relation: J.B., Kurth, 1998.

These & other relations hold widely independent of their **Value** and **Type**.

Determined by :
- Index Structure
- Multiplication Relation

The Formalism applies as well to the Harmonic Polylogarithms. Remiddi, Vermaseren, 1999.

Application to QED: T. Riemann et al., 2004

J.B., Comput. Phys. Commun. **159** (2004) 19.

Linear Representations of Mellin Transform by Harmonic Sums:

$$\mathbf{M}[F_w(x)](N) = S^{w}_{k_1,\ldots,k_m}(N) + P\left(S^{\tau'}_{k_1,\ldots,k_r}, \sigma^{\tau''}_{k_1,\ldots,k_p}\right)$$

$w = \sum_{i=1}^{m} |k_i|$ Weight

$\tau', \tau'' < w$ P is a polynomial.

w	#	Σ	
1	2	2	
2	6	8	
3	18	26	2 Loop anom. Dimensions
4	54	80	2 Loop Wilson Coefficients
5	162	242	3 Loop anom. Dimensions
6	486	728	3 Loop Wilson Coefficients
	$2 \cdot 3^{w-1}$	$3^w - 1$	

Shuffle Products

Depth 2:

$$S_{a_1}(N) \sqcup\!\sqcup\, S_{a_2}(N) = S_{a_1,a_2}(N) + S_{a_2,a_1}(N)$$

Depth 3:

$$S_{a_1}(N) \sqcup\!\sqcup\, S_{a_2,a_3}(N) = S_{a_1,a_2,a_3}(N) + S_{a_2,a_1,a_3}(N) + S_{a_2,a_3,a_1}(N)$$

Depth 4:

$$\begin{aligned}
S_{a_1}(N) \sqcup\!\sqcup\, S_{a_2,a_3,a_4}(N) &= S_{a_1,a_2,a_3,a_4}(N) + S_{a_2,a_1,a_3,a_4}(N) + S_{a_2,a_3,a_1,a_4}(N) \\
&+ S_{a_2,a_3,a_4,a_1}(N) \\
S_{a_1,a_2}(N) \sqcup\!\sqcup\, S_{a_3,a_4}(N) &= S_{a_1,a_2,a_3,a_4}(N) + S_{a_1,a_3,a_2,a_4}(N) + S_{a_1,a_3,a_4,a_2}(N) \\
&+ S_{a_3,a_4,a_1,a_2}(N) + S_{a_3,a_1,a_4,a_2}(N) + S_{a_3,a_1,a_2,a_4}(N)
\end{aligned}$$

Depth 5:

$$\begin{aligned}
S_{a_1}(N) \sqcup\!\sqcup\, S_{a_2,a_3,a_4,a_5}(N) &= S_{a_1,a_2,a_3,a_4,a_5}(N) + S_{a_2,a_1,a_3,a_4,a_5}(N) \\
&+ S_{a_2,a_3,a_1,a_4,a_5}(N) + S_{a_2,a_3,a_4,a_1,a_5}(N) \\
&+ S_{a_2,a_3,a_4,a_5,a_1}(N) \\
S_{a_1,a_2}(N) \sqcup\!\sqcup\, S_{a_3,a_4,a_5}(N) &= S_{a_1,a_2,a_3,a_4,a_5}(N) + S_{a_1,a_3,a_2,a_4,a_5}(N) \\
&+ S_{a_1,a_3,a_4,a_2,a_5}(N) + S_{a_1,a_3,a_4,a_5,a_2}(N) \\
&+ S_{a_3,a_1,a_2,a_4,a_5}(N) + S_{a_3,a_1,a_4,a_2,a_5}(N) \\
&+ S_{a_3,a_1,a_4,a_5,a_2}(N) + S_{a_3,a_4,a_5,a_1,a_2}(N) \\
&+ S_{a_3,a_4,a_1,a_5,a_2}(N) + S_{a_3,a_4,a_1,a_2,a_5}(N)
\end{aligned}$$

Depth 6: .....

Algebraic Equations

Depth 2:

$$S_{a_1}(N) ⧢ S_{a_2}(N) - S_{a_1}(N)S_{a_2}(N) - S_{a_1\wedge a_2}(N) = 0$$

Depth 3:

$$S_{a_1}(N) ⧢ S_{a_2,a_3}(N) \quad - \quad S_{a_1}(N)S_{a_2,a_3}(N) - S_{a_1\wedge a_2,a_3}(N) - S_{a_2,a_1\wedge a_3}(N) = 0$$

Depth 4:

$$\begin{aligned} S_{a_1}(N) ⧢ S_{a_2,a_3,a_4}(N) &- S_{a_1}(N)S_{a_2,a_3,a_4}(N) - S_{a_1\wedge a_2,a_3,a_4}(N) \\ &- S_{a_2,a_1\wedge a_3,a_4}(N) - S_{a_2,a_3,a_1\wedge a_4}(N) = 0 \\ S_{a_1,a_2}(N) ⧢ S_{a_3,a_4}(N) &- S_{a_1,a_2}(N)S_{a_3,a_4}(N) - S_{a_1,a_2\wedge a_3,a_4}(N) \\ &- S_{a_1,a_3,a_2\wedge a_4}(N) - S_{a_3,a_1\wedge a_4,a_2}(N) \\ &- S_{a_3,a_1,a_2\wedge a_4}(N) - S_{a_1\wedge a_3,a_2,a_4}(N) \\ &- S_{a_1\wedge a_3,a_4,a_2}(N) + S_{a_1\wedge a_3,a_2\wedge a_4} = 0 \end{aligned}$$

Depth 5: .....

\# Basic Sums = # Permutations - # Independent Equations

Some Solution for $d = 6$

$$
\begin{aligned}
&S_{a,a,a,a,b,b} = \\
&-\frac{1}{4}S_aS_{b,a,a,a,b} + \frac{3}{4}S_{a\wedge b,a,a,a,b} - \frac{1}{4}S_{b,a,a,a,a\wedge b} + \frac{1}{12}S_aS_{a,a,b,b,a} \\
&+S_{a,a,a,a,b\wedge b} - \frac{1}{12}S_{a\wedge a,b,b,a,a} - \frac{1}{12}S_{a,b,b,a\wedge a,a} - \frac{1}{12}S_{a,b,b,a,a\wedge a} \\
&-\frac{1}{4}S_{b,a\wedge a,a,a,b} - \frac{1}{4}S_{b,a,a\wedge a,a,b} - \frac{1}{4}S_{b,a,a,a\wedge a,b} - \frac{1}{4}S_{a,a\wedge a,a,b,b} \\
&-\frac{1}{4}S_{a,a,a\wedge a,b,b} - \frac{1}{4}S_{a,b,a\wedge a,a,b} - \frac{1}{4}S_{a,b,a,a\wedge a,b} + \frac{1}{12}S_{a\wedge a,b,a,b,a} \\
&+\frac{1}{12}S_{a,b,a\wedge a,b,a} - \frac{1}{4}S_{a,a,a,b,a\wedge b} - \frac{1}{4}S_{a,a,b,a,a\wedge b} + \frac{1}{12}S_{a,a,a\wedge b,b,a} \\
&+\frac{3}{4}S_{a,a,a\wedge b,a,b} - S_{b,b,a,a,a,a} + \frac{1}{4}S_{b,b,a,a\wedge a,a} - \frac{1}{4}S_{a\wedge a,a,a,b,b} \\
&+\frac{1}{12}S_{a,b,a,b,a\wedge a} + \frac{1}{12}S_{b,a\wedge a,a,b,a} + \frac{1}{12}S_{b,a,a\wedge a,b,a} + \frac{1}{12}S_{b,a,a,b,a\wedge a} \\
&-\frac{1}{12}S_{b,a\wedge a,b,a,a} - \frac{1}{12}S_{b,a,b,a\wedge a,a} + \frac{1}{4}S_{b,b,a\wedge a,a,a} + \frac{1}{4}S_{b,b,a,a,a\wedge a} \\
&-\frac{1}{4}S_{a\wedge a,a,b,a,b} - \frac{1}{4}S_{a,a\wedge a,b,a,b} - \frac{1}{4}S_{a,a,b,a\wedge a,b} + \frac{1}{12}S_{a\wedge a,a,b,b,a} \\
&+\frac{1}{12}S_{a,a\wedge a,b,b,a} + \frac{1}{12}S_{a,a,b,b,a\wedge a} - \frac{1}{4}S_{a\wedge a,b,a,a,b} - \frac{1}{12}S_{b,a,b,a,a\wedge a} \\
&+\frac{1}{12}S_{a\wedge b,a,a,b,a} + \frac{1}{12}S_{b,a,a,a\wedge b,a} - \frac{1}{12}S_{a\wedge b,a,b,a,a} + \frac{1}{4}S_{a\wedge b,b,a,a,a} \\
&+\frac{1}{4}S_{b,a\wedge b,a,a,a} - \frac{1}{12}S_{b,a,a\wedge b,a,a} + \frac{3}{4}S_{a,a,a,a\wedge b,b} + \frac{1}{12}S_{a,a,b,a\wedge b,a} \\
&+\frac{3}{4}S_{a,a\wedge b,a,a,b} - \frac{1}{4}S_{a,b,a,a,a\wedge b} + \frac{1}{12}S_{a,a\wedge b,a,b,a} + \frac{1}{12}S_{a,b,a,a\wedge b,a} \\
&-\frac{1}{12}S_{a,a\wedge b,b,a,a} - \frac{1}{12}S_{a,b,a\wedge b,a,a} - \frac{1}{4}S_aS_{a,a,a,b,b} - \frac{1}{12}S_aS_{a,b,b,a,a} \\
&+\frac{1}{12}S_aS_{a,b,a,b,a} - \frac{1}{12}S_aS_{b,a,b,a,a} + \frac{1}{12}S_aS_{b,a,a,b,a} - \frac{1}{4}S_aS_{a,b,a,a,b} \\
&+S_bS_{a,a,a,a,b} + \frac{1}{4}S_aS_{b,b,a,a,a} - \frac{1}{4}S_aS_{a,a,b,a,b}
\end{aligned}
$$

Depth $d = 3$

Index Set	Number	Dep. Sums of Depth 3	min. Weight	Fraction of fund. Sums
$\{a, a, a\}$	1	1	3	0
$\{a, a, b\}$	3	2	3	1/3
$\{a, b, c\}$	6	4	4	1/3

Depth $d = 4$

Index Set	Number	Dep. Sums of Depth 4	min. Weight	Fraction of fund. Sums
$\{a, a, a, a\}$	1	1	4	0
$\{a, a, a, b\}$	4	3	4	1/4
$\{a, a, b, b\}$	6	5	4	1/6
$\{a, a, b, c\}$	12	9	5	1/4
$\{a, b, c, d\}$	24	18	6	1/4

Depth $d = 6$

Index Set	Number	Dep. Sums of Depth 6	min. Weight	Fraction of fund. Sums
$\{a, a, a, a, a, a\}$	1	1	6	0
$\{a, a, a, a, a, b\}$	6	5	6	1/6
$\{a, a, a, a, b, b\}$	15	13	6	2/15
$\{a, a, a, b, b, b\}$	20	17	6	3/20
$\{a, a, a, a, b, c\}$	30	25	7	1/6
$\{a, a, a, b, b, c\}$	60	50	7	1/6
$\{a, a, b, b, c, c\}$	90	76	8	7/45
$\{a, a, a, b, c, d\}$	120	100	8	1/6
$\{a, a, b, b, c, d\}$	180	150	8	1/6
$\{a, a, b, c, d, e\}$	360	300	10	1/6
$\{a, b, c, d, e, f\}$	720	600	12	1/6

Theory of Words

Can we count the Basis in simpler way ? $\Longrightarrow$ YES.

Free Algebras and Elements of the Theory of Codes
$\Longrightarrow$ **Particle Physics**

Only the multiplication relation and the Index structure matters

$\mathfrak{A} = \{a, b, c, d, \ldots\}$ **Alphabet**

$a < b < c < d < \ldots$ **ordered**

$\mathfrak{A}^*(\mathfrak{A})$ **Set of all words W**

$W = a_1 \cdot a_2 \cdot a_{27} \ldots a_{532} \equiv$ **concatenation product (nc)**

$W = p \cdot x \cdot s$ **p = prefix; s = suffix**

Definition:

A Lyndon word is smaller than any of its suffixes.

Theorem:[Radford, 1979]

The shuffle algebra $K\langle\mathfrak{A}\rangle$ is freely generated by the Lyndon words.
I.e. the number of Lyndon words yields the number of basic elements.

Examples :

$\{a, a, \ldots, a, b\} = aaa \ldots ab$ 1 Lyndon word for these sets

$n \quad a's: \quad n_{basic}/n_{all} = 1/n$ $n \equiv$ depth of the sums

$\{a, a, a, b, b, b\}$ $aaabbb, aababb, aabbab$ 3 Lyndon words

$n_{basic}/n_{all} = 3/20 < 1/6$. Symmetries lead to a smaller fraction.

Is there a general Counting Relation ?

E. Witt, 1937

$$l_n(n_1, \ldots, n_q) = \frac{1}{n} \sum_{d \| n_i} \mu(d) \frac{(n/d)!}{(n_1/d)! \ldots (n_q/d)!}, \qquad \sum_i n_i = n$$

$\mu(k)$ Möbius function

2nd Witt formula.

The Length of the Basis is a function mainly of the Depth.

$$l_6(\{a, a, a, b, b, b\}) = \frac{1}{6} \left[\mu(1) \frac{6!}{3!3!} + \mu(3) \frac{2!}{1!1!} \right] = 3$$

$$n_6(\{a, a, a, b, b, b\}) = \frac{6!}{2!3!} = 20$$

Weight	# Sums	Cum. # Sums	# Basic Sums	Cum. # Basic Sums	Cum. Fraction
1	2	2	0	0	0.0
2	6	8	1	1	0.1250
3	18	26	6	7	0.2692
4	54	80	16	23	0.2875
5	162	242	46	69	0.2851
6	486	728	114	183	0.2513

⇑ 2nd Witt formula

Structural Relations

Seek for further Reduction:

Relations using the Value of the Objects [DESY 04-064]

Use Relations like:

$$\frac{1}{2}\frac{\mathrm{Li}_2(x^2)}{1-x^2} = \frac{\mathrm{Li}_2(x)}{1-x} + \frac{\mathrm{Li}_2(x)}{1+x} + \frac{\mathrm{Li}_2(-x)}{1-x} + \frac{\mathrm{Li}_2(-x)}{1+x}$$

$$\begin{aligned}\frac{1}{4}\mathbf{M}\left[\left(\frac{\mathrm{Li}_4(x)}{1-x}\right)_+\right]\left(\frac{N}{2}\right) &= \mathbf{M}\left[\left(\frac{\mathrm{Li}_4(x)}{1-x}\right)_+\right](N) + \mathbf{M}\left[\left(\frac{\mathrm{Li}_4(-x)}{1-x}\right)_+\right](N) \\ &+\mathbf{M}\left[\frac{\mathrm{Li}_4(x)}{1+x}\right](N) + \mathbf{M}\left[\frac{\mathrm{Li}_4(-x)}{1+x}\right](N) \\ &+\frac{1}{8}\zeta(4)\ln(2) - \frac{3}{4}\zeta(2)\zeta(3) + \frac{41}{128}\zeta(5)\,.\end{aligned}$$

and similar ones.

Apply Symmetries among Mellin-Transforms of Nielsen Integrals.

Since all harmonic sums are meromorphic functions for $N \in \mathbf{C}$ since they may be represented by **Factorial Series** Derivatives are not essentially new functions.

$$\mathbf{M}[\ln^k(x)f(x)](N) = \frac{\partial^k}{\partial N^k}\mathbf{M}[f(x)](N)$$

Further Reduction due to the Structure of Feynman Amplitudes

The Lord is mercy, after all!

Analytic Continuation

The Harmonic Sums and Mellin Transforms have to be represented such, that the outer summation index can be analytically continued to $N \in \mathbb{C}$ [J.B., Comput.Phys.Commun.**133** (2000) 76.]

- Use precise, adaptive Representations in analytic Form
- Refer to the Representation through Factorial Series etc.
- The **Residue Theorem** is used to get back to x space

4. The Length of the Basis

The anomalous dimensions and Wilson coefficients for $m_i = 0$ can be expressed in terms of multiple harmonic sums to 3–loop order.

What are the irreducible functions behind this representation ?

We will not count Euler's Γ-function neither all derivations of the functions occurring.

The final set of functions:

Trivial functions:

$$S_{\pm k}(N) \quad \longrightarrow \quad \psi^{(k-1)}(N+1)$$

For $w = 1, 2$ no non–trivial functions contribute to the anomalous dimensions and Wilson coefficients.

Non–trivial functions:

$N = 3$: Two–Loop anomalous dimensions

$$\mathbf{M}\left[\frac{\mathrm{Li}_2(x)}{1+x}\right](N)$$

Yndurain et al., 1980

$N = 4$: Two–Loop Wilson Coefficients

$$\mathbf{M}\left[\frac{\ln(1+x)}{1+x}\right](N), \quad \mathbf{M}\left[\frac{\mathrm{Li}_2(x)}{1-x}\right](N), \quad \mathbf{M}\left[\frac{\mathrm{S}_{1,2}(x)}{1\pm x}\right](N)$$

J.B., S. Moch, 2003,

J.B., V. Ravindran, 2004.

$N = 5$: Three–Loop Anomalous Dimensions

$$\mathbf{M}\left[\frac{\mathrm{Li}_4(x)}{1 \pm x}\right](N), \quad \mathbf{M}\left[\frac{S_{1,3}(x)}{1 + x}\right](N),$$

$$\mathbf{M}\left[\frac{S_{2,2}(x)}{1 \pm x}\right](N), \quad \mathbf{M}\left[\frac{S_{2,2}(-x) - \mathrm{Li}_2^2(-x)/2}{1 \pm x}\right](N),$$

$$\mathbf{M}\left[\frac{\mathrm{Li}_2^2(x)}{1 + x}\right](N)$$

J.B., S. Moch, 2004.

Essentially 14 Functions seem to rule the single scale processes of massless QCD.

This is a rather small number if compared to the number of possible harmonic sums $3^w - 1$.

5. The 16th Moment of the 3-Loop Non–Singlet Anomalous Dimension of $F_{2,L}(x, Q^2)$

Seek for another, blind check of the complete calculation of the NS–anomalous dimension by Moch, Vermaseren, and Vogt, 2004.

The calculation was started far before the complete calculation was completed and is based on the MINCER algorithm used before by Larin, Noguiera, van Ritbergen, Vermaseren and Retey, 1994–2000

Moment	CPU time [days]	
	$g_{\mu\nu}$	$P_\mu P_\nu$
2	0.002567	0.002190
4	0.012562	0.020027
6	0.057144	0.059320
8	0.303415	0.332731
10	1.108047	1.219046
.	.	.
16	236.236	327.542

J.B., J. Vermaseren, hep-ph/0411111

Set-up used: 1 XEON-Dual PC 3 GHz (Drittmittel) and 1 XEON-Dual 2.6 GHz PC (borrowed from AMANDA) linked to a 4.2 Tbyte RAID system; partly 1 64-bit OPTERON 4Gbyte (dual)

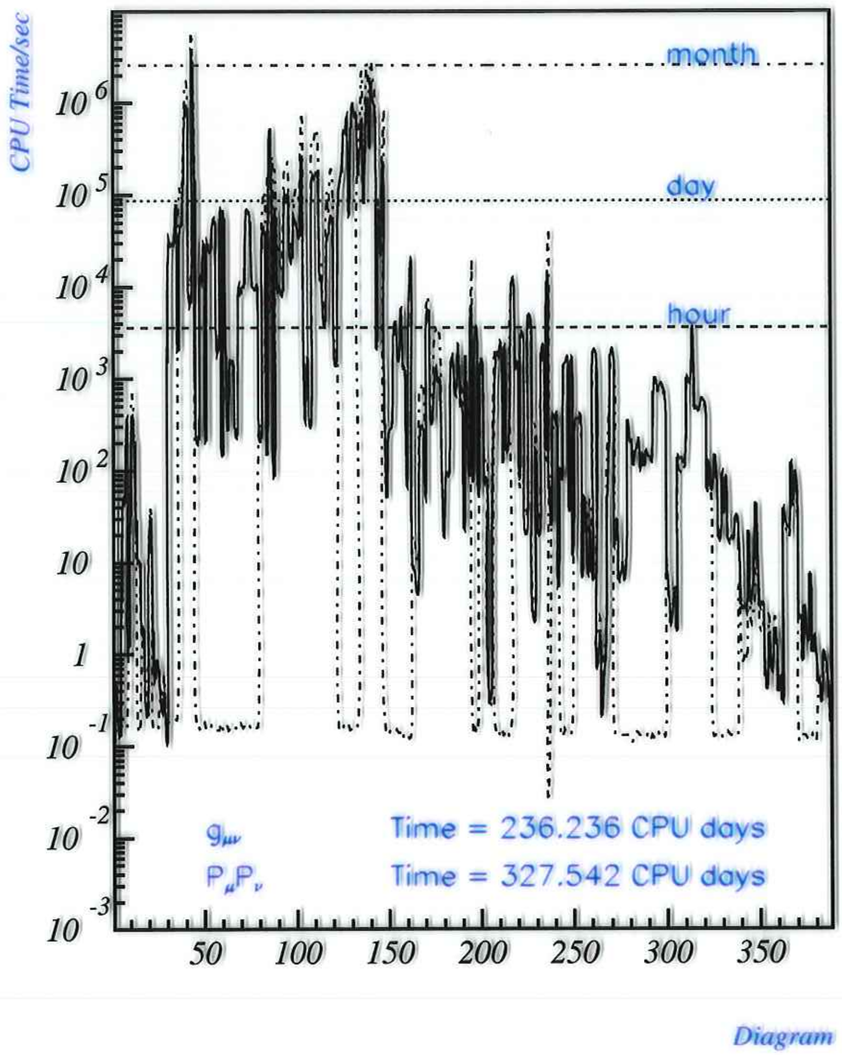
Results after 564 CPU days

Many Thanks to: P. Wegner, S. Wiesand, U. Gensch & C. Spiering for supporting this project.

⇒ **DESY-Z urgently needs a Parallel PC-Facility** ⇐

for FORM Formula Manipulation

Run-time statistics :



Several diagrams stayed in the CPU for 25–40 days each.

Results :

$$\gamma_{16}^{(0)} = \frac{64419601}{6126120} C_F$$

$$\gamma_{16}^{(1)} = -\frac{11765253738403003}{112588038763200} C_F N_F + \frac{21546159166129889}{484994628518400} C_F C_A - \frac{3689024452928781382877}{4598185573520098560000} C_F{}^2$$

$$\gamma_{16}^{(2)} =$$
$$\left(\frac{592905127681 43}{1563722760600}\zeta_3 - \frac{5855293027065230088677870506342 9867}{34513379706124525343170966732 80000}\right) C_F{}^3$$
$$+\left(-\frac{1501842182406038865943 6559}{5793713822635324185600 00} - \frac{64419601}{765765}\zeta_3\right) C_F C_A N_F$$
$$+\left(\frac{16704237280839842078788 25467}{648895948135156308787 2000} + \frac{59290512768143}{3127445521200}\zeta_3\right) C_F C_A{}^2$$
$$-\frac{555946634983457315725 1}{2069183508084044352000} C_F N_F{}^2$$
$$+\left(-\frac{122979464600077578112785606 4477}{3033588557531855743580160 0000} - \frac{59290512768143}{1042481840400}\zeta_3\right) C_F{}^2 C_A$$
$$+\left(-\frac{71543599677985155342551355451}{93896788685509820634624 0000} + \frac{64419601}{765765}\zeta_3\right) C_F{}^2 N_F$$

Agreement with : Moch, Vermaseren, Vogt, hep-ph/0403192.

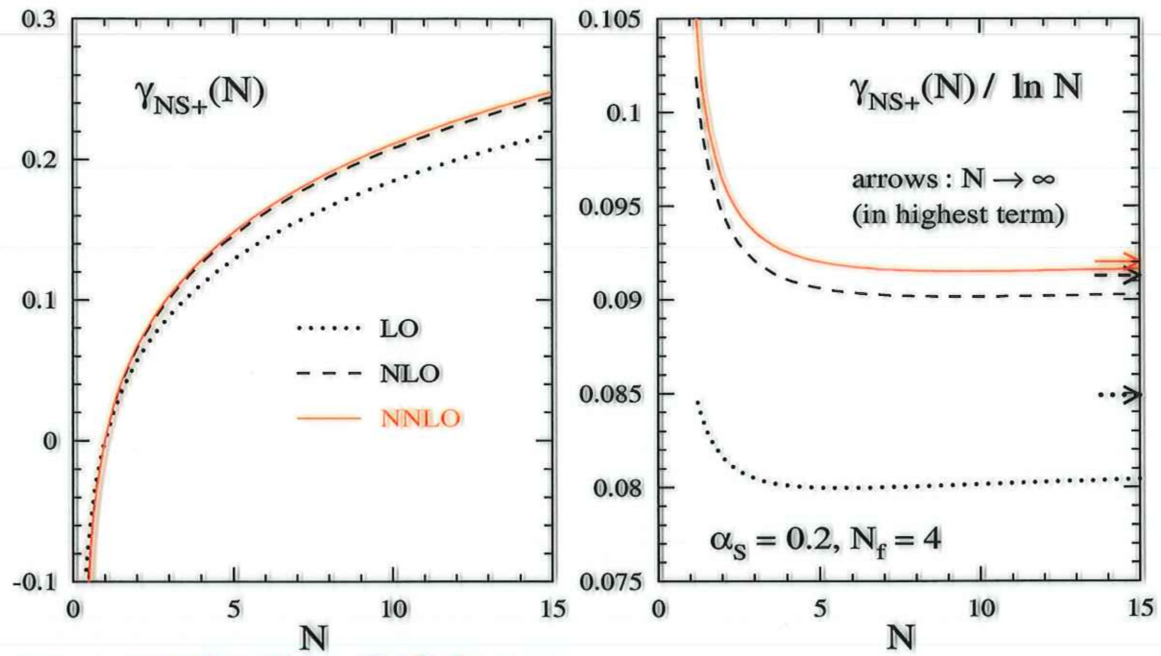
$$
\begin{aligned}
C_2^{\mathrm{NS},16}(x,a_s) \;=\; & \frac{4047739719}{190590400}\, C_F\, a_s \\
+ & \left(\left(\frac{444266741630444288793669701 27}{321931846921747956461568000}\; \frac{24439538}{255255}\,\zeta_3\right) {C_F}^2\right. \\
+ & \left(\frac{179183084084982942227830 87}{59422705873182812160000} - \frac{113298677}{1021020}\,\zeta_3\right) C_F\, C_A \\
- & \left.\frac{143568372761907472111177}{27589113441120591360 00}\, C_F\, N_F\right) {a_s}^2 \\
+ & \left(\left(\frac{592905127681 43}{3127445521200}\,\zeta_4 - \frac{27643576}{21879}\,\zeta_5\right.\right. \\
+ & \frac{30368133975995097250846772938425059765591616 89}{803445801604077593342164786340334796 8000000} \\
+ & \left.\frac{149434192694045086538740 3}{595674040206012768000}\,\zeta_3\right) {C_F}^3 \\
+ & \left(\frac{59290512768143}{6254891042400}\,\zeta_4 + \frac{26286537788347572655880093551503319 0333}{566468058525038486710210437120 00000}\right. \\
+ & \left.\frac{47187263}{51051}\,\zeta_5 - \frac{1535505046917148231 3}{4991403051835200}\,\zeta_3\right) C_F\, {C_A}^2 \\
+ & \left(\frac{72273849359996703123187898849 99}{760563988352629547140454 40000} + \frac{64419601}{20675655}\,\zeta_3\right) C_F\, {N_F}^2 \\
+ & \left(\frac{77500266271187687528450917608900514652427 41}{16525006203292422734310258871664640 00000}\right. \\
- & \frac{2849482004138921491531}{6741167121672984000}\,\zeta_3 + \frac{983963}{21879}\,\zeta_5 \\
- & \left.\frac{59290512768143}{2084963680800}\,\zeta_4\right) {C_F}^2 ca + \left(-\frac{552298563960959}{4021001384400}\,\zeta_3\right. \\
- & \left.\frac{40732072413484931961522220799335575 29}{352977746994455372827884887040 0000} + \frac{64419601}{1531530}\,\zeta_4\right) {C_F}^2 N_F \\
+ & \left(\frac{598788865585667}{1850495446800}\,\zeta_3 - \frac{64419601}{1531530}\,\zeta_4\right. \\
- & \left.\frac{582811634921542995647179358698536547}{404620041803598919078721740800000}\right) C_F\, C_A\, N_F \\
+ & \left.\langle e\rangle \left(\frac{705894258514655486993}{324842983135070400 0} + \frac{38404365803}{1533061530}\,\zeta_3 - \frac{14560}{51}\,\zeta_5\right) \frac{d_{abc}^2}{N_c}\, N_F\right) {a_s}^3
\end{aligned}
$$

$$
\langle e\rangle = \frac{3}{N_F} \sum_{k=1}^{N_F} e_f
$$

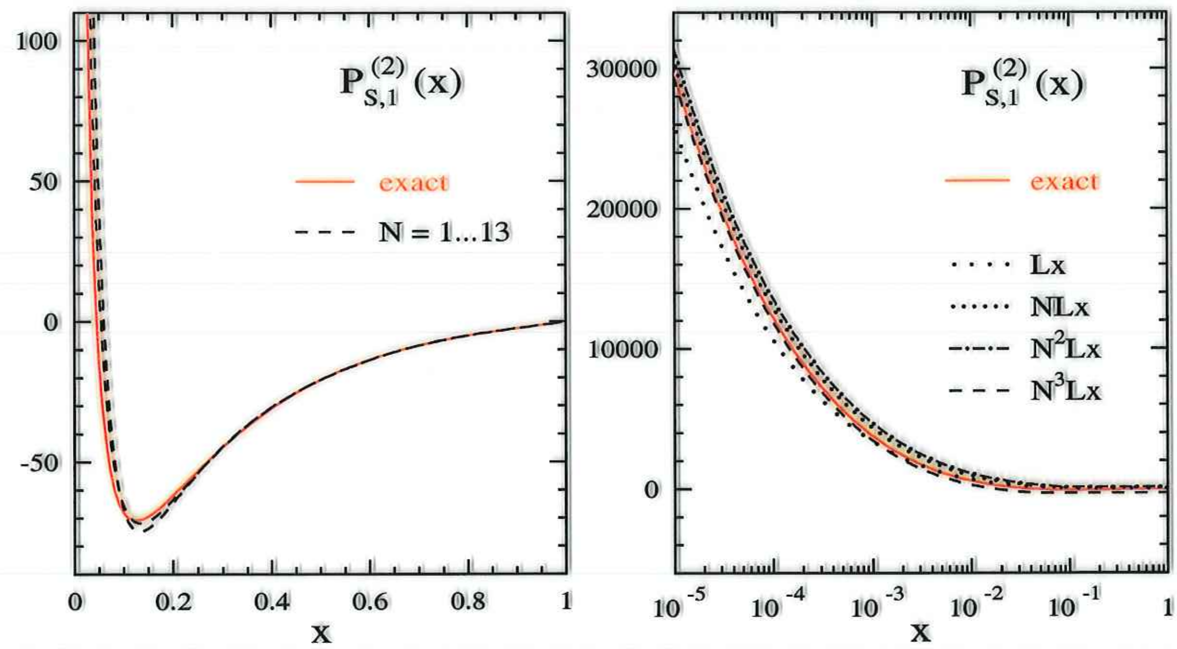
$$
\begin{aligned}
C_L^{\mathrm{NS},16}(x,a_s) \; = \; & \frac{4}{17} C_F\, a_s \\
& + \Bigg[-\frac{29393927457809}{44659922042736} {C_F}^2 - \frac{39366889}{39054015} C_F\, N_F \\
& - \frac{48}{17} \zeta_3\, C_F\, C_A + \frac{55969347000169}{8209544493150} C_F\, C_A + \frac{96}{17} \zeta_3\, {C_F}^2 \Bigg] {a_s}^2 \\
& + \Bigg[\Bigg(\frac{39360}{17} \zeta_5 - \frac{196256899828170631}{133698296031300} \zeta_3 \\
& - \frac{75082818212767714981264472901109 19}{1364789823543885242924259840 0000} \Bigg) {C_F}^3 \\
& + \Bigg(\frac{2960455010101335653220392071596 77}{9366204671379604608303744000 00} \\
& - \frac{40160}{17} \zeta_5 + \frac{2253147763389895}{1188429298056} \zeta_3 \Bigg) C_A\, {C_F}^2 \\
& + \Bigg(\frac{35291373463211704531604 63}{1367960208122229321600 00} - \frac{44651224}{765765} \zeta_3 \Bigg) N_F\, {C_F}^2 \\
& + \Bigg(-\frac{1634895686765221}{2673965920626} \zeta_3 \\
& + \frac{146079249942710013949328037 1}{825604219725533696448000 0} \\
& + \frac{10240}{17} \zeta_5 \Bigg) {C_A}^2 C_F + \frac{895967716232}{209134250325} C_F\, {N_F}^2 \\
& + \Bigg(-\frac{44958051446585653855015736 89}{577922953807873587513600 00} \\
& + \frac{435943306 72}{1249937325} \zeta_3 \Bigg) C_A\, N_F\, C_F \\
& + \langle e \rangle \Bigg(-\frac{1798450729620489619601}{182724178013477100 00} - \frac{28854977192}{547521975} \zeta_3 \\
& + \frac{2560}{17} \zeta_5 \Bigg) \frac{d_{abc}^2}{N_c} N_F \Bigg] {a_s}^3
\end{aligned}
$$

6. Phenomenological Results

S. Moch, J. Vermaseren, A. Vogt: non-singlet: hep-ph/0403192 :

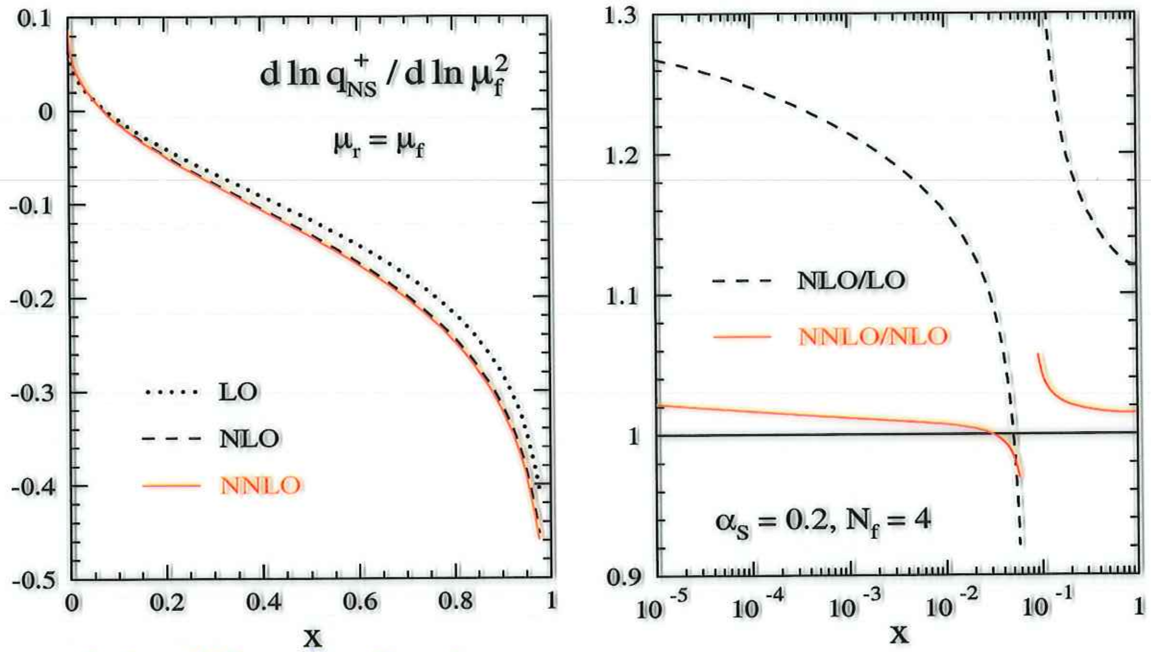


A new contribution @ 3 loops :

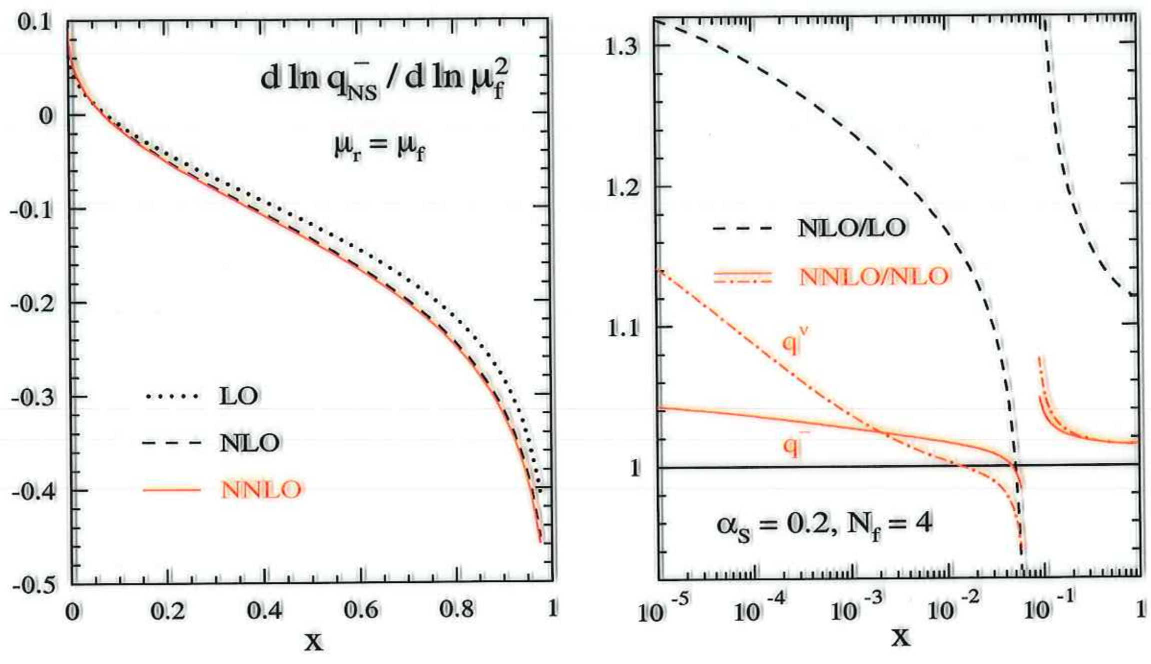


S. Moch, J. Vermaseren, A. Vogt: non-singlet: hep-ph/0403192 :

Slope of the NS$^+$ distribution



Slope of the NS$^-$ distribution



7. Conclusions

- Mellin space expressions of anomalous dimensions and Wilson coefficients are of much simpler structure than the x–space results.
- Index-based algebraic relations of harmonic sums, structural relations of Mellin–transforms of Nielsen–integrals and the specific structure of Feynman amplitudes cause this reduction.
- All algebraic relations are derived in explicit form up to weight $w = 6$ and apply to harmonic sums, harmonic polylogarithms and all other objects in the corresponding equivalence class. The structural relations are worked out up to $w = 4$ and the anomalous dimensions for $w = 5$.
- The number of multiple harmonic sums of weight w is $2 \cdot 3^{w-1}$. The number of the harmonic sums after algebraic reduction is given by the Witt formula(e) yielding a reduction to $\approx 1/4$. Further reductions result from structural relations.
- The number of non–trivial basic functions for $w \leq 5$ which are needed to express the known anomalous dimensions and the (space– and time–like) Wilson coefficients for $m_i = 0$ is given by the sequence :

{0, 0, 1, 4, 8, ...}

to compared to

{2, 6, 18,54,162,486, ...}

- An independent calculation of the 16th moment of the non–singlet structure function $F_{2,L}(x, Q^2)$ shows agreement with the complete calculation and upcoming results for the anomalous dimension and the Wilson coefficients.