Les Arcs
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$k_\perp$ dependent parton densities in the proton and photon

Johannes Blümlein

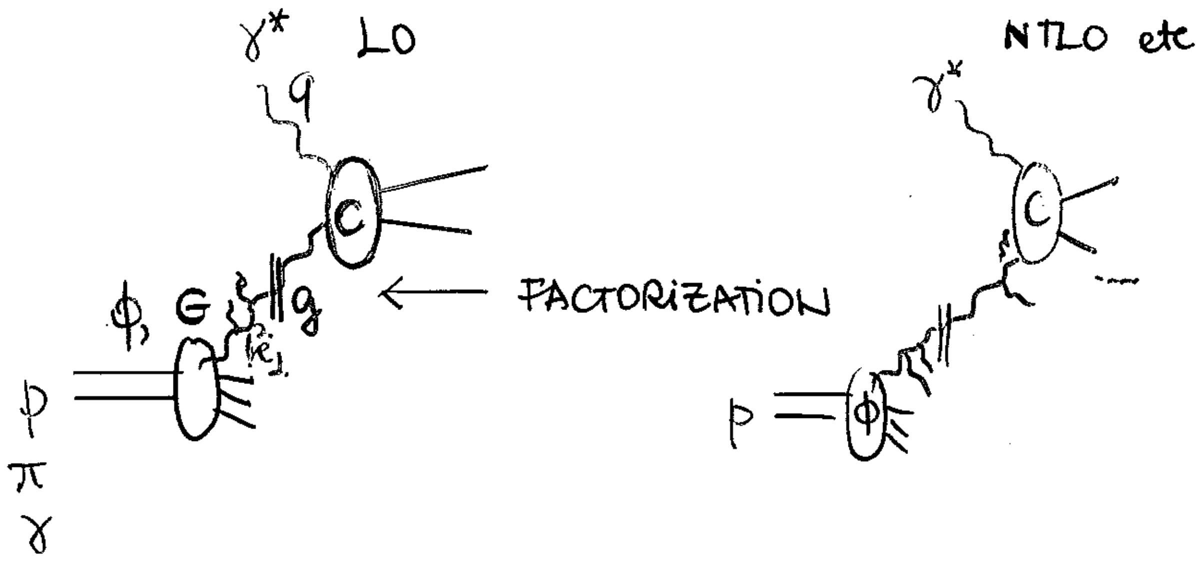
DESY

1. Introduction
2. Phenomenological approaches
3. $k_\perp$ factorization & $k_\perp$ dependent gluon densities
4. xG and $F_L(x, Q^2)$

1 Introduction

- NOVEL BEHAVIOUR AT SMALL X:
  - NON STRONG $k_\perp$ ORDERING
  - EVENTUALLY SCREENING (DRELL, YAN)
  - ⋮
  - $\phi(x, k_\perp^2, Q^2)$ RATHER THAN $\int^{\mu^2} dk_\perp^2\, \phi(x, k_\perp^2, Q^2)$.
  - $k_\perp$ dependent COEFFICIENT FUNCTIONS

→ LARGE X COLLINEAR PICTURE RESTORED



2 Phenomenological approaches

'SMALL x' SOLUTIONS

$$F(x, k^2) \sim \frac{1}{2\pi} e^{-\lambda \ln x} (k^2)^{-\frac{1}{2}}$$

LIPATOV et al.

$$\lambda = 4 \bar{\alpha}_s \ln 2$$

$$F(x, k^2) = \phi_0 (1-x)^3 f_1(x, k^2) \frac{0.05}{x + 0.05}$$

Levin, Ryskin (Salcedo et al.)

$$f_1(x, k^2) = \begin{cases} 1 & q_T^2 \le q_0^2 \\ q_0^2(x)/q_T^2 & , q_T \equiv k \text{ else} \end{cases}$$

$$F(x, k^2) = N \frac{x^{-\lambda}}{\sqrt{k^2}\left(\sqrt{\ln 1/x}\right)} \exp\left[- \frac{\ln^2 (k^2/k_0^2)}{\Delta \lambda \ln 1/x}\right]$$

$$\Delta = 14 \zeta_3 / \ln 2$$

LEVIN, RYSKIN (FORSHAW et al.) (FORSHAW, ROBERTS)

$$F(x, k^2) \sim \left.\frac{\partial G}{\partial Q^2}\right|_{Q^2 = k^2}.$$

3 $k_\perp$ factorization & $k_\perp$ dependent gluon distributions

$$\widetilde{F}(j,k^2,\mu) = \gamma_c(j,\overline{\alpha}_s)\frac{1}{k^2}\left(\frac{k^2}{\mu^2}\right)^{\gamma_c(j,\overline{\alpha}_s)}\widetilde{f}(j,\mu)$$

$$\overline{\alpha}_s = \frac{N_c}{\pi}\alpha_s(\mu^2)$$

COLLINS, ELLIS.

with

$$\begin{aligned}\widetilde{G}(j) &\equiv \mathcal{M}[G](j) = \int_0^1 dx x^{j-1} G(x) \\ G(x) &\equiv \mathcal{M}^{-1}[\widetilde{G}](x) = \frac{1}{2\pi i}\int_C dz x^{-z}\widetilde{G}(z)\end{aligned}$$

where $G = F, f$.

$$F(x,k^2,\mu) = \mathcal{G}(x,k^2,\mu)\otimes f(x,\mu) \tag{1}$$

$$\int_0^{\mu^2} dk^2 \widetilde{F}(j,k^2,\mu) = \widetilde{f}(j,\mu)$$

$$\int_0^{\mu^2} dk^2 \mathcal{G}(x,k^2,\mu) = \delta(1-x)$$

The exponent $\gamma_c(j,\overline{\alpha}_s)$ is determined by the Lipatov (BFKL) equation:

$$\boxed{j-1 = \overline{\alpha}_s\chi\left(\gamma_c(j,\overline{\alpha}_s)\right)}$$

with

$$\chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma)$$

- multivalued function $\chi(\gamma) \to$ define the 'perturbative' sheet for $\gamma_c(z,\overline{\alpha}_s)$
- derive analytic expansions for $Re\gamma \to 0$ and $Re\gamma \to 1/2$.

FIG.

PHYS.

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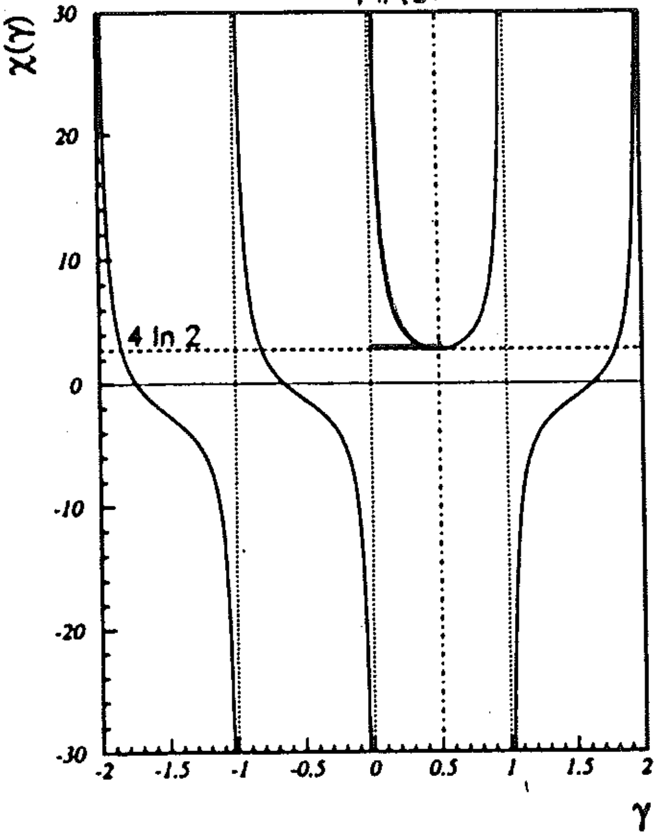
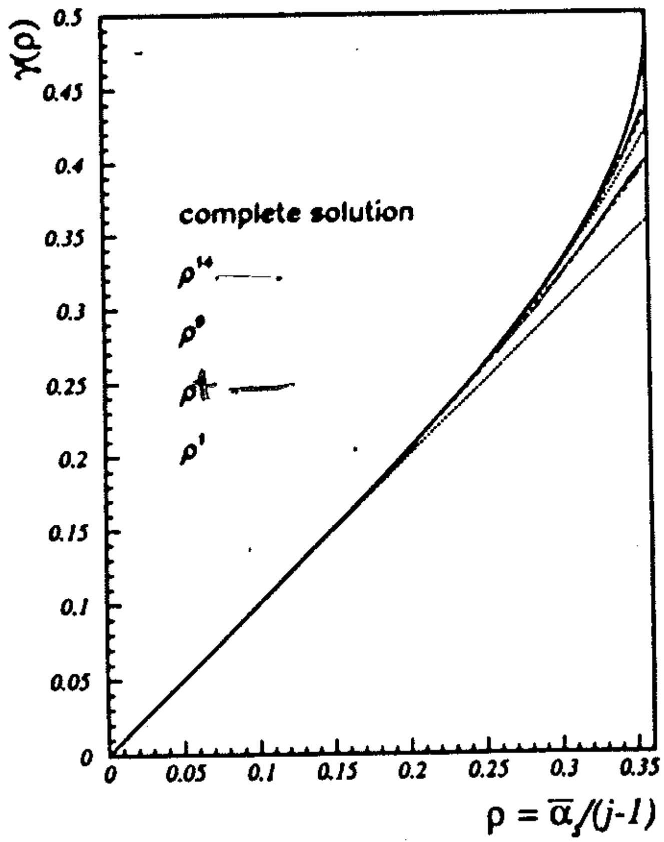
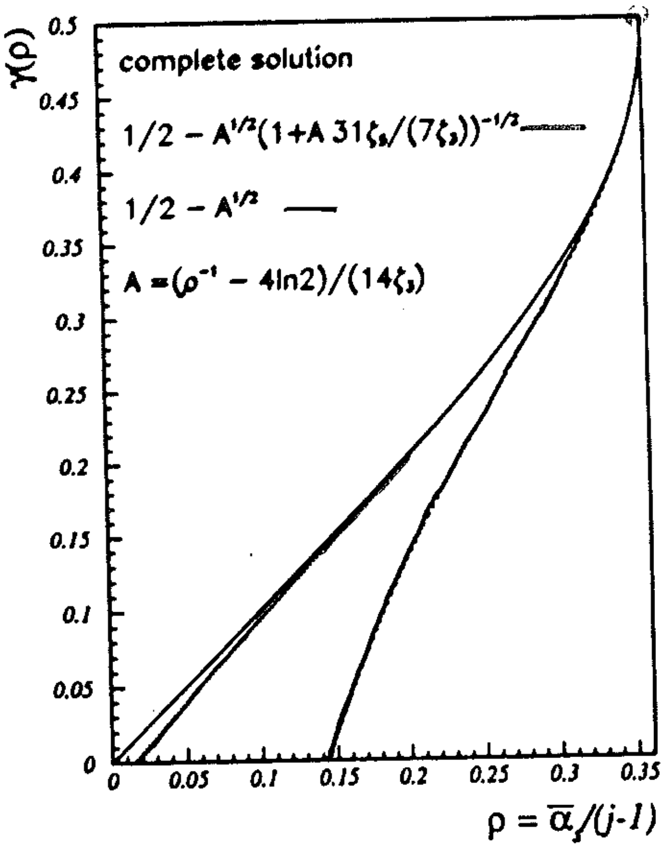


χ(γ)
4 ln 2
γ



γ(ρ)
complete solution
ρ14
ρ8
ρ4
ρ1
ρ = ᾱs/(j-1)



γ(ρ)
complete solution
1/2 − A1/2(1+A 31ζ5/(7ζ3))−1/2
1/2 − A1/2
A =(ρ−1 − 4ln2)/(14ζ3)
ρ = ᾱs/(j-1)

Solution for $\gamma \to 0$:

$$\gamma_c(j, \overline{\alpha}_s) = \frac{\overline{\alpha}_s}{j-1} \left\{ 1 + 2 \sum_{k=1}^{\infty} \zeta_{2k+1} \gamma_c^{2k+1}(j, \overline{\alpha}_s) \right\}$$

$$\gamma_c(j, \overline{\alpha}_s) = \sum_{l=1}^{\infty} g_l A^l; \qquad A = \frac{\overline{\alpha}_s}{j-1}$$

CATANI, FIORANI, MARCHESINI

$$\begin{aligned}
g_1 &= 1 \\
g_2 &= 0 \\
g_3 &= 0 \\
g_4 &= 2\zeta_3 \\
g_5 &= 0 \\
g_6 &= 2\zeta_5 \\
g_7 &= 12\zeta_3^2 \\
g_8 &= 2\zeta_7 \\
g_9 &= 32\zeta_3\zeta_5 \\
g_{10} &= 2[48\zeta_3^2 + \zeta_9] \\
g_{11} &= 2[20\zeta_7\zeta_3 + 10\zeta_5^2] \\
g_{12} &= 2[220\zeta_5\zeta_3^2 + \zeta_{11}] \\
g_{13} &= 2[440\zeta_3^4 + 24\zeta_9\zeta_3 + \zeta_7\zeta_5] \\
g_{14} &= 2[312\zeta_7\zeta_3^2 + 312\zeta_5^2\zeta_3 + \zeta_{13}]
\end{aligned}$$

$$
\begin{aligned}
k^2\widetilde{\mathcal{G}}(j,k^2,\mu) &= A\exp(AL)\left[1+2A(1+AL)\sum_{k=1}^{\infty}\zeta_{2k+1}A^{2k}\right] \\
&+ 12A^7\zeta_3^2\left[1+\frac{7}{3}AL+2(AL)^2+(AL)^3\right] \\
&+ 32A^9\zeta_3\zeta_5\left[1+\frac{9}{4}AL\right] \\
&+ 96A^{10}\zeta_3^3 \\
&+ \mathcal{O}(A^{11})
\end{aligned}
$$

THIS SERIES CAN BE REWRITTEN AS ∞ A exp (AL) [...] EVENTUALLY.

$$L := \log\left(\frac{k^2}{\mu^2}\right)$$

The x dependent function is uniquely determined by the moments due to CARLSON's theorem.

One has:

$$
\begin{aligned}
\frac{\overline{\alpha}_s}{(j-1)} &= \int_0^1 dx x^{j-1}\frac{\overline{\alpha}_s}{x} \\
\frac{\overline{\alpha}_s^l}{(j-1)^l} &= \int_0^1 dx x^{j-1}\left[\otimes_{k=1}^l\frac{1}{x}\right]\overline{\alpha}_s^l \\
\overline{\alpha}_s^l\otimes_{k=1}^l\frac{1}{x} &= \frac{1}{x}\frac{\overline{\alpha}_s^l}{(l-1)!}\log^{l-1}\left(\frac{1}{x}\right), l\geq 1
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{M}^{-1}\left\{A\exp[AL]\right\}(x) &= \frac{\overline{\alpha}_s}{x}I_0\left(2\sqrt{\overline{\alpha}_s\log(1/x)L}\right) \qquad L>0 \\
&= \frac{\overline{\alpha}_s}{x}J_0\left(2\sqrt{\overline{\alpha}_s\log(1/x)|L|}\right) \qquad L<0
\end{aligned}
$$

$$
\lim_{k^2\to 0}\mathcal{M}^{-1}\left\{A\exp[AL]\right\}(x) = \lim_{|L|\to\infty}\frac{\cos(2\sqrt{\overline{\alpha}_s|L|\log(1/x)})}{\sqrt{\pi\sqrt{\overline{\alpha}_s|L|\log(1/x)}}} = 0
$$

$$\mathcal{M}^{-1}\left\{A^{\sigma}\exp[AL]\right\}(x)=\frac{\overline{\alpha}_s}{x}\left(\frac{\overline{\alpha}_s\log(1/x)}{L}\right)^{(\sigma-1)/2} I_{\sigma-1}(2\sqrt{\overline{\alpha}_s L\log(1/x)})$$

For $k^2\to 0$ the resummed expression behaves as:

$$\begin{aligned}&\mathcal{M}^{-1}\left\{A^{\rho}\exp[AL][...+(AL)^{\kappa}...]\right\}(x)\\ =\ &...\frac{\overline{\alpha}_s}{x}\left(\overline{\alpha}_s\log(1/x)\right)^{(\rho+\kappa-1)/2} J_{\rho+\kappa-1}(2\sqrt{\overline{\alpha}_s|L|\log(1/x)})\left(\frac{1}{|L|}\right)^{(\rho-1)/2}\\ \propto\ &\left(\frac{1}{|L|}\right)^{3/4+\delta}, \qquad \rho\geq 2,\delta\geq 0\end{aligned}$$

- AP equ. (GLUONS only) $\quad k^2\sim Q^2$

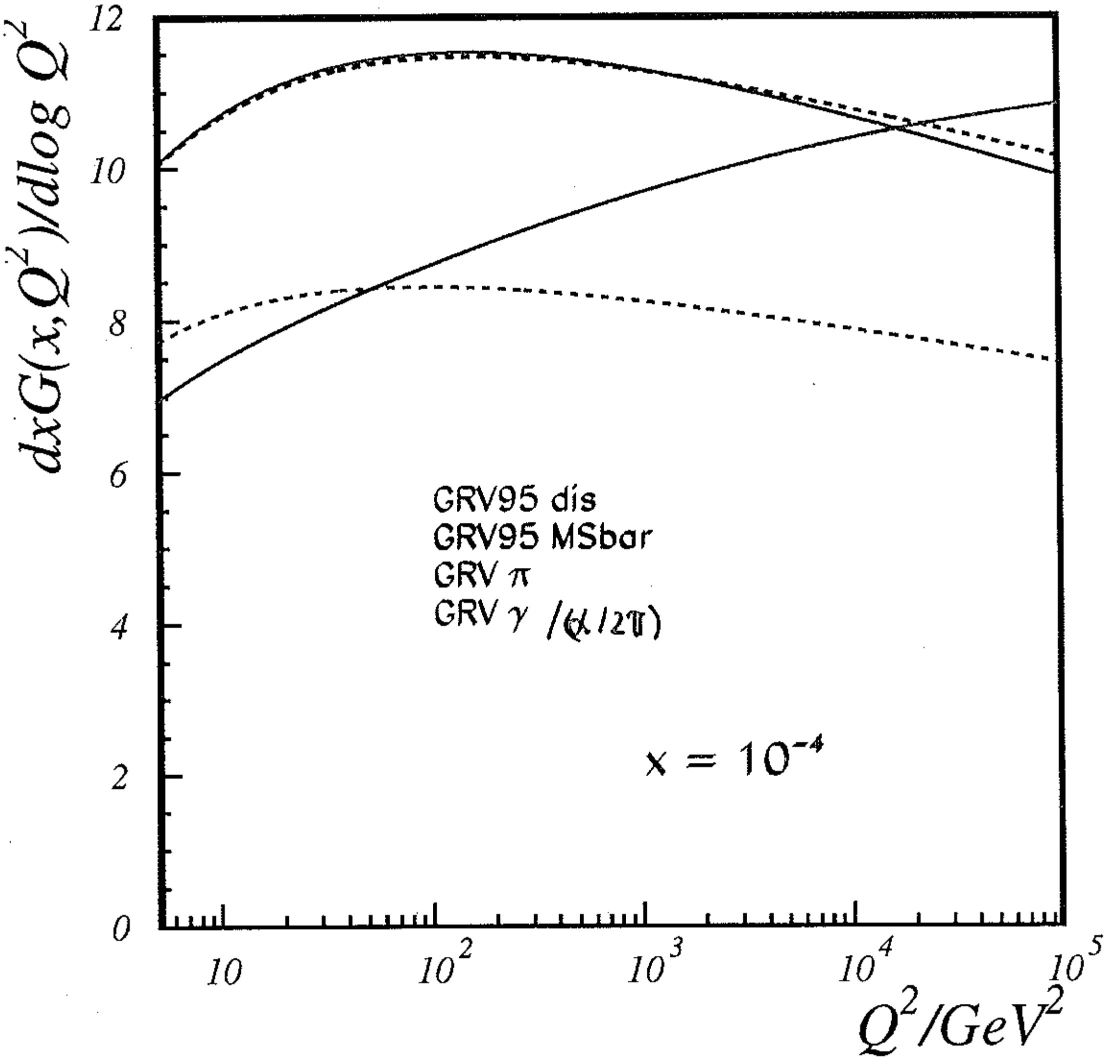
$$\left\langle k^2\frac{\partial G}{\partial k^2}\right\rangle_j \approx \underbrace{\frac{\alpha_s}{2\pi}}_{\overline{\alpha}_s}\overbrace{2C_G\cdot\frac{1}{j-1}}^{\equiv A}\langle G\rangle_j$$

$\mathcal{M}$: $$k^2\frac{\partial G(x,k^2)}{\partial k^2}\approx\overline{\alpha}_s\cdot\frac{1}{x}\otimes G(x,k^2)$$

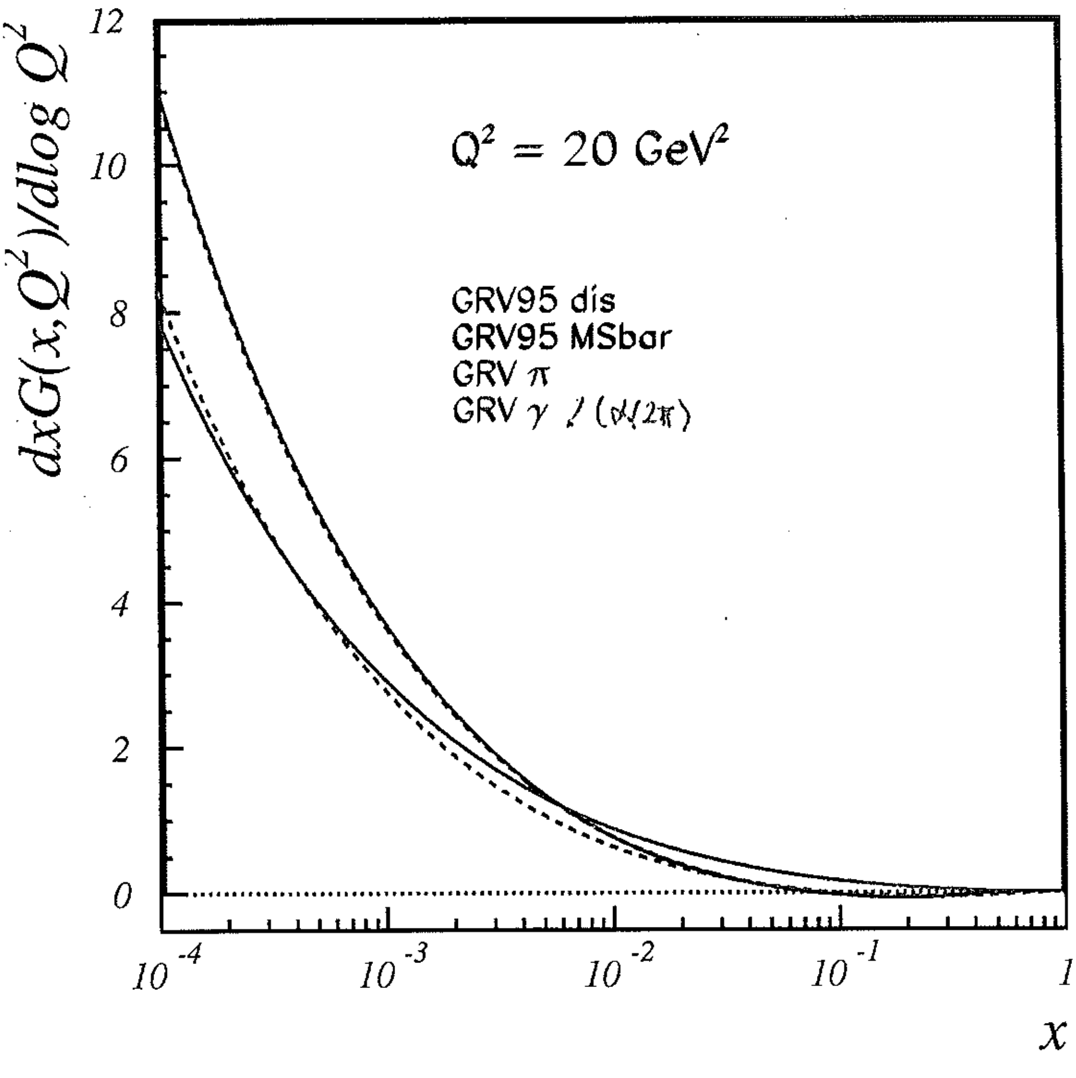
(i.e. the lowest order term above).

$$\equiv\ \underline{k^2\phi(x,k^2,\mu)}$$

Fig.



$dxG(x,Q^2)/dlog\ Q^2$
12
10
8
6
4
2
0
GRV95 dis
GRV95 MSbar
GRV π
GRV γ /(α/2π)
x = 10^{-4}
10
10^2
10^3
10^4
10^5
Q^2/GeV^2



$dxG(x,Q^2)/dlog\,Q^2$
12
10
8
6
4
2
0
$Q^2 = 20$ GeV^2
GRV95 dis
GRV95 MSbar
GRV π
GRV γ / (α/2π)
10^{-4}
10^{-3}
10^{-2}
10^{-1}
1
x

$$\mathcal{G}(x,k^2,\mu) = \frac{\overline{\alpha}_s}{x}\Big\{I_0(B) \;+\; 2\zeta_3\big[C^{3/2}I_3(B) + C^2 L I_4(B)\big] \\ \;+\; 2\zeta_5\left[C^{5/2}I_5(B) + C^3 L I_6(B)\right]\Big\} + \mathcal{O}(\overline{\alpha}_s^7)$$

AP

$$B = 2\sqrt{\overline{\alpha}_s L \log(1/x)}$$
$$C = \overline{\alpha}_s \frac{\log(1/x)}{L}$$

If $L < 0 : I_\nu(z) \to J_\nu(z)$.

One obtains:

$$k^2\Phi(x,k^2,\mu) = \mathcal{G}(x,k^2,\mu) \otimes G(x,\mu^2)$$

(for $\overline{\alpha}_s \ll 1$).

Solution for $\gamma \to 1/2$:

$$\frac{1}{\rho} := \frac{j-1}{\overline{\alpha}_s} \qquad \gamma := \frac{1}{2} - \alpha$$

$$\frac{1}{\rho} = 2\psi(1) - \psi\left(\frac{1}{2} - \alpha\right) - \psi\left(\frac{1}{2} + \alpha\right)$$

$$\frac{1}{\rho} = 4\log 2 + \sum_{n=1}^{\infty} \zeta_{2n+1}\left(2^{2(n+1)} - 2\right)\alpha^{2n}$$

$$\alpha_{(0)} \approx 0 \qquad \gamma_c^{(0)} \approx \frac{1}{2}$$

FiG.

$$\gamma_c^{(1)} \approx \frac{1}{2} - \sqrt{\left(\frac{1}{\rho} - 4\log 2\right)\frac{1}{14\zeta_3}} = \frac{1}{2} - \alpha_{(1)}$$

$$\gamma_c^{(2)} \approx \frac{1}{2} - \frac{\alpha_{(1)}(\rho)}{\sqrt{1 + (31\zeta_5/7\zeta_3)\alpha_{(1)}^2(\rho)}}$$

$$\rho \to \rho(z)$$

$$\mathcal{G}(x,\mu,k^2) = \frac{1}{2\pi i}\int_C dz\, x^{-z}\, \frac{1}{k^2}\, \gamma_c(z,\overline{\alpha}_s)\left(\frac{k^2}{\mu^2}\right)^{\gamma_c(z)}$$

Solution of the BFKL equation (for large $\ln(1/x)$):

(saddle point method)

→ **Green's function for a δ–source:**

$$\Phi(x,k^2,\mu) = \frac{N(x,\mu)}{\mu^2}\frac{x^{-\chi(0)}}{\sqrt{\ln(1/x)}}\sqrt{\frac{\mu^2}{k^2}}\exp\left[-\frac{\ln^2(k^2/\mu^2)}{2\chi''(0)\ln(1/x)}\right]$$

with

$$\chi(0) = \overline{\alpha}_s 4\ln 2 \qquad\qquad \chi''(0) = \overline{\alpha}_s 28\zeta_3$$

The normalization factor $N(x,\mu)$ cancles the $x^{-\chi(0)}$ dependence!

$$N(x,\mu) = x^{+\chi(0)}\sqrt{\ln(1/x)}I^{-1}(a)G(x,\mu^2)$$

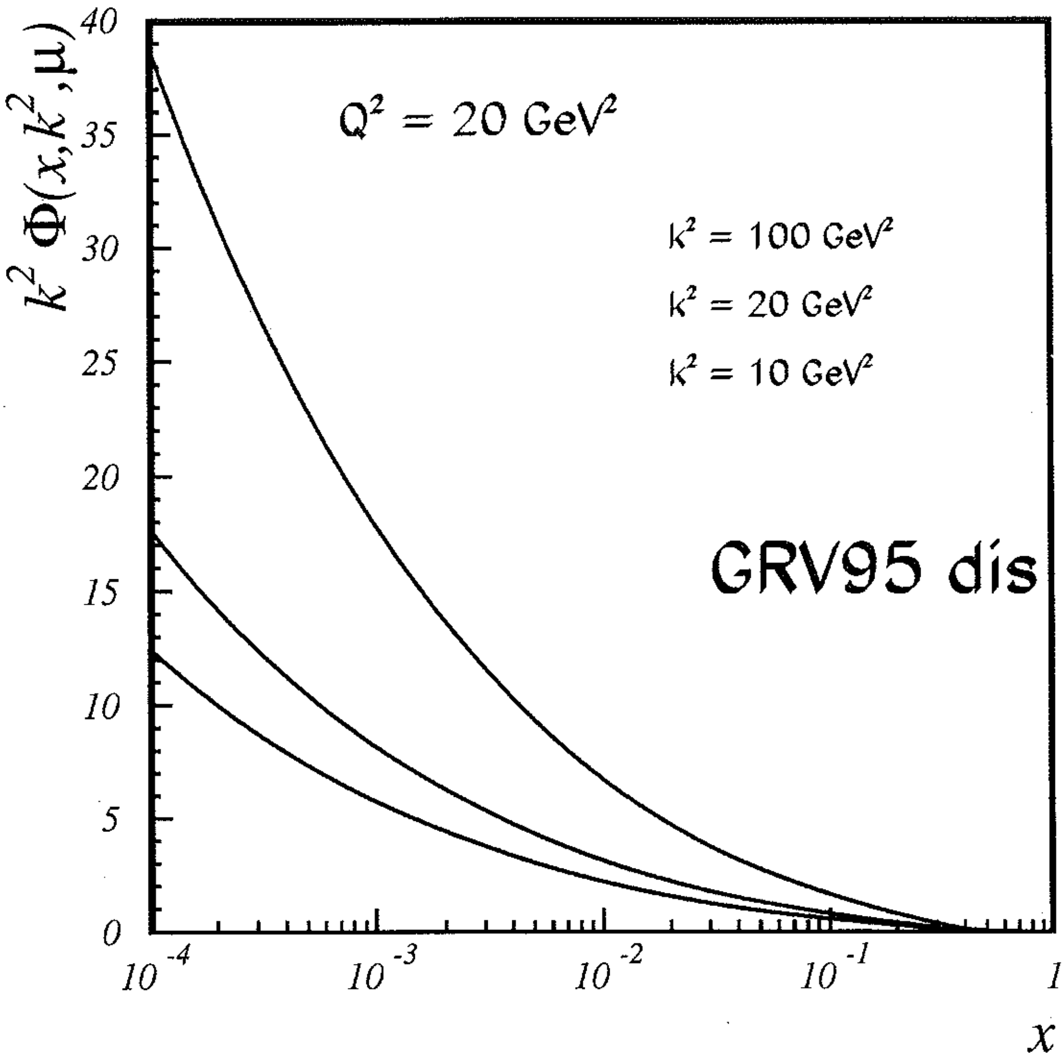
with (↑ rising behaviour through evolution + BFKL)

$$I(a) = a(\sqrt{\pi}/2)\exp(a^2/16)\mathrm{erfc}(a/4), \qquad a = \sqrt{2\chi''(0)\ln(1/x)}.$$

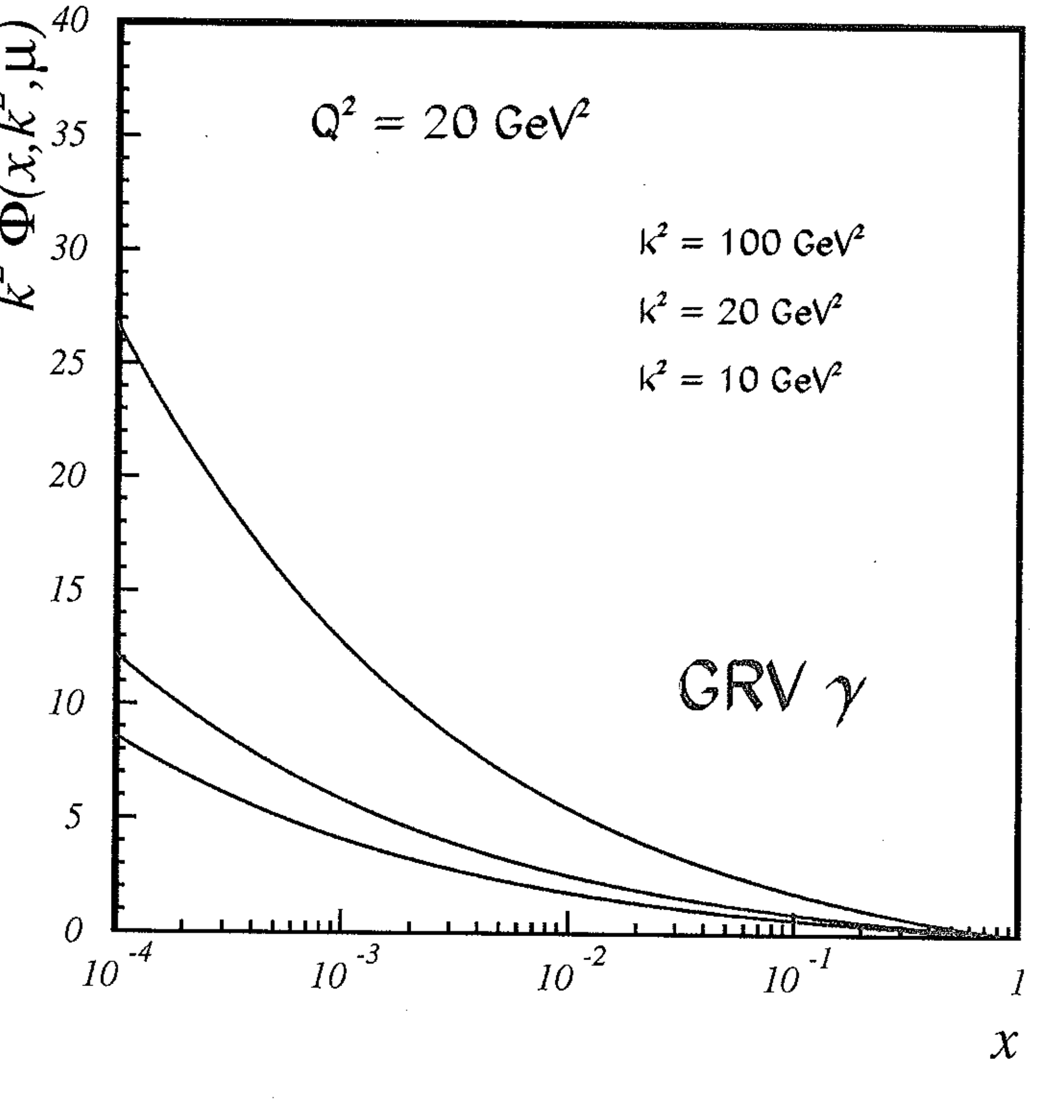
Since for $a \gg 1$

$$I(a) = 2\left[1 + \sum_{m=1}^{\infty}\frac{(-1)^m(2m)!2^{2m}}{m!a^{2m}}\right] \approx 2 + \mathcal{O}(1/\ln(1/x))$$

$$\begin{aligned}\Phi(x,k^2,\mu) &= \frac{2}{\sqrt{\pi}\mu^2}\sqrt{\frac{\mu^2}{k^2}}\frac{G(x,\mu^2)}{a\exp(a^2/16)\mathrm{erfc}(a/4)}\exp\left[-\frac{\ln^2(k^2/\mu^2)}{2\chi''(0)\ln(1/x)}\right] \\ &\approx \frac{1}{2\mu^2}\sqrt{\frac{\mu^2}{k^2}}\exp\left[-\frac{\ln^2(k^2/\mu^2)}{2\chi''(0)\ln(1/x)}\right]G(x,\mu^2)\end{aligned}$$



$k^2\,\Phi(x,k^2,\mu)$
$Q^2 = 20\ \mathrm{GeV}^2$
$k^2 = 100\ \mathrm{GeV}^2$
$k^2 = 20\ \mathrm{GeV}^2$
$k^2 = 10\ \mathrm{GeV}^2$
GRV95 dis
0
5
10
15
20
25
30
35
40
10^{-4}
10^{-3}
10^{-2}
10^{-1}
1
x


$k^2\,\Phi(x,k^2,\mu)$
$Q^2 = 20\ \mathrm{GeV}^2$
$k^2 = 100\ \mathrm{GeV}^2$
$k^2 = 20\ \mathrm{GeV}^2$
$k^2 = 10\ \mathrm{GeV}^2$
GRV γ
40
35
30
25
20
15
10
5
0
10^{-4}
10^{-3}
10^{-2}
10^{-1}
1
x

4 xG and $F_L(x, Q^2)$

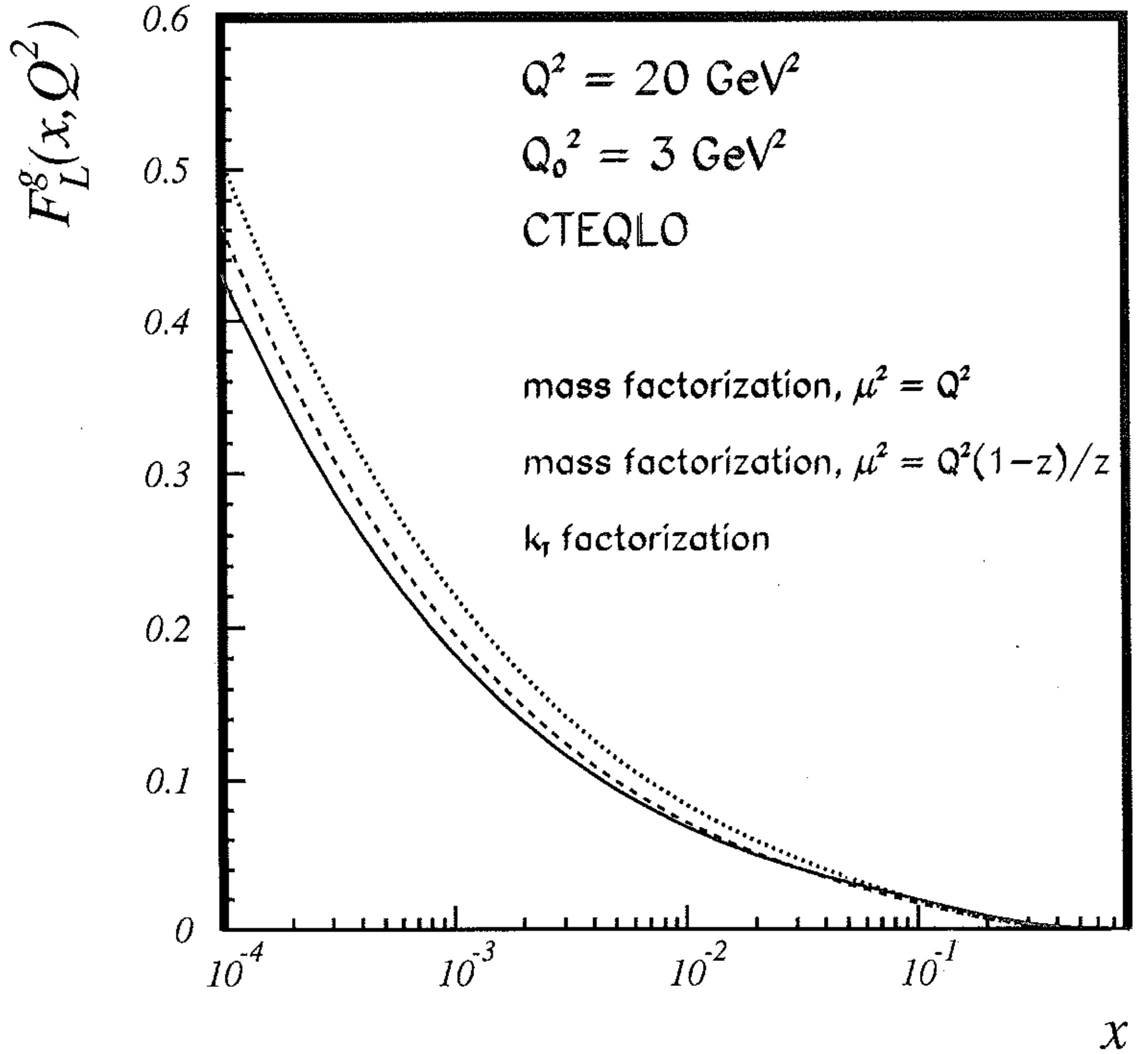
- $F_L(x, Q^2)$ — xG DOMINATED AT SMALL x
  ↪ UNFOLDING

⟶ DIFFERENT EFFECTS

- COLLINEAR RESULT (AP) $O(\alpha_s) \times G_{coll}^{LO}$
  $O(\alpha_s^2) \times G_{coll}^{NLO}$

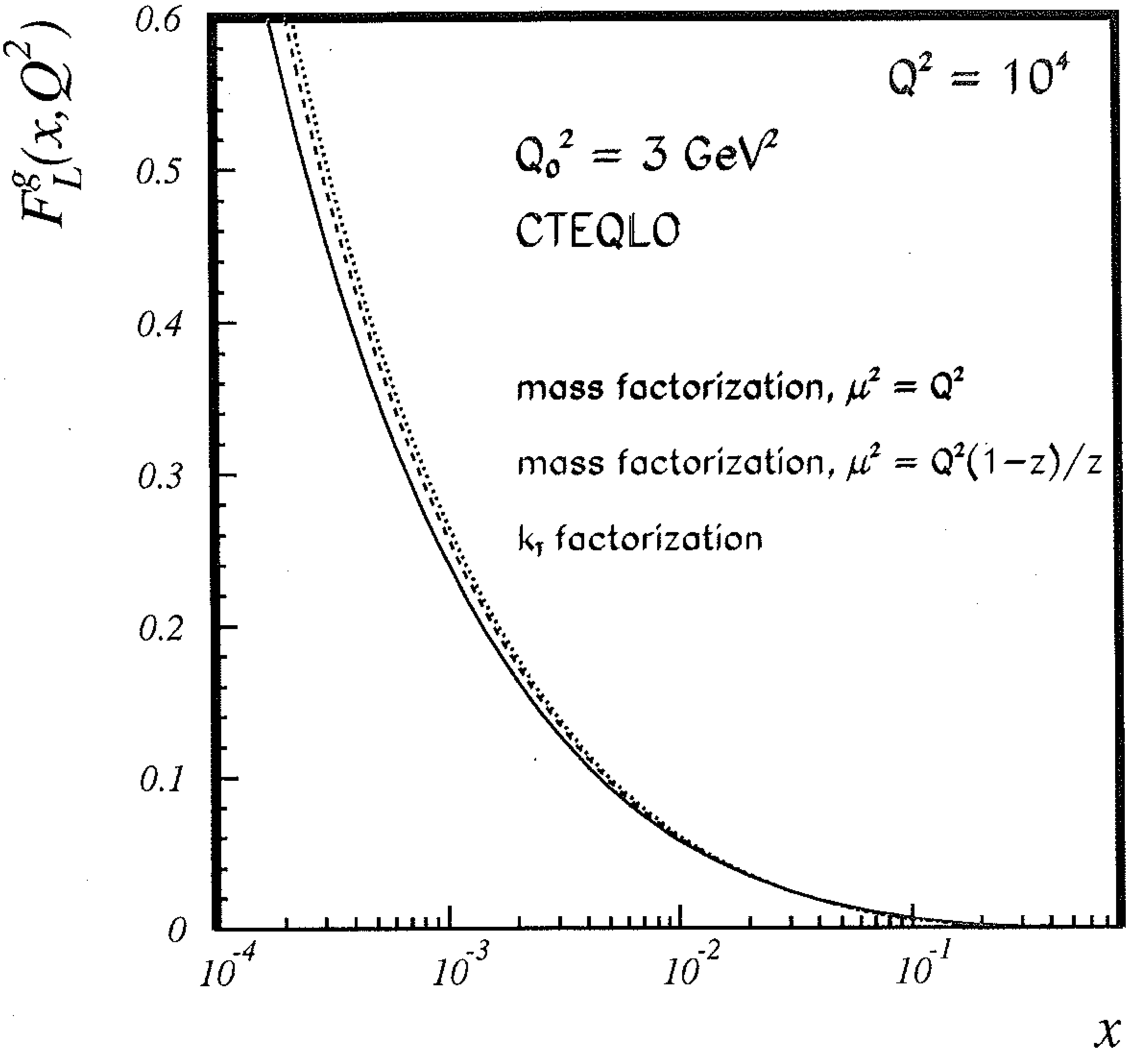
- KT RESULT : $\underline{C^{LO}(k_\perp)} \times \underbrace{\phi^{LO}(x, k_\perp^2, Q^2)}_{\text{'BFKL'}}$.

⟷ OBS. EFFECTS

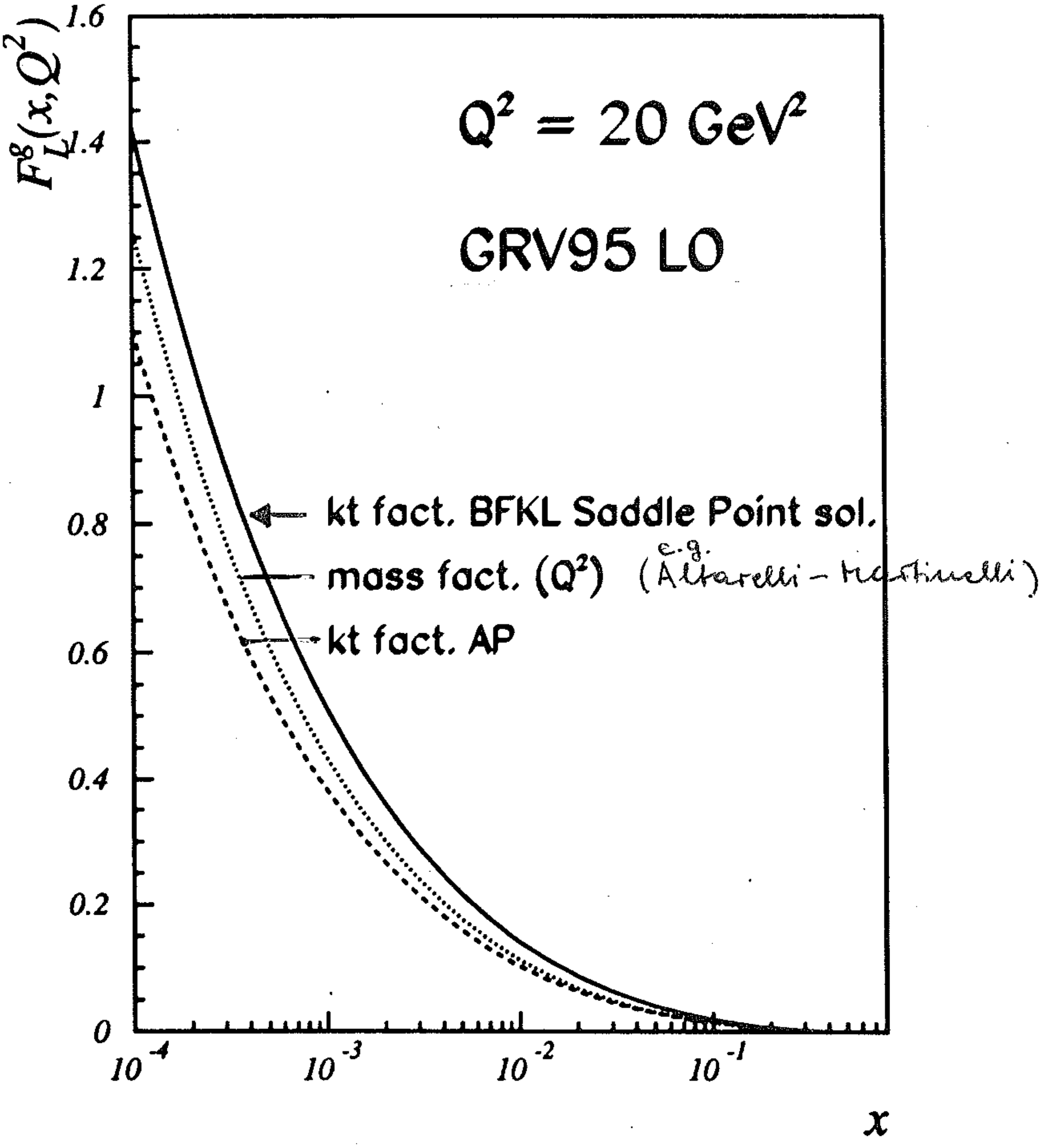


$\frac{\partial G}{\partial k^2}$: AP

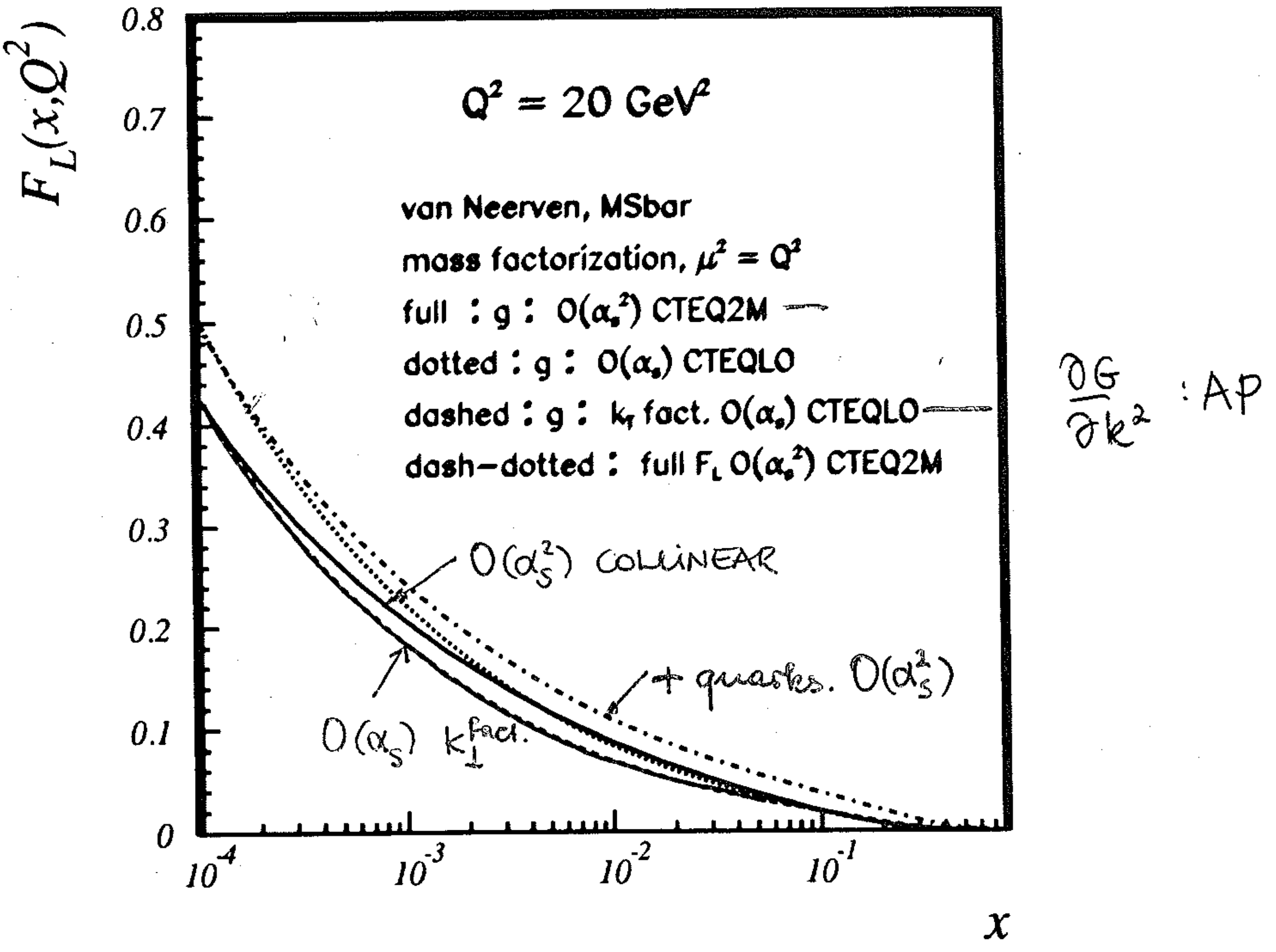
't



$\frac{\partial G}{\partial k^2}$: AP



$F_L^g(x, Q^2)$
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
$Q^2 = 20$ GeV^2
GRV95 LO
kt fact. BFKL Saddle Point sol.
mass fact. (Q^2)
(e.g. Altarelli – Martinelli)
kt fact. AP
10^{-4}
10^{-3}
10^{-2}
10^{-1}
x



$F_L(x,Q^2)$
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
$Q^2 = 20\ GeV^2$
van Neerven, MSbar
mass factorization, $\mu^2 = Q^2$
full : g : $O(\alpha_s^2)$ CTEQ2M
dotted : g : $O(\alpha_s)$ CTEQLO
dashed : g : k_T fact. $O(\alpha_s)$ CTEQLO
dash-dotted : full F_L $O(\alpha_s^2)$ CTEQ2M
$O(\alpha_s^2)$ COLLINEAR
+ quarks. $O(\alpha_s^2)$
$O(\alpha_s)$ $k_\perp^{fact.}$
10^{-4}
10^{-3}
10^{-2}
10^{-1}
x
$\frac{\partial G}{\partial k^2}$: AP

5. CONCLUSIONS

- THE DESCRIPTION OF STRUCTUREFUNCTIONS REQUIRES TO TAKE INTO ACCOUNT ALL RELEVANT DYNAMICAL TERMS (LO, NTLO, NNTLO ...
  - · BFKL SMALLX
  - · MARCH. EQU, ...)

- THE UNFOLDED NONPERTURBATIVE INPUT WILL DIFFER IN LO, NTLO ; BFKL / NO BFKL - ANALYSES TO SOME EXTEND LEADING NOT TO DEFINITE CONCLUSION ON THE DYNAMICS ENCOUNTEREDIN THE 2-DIM. ANALYSIS.

$\Longrightarrow$

UNFOLD : $\phi(x, Q^2, k_\perp^2)$ AT LEAST IN SOME RANGE FOR $k_\perp^2$.

$\longrightarrow$ MORE DEFINITE CONCLUSION ON THE RELEVANCE OF BFKL DYNAMICS etc.