



The 3-loop anomalous dimensions from off-shell operator matrix elements

Johannes Blümlein, KMPB seminar, Berlin, D | December 13, 2021

DESY

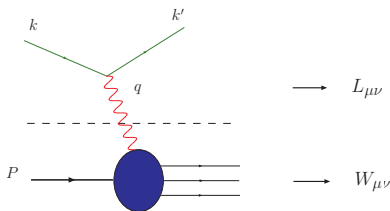
in collaboration with: P. Marquard, K. Schönwald, and C. Schneider

- 1 Introduction
- 2 The status of the QCD anomalous dimensions
- 3 Computation Methods at Three Loops
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- 5 Results
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Based on:

Nucl.Phys.B 922 (2017) 1-40, Nucl.Phys.B 948 (2019) 114753, Nucl.Phys.B 971 (2021) 115542, 2111.12401 [hep-ph] and in preparation

Theory of deep inelastic scattering



- Kinematic invariants:

$$Q^2 = -q^2, \quad x = \frac{Q^2}{2P \cdot q}$$

- The cross section factorizes into leptonic and hadronic tensor:

$$\frac{d^2\sigma}{dQ^2 dx} \sim L_{\mu\nu} W^{\mu\nu}$$

- The hadronic tensor can be expressed through structure functions:

$$\begin{aligned} W_{\mu\nu} &= \frac{1}{4\pi} \int d^4\xi \exp(iq\xi) \langle P, | [J_\mu^{\text{em}}(\xi), J_\nu^{\text{em}}(\xi)] | P \rangle \\ &= \frac{1}{2x} \left(g_{\mu\nu} + \frac{q_\mu q_\nu}{Q^2} \right) F_L(x, Q^2) + \frac{2x}{Q^2} \left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu} \right) F_2(x, Q^2) \\ &\quad + i\epsilon_{\mu\nu\rho\sigma} \frac{q^\rho S^\sigma}{q \cdot P} g_1(x, Q^2) + i\epsilon_{\mu\nu\rho\sigma} \frac{q^\rho (q \cdot PS^\sigma - q \cdot SP^\sigma)}{(q \cdot P)^2} g_2(x, Q^2) \end{aligned}$$

- F_L, F_2, g_1 and g_2 contain contributions from both, charm and bottom quarks.

Light and Heavy Flavor Contributions are both important.

- They form a significant contribution to all structure functions particularly at small x and high Q^2 .
- Concise 3-loop corrections are needed to determine $\alpha_s(M_Z)$, m_c and perhaps m_b .
- The accuracy of measurements at the LHC reaches a level of precision requiring 3-loop VFNS matching.

NNLO: [S. Alekhin, J. Blümlein, S. Moch and R. Placakyte (Phys. Rev. D96 (2017))]

$$\alpha_s(M_Z^2) = 0.1147 \pm 0.0008$$

$$m_c(m_c) = 1.252 \pm 0.018(\text{exp}) \begin{matrix} +0.03 \\ -0.02 \end{matrix} (\text{scale}), \begin{matrix} +0.00 \\ -0.07 \end{matrix} (\text{thy})\text{GeV} \quad (\overline{\text{MS}}\text{-scheme})$$

Yet approximate NNLO treatment for F_2 (non-negligible error) [H. Kawamura et al. (Nucl. Phys. B864 (2012))]

NS & PS corrections are exact [J. Ablinger et al. (Nucl. Phys. B886 & B890 (2014))]

EIC: many more high precision data ahead for various detailed unpolarized and polarized precision measurements.

One important ingredient to the description of the scaling violations are the anomalous dimensions.

The status of the QCD anomalous dimensions

Unpolarized anomalous dimensions



1 Loop:

- Gross, Wilczek 1974a,b
- Georgi, Politzer 1974

2 Loop:

- Floratos, D. Ross, Sachrajda 1977, 1978
- Gonzalez-Arroyo, Lopez, Yndurain 1979, Gonzalez-Arroyo and Lopez 1980
- Curci, Furmanski and Petronzio 1980, Furmanski and Petronzio 1980
- Floratos, Kounnas and Lacaze 1981
- Hamberg and van Neerven 1992 [all former flaws clarified.]
- Ellis and Vogelsang 1996; Matiounine, Smith and van Neerven 1998; Moch and Vermaseren 1999

3 Loop:

- Moch, Vermaseren, Vogt 2004a,b
- Ablinger et al. 2010, 2014abc, 2017
- Anastasiou et al. 2015, Duhr, Dulat and Mistlberger, 2020 (implicit)
- Blümlein, Marquard, Schönwald and Schneider 2021

The status of the QCD anomalous dimensions



During the 1990ies until 2004 moments of the 3-loop anomalous dimensions were calculated up to $N = 16$ in the non-singlet case in 2004, which has been quite an effort at that time.

Polarized anomalous dimensions

1 Loop:

- Sasaki 1975
- Ahmed and Ross 1975

2 Loop:

- Mertig and van Neerven 1995
- Vogelsang 1995
- Matiounine, Smith and van Neerven 1998

3 Loop:

- Moch, Vermaseren, Vogt 2014, 2015
- Behring et al. 2019
- Blümlein, Marquard, Schönwald and Schneider 2021

The status of the QCD anomalous dimensions



Transversity

1 Loop:

- Artru and Mekhfi 1990
- Ioffe and Khojamirian 1995
- Blümlein 2001 [and many other papers]

2 Loop:

- Hayashigaki, Kanazawa and Koike 1995
- Kumano and Miyama 1997
- Vogelsang 1998
- Blümlein, Klein, Tödtli 2009 (including 3-loop TF moments)

3 Loop:

- Velizhanin 2012 (diophantine guess)
- Blümlein, Marquard, Schönwald and Schneider 2021



The status of the QCD anomalous dimensions

Large N_f and Small x predictions [on all these quantities]

Leading N_f series

- Gracey 1994, 1996, 2000, 2003, 2006 a,b
- Gracey and Bennett 1998 a,b

Leading small x series

- Fadin, Kuraev and Lipatov 1975, Balitsky and Lipatov 1978
- Jaroszewicz 1982
- Kirschner and Lipatov 1983
- Catani and Hautmann, 1994
- Bartels, Ermolaev and Ryskin, 1995, 1996
- Blümlein and Vogt 1995, 1996

Next-to-leading order small x series

- Fadin and Lipatov 1998

4-loop moments

- Baikov and Chetyrkin 2006, Baikov, Chetyrkin and Kühn 2015
- Velizhanin 2012, 2014
- Ruijl et al. 2016, Davies et al., Moch et al., 2017, 2021

5-loop moments

- Herzog et al. 2019

The evolution equations



The different anomalous dimensions

- 3 non-singlet anomalous dimensions (starting at 1-, 2-, and 3-loop) $\gamma_{qq}^{NS,\pm}, \gamma_{qq}^{NS,s}$
- 4 singlet anomalous dimensions $\gamma_{qq}^{PS}, \gamma_{qg}, \gamma_{gq}, \gamma_{gg}$ [γ_{qq}^{PS} contributes from 2-loops.]
- both in the unpolarized and polarized case: $\gamma_{ij} \rightarrow \Delta\gamma_{ij}$
- + expansion coefficients for the so called alien operators, including **new** anomalous dimensions.

Evolution equations

$$\Sigma(N) = \sum_{k=1}^{N_F} q_k(N) + \bar{q}_k(N)$$

$$\gamma_{ij} = \sum_{k=0}^{\infty} a_s^{k+1} \gamma_{ij}^{(k)}$$

$$\frac{dq^{NS,\pm,(s)}(N, a_s)}{d \ln(\mu^2)} = \sum_{k=0}^{\infty} a_s^{k+1} \gamma_{qq}^{(k),NS,\pm,(s)}(N) \cdot q^{NS,\pm,(s)}(N, a_s)$$

$$\frac{d}{d \ln(\mu^2)} \begin{pmatrix} \Sigma(N) \\ G(N) \end{pmatrix} = \sum_{k=0}^{\infty} a_s^{k+1} \begin{pmatrix} \gamma_{qq}^{(k)} & \gamma_{qg}^{(k)} \\ \gamma_{gq}^{(k)} & \gamma_{gg}^{(k)} \end{pmatrix} \cdot \begin{pmatrix} \Sigma(N) \\ G(N) \end{pmatrix}$$

Computation Methods at Three Loops



Forward Compton Amplitude:

- Moch, Vermaseren and Vogt 2004a,b, 2014; based on Mincer Gorishnii, Larin, Surguladze, 1989.
- straightforward in the quarkonic case; gauge invariant framework
- necessitates scalar currents to obtain the gluonic anomalous dimensions in the polarized case
- polarized case: gravitational currents in the case of the gluonic anomalous

Massive On-Shell Operator Matrix Elements:

- Blümlein at al. 2014 a,b,c, 2015, 2017, 2019
- allows to calculate all contributions $\propto T_F$
- gauge invariant framework


Massless Off-Shell Operator Matrix Elements:

- Blümlein, Marquard, Schönwald and Schneider 2021 a,b
- allows to calculate all anomalous dimensions
- gauge variant framework
- no further problems in the non-singlet and polarized singlet case
- mixing with new (alien) operator matrix elements in the unpolarized case
- the previous 2-loop studies by Matiounine, van Neerven and Smith 1998 contain some errors


Calculation of the 3-loop operator matrix elements



The OMEs are calculated using the QCD Feynman rules together with the following operator insertion Feynman rules + 1 more five-gluon graph. $p^2 < 0$ implies breaking of gauge invariance.




$$\delta^{ij} \Delta_{\gamma_{\pm}} (\Delta \cdot p)^{N-1}, \quad N \geq 1$$

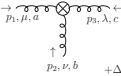


$$\frac{1+(-1)^N}{2} g^{ab} (\Delta \cdot p)^{N-2}$$


$$\left[g_{\mu\nu} (\Delta \cdot p)^2 - (\Delta_{\mu} p_{\nu} + \Delta_{\nu} p_{\mu}) \Delta \cdot p + p^2 \Delta_{\mu} \Delta_{\nu} \right], \quad N \geq 2$$



$$g_{\mu a}^{\mu a} \Delta_{\gamma_{\pm}} \sum_{j=0}^{N-2} (\Delta \cdot p_1)^j (\Delta \cdot p_2)^{N-j-2}, \quad N \geq 2$$



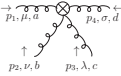
$$-i g \frac{1+(-1)^N}{2} f^{abc} \left(\begin{aligned} & [(\Delta_{\nu} g_{\lambda\mu} - \Delta_{\lambda} g_{\mu\nu}) \Delta \cdot p_1 + \Delta_{\mu} (p_{1,\nu} \Delta_{\lambda} - p_{1,\lambda} \Delta_{\nu})] (\Delta \cdot p_1)^{N-2} \\ & + \Delta_{\lambda} [\Delta \cdot p_1 p_{2,\mu} \Delta_{\nu} + \Delta \cdot p_2 p_{1,\nu} \Delta_{\mu} - \Delta \cdot p_1 \Delta \cdot p_2 g_{\mu\nu} - p_1 \cdot p_2 \Delta_{\mu} \Delta_{\nu}] \\ & \times \sum_{j=0}^{N-3} (-\Delta \cdot p_1)^j (\Delta \cdot p_2)^{N-3-j} \\ & + \left\{ \begin{aligned} & p_1 \rightarrow p_2 \rightarrow p_3 \rightarrow p_1 \\ & \mu \rightarrow \nu \rightarrow \lambda \rightarrow \mu \end{aligned} \right\} \right), \quad N \geq 2 \end{aligned}$$



$$g^2 \Delta_{\mu} \Delta_{\nu} \Delta_{\gamma_{\pm}} \sum_{j=0}^{N-3} \sum_{l=j+1}^{N-2} (\Delta p_2)^j (\Delta p_1)^{N-l-2}$$

$$\left[(t^a t^b)_{ji} (\Delta p_1 + \Delta p_4)^{l-j-1} + (t^b t^a)_{ji} (\Delta p_1 + \Delta p_3)^{l-j-1} \right],$$


$$N \geq 3$$



$$g^2 \frac{1+(-1)^N}{2} \left(f^{abc} f^{cde} O_{\mu\nu\lambda\sigma}(p_1, p_2, p_3, p_4) \right. \\ \left. + f^{ace} f^{bde} O_{\mu\nu\lambda\sigma}(p_1, p_3, p_2, p_4) + f^{abc} f^{cde} O_{\mu\nu\lambda\sigma}(p_1, p_4, p_2, p_3) \right),$$

$$O_{\mu\nu\lambda\sigma}(p_1, p_2, p_3, p_4) = \Delta_{\nu} \Delta_{\lambda} \left\{ -g_{\mu\sigma} (\Delta \cdot p_3 + \Delta \cdot p_4)^{N-2} \right. \\ \left. + [p_{4,\mu} \Delta_{\sigma} - \Delta \cdot p_4 g_{\mu\sigma}] \sum_{i=0}^{N-3} (\Delta \cdot p_3 + \Delta \cdot p_4)^i (\Delta \cdot p_1)^{N-3-i} \right. \\ \left. - [p_{1,\sigma} \Delta_{\mu} - \Delta \cdot p_1 g_{\mu\sigma}] \sum_{i=0}^{N-3} (-\Delta \cdot p_1)^i (\Delta \cdot p_3 + \Delta \cdot p_4)^{N-3-i} \right. \\ \left. + [\Delta \cdot p_1 \Delta \cdot p_4 g_{\mu\sigma} + p_1 \cdot p_4 \Delta_{\mu} \Delta_{\sigma} - \Delta \cdot p_4 p_{1,\sigma} \Delta_{\mu} - \Delta \cdot p_1 p_{4,\sigma} \Delta_{\mu}] \right. \\ \left. \times \sum_{i=0}^{N-4} \sum_{j=0}^i (-\Delta \cdot p_1)^{N-4-i} (\Delta \cdot p_3 + \Delta \cdot p_4)^{i-j} (\Delta \cdot p_4)^j \right\}$$

$$- \left\{ \begin{aligned} & p_1 \leftrightarrow p_2 \\ & \mu \leftrightarrow \nu \end{aligned} \right\} - \left\{ \begin{aligned} & p_2 \leftrightarrow p_3 \\ & \lambda \leftrightarrow \sigma \end{aligned} \right\} + \left\{ \begin{aligned} & p_1 \leftrightarrow p_2, p_3 \leftrightarrow p_4 \\ & \mu \leftrightarrow \nu, \lambda \leftrightarrow \sigma \end{aligned} \right\}, \quad N \geq 2$$



$$g^2 \Delta_{\mu} \Delta_{\nu} \Delta_{\rho} \Delta_{\gamma_{\pm}} \sum_{j=0}^{N-4} \sum_{l=j+1}^{N-3} \sum_{m=l+1}^{N-2} (\Delta \cdot p_2)^j (\Delta \cdot p_1)^{N-m-2}$$

$$\left[\begin{aligned} & (t^a t^b t^c)_{jil} (\Delta p_4 + \Delta p_5 + \Delta p_1)^{l-j-1} (\Delta p_5 + \Delta p_1)^{m-l-1} \\ & + (t^a t^b t^c)_{jil} (\Delta p_4 + \Delta p_5 + \Delta p_1)^{l-j-1} (\Delta p_4 + \Delta p_1)^{m-l-1} \\ & + (t^b t^a t^c)_{jil} (\Delta p_3 + \Delta p_5 + \Delta p_1)^{l-j-1} (\Delta p_5 + \Delta p_1)^{m-l-1} \\ & + (t^b t^a t^c)_{jil} (\Delta p_3 + \Delta p_5 + \Delta p_1)^{l-j-1} (\Delta p_3 + \Delta p_1)^{m-l-1} \\ & + (t^c t^a t^b)_{jil} (\Delta p_3 + \Delta p_4 + \Delta p_1)^{l-j-1} (\Delta p_4 + \Delta p_1)^{m-l-1} \\ & + (t^c t^a t^b)_{jil} (\Delta p_3 + \Delta p_4 + \Delta p_1)^{l-j-1} (\Delta p_3 + \Delta p_1)^{m-l-1} \end{aligned} \right],$$

$$N \geq 4$$

$$\gamma_+ = 1, \quad \gamma_- = \gamma_+.$$

Introduction	The status of the QCD anomalous dimensions	Computation Methods at Three Loops	Massless off-shell operator matrix elements	Results	Conclusions and Outlook
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The unpolarized massless OMEs



Vector case:

$$O_{q,r;\mu_1\dots\mu_N}^{\text{NS}} = i^{N-1} \mathbf{s} \left[\bar{\psi} \gamma_{\mu_1} D_{\mu_2} \dots D_{\mu_N} \frac{\lambda_r}{2} \psi \right] - \text{trace terms},$$

$$O_{q,r;\mu_1\dots\mu_N}^{\text{NS},5} = i^{N-1} \mathbf{s} \left[\bar{\psi} \gamma_5 \gamma_{\mu_1} D_{\mu_2} \dots D_{\mu_N} \frac{\lambda_r}{2} \psi \right] - \text{trace terms}$$

Transversity case:

$$O_{q,r;\mu\mu_1\dots\mu_N}^{\text{NS,tr}} = i^{N-1} \mathbf{s} \left[\bar{\psi} \sigma_{\mu\mu_1} D_{\mu_2} \dots D_{\mu_N} \frac{\lambda_r}{2} \psi \right] - \text{trace terms},$$

where $\sigma_{\mu\nu} = (i/2)[\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu]$.

$$\Delta \hat{A}_{ij} = \langle q(p), j | O_i^{\text{S},(5)} | q(p), j \rangle.$$

The unpolarized massless OMEs



The operator matrix elements have the representation

$$\hat{A}_{qq}^{\text{NS}} = \left[\hat{\Delta} \hat{A}_{qq}^{\text{NS,phys}} + \not{p} \frac{\Delta \cdot p}{p^2} \hat{A}_{qq}^{\text{NS,EOM}} \right] (\Delta \cdot p)^{N-1}.$$

with $\Delta \cdot \Delta = 0$.

$$\hat{A}_{qq}^{\text{NS,phys}} = \frac{1}{4(\Delta \cdot p)^N} \text{tr} \left[\left(\not{p} - \frac{p^2}{\Delta \cdot p} \hat{\Delta} \right) \hat{A}_{qq}^{\text{NS}} \right],$$

$$\hat{A}_{qq}^{\text{NS,EOM}} = \frac{1}{4(\Delta \cdot p)^N} \text{tr} \left[\hat{\Delta} \hat{A}_{qq}^{\text{NS}} \right].$$

$$\hat{A}_{qq}^{\text{NS,5,phys}} = \frac{1}{4(\Delta \cdot p)^N} \text{tr} \left[\left(\not{p} - \frac{p^2}{\Delta \cdot p} \hat{\Delta} \right) \gamma_5 \hat{A}_{qq}^{\text{NS,5}} \right],$$

$$\hat{A}_{qq}^{\text{NS,5,EOM}} = \frac{1}{4(\Delta \cdot p)^N} \text{tr} \left[\hat{\Delta} \gamma_5 \hat{A}_{qq}^{\text{NS,5}} \right].$$

The polarized massless OMEs



$$O_{q;\mu_1 \dots \mu_N}^{S,5} = i^{N-1} \mathbf{S} \left[\bar{\psi} \gamma_5 \gamma_{\mu_1} D_{\mu_2} \dots D_{\mu_N} \psi \right] - \text{trace terms,}$$

$$O_{g;\mu_1 \dots \mu_N}^{S,5} = 2i^{N-2} \mathbf{S} \text{tr} \left[\frac{1}{2} \varepsilon^{\mu_1 \alpha \beta \gamma} F_{\beta \gamma}^a D_{\mu_2} \dots D_{\mu_{N-1}} F_{\alpha, a}^{\mu_N} \right] - \text{trace terms}$$

$$\Delta \hat{A}_{qq}^{\text{PS}} = \left[\gamma_5 \Delta \Delta \hat{A}_{qq}^{\text{PS,phys}} + \gamma_5 \not{p} \frac{\Delta \cdot p}{p^2} \Delta \hat{A}_{qq}^{\text{PS,EOM}} \right] (\Delta \cdot p)^{N-1}$$

$$\Delta \hat{A}_{qq} = \varepsilon_{\mu\nu\alpha\beta} \Delta^\alpha p^\beta \frac{1}{\Delta \cdot p} \Delta \hat{A}_{qq}^{\text{phys}} (\Delta \cdot p)^{N-1}$$

$$\Delta \hat{A}_{gq} = \left[\gamma_5 \Delta \Delta \hat{A}_{gq}^{\text{phys}} + \gamma_5 \not{p} \frac{\Delta \cdot p}{p^2} \hat{A}_{gq}^{\text{EOM}} \right] (\Delta \cdot p)^{N-1}$$

$$\Delta \hat{A}_{gg,\mu\nu} = \varepsilon_{\mu\nu\alpha\beta} \Delta^\alpha p^\beta \frac{1}{\Delta \cdot p} \Delta \hat{A}_{gq}^{\text{phys}} (\Delta \cdot p)^{N-1},$$

The polarized massless OMEs



$$\begin{aligned}\Delta\hat{A}_{iq}^{\text{phys}} &= -\frac{1}{4(D-2)(D-3)}\varepsilon_{\mu\nu\rho\sigma}p^\rho\Delta^\sigma\text{tr}[\not{p}\gamma^\mu\gamma^\nu\Delta\hat{A}_{iq}](\Delta\cdot p)^{-N-1} \\ &\quad -\frac{p^2}{4(D-2)(D-3)}(\Delta\cdot p)^{-N-2}\varepsilon_{\mu\nu\rho\sigma}p^\rho\Delta^\sigma\text{tr}[\not{\Delta}\gamma^\mu\gamma^\nu\Delta\hat{A}_{iq}], \\ \Delta\hat{A}_{ig}^{\text{phys}} &= \frac{1}{(D-2)(D-3)}\varepsilon_{\mu\nu\rho\sigma}\Delta^\rho p^\sigma(\Delta\cdot p)^{-N-1}\Delta\hat{A}_{ig}^{\mu\nu},\end{aligned}$$

We use the Larin scheme [Larin 1993](#), being a consistent scheme, to describe γ^5 in $D = 4 + \varepsilon$ -dimensions:

$$\begin{aligned}\gamma^5 &= \frac{i}{24}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma, \\ \not{\Delta}\gamma^5 &= \frac{i}{6}\varepsilon_{\mu\nu\rho\sigma}\Delta^\mu\gamma^\nu\gamma^\rho\gamma^\sigma.\end{aligned}$$

The Levi-Civita symbols are now contracted in D dimensions,

$$\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\alpha\lambda\tau\gamma} = -\text{Det}[g_\omega^\beta], \quad \beta = \alpha, \lambda, \tau, \gamma; \quad \omega = \mu, \nu, \rho, \sigma.$$

OME structure: Renormalization

- coupling constant renormalization
- gauge parameter renormalization
- renormalization of the local operator
- The OMEs remain gauge-dependent (off-shellness).

The partly renormalized polarized singlet OMEs, $\tilde{A}_{ij}^{\text{phys}}$, read

$$\tilde{A}_{ij}^{\text{phys}} = 1 + a \left[\frac{a_{ij}^{(1,-1)}}{\epsilon} + a_{ij}^{(1,0)} + a_{ij}^{(1,1)} \epsilon \right] + a^2 \left[\frac{a_{ij}^{(2,-2)}}{\epsilon^2} + \frac{a_{ij}^{(2,-1)}}{\epsilon} + a_{ij}^{(2,0)} \right] + a^3 \left[\frac{a_{ij}^{(3,-3)}}{\epsilon^3} + \frac{a_{ij}^{(3,-2)}}{\epsilon^2} + \frac{a_{ij}^{(3,-1)}}{\epsilon} \right].$$

$$Z_{ij}^S = \delta_{ij} + a \frac{\gamma_{ij}^{(0)}}{\epsilon} + a^2 \left[\frac{1}{\epsilon^2} \left(\frac{1}{2} \gamma_{il}^{(0)} \gamma_{lj}^{(0)} + \beta_0 \gamma_{ij}^{(0)} \right) + \frac{1}{2\epsilon} \gamma_{ij}^{(1)} \right] + a^3 \left[\frac{1}{\epsilon^3} \left(\frac{1}{6} \gamma_{il}^{(0)} \gamma_{lk}^{(0)} \gamma_{kj}^{(0)} + \beta_0 \gamma_{il}^{(0)} \gamma_{lj}^{(0)} + \frac{4}{3} \beta_0^2 \gamma_{ij}^{(0)} \right) \right. \\ \left. + \frac{1}{\epsilon^2} \left(\frac{1}{6} \gamma_{il}^{(1)} \gamma_{lj}^{(0)} + \frac{1}{3} \gamma_{il}^{(0)} \gamma_{lj}^{(1)} + \frac{2}{3} \beta_0 \gamma_{ij}^{(1)} + \frac{2}{3} \beta_1 \gamma_{ij}^{(0)} \right) + \frac{1}{3\epsilon} \gamma_{ij}^{(2)} \right].$$

$$A_{ij}^{\text{phys}} = (Z_{ik}^S)^{-1} \tilde{A}_{kj}^{\text{phys}}.$$

Technical aspects of the computation



The program chain:

- The 3-loop calculation is by now fully automated.
- Diagram generation: QGRAF, [Nogueira, 1991](#).
- Lorentz algebra: FORM, [Vermaseren and Tentyukov, 2000, 2010](#).
- Color algebra: Color, [van Ritbergen, Schellekens, Vermaseren 1999](#).
- IBP reductions: Crusher, [Marquard & Seidel](#).
- Initial values, [Chetyrkin, Tkachov, 1981](#), [Tkachov, 1981](#), [Gehrmann et al. 2010](#); [Lee, Smirnov, Smirnov 2010, 2011](#).
- Method of arbitrary high moments to generated large sets of Mellin moments, here 3000: SolveCoupledSystems, [Blümlein and Schneider, 2017](#).
- Determination of the recurrences for the OMEs: Guess in Sage, [Blümlein, Kauers, Schneider, 2009](#); [Kauers et al. 2009, 2015](#).
- Solution of the recurrences: Sigma, [Schneider, 2007–](#)
- Reduction of the solution HarmonicSums: [Ablinger 2010–](#) using shuffle-algebras [Blümlein, 2003](#) and structural relations [i.e. double integer and differential relations] [Blümlein, 2009](#).

Mass production of moments is no problem since 2017 anymore. Various physical problems have already been computed by us based on 8000 moments.

Technical aspects of the computation

Expression of the results in Mellin N and z space

Harmonic Sums:

Vermaseren 1998, Blümlein and Kurth, 1998

$$S_{b,\bar{a}}(N) = \sum_{k=1}^N \frac{(\text{sign}(b))^k}{k^{|b|}} S_{\bar{a}}(k), \quad S_{\emptyset} = 1, \quad b, a_i \in \mathbb{Z} \setminus \{0\}, N \in \mathbb{N} \setminus \{0\}.$$

Their Mellin inversion to the splitting functions $P_{qq}(z)$

$$\gamma_{qq}(N) = - \int_0^1 dz z^{N-1} P_{qq}(z)$$

Harmonic Polylogarithms:

Remiddi and Vermaseren, 1999

$$H_{b,\bar{a}}(z) = \int_0^z dx f_b(x) H_{\bar{a}}(x), \quad H_{\emptyset} = 1, \quad b, a_i \in \{-1, 0, 1\},$$

with the alphabet of letters

$$\mathfrak{A}_H = \left\{ f_0(z) = \frac{1}{z}, \quad f_{-1}(z) = \frac{1}{1+z}, \quad f_1(z) = \frac{1}{1-z} \right\}.$$

Technical aspects of the computation



- Consider the example of the unpolarized 3-loop singlet case
- The quoted quantities are all less in the different non-singlet and polarized singlet cases
- Number of 1PI diagrams: 3324
- Number of moments needed:
 - $\gamma_{qq}^{(2),PS}$: 588
 - $\gamma_{qg}^{(2)}$: 1248
 - $\gamma_{gq}^{(2)}$: 1274
 - $\gamma_{gg}^{(2)}$: 1860
- largest difference equation: $d = 373$, $o = 16$. [Far larger difference equations can be solved by Sigma.]
- overall computation time: ~ 3 weeks.

All 3-loop anomalous dimensions can be represented by 10 basic harmonic sums only after using structural relations (in the complex plane):

$$\left\{ S_1, S_{2,1}, S_{-2,1}, S_{-3,1}, S_{-4,1}, S_{2,1,1}, S_{-2,1,1}, S_{2,1,-2}, S_{-3,1,1}, S_{-2,1,1,1} \right\}.$$

The number of moments obtained by the method of arbitrary high moments is several orders of magnitude higher than the one which can be produced using e.g. Mincer.

Technical aspects of the computation



The Larin and M-scheme in the polarized singlet case Matiounine, van Neerven and Smith, 1998

$$\begin{aligned}
 \Delta\gamma_{ij}^{(0),M} &= \Delta\gamma_{ij}^{(0),L} \\
 \Delta\gamma_{qq}^{(1),NS,M} &= \Delta\gamma_{qq}^{(1),NS,L} + 2\beta_0 z_{qq}^{(1)}, \\
 \Delta\gamma_{qq}^{(1),PS,M} &= \Delta\gamma_{qq}^{(1),PS,L}, \\
 \Delta\gamma_{qg}^{(1),M} &= \Delta\gamma_{qg}^{(1),L} + \Delta\gamma_{qg}^{(0)} z_{qq}^{(1)}, \\
 \Delta\gamma_{gq}^{(1),M} &= \Delta\gamma_{gq}^{(1),L} - \Delta\gamma_{gq}^{(0)} z_{qq}^{(1)}, \\
 \Delta\gamma_{gg}^{(1),M} &= \Delta\gamma_{gg}^{(1),L}. \\
 \Delta\gamma_{qq}^{(2),NS,M} &= \Delta\gamma_{qq}^{(2),NS,L} - 2\beta_0 \left((z_{qq}^{(1)})^2 - 2z_{qq}^{(2),NS} \right) + 2\beta_1 z_{qq}^{(1)}, \\
 \Delta\gamma_{qq}^{(2),PS,M} &= \Delta\gamma_{qq}^{(2),PS,L} + 4\beta_0 z_{qq}^{(2),PS}, \\
 \Delta\gamma_{qg}^{(2),M} &= \Delta\gamma_{qg}^{(2),L} + \Delta\gamma_{qg}^{(1),M} z_{qq}^{(1)} + \Delta\gamma_{qg}^{(0)} \left(z_{qq}^{(2)} - (z_{qq}^{(1)})^2 \right), \\
 \Delta\gamma_{gq}^{(2),M} &= \Delta\gamma_{gq}^{(2),L} - \Delta\gamma_{gq}^{(1),M} z_{qq}^{(1)} - \Delta\gamma_{gq}^{(0)} z_{qq}^{(2)}, \\
 \Delta\gamma_{gg}^{(2),M} &= \Delta\gamma_{gg}^{(2),L},
 \end{aligned}$$

$$\begin{aligned}
 z_{qq}^{(1)} &= -\frac{8C_F}{N(N+1)}, \\
 z_{qq}^{(2),\text{NS}} &= C_F T_F N_F \frac{16(-3-N+5N^2)}{9N^2(1+N)^2} + C_A C_F \left\{ -\frac{4R_1}{9N^3(1+N)^3} - \frac{16}{N(1+N)} S_{-2} \right\} \\
 &\quad + C_F^2 \left\{ \frac{8(2+5N+8N^2+N^3+2N^4)}{N^3(1+N)^3} + \frac{16(1+2N)}{N^2(1+N)^2} S_1 \right. \\
 &\quad \left. + \frac{16}{N(1+N)} S_2 + \frac{32}{N(1+N)} S_{-2} \right\}, \\
 z_{qq}^{(2),\text{PS}} &= 8C_F T_F N_F \frac{(N+2)(1+N-N^2)}{N^3(N+1)^3}, \\
 z_{qq}^{(2)} &= z_{qq}^{(2),\text{NS}} + z_{qq}^{(2),\text{PS}}
 \end{aligned}$$

The simplest 3-loop anomalous dimensions: $\gamma_{qq}^{\text{NS},(s)}$ and $\Delta\gamma_{qq}^{\text{NS},(s)}$



- Moch, Vermaseren, Vogt, 2004a; Blümlein, Marquard, Schönwald and Schneider 2021a
- $\gamma_{qq}^{\text{NS},(s)}$ stems from considering the **odd** moments of the unpolarized pure singlet OME.

$$\begin{aligned} \gamma_{\text{NS}}^{(2),s} = & 4N_F \frac{d_{abc}d^{abc}}{N_C} \left[\frac{2P_{60}}{(-1+N)N^5(1+N)^5(2+N)} - \frac{P_{61}}{(-1+N)N^4(1+N)^4(2+N)} S_1 \right. \\ & + \left(-\frac{2P_{62}}{(-1+N)N^3(1+N)^3(2+N)} - \frac{4(2+N+N^2)^2}{(-1+N)N^2(1+N)^2(2+N)} S_1 \right) S_{-2} \\ & \left. - \frac{(2+N+N^2)}{N^2(1+N)^2} [S_3 - 2S_{-3} + 4S_{-2,1}] \right], \end{aligned}$$

The simplest 3-loop anomalous dimensions: $\gamma_{qq}^{\text{NS},(s)}$ and $\Delta\gamma_{qq}^{\text{NS},(s)}$



- $\Delta\gamma_{qq}^{\text{NS},(s)}$ rules the evolution of the polarized electro-weak interference structure function $g_5^-(x, Q^2)$, cf. [Blümlein and Kochelev, 1996](#).
- It has only even moments (although being polarized).
- It is best calculated using the forward Compton amplitude for a vector and axial-vector current.
- [Moch, Vermaseren, Vogt, 2015](#); [Blümlein, Marquard, Schönwald and Schneider 2021b](#)

$$\Delta\gamma_{\text{NS}}^{(2),s} = -16 \frac{1 + (-1)^N}{2} N_F \frac{d^{abc} d_{abc}}{N_c} \left[\frac{1}{N^2} + \frac{2Q_1}{N^4(1+N)^4} S_1 + \frac{2(2+3N+3N^2)}{N^2(1+N)^2} \right. \\ \times [S_3 - 2S_{-3} + 4S_{-2,1}] + \left(\frac{4(2+4N+4N^2+N^3+N^4)}{N^3(1+N)^3} + \frac{8(-1+N)(2+N)}{N^2(1+N)^2} S_1 \right) \\ \left. \times S_{-2} \right]$$

The non-singlet anomalous dimension $\gamma_{qq}^{+,NS}$



$$\begin{aligned}
 & \gamma_{88}^{(2)+} = \frac{1}{2} (1 + (-1)^N) \\
 & \times \left\{ C_F^2 \left[C_A \left[\frac{72P_3}{N^2(1+N)^2} G_6 + \frac{32P_{12}}{9N^2(1+N)^2} S_{-2,1} - \frac{16P_{17}}{9N^2(1+N)^2} S_3 + \frac{P_{33}}{18N^4(1+N)^4} \right. \right. \right. \\
 & + \left. \left. \left. \frac{16P_{29}}{9N^4(1+N)^4} - \frac{428S_2}{9} S_2 + \frac{64(-12+31N+31N^2)}{3N(1+N)} S_3 + 320S_4 - 1024S_{3,1} \right. \right. \\
 & + \left. \left. \frac{64(-84+31N+31N^2)}{3N(1+N)} S_{-2,1} + 3712S_{-2,2} + 3840S_{-3,1} - 7168S_{-2,1,1} \right] S_1 + \left(256S_3 \right. \right. \\
 & + 1792S_{-2,1} \left. \right) S_1^2 + \left(\frac{4P_{39}}{9N^2(1+N)^2} - 832S_3 - 5248S_{-2,1} \right) S_2 + \frac{352}{3} S_2^2 \\
 & + \frac{16(-30+151N+151N^2)}{3N(1+N)} S_4 + \left(-\frac{16P_{22}}{9N^2(1+N)^2} + \left(-\frac{64P_9}{9N^2(1+N)^2} - 256S_2 \right) S_1 \right. \\
 & + \left. \frac{32(12+31N+31N^2)S_2}{3N(1+N)} + 64S_3 + 5376S_{2,1} - 384S_{-2,1} + 576G_5 \right) S_{-2} \\
 & + \left(-\frac{32(8+3N+3N^2)}{N(1+N)} + 512S_1 \right) S_{-2}^2 + \left(\frac{32(108+31N+31N^2)}{3N(1+N)} S_1 - \frac{16P_{16}}{9N^2(1+N)^2} \right. \\
 & - 1152S_2^2 + 2624S_2 + 960S_{-2} \left. \right) S_{-3} + \left(\frac{16(138+35N+35N^2)}{3N(1+N)} - 1472S_1 \right) S_{-4} \\
 & + 2304S_{-5} + 768S_{2,3} + 2688S_{2,-3} - \frac{64(-24+29N+29N^2)}{3N(1+N)} S_{3,1} - 768S_{4,1} \\
 & + \frac{32(-174+31N+31N^2)}{3N(1+N)} S_{-2,2} - 3648S_{-2,3} - \frac{1920}{N(1+N)} S_{-3,1} + 1728S_{-4,1} \\
 & - 5376S_{2,-2} + 1536S_{3,1,1} - \frac{128(-84+31N+31N^2)}{3N(1+N)} S_{-2,1,1} - 1536S_{-2,1,-2} \\
 & - 5376S_{-2,2,1} - 5376S_{-3,1,1} + 10752S_{-2,1,1,1} \left. \right\} + T_F N_f \left[-\frac{16P_3}{9N^2(1+N)^2} S_2 + \frac{4P_{34}}{9N^4(1+N)^4} \right. \\
 & + \left(\frac{8P_{13}}{9N^2(1+N)^2} + \frac{1280}{9} S_2 - \frac{512}{3} S_3 - \frac{512}{3} S_{-2,1} + 128G_5 \right) S_1 - \frac{128}{3} S_2^2 \\
 & + \frac{64(12+29N+29N^2)}{9N(1+N)} S_3 - \frac{512}{3} S_4 + \left(-\frac{128(-3+10N+16N^2)}{9N^2(1+N)^2} + \frac{2560}{9} S_1 - \frac{256}{3} S_2 \right) \\
 & \times S_{-2} + \left(\frac{128(3+10N+10N^2)}{9N(1+N)} - \frac{256}{3} S_1 \right) S_{-3} - \frac{256}{3} S_{-4} - \frac{256(-3+10N+10N^2)}{9N(1+N)} S_{-2,1} \\
 & + \frac{256}{3} S_{3,1} - \frac{256}{3} S_{-2,2} + \frac{1024}{3} S_{-2,1,1} \left. \right\} \left. \right\} \left[\frac{32(2+3N+3N^2)}{N(1+N)} G_6 \right] \\
 & + C_F \left\{ T_F^2 N_f^2 \left[\frac{8P_{28}}{27N^3(1+N)^3} - \frac{128}{27} S_1 - \frac{640}{27} S_2 + \frac{128}{9} S_3 \right] + C_A^2 \left[-\frac{24P_9}{N^3(1+N)^3} G_6 \right. \right. \\
 & \left. \left. \frac{32P_{11}}{9N^2(1+N)^2} S_{-2,1} + \frac{8P_{16}}{9N^2(1+N)^2} S_3 + \frac{P_{32}}{54N^2(1+N)^2} + \left(\frac{4P_{30}}{3N^4(1+N)^4} \right. \right. \right. \\
 & - \frac{16(-8+11N+11N^2)}{N(1+N)} S_3 - 256S_4 + 512S_{3,1} - \frac{64(-24+11N+11N^2)}{3N(1+N)} S_{-2,1} \\
 & - 1024S_{-2,2} - 1024S_{-3,1} + 2048S_{-2,1,1} \left. \right) S_1 + \left(-128S_3 - 512S_{-2,1} \right) S_1^2 + \left(-\frac{8344}{27} \right. \\
 & + 384S_3 + 1536S_{-2,1} \left. \right) S_2 - \frac{16(-24+55N+55N^2)}{3N(1+N)} S_4 + 64S_5 + \left(\frac{32P_{19}}{9N^2(1+N)^2} S_1 \right. \\
 & + \frac{16P_{27}}{9N^3(1+N)^3} - \frac{352}{3} S_2 - 64S_3 - 1536S_{2,1} + 128S_{-2,1} - 192G_5 \left. \right) S_{-2} \\
 & + \left(\frac{48(2+N+N^2)}{N(1+N)} - 192S_1 \right) S_{-2}^2 + \left(\frac{16P_{12}}{9N^2(1+N)^2} - \frac{32(24+11N+11N^2)}{3N(1+N)} S_1 \right. \\
 & + 256S_2^2 - 768S_2 - 320S_{-2} \left. \right) S_{-3} + \left(-\frac{16(30+13N+13N^2)}{3N(1+N)} + 320S_1 \right) S_{-4} \\
 & - 704S_{-5} - 384S_{2,3} - 768S_{-3} + \frac{64(-12+11N+11N^2)}{3N(1+N)} S_{3,1} + 384S_{4,1} \\
 & - \frac{32(-48+11N+11N^2)}{3N(1+N)} S_{-2,2} + 1088S_{-2,3} + \frac{512}{N(1+N)} S_{-3,1} - 448S_{-4,1} \\
 & + 1536S_{2,1,-2} - 768S_{3,1,1} + \frac{128(-24+11N+11N^2)}{3N(1+N)} S_{-2,1,1} + 512S_{-2,1,-2} + 1536(S_{-2,2,1} \\
 & + S_{-3,1,1}) - 3072S_{-2,1,1,1} \left. \right\} + C_A T_F N_f \left[-\frac{8P_{13}}{27N^3(1+N)^3} + \left(-\frac{16P_{39}}{27N^3(1+N)^3} + 64S_3 \right. \right. \\
 & + \frac{256}{3} S_{-2,1} - 128G_5 \left. \right) S_1 + \frac{5344}{27} S_2 - \frac{32(3+14N+14N^2)}{3N(1+N)} S_3 + \frac{320}{3} S_4 + \left(\frac{1280}{9} S_1 \right. \\
 & + \frac{64(-3+10N+16N^2)}{9N^2(1+N)^2} + \frac{128}{3} S_2 \left. \right) S_{-2} + \left(-\frac{64(3+10N+10N^2)}{9N(1+N)} + \frac{128}{3} S_1 \right) S_{-3} \\
 & + \frac{128}{3} S_{-4} - \frac{256}{3} S_{3,1} + \frac{128(-3+10N+10N^2)}{9N(1+N)} S_{-2,1} + \frac{128}{3} S_{-2,2} - \frac{512}{3} S_{-2,1,1} \\
 & + \left. \left. \frac{32(2+3N+3N^2)}{N(1+N)} G_6 \right] + C_F^2 \left[-\frac{48P_3}{N^2(1+N)^2} G_6 + \frac{8P_9}{N^2(1+N)^2} S_3 + \frac{P_{36}}{N^3(1+N)^3} \right. \right. \\
 & + \left(\frac{8P_{36}}{N^4(1+N)^4} - \frac{128(1+2N)}{N^2(1+N)^2} S_2 + 128S_2^2 - 384S_3 + 128S_4 + 512S_{3,1} - 3328S_{-2,2} \right. \\
 & - \frac{384(-4+N+N^2)}{N(1+N)} S_{-2,1} - 3584S_{-3,1} + 6144S_{-2,1,1} \left. \right) S_1 + \left(-\frac{64(1+3N+3N^2)}{N^3(1+N)^3} \right. \\
 & - 1536S_{-2,1} \left. \right) S_1^2 + \left(\frac{4P_{20}}{N^4(1+N)^4} + 512S_2 + 4352S_{-2,1} \right) S_2 - \frac{32(2+3N+3N^2)}{N(1+N)} S_2^2
 \end{aligned}$$

The non-singlet anomalous dimension $\gamma_{qq}^{-,NS}$



$$\begin{aligned}
 \gamma_{qq}^{-,NS} = & \frac{1}{2} [1 - (-1)^N] \\
 & \times \left\{ C_3^q \left[C_4 \left[\frac{16(-126 + 6N + 427N^2 + 770N^3 + 385N^4)}{9N^2(1+N)^2} [1 - S_1] + \frac{72C_3 P_{37}}{N^2(1+N)^2} \right. \right. \right. \\
 & + \frac{32P_{33}}{9N^2(1+N)^2} S_{-2,1} - \frac{16P_{37}}{9N^2(1+N)^2} + \frac{P_{37}}{18N^3(1+N)^3} + \left. \left. \left(-\frac{16P_{60}}{9N^4(1+N)^4} - \frac{4288}{9} S_2 \right. \right. \right. \\
 & + \frac{64(-12 + 31N + 31N^2)}{3N(1+N)} S_3 + 320S_4 - 1024S_{2,1} + \frac{64(-84 + 31N + 31N^2)}{3N(1+N)} S_{-2,1} \\
 & + 3712S_{-2,2} + 3840S_{-3,1} - 7168S_{-2,1,1} \left. \right\} S_1 + \left(256S_5 + 1792S_{-2,1} \right) S_1^2 + \left(\frac{4P_{33}}{9N^2(1+N)^2} \right. \\
 & - 832S_5 - 5248S_{-2,1} \left. \right) S_2 + \frac{352}{3} S_2^2 + \frac{16(-30 + 151N + 151N^2)}{3N(1+N)} S_4 + \left(-\frac{16P_{60}}{9N^3(1+N)^3} \right. \\
 & + \left. \left(-\frac{64P_{60}}{9N^2(1+N)^2} - 256S_2 \right) S_1 + \frac{32(12 + 31N + 31N^2)}{3N(1+N)} S_2 + 64S_5 + 5376S_{2,1} - 384S_{-2,1} \right. \\
 & + 576C_3 \left. \right\} S_{-2} + \left(-\frac{32(8 + 3N + 3N^2)}{N(1+N)} + 512S_1 \right) S_{-2}^2 + \left(\frac{32(108 + 31N + 31N^2)}{3N(1+N)} S_1 \right. \\
 & - \frac{16P_{64}}{9N^2(1+N)^2} - 1152S_1^2 + 2624S_2 + 960S_{-2} \left. \right\} S_{-3} + \left(\frac{16(138 + 35N + 35N^2)}{3N(1+N)} - 1472S_1 \right) \\
 & \times S_{-4} + 2304S_{-3} + 768S_{2,3} + 2688S_{2,-3} - \frac{64(-24 + 29N + 29N^2)}{3N(1+N)} S_{3,1} - 768S_{4,1} \\
 & + \frac{32(-174 + 31N + 31N^2)}{3N(1+N)} S_{-2,2} - 3648S_{-2,3} - \frac{1920S_{-3,1}}{N(1+N)} + 1728S_{-4,1} - 5376S_{2,1,-2} \\
 & + 1536S_{3,1,1} - \frac{128(-84 + 31N + 31N^2)}{3N(1+N)} S_{-2,1,1} - 1536S_{-2,1,-2} - 5376S_{-2,2,1} - 5376S_{-3,1,1} \\
 & + 10752S_{-2,1,1,1} \left. \right\} + T_F N_F \left[-\frac{16P_5}{9N^2(1+N)^2} S_2 + \frac{4P_{54}}{9N^4(1+N)^4} + \left(-\frac{8P_{34}}{9N^3(1+N)^3} + \frac{1280}{9} S_2 \right. \right. \\
 & - \frac{512}{3} S_3 - \frac{512}{3} S_{-2,1} + 128C_3 \left. \right\} S_1 - \frac{128}{3} S_2^2 + \frac{64(12 + 29N + 29N^2)}{9N(1+N)} S_5 - \frac{512}{3} S_4 \\
 & + \left(-\frac{128(-3 + 10N + 16N^2)}{9N^2(1+N)^2} + \frac{2560}{9} S_1 - \frac{256}{3} S_2 \right) S_{-2} + \left(\frac{128(3 + 10N + 10N^2)}{9N(1+N)} \right. \\
 & - \frac{256}{3} S_3 \left. \right\} S_{-3} - \frac{256}{3} S_{-4} + \frac{256}{3} S_{3,1} - \frac{256(-3 + 10N + 10N^2)}{9N(1+N)} S_{-2,1} - \frac{256}{3} S_{-2,2} \\
 & + \frac{1024}{3} S_{-2,1,1} - \frac{32(2 + 3N + 3N^2)}{N(1+N)} C_4 \left. \right\} + C_4 \left[T_F^2 N_F^2 \left[\frac{8P_{26}}{27N^3(1+N)^3} - \frac{128}{27} S_1 \right. \right. \\
 & - \frac{640}{27} S_2 + \frac{128}{9} S_3 \left. \right] + C_4^2 \left[-\frac{24P_{37}}{N^2(1+N)^2} C_3 - \frac{32P_{41}}{9N^3(1+N)^3} S_{-2,1} + \frac{8P_{34}}{9N^2(1+N)^2} S_3 \right. \\
 & + \frac{P_{30}}{54N^3(1+N)^3} + \left(\frac{4P_{35}}{3N^4(1+N)^4} - \frac{16(-8 + 11N + 11N^2)}{N(1+N)} S_1 - 256S_5 + 512S_{3,1} \right. \\
 & - \frac{64(-24 + 11N + 11N^2)}{3N(1+N)} S_{-2,1} - 1024S_{-2,2} - 1024S_{-3,1} + 2048S_{-2,1,1} \left. \right\} S_1 \\
 & + \left(-128S_5 - 512S_{-2,1} \right) S_1^2 + \left(-\frac{8344}{27} + 384S_5 + 1536S_{-2,1} \right) S_2 + 64S_5 \\
 & - \frac{16(-24 + 55N + 55N^2)}{3N(1+N)} S_4 + \left(\frac{32P_{30}}{9N^3(1+N)^3} S_1 + \frac{16P_{49}}{9N^2(1+N)^2} - \frac{352}{3} S_2 - 64S_5 \right. \\
 & - 1536S_{3,1} + 128S_{-2,1} - 192C_3 \left. \right\} S_{-2} + \left(\frac{48(2 + N + N^2)}{N(1+N)} - 192S_1 \right) S_{-2}^2 + \left(\frac{16P_{62}}{9N^2(1+N)^2} \right. \\
 & - \frac{32(24 + 11N + 11N^2)}{3N(1+N)} S_1 + 256S_2^2 - 768S_3 - 320S_{-2} \left. \right\} S_{-3} + \left(-\frac{16(30 + 13N + 13N^2)}{3N(1+N)} \right. \\
 & + 320S_1 \left. \right\} S_{-4} - 704S_{-3} - 384S_{2,3} - 768S_{2,-3} + \frac{64(-12 + 11N + 11N^2)}{3N(1+N)} S_{3,1} \\
 & + 384S_{4,1} - \frac{32(-48 + 11N + 11N^2)}{3N(1+N)} S_{-2,2} + 1088S_{-2,3} + \frac{512S_{-3,1}}{N(1+N)} - 448S_{-4,1} \\
 & + 1536S_{3,1,-2} - 768S_{3,1,1} + \frac{128(-24 + 11N + 11N^2)}{3N(1+N)} S_{-2,1,1} + 512S_{-2,1,-2} + 1536[S_{-2,2,1} \\
 & + S_{-3,1,1}] - 3072S_{-2,1,1,1} \left. \right\} + C_4 T_F N_F \left[-\frac{8P_{26}}{27N^4(1+N)^4} + \left(-\frac{16P_{62}}{27N^3(1+N)^3} + 64S_5 \right. \right. \\
 & + \frac{256}{3} S_{-2,1} - 128C_3 \left. \right\} S_1 + \frac{5344}{27} S_2 - \frac{32(3 + 14N + 14N^2)}{3N(1+N)} S_3 + \frac{320}{3} S_4 \\
 & + \left(\frac{64(-3 + 10N + 16N^2)}{9N^2(1+N)^2} - \frac{1280}{9} S_1 + \frac{128}{3} S_2 \right) S_{-2} + \left(-\frac{64(3 + 10N + 10N^2)}{9N(1+N)} \right. \\
 & + \frac{128}{3} S_3 \left. \right\} S_{-3} + \frac{128}{3} S_{-4} - \frac{256}{3} S_{3,1} + \frac{128(-3 + 10N + 10N^2)}{9N(1+N)} S_{-2,1} + \frac{128}{3} S_{-2,2} \\
 & - \frac{512}{3} S_{-2,1,1} + \frac{32(2 + 3N + 3N^2)}{N(1+N)} C_3 \left. \right\} + C_5^q \left[-\frac{48C_3 P_{37}}{N^2(1+N)^2} + \frac{8P_{38}}{N^2(1+N)^2} S_1 \right. \\
 & + \frac{P_{38}}{N^3(1+N)^3} + \left(\frac{8P_{38}}{N^4(1+N)^4} - \frac{128(1 + 2N)}{N^2(1+N)^2} S_2 + 128S_2^2 - 384S_5 + 128S_4 + 512S_{3,1} \right. \\
 & - \frac{384(-4 + N + N^2)}{N(1+N)} S_{-2,1} - 3328S_{-2,2} - 3584S_{-3,1} + 6144S_{-2,1,1} \left. \right\} S_1 \\
 & + \left(-\frac{64(1 + 3N + 3N^2)}{N^3(1+N)^3} - 1536S_{-2,1} \right) S_2^2 + \left(\frac{4P_{47}}{3N^2(1+N)^2} + 512S_5 + 4352S_{-2,1} \right) S_2 \\
 & + S_{-3,1,1} - 9216S_{-2,1,1,1} \left. \right\} \left. \right\}
 \end{aligned}$$

The non-singlet anomalous dimension for transversity

$\gamma_{qq}^{(2),+,NS,lr}$



$$\begin{aligned}
 \gamma_{NS}^{(2),+,NS} &= \frac{1}{2} [1 + (-1)^N] \\
 &\times \left\{ C_F \left[T_F^2 N_F^2 \left[\frac{8(-8+17N+17N^2)}{9N(1+N)} - \frac{128}{27} S_1 - \frac{640}{27} S_2 + \frac{128}{9} S_3 \right] \right. \right. \\
 &+ C_A T_F N_F \left[-\frac{16(-22+45N+45N^2)}{9N(1+N)} + \left(-\frac{16(9+209N+209N^2)}{27N(1+N)} + 64S_3 \right. \right. \\
 &+ \frac{256}{3} S_{-2,1} - 128C_3 \Big] S_1 + \frac{5344}{27} S_2 - \frac{448}{3} S_3 + \frac{320}{3} S_4 + \left(-\frac{1280}{9} S_1 + \frac{128}{3} S_2 \right) S_{-2} \\
 &+ \left(-\frac{640}{9} + \frac{128}{3} S_1 \right) S_{-3} + \frac{128}{3} S_{-4} - \frac{256}{3} S_{3,1} + \frac{1280}{9} S_{-2,1} + \frac{128}{3} S_{-2,2} - \frac{512}{3} S_{-2,1,1} \\
 &+ 96C_3 \Big] + C_A^2 \left[\frac{-968+1657N+1657N^2}{18N(1+N)} + \left(\frac{4P_{14}}{3(-1+N)N(1+N)(2+N)} - 176S_3 \right. \right. \\
 &- 256S_4 + 512S_{3,1} - \frac{704}{3} S_{-2,1} - 1024S_{-2,2} - 1024S_{-3,1} + 2048S_{-2,1,1} \Big] S_1 + \left(-128S_3 \right. \\
 &- 512S_{-2,1} \Big) S_1^2 + \left(-\frac{8344}{27} + 384S_3 + 1536S_{-2,1} \right) S_2 + \frac{3112}{9} S_3 - \frac{880}{3} S_4 + 64S_5 \\
 &+ \left(\frac{16P_2}{(-1+N)N(1+N)(2+N)} + \frac{32(-241+134N+134N^2)S_1}{9(-1+N)(2+N)} - \frac{352}{3} S_2 - 64S_3 \right. \\
 &- 1536S_{2,1} + 128S_{-2,1} - 192C_3 \Big] S_{-2} + \left(48 - 192S_1 \right) S_{-2}^2 + \left(256S_1^2 - 768S_2 - 320S_{-2} \right. \\
 &+ \frac{32(-107+67N+67N^2)}{9(-1+N)(2+N)} - \frac{352}{3} S_1 \Big) S_{-3} + \left(-\frac{208}{3} + 320S_1 \right) S_{-4} - 704S_{-5} - 384S_{2,3} \\
 &- 768S_{-2,3} + \frac{704}{3} S_{3,1} + 384S_{4,1} - \frac{64(-107+67N+67N^2)S_{-2,1}}{9(-1+N)(2+N)} - \frac{352}{3} S_{-2,2} \\
 &+ 1088S_{-2,3} - 448S_{-4,1} + 1536[S_{2,1,-2} + S_{-2,2,1} + S_{-3,1,1}] - 768S_{3,1,1} \\
 &+ \left. \frac{1408}{3} S_{-2,1,1} + 512S_{-2,1,-2} - 3072S_{-2,1,1,1} - \frac{24(-6+5N+5N^2)C_3}{(-1+N)(2+N)} \right\} \\
 &+ C_F^2 \left[T_F N_F \left[92 + \left(-\frac{8(-8+55N+55N^2)}{3N(1+N)} + \frac{1280}{9} S_2 - \frac{512}{3} S_3 - \frac{512}{3} S_{-2,1} + 128C_3 \right) \right. \right. \\
 &\times S_1 - \frac{80}{3} S_2 - \frac{128}{3} S_2^2 + \frac{1856}{9} S_3 - \frac{512}{3} S_4 + \left(\frac{2560}{9} S_1 - \frac{256}{3} S_2 \right) S_{-2} + \left(\frac{1280}{9} \right. \\
 &- \left. \frac{256}{3} S_1 \right) S_{-3} - \frac{256}{3} S_{-4} + \frac{256}{3} S_{3,1} - \frac{2560}{9} S_{-2,1} - \frac{256}{3} S_{-2,2} + \frac{1024}{3} S_{-2,1,1} - 96C_3 \Big] \\
 &+ C_A \left[-\frac{151}{2} + \left(-\frac{8(-206+211N+211N^2)}{3(-1+N)N(1+N)(2+N)} - \frac{4288}{9} S_2 + \frac{1984}{3} S_3 + 320S_4 - 1024S_{3,1} \right. \right.
 \end{aligned}$$

$$\begin{aligned}
 &+ \frac{1984}{3} S_{-2,1} + 3712S_{-2,2} + 3840S_{-3,1} - 7168S_{-2,1,1} \Big] S_1 + \left(256S_3 + 1792S_{-2,1} \right) S_1^2 \\
 &+ \left(\frac{604}{3} - 832S_3 - 5248S_{-2,1} \right) S_2 + \frac{352}{3} S_2^2 - \frac{6160}{9} S_3 + \frac{2416}{3} S_4 \\
 &+ \left(-\frac{48P_2}{(-1+N)N(1+N)(2+N)} + \left(-\frac{64P_2}{9(-1+N)N(1+N)(2+N)} - 256S_2 \right) S_1 \right. \\
 &+ \frac{992}{3} S_2 + 64S_3 + 5376S_{2,1} - 384S_{-2,1} + 576C_3 \Big) S_{-2} + \left(-96 + 512S_1 \right) S_{-2}^2 \\
 &+ \left(-\frac{32(-187+134N+134N^2)}{9(-1+N)(2+N)} + \frac{992}{3} S_1 - 1152S_1^2 + 2624S_2 + 960S_{-2} \right) S_{-3} \\
 &+ \left(\frac{560}{3} - 1472S_1 \right) S_{-4} + 2304S_{-5} + 768S_{2,3} + 2688S_{-2,-3} - \frac{1856}{3} S_{3,1} - 768S_{4,1} \\
 &+ \frac{64(-187+134N+134N^2)S_{-2,1}}{9(-1+N)(2+N)} + \frac{992}{3} S_{-2,2} - 3648S_{-2,3} + 1728S_{-4,1} \\
 &- 5376[S_{2,1,-2} + S_{-2,2,1} + S_{-3,1,1}] + 1536[S_{3,1,1} - S_{-2,1,-2}] - \frac{3968}{3} S_{-2,1,1} \\
 &+ 10752S_{-2,1,1,1} + \left. \frac{72(-6+5N+5N^2)C_3}{(-1+N)(2+N)} \right\} \\
 &+ C_F^3 \left\{ -29 + \left(\frac{384(-1+N+N^2)}{(-1+N)N(1+N)(2+N)} + 128S_2^2 - 384S_3 + 128S_4 + 512S_{3,1} \right. \right. \\
 &- 384S_{-2,1} - 3328S_{-2,2} - 3584S_{-3,1} + 6144S_{-2,1,1} \Big) S_1 - 256S_2^2 S_1 + \left(12 + 512S_3 \right. \\
 &+ 4352S_{-2,1} \Big) S_2 - 96S_2^2 + 104S_3 - 480S_4 + \left(\frac{32P_2}{(-1+N)N(1+N)(2+N)} \right. \\
 &+ \left(\frac{384}{(-1+N)N(1+N)(2+N)} + 512S_2 \right) S_1 - 192S_2 + 128S_3 - 4608S_{2,1} + 256S_{-2,1} \\
 &- 384C_3 \Big) S_{-2} + \left(\frac{192}{(-1+N)(2+N)} - 192S_1 + 1280S_1^2 - 2176S_2 - 640S_{-2} \right) S_{-3} \\
 &+ \left(-96 + 1664S_1 \right) S_{-4} - 1792S_{-5} + 384[-S_{2,3} + S_{3,1} + S_{4,1}] - 2304S_{2,-3} \\
 &- \frac{384S_{-2,1}}{(-1+N)(2+N)} - 1536S_1^2 S_{-2,1} - 192S_{-2,2} + 2944S_{-2,3} - 1664S_{-4,1} + 4608S_{2,1,-2} \\
 &- 768S_{3,1,1} + 768S_{-2,1,1} + 1024S_{-2,1,-2} + 4608[S_{-2,2,1} + S_{-3,1,1}] - 9216S_{-2,1,1,1} \\
 &- \left. \frac{48(-6+5N+5N^2)C_3}{(-1+N)(2+N)} \right\}.
 \end{aligned}$$

The non-singlet anomalous dimension for transversity

$\gamma_{qq}^{(2),-,\text{NS},\text{tr}}$



$$\begin{aligned}
 & \gamma_{\text{NS}}^{(2),\text{tr},-} = \frac{1}{2} [1 - (-1)^N] \left\{ C_F \left\{ T_F^2 N_F^2 \left[\frac{8(-8+17N+17N^2)}{9N(1+N)} - \frac{128}{27} S_1 - \frac{640}{27} S_2 + \frac{128}{9} S_3 \right] \right. \right. \\
 & + C_A T_F N_F \left[-\frac{16(-8+49N+90N^2+45N^3)}{9N(1+N)^2} + \left(-\frac{16(-27+209N+209N^2)}{27N(1+N)} \right. \right. \\
 & + 64S_3 + \frac{256}{3} S_{-2,1} - 128\zeta_3 \Big) S_1 + \frac{5344}{27} S_2 - \frac{448}{3} S_3 + \frac{320}{3} S_4 + \left(-\frac{1280}{9} S_1 + \frac{128}{3} S_2 \right) S_{-2} \\
 & + \left(-\frac{640}{9} + \frac{128}{3} S_1 \right) S_{-3} + \frac{128}{3} S_{-4} - \frac{256}{3} S_{3,1} + \frac{1280}{9} S_{-2,1} + \frac{128}{3} S_{-2,2} - \frac{512}{3} S_{-2,1,1} \\
 & + 96\zeta_3 \Big] + C_A^2 \left[\frac{P_{23}}{18(-1+N)N(1+N)^2(2+N)} + \left(\frac{4(12+245N+245N^2)}{3N(1+N)} - 176S_3 \right. \right. \\
 & - 256S_4 + 512S_{3,1} - \frac{704}{3} S_{-2,1} - 1024S_{-2,2} - 1024S_{-3,1} + 2048S_{-2,1,1} \Big) S_1 + \left(-128S_3 \right. \\
 & - 512S_{-2,1} \Big) S_1^2 + \left(-\frac{8344}{27} + 384S_3 + 1536S_{-2,1} \right) S_2 + \frac{3112}{9} S_3 - \frac{880}{3} S_4 + 64S_5 \\
 & + \left(\frac{16(-5+3N+3N^2)}{(-1+N)(2+N)} + \frac{32(81+134N+134N^2)}{9N(1+N)} S_1 - \frac{352}{3} S_2 - 64S_3 - 1536S_{2,1} \right. \\
 & + 128S_{-2,1} - 192\zeta_3 \Big) S_{-2} + \left(48 - 192S_1 \right) S_{-2}^2 + \left(\frac{32(81+67N+67N^2)}{9N(1+N)} - \frac{352}{3} S_1 \right. \\
 & + 256S_{-2,1}^2 - 768S_2 - 320S_{-2} \Big) S_{-3} + \left(-\frac{208}{3} + 320S_1 \right) S_{-4} - 704S_{-5} - 384S_{2,3} - 768S_{2,-3} \\
 & + \frac{704}{3} S_{3,1} + 384S_{4,1} - \frac{64(81+67N+67N^2)S_{-2,1}}{9N(1+N)} - \frac{352}{3} S_{-2,2} + 1088S_{-2,3} - 448S_{-4,1} \\
 & + 1536S_{2,-2} - 768S_{3,1,1} + \frac{1408}{3} S_{-2,1,1} + 512S_{-2,1,-2} + 1536[S_{-2,2,1} + S_{-3,1,1}] \\
 & \left. - 3072S_{-2,1,1,1} - \frac{24(12+5N+5N^2)\zeta_3}{N(1+N)} \right\} \\
 & + C_F^2 \left\{ T_F N_F \left[\frac{4(112+415N+414N^2+207N^3)}{9N(1+N)^2} + \left(-\frac{8(8+55N+55N^2)}{3N(1+N)} + \frac{1280}{9} S_2 \right. \right. \right. \\
 & - \frac{512}{3} S_3 - \frac{512}{3} S_{-2,1} + 128\zeta_3 \Big) S_1 - \frac{80}{3} S_2 - \frac{128}{3} S_3 + \frac{1856}{9} S_4 - \frac{512}{3} S_4 + \left(\frac{2560}{9} S_1 \right. \\
 & - \frac{256}{3} S_2 \Big) S_{-2} + \left(\frac{1280}{9} - \frac{256}{3} S_1 \right) S_{-3} - \frac{256}{3} S_{-4} + \frac{256}{3} S_{3,1} - \frac{2560}{9} S_{-2,1} - \frac{256}{3} S_{-2,2} \\
 & + \frac{1024}{3} S_{-2,1,1} - 96\zeta_3 \Big] + C_A \left[\frac{P_{20}}{18(-1+N)N(1+N)^2(2+N)} \right. \\
 & + \left(-\frac{8P_6}{3(-1+N)N^2(1+N)^2(2+N)} - \frac{4288}{9} S_2 + \frac{1984}{3} S_3 + 320S_4 - 1024S_{3,1} \right. \\
 & + \frac{1984}{3} S_{-2,1} + 3712S_{-2,2} + 3840S_{-3,1} - 7168S_{-2,1,1} \Big) S_1 + \left(256S_3 + 1792S_{-2,1} \right) S_1^2 \\
 & + \left(\frac{4(-24+151N+151N^2)}{3N(1+N)} - 832S_3 - 5248S_{-2,1} \right) S_2 + \frac{352}{3} S_2^2 - \frac{6160}{9} S_3 + \frac{2416}{3} S_4 \\
 & + \left(-\frac{48(-5+3N+3N^2)}{(-1+N)(2+N)} + \left(-\frac{64P_8}{9(-1+N)N(1+N)(2+N)} - 256S_2 \right) S_1 + \frac{992}{3} S_2 \right. \\
 & + 64S_3 + 5376S_{2,1} - 384S_{-2,1} + 576\zeta_3 \Big) S_{-2} + \left(-96 + 512S_1 \right) S_{-2}^2 \\
 & + \left(-\frac{32(243+134N+134N^2)}{9N(1+N)} + \frac{992}{3} S_1 - 1152S_1^2 + 2624S_2 + 960S_{-2} \right) S_{-3} + \left(\frac{560}{3} \right. \\
 & - 1472S_1 \Big) S_{-4} + 2304S_{-5} + 768S_{2,3} + 2688S_{2,-3} - \frac{1856}{3} S_{3,1} - 768S_{4,1} \\
 & + \frac{64(243+134N+134N^2)}{9N(1+N)} S_{-2,1} + \frac{992}{3} S_{-2,2} - 3648S_{-2,3} + 1728S_{-4,1} - 5376S_{2,1,-2} \\
 & - 1536[S_{-2,1,-2} - S_{3,1,1}] - 5376[S_{-2,2,1} + S_{-3,1,1}] - \frac{3968}{3} S_{-2,1,1} + 10752S_{-2,1,1,1} \\
 & \left. + \frac{72(12+5N+5N^2)\zeta_3}{N(1+N)} \right\} + C_F^3 \left\{ \frac{P_{31}}{(-1+N)N(1+N)^2(2+N)} \right. \\
 & + \left(-\frac{32P_1}{(-1+N)N^2(1+N)^2(2+N)} + 128S_2^2 - 384[S_3 + S_{-2,1}] + 128S_4 + 512S_{3,1} \right. \\
 & - 3328S_{-2,2} - 3584S_{-3,1} + 6144S_{-2,1,1} \Big) S_1 - 256S_{-2,1}^2 S_2 + \left(\frac{4(16+3N+3N^2)}{N(1+N)} \right. \\
 & + 512S_3 + 4352S_{-2,1} \Big) S_2 - 96S_2^2 + 104S_3 - 480S_4 + \left(\frac{32(-5+3N+3N^2)}{(-1+N)(2+N)} \right. \\
 & + \left(\frac{128(-9+4N+4N^2)}{(-1+N)N(1+N)(2+N)} + 512S_2 \right) S_1 - 192S_2 + 128S_3 - 4608S_{2,1} \\
 & + 256S_{-2,1} - 384\zeta_3 \Big) S_{-2} + \left(\frac{576}{N(1+N)} - 192S_1 + 1280S_1^2 - 2176S_2 - 640S_{-2} \right) S_{-3} \\
 & + \left(-96 + 1664S_1 \right) S_{-4} - 1792S_{-5} - 384[S_{2,3} - S_{3,1} - S_{4,1}] - 2304S_{2,-3} - \frac{1152}{N(1+N)} S_{-2,1} \\
 & - 1536S_{-2,1}^2 S_{-2} - 192S_{-2,2} + 2944S_{-2,3} - 1664S_{-4,1} + 4608S_{2,1,-2} - 768[S_{3,1,1} - S_{-2,1,1}] \\
 & + 1024S_{-2,1,-2} + 4608[S_{-2,2,1} + S_{-3,1,1} - S_{-2,1,1,1}] - \frac{48(12+5N+5N^2)\zeta_3}{N(1+N)} \Big\}
 \end{aligned}$$

The polarized singlet anomalous dimension $\Delta\gamma_{qq}^{(2),PS}$



$$\begin{aligned}
 \Delta\gamma_{qq}^{(2),PS} = & C_F \left[T_F^2 N_F^2 \left[-\frac{64(2+N)P_{30}}{27N^4(1+N)^4} + \frac{64(2+N)(6+10N-3N^2+11N^3)}{9N^3(1+N)^3} S_1 \right. \right. \\
 & \left. \left. - \frac{32(N-1)(2+N)}{3N^2(1+N)^2} [S_1^2 + S_2] \right] + C_A T_F N_F \left[\frac{8P_9}{3N^3(1+N)^3} S_1^2 + \frac{8P_{10}}{3N^3(1+N)^3} S_2 \right. \right. \\
 & \left. \left. + \frac{16P_{61}}{27N^5(1+N)^5} + \left(-\frac{16P_{51}}{9N^4(1+N)^4} + \frac{32(N-1)(2+N)}{N^2(1+N)^2} S_2 \right) S_1 \right. \right. \\
 & \left. \left. - \frac{32(N-1)(2+N)}{3N^2(1+N)^2} S_1^3 + \frac{16(-58+23N+23N^2)}{3N^2(1+N)^2} S_3 + \left(-\frac{32P_1}{N^3(1+N)^3} \right. \right. \right. \\
 & \left. \left. \left. + \frac{64(N-1)(2+N)}{N^2(1+N)^2} S_1 \right) S_{-2} + \frac{32(-10+7N+7N^2)}{N^2(1+N)^2} S_{-3} - \frac{64(N-1)(2+N)}{N^2(1+N)^2} S_{2,1} \right. \right. \\
 & \left. \left. - \frac{64(-2+3N+3N^2)}{N^2(1+N)^2} S_{-2,1} - \frac{192(N-1)(2+N)}{N^2(1+N)^2} \zeta_3 \right] \right] \\
 & + C_F^2 T_F N_F \left[-\frac{16(2+N)P_{54}}{N^5(1+N)^5} + \left(\frac{16(2+N)P_{27}}{N^4(1+N)^4} - \frac{32(N-1)(2+N)}{N^2(1+N)^2} S_2 \right) S_1 \right. \\
 & \left. - \frac{8(N-1)(2+N)(2+3N+3N^2)}{N^3(1+N)^3} S_1^2 + \frac{32(N-1)(2+N)}{3N^2(1+N)^2} S_1^3 \right. \\
 & \left. - \frac{8(2+N)(14+23N+11N^3)}{N^3(1+N)^3} S_2 - \frac{224(N-1)(2+N)}{3N^2(1+N)^2} S_3 \right. \\
 & \left. \left. + \frac{64(N-1)(2+N)}{N^2(1+N)^2} S_{2,1} + \frac{192(N-1)(2+N)}{N^2(1+N)^2} \zeta_3 \right]
 \end{aligned}$$

The polynomials in the non-singlet cases are different from those in the polarized singlet case.

The polarized singlet anomalous dimension $\Delta\gamma_{qg}^{(2)}$



$$\begin{aligned}
 \Delta\gamma_{qg}^{(2)} = & C_F T_F^2 N_F^2 \left[\frac{4P_{44}}{27N^3(1+N)^5} + \left(-\frac{32(-24+4N+47N^2)}{27N^2(1+N)} - \frac{32(N-1)}{3N(1+N)} S_2 \right) S_1 \right. \\
 & + \left. \frac{32(N-1)(3+10N)}{9N^2(1+N)} S_1^2 - \frac{32(N-1)}{9N(1+N)} S_1^3 + \frac{32(5N-1)}{3N^2(1+N)} S_2 + \frac{320(N-1)}{9N(1+N)} S_3 \right] \\
 & + C_A C_F T_F N_F \left[\frac{8P_{20}}{3N^3(1+N)^3} S_2 + \frac{P_{60}}{27N^3(1+N)^3(2+N)} + \left(-\frac{384(N-1)}{N(1+N)} S_{2,1} \right. \right. \\
 & + \frac{16P_{30}}{27N^3(1+N)^3(2+N)} + \frac{16(75+14N+18N^2+N^3)S_2}{3N^2(1+N)^2} + \frac{640(N-1)}{3N(1+N)} S_3 \\
 & - \left. \frac{192(N-1)}{N(1+N)} G_3 \right) S_1 + \left(-\frac{8P_{20}}{9N^3(1+N)^3} + \frac{160(N-1)}{N(1+N)} S_2 \right) S_1^2 \\
 & + \frac{16(3-31N-18N^2+10N^3)}{9N^2(1+N)^2} S_1^3 + \frac{32(N-1)}{3N(1+N)} S_1^4 - \frac{64(N-1)}{N(1+N)} S_2^2 \\
 & - \frac{16(N-1)(240-17N+19N^2)}{9N^2(1+N)^2} S_3 + \left(\frac{128(N-1)(-4-N+N^2)}{N^2(1+N)^2(2+N)} S_1 \right. \\
 & - \frac{32P_{34}}{N^3(1+N)^3(2+N)} + \frac{192(N-1)}{N(1+N)} S_1^2 \left. \right) S_{-2} + \frac{96(N-1)}{N(1+N)} S_{-2}^2 \\
 & + \frac{32(N-1)(2+N)(-1+3N)}{N^2(1+N)^2} S_{-3} + \frac{96(N-1)(4+N+N^2)}{N^2(1+N)^2} S_{2,1} \\
 & + \frac{160(N-1)}{N(1+N)} S_{-4} + \frac{64(N-1)}{N(1+N)} S_{3,1} - \frac{128(N-1)^2}{N^2(1+N)^2} S_{-2,1} + \frac{64(N-1)}{N(1+N)} S_{-2,2} \\
 & + \left. \frac{192(N-1)}{N(1+N)} S_{2,1,1} - \frac{256(N-1)}{N(1+N)} S_{-2,1,1} - \frac{192(N-1)(-5+3N+3N^2)}{N^2(1+N)^2} G_3 \right] \\
 & + C_A T_F^2 N_F \left[\frac{16P_{35}}{27N^4(1+N)^4} + \left(\frac{64(23+50N+10N^2+19N^3)}{27N(1+N)^3} - \frac{32(N-1)}{3N(1+N)} \right. \right. \\
 & \times S_2 \left. \right) S_1 - \frac{64(-2+5N^2)}{9N(1+N)^2} S_1^2 + \frac{32(N-1)}{9N(1+N)} S_1^3 - \frac{64(-2+6N+5N^2)}{9N(1+N)^2} S_2 \\
 & + \left. \frac{64(N-1)}{9N(1+N)} S_3 - \frac{128(-2+5N)}{9N(1+N)} S_{-2} + \frac{128(N-1)}{3N(1+N)} [S_{-3} + S_{2,1}] \right] \\
 & + C_A^2 N_F T_F \left[\frac{16P_{34}}{9N^3(1+N)^3} S_2 - \frac{8P_{70}}{27N^3(1+N)^3(2+N)} + \left(-\frac{8P_{30}}{27N^4(1+N)^4} \right. \right. \\
 & + \frac{8(-72+181N-48N^2+11N^3)}{3N^2(1+N)^2} S_2 - \frac{704(N-1)}{3N(1+N)} S_3 + \frac{128(N-1)}{N(1+N)} S_{2,1} \\
 & + \left. \frac{512(N-1)}{N(1+N)} S_{-2,1} + \frac{192(N-1)}{N(1+N)} G_3 \right) S_1 + \left(\frac{16P_{31}}{9N^3(1+N)^3} - \frac{160(N-1)}{N(1+N)} S_2 \right) S_1^2 \\
 & - \frac{8(-24-59N+11N^3)}{9N^2(1+N)^2} S_1^3 - \frac{16(N-1)}{3N(1+N)} S_1^4 - \frac{16(N-1)}{N(1+N)} S_2^2 \\
 & - \frac{16(345-428N+11N^3)}{9N^2(1+N)^2} S_3 - \frac{32(N-1)}{N(1+N)} S_4 + \left(\frac{32P_{60}}{9N^3(1+N)^3(2+N)} \right. \\
 & - \frac{64(-5+N)(-1+2N)}{N^2(1+N)^2} S_1 - \frac{192(N-1)}{N(1+N)} S_1^2 - \frac{128(N-1)}{N(1+N)} S_2 \\
 & - \frac{96(N-1)}{N(1+N)} S_{-2}^2 + \left(-\frac{32(69-92N+11N^3)}{3N^2(1+N)^2} - \frac{512(N-1)}{N(1+N)} S_1 \right) S_{-3} \\
 & - \frac{352(N-1)}{N(1+N)} S_{-4} - \frac{32(N-1)(24+11N+11N^2)}{3N^2(1+N)^2} S_{2,1} - \frac{128(N-1)}{N(1+N)} S_{3,1} \\
 & - \frac{64(-7+11N)}{N^2(1+N)^2} S_{-2,1} + \frac{448(-1+N)}{N(1+N)} S_{-2,2} + \frac{512(-1+N)}{N(1+N)} S_{-3,1} \\
 & - \left. \frac{768(-1+N)}{N(1+N)} S_{-2,1,1} + \frac{96(-1+N)(-8+3N+3N^2)}{N^2(1+N)^2} G_3 \right] \\
 & + C_F^2 T_F N_F \left[-\frac{8P_{20}}{N^3(1+N)^3} S_1^2 + \frac{8P_{20}}{N^3(1+N)^3} S_2 + \frac{P_{30}}{N^4(1+N)^3(2+N)} \right. \\
 & + \left(-\frac{8P_{33}}{N^4(1+N)^4} - \frac{8(-6+7N+28N^2+3N^3)}{N^2(1+N)^2} S_2 - \frac{704(N-1)}{3N(1+N)} S_3 \right. \\
 & + \left. \frac{256(N-1)}{N(1+N)} S_{2,1} \right) S_1 - \frac{8(N-1)(-10-9N+3N^2)}{3N^2(1+N)^2} S_1^2 - \frac{16(N-1)}{3N(1+N)} S_1^3 \\
 & + \frac{48(N-1)}{N(1+N)} S_{-2}^2 - \frac{16(N-1)(-22+27N+3N^2)}{3N^2(1+N)^2} S_3 - \frac{160(N-1)}{N(1+N)} S_4 \\
 & + \left(\frac{64P_{21}}{N^2(1+N)^2(2+N)} - \frac{256(N-1)}{N(1+N)^2} S_1 - \frac{128(N-1)}{N(1+N)} S_2 \right) S_{-2} - \frac{64(N-1)}{N(1+N)} S_{-2}^2 \\
 & + \left(-\frac{128(N-1)^2}{N^2(1+N)^2} - \frac{256(N-1)}{N(1+N)} S_1 \right) S_{-3} - \frac{320(N-1)}{N(1+N)} S_{-4} - \frac{128(N-1)}{N^2(1+N)^2} S_{2,1} \\
 & + \frac{64(N-1)}{N(1+N)} S_{3,1} + \frac{256(N-1)}{N(1+N)^2} S_{-2,1} + \frac{128(N-1)}{N(1+N)} S_{-2,2} + \frac{256(N-1)}{N(1+N)} S_{-3,1} \\
 & - \left. \frac{192(N-1)}{N(1+N)} S_{2,1,1} + \frac{96(N-1)(-2+3N+3N^2)}{N^2(1+N)^2} G_3 \right]
 \end{aligned}$$

The polarized singlet anomalous dimension $\Delta\gamma_{gg}^{(2)}$



$$\Delta\gamma_{gg}^{(2)} = C_F^2 \left[C_A \left[\frac{4P_{30}}{9N^4(1+N)^2} S_2 + \frac{P_{23}}{54(N-1)N^3(1+N)^5} + \left(-\frac{4P_{36}}{27N^4(1+N)^4} \right. \right. \right. \\ \left. \left. \left. - \frac{8(30+203N+177N^2+49N^3)}{3N^2(1+N)^2} S_2 - \frac{640(2+N)}{3N(1+N)} S_3 + \frac{64(2+N)}{N(1+N)} S_{2,1} \right. \right. \right. \\ \left. \left. \left. + \frac{128(2+N)}{N(1+N)} S_{-2,1} + \frac{288(2+N)}{N(1+N)} C_3 \right) S_1 + \left(\frac{4P_{36}}{9N^3(1+N)^3} - \frac{16(2+N)}{N(1+N)} \right) S_2 \right] S_1^2 \\ - \frac{8(6+85N+132N^2+50N^3)}{9N^2(1+N)^2} S_1^3 + \frac{8(2+N)(102+65N+29N^2)}{9N^2(1+N)^2} S_3 \\ + \frac{16(2+N)}{3N(1+N)} S_1^4 - \frac{32(2+N)}{N(1+N)} S_4 + \left(-\frac{16P_{34}}{(N-1)N^3(1+N)^3} + \frac{64(7+3N)}{N(1+N)} \right) S_1 \\ - \frac{96(2+N)}{N(1+N)} S_1^2 S_2 + \left(\frac{32(2+N)(4+3N)}{N(1+N)^2} - \frac{64(2+N)}{N(1+N)} \right) S_1 S_3 \\ + \frac{80(2+N)}{N(1+N)} [S_{-2}^2 + S_{-4}] + \frac{16(2+N)(-6+11N+11N^2)}{3N^2(1+N)^2} S_{2,1} + \frac{224(2+N)}{N(1+N)} S_{3,1} \\ - \frac{32(2+N)(5+3N)}{N(1+N)^2} S_{-2,1} + \frac{32(2+N)}{N(1+N)} S_{-2,2} - \frac{96(2+N)}{N(1+N)} S_{2,1,1} \\ - \frac{128(2+N)}{N(1+N)} S_{-2,1,1} - \frac{432(2+N)C_3}{N(1+N)} + T_F T_F \left[\frac{2P_{71}}{27(N-1)N^3(1+N)^3} \right. \\ \left. + \left(\frac{32(2+N)P_{14}}{27N^3(1+N)^3} + \frac{208(2+N)}{3N(1+N)} \right) S_1 - \frac{16(2+N)(-3+16N+37N^2)}{9N^2(1+N)^2} S_1^2 \right. \\ \left. + \frac{80(2+N)}{9N(1+N)} S_1^3 - \frac{16(2+N)(9+46N+67N^2)}{9N^2(1+N)^2} S_2 + \frac{256(2+N)}{9N(1+N)} S_3 \right. \\ \left. + \frac{256}{(N-1)N^2(1+N)^2} S_{-2} - \frac{64(2+N)}{3N(1+N)} S_{2,1} - \frac{128(2+N)}{N(1+N)} C_3 \right] \\ + C_F \left[T_F^2 N_F^2 \left[\frac{64(2+N)(3+7N+N^2)}{9N(1+N)^3} + \frac{64(2+N)(2+5N)}{9N(1+N)^2} S_1 \right. \right. \\ \left. \left. - \frac{32(2+N)}{3N(1+N)} [S_2^2 + S_3] + C_4 T_F N_F \left[\frac{8P_{71}}{27(N-1)N^3(1+N)^4} \right. \right. \right. \\ \left. \left. + \left(-\frac{16P_{37}}{27N^3(1+N)^3} + \frac{80(2+N)S_2}{3N(1+N)} \right) S_1 + \frac{16(18+116N+129N^2+43N^3)}{9N^2(1+N)^2} S_1^2 \right. \right. \\ \left. \left. - \frac{80(2+N)}{9N(1+N)} S_1^3 + \frac{16(-2+16N+9N^2+N^3)}{3N^2(1+N)^2} S_2 + \frac{512(2+N)}{9N(1+N)} S_3 \right. \right. \\ \left. \left. + \left(-\frac{64P_7}{3(-1+N)N^2(1+N)^2} + \frac{256(2+N)S_1}{3N(1+N)} \right) S_{-2} + \frac{128(2+N)}{3N(1+N)} [S_{-3} - S_{-2,1}] \right. \right. \\ \left. \left. + \frac{128(2+N)}{N(1+N)} C_3 \right] + C_A^2 \left[\frac{2P_{35}}{3N^3(1+N)^3} S_2 - \frac{4P_{72}}{27(N-1)N^3(1+N)^5} \right. \right. \\ \left. \left. + \left(\frac{4P_{32}}{27(N-1)N^4(1+N)^4} - \frac{4(120+158N+141N^2+55N^3)}{3N^2(1+N)^2} S_2 + \frac{128(2+N)}{3N(1+N)} S_3 \right. \right. \\ \left. \left. + \frac{128(2+N)}{N(1+N)} S_{-2,1} - \frac{96(2+N)}{N(1+N)} C_3 \right) S_1 + \left(-\frac{2P_{36}}{9N^3(1+N)^3} + \frac{48(2+N)}{N(1+N)} \right) S_2 \right] S_1^2 \\ + \frac{4(24+158N+165N^2+55N^3)}{9N^2(1+N)^2} S_1^3 - \frac{8(2+N)}{3N(1+N)} S_1^4 - \frac{40(2+N)}{N(1+N)} S_2^2 \\ - \frac{8(-186+295N+528N^2+176N^3)}{9N^2(1+N)^2} S_3 - \frac{16(2+N)}{N(1+N)} S_4 \\ + \left(-\frac{32P_{32}}{3(N-1)N^2(1+N)^2} S_1 + \frac{16P_{71}}{3(N-1)N^3(1+N)^3} + \frac{96(2+N)}{N(1+N)} \right) S_1^2 \\ - \frac{64(2+N)}{N(1+N)} S_2 S_1 - \frac{16(2+N)}{N(1+N)} S_{-2}^2 + \left(-\frac{16(-126-13N+66N^2+22N^3)}{3N^2(1+N)^2} \right. \\ \left. - \frac{192(2+N)}{N(1+N)} S_1 \right) S_{-3} - \frac{176(2+N)}{N(1+N)} S_{-4} + \frac{32(-30+13N+33N^2+11N^3)}{3N^2(1+N)^2} S_{-2,1} \\ - \frac{64(2+N)}{N(1+N)} S_{3,1} + \frac{224(2+N)}{N(1+N)} S_{-2,2} + \frac{256(2+N)}{N(1+N)} S_{-3,1} - \frac{384(2+N)}{N(1+N)} S_{-2,1,1} \\ + \frac{144(2+N)}{N(1+N)} C_3 \left. \right] + C_F^2 \left[-\frac{2(2+N)P_{16}}{N^3(1+N)^3} S_2 + \frac{P_{66}}{2(N-1)N^3(1+N)^5} \right. \\ \left. + \left(-\frac{4(2+N)P_{43}}{N^4(1+N)^4} + \frac{4(2+N)(-2+19N+39N^2)}{N^2(1+N)^2} S_2 + \frac{128(2+N)}{3N(1+N)} S_3 \right. \right. \\ \left. \left. - \frac{64(2+N)}{N(1+N)} S_{2,1} - \frac{192(2+N)}{N(1+N)} C_3 \right) S_1 + \left(\frac{2(2+N)P_2}{N^3(1+N)^3} - \frac{32(2+N)}{N(1+N)} \right) S_2 \right] S_1^2 \\ + \frac{4(2+N)(-2+3N+15N^2)}{3N^2(1+N)^2} S_1^3 - \frac{8(2+N)}{3N(1+N)} S_1^4 - \frac{24(2+N)}{N(1+N)} S_2^2 \\ + \frac{64(2+N)(-1+3N^2)}{3N^2(1+N)^2} S_3 - \frac{48(2+N)}{N(1+N)} S_4 + \left(\frac{32P_{22}}{(N-1)N^2(1+N)^3} \right. \\ \left. - \frac{128(2+N)}{N(1+N)^2} S_1 - \frac{64(2+N)}{N(1+N)} S_2 \right) S_{-2} - \frac{16(2+N)(-2+3N+3N^2)}{N^2(1+N)^2} S_{3,1} \\ - \frac{96(2+N)}{N(1+N)} S_{-2}^2 + \left(-\frac{64(N-1)(2+N)}{N^2(1+N)^2} - \frac{128(2+N)}{N(1+N)} \right) S_1 S_{-3} \\ - \frac{160(2+N)}{N(1+N)} [S_{-4} + S_{3,1}] + \frac{128(2+N)}{N(1+N)^2} [S_{-2,1} + (1+N)S_{-3,1}] \\ + \frac{64(2+N)}{N(1+N)} S_{-2,2} + \frac{96(2+N)}{N(1+N)} S_{2,1,1} + \frac{288(2+N)C_3}{N(1+N)} \left. \right] \\ \left. \right]$$

The polarized singlet anomalous dimension $\Delta\gamma_{gg}^{(2)}$



$$\begin{aligned} \Delta\gamma_{gg}^{(2)} = & C_A T_F^2 N_F^2 \left[-\frac{16P_{38}}{27N^2(1+N)^2} S_1 - \frac{4P_{38}}{27N^3(1+N)^3} \right] + C_F \left[T_F^2 N_F^2 \left[-\frac{8P_{39}}{27N^4(1+N)^4} \right. \right. \\ & + \frac{64(N-1)(2+N)(-6-8N+N^2)}{9N^3(1+N)^3} S_1 + \frac{32(N-1)(2+N)}{3N^2(1+N)^2} S_1^2 \\ & - \frac{32(N-1)(2+N)}{N^2(1+N)^2} S_2 \left. \right] + C_A T_F N_F \left[\frac{8P_6}{N^3(1+N)^3} S_2 - \frac{8P_5}{3N^3(1+N)^3} S_1^2 \right. \\ & + \frac{2P_{77}}{27(N-1)N^3(1+N)^3(2+N)} + \left(-\frac{8P_{67}}{9(-1+N)N^4(1+N)^4(2+N)} \right. \\ & - \frac{32(N-1)(2+N)}{N^2(1+N)^2} S_2 + 128C_3 \left. \right) S_1 + \frac{32(N-1)(2+N)}{3N^2(1+N)^2} S_1^2 - \frac{32(34+N+N^2)}{3N^2(1+N)^2} \\ & \times S_3 + \left(\frac{128P_2}{(N-1)N^2(1+N)^2(2+N)} S_1 - \frac{32P_{25}}{(N-1)N^2(1+N)^3(2+N)} \right) S_2 \\ & - \frac{192(4-N-N^2)}{N^2(1+N)^2} S_3 + \frac{64(N-1)(2+N)}{N^2(1+N)^2} S_{2,1} - \frac{128(-8+N+N^2)}{N^2(1+N)^2} S_{-2,1} \\ & - \left. \frac{64(-3+N)(4+N)C_3}{N^2(1+N)^2} \right] + C_A^3 \left[\frac{64P_{36}}{9N^2(1+N)^2} S_{-2,1} - \frac{32P_{18}}{9N^2(1+N)^2} S_3 \right. \\ & + \frac{P_{74}}{27(N-1)N^3(1+N)^3(2+N)} + \left(\frac{4P_{109}}{9(N-1)N^4(1+N)^4(2+N)} \right. \\ & - \frac{64P_{17}}{9N^2(1+N)^2} S_2 + 128S_1^2 + \frac{16(-96+11N+11N^2)}{3N(1+N)} S_3 + 192S_4 \\ & + \frac{1024}{N(1+N)} S_{-2,1} - 640S_{-2,2} - 768S_{-3,1} + 1024S_{-2,1,1} \left. \right) S_1 \\ & + \left(-\frac{256(1+3N+3N^2)}{N^3(1+N)^3} + 128S_3 - 256S_{-2,1} \right) S_1^2 + \left(-\frac{16P_{41}}{9N^3(1+N)^3} \right. \\ & + 64S_3 + 640S_{-2,1} \left. \right) S_2 - \frac{256}{N(1+N)} S_2^2 - \frac{384}{N(1+N)} S_4 + 64S_5 \\ & + \left(\frac{32P_{32}}{9(N-1)N^3(1+N)^3(2+N)} + \left(-\frac{64P_{32}}{9(-1+N)N(1+N)^2(2+N)} + 256S_2 \right) \right. \\ & \times S_1 - \frac{512}{N(1+N)} S_2 + 128S_3 - 768S_{2,1} \left. \right) S_{-2} + \left(-\frac{16(24+11N+11N^2)}{3N(1+N)} \right. \\ & + 64S_1 \left. \right) S_{-2}^2 + \left(-\frac{32P_{15}}{9N^2(1+N)^2} - \frac{1536}{N(1+N)} S_1 + 384S_1^2 - 320S_2 \right) S_{-3} \\ & + \left(-\frac{1024}{N(1+N)} + 512S_1 \right) S_{-4} - 192S_{-3} - 384S_{-3,3} + \frac{1280}{N(1+N)} S_{-2,2} \\ & + 384S_{-2,3} + \frac{1536}{N(1+N)} S_{-3,1} - 384S_{-4,1} + 768S_{2,1,-2} - \frac{2048}{N(1+N)} S_{-2,1,1} \end{aligned}$$

$$\begin{aligned} & + 768[S_{-2,2,1} + S_{-3,1,1}] - 1536S_{-2,1,1} \left. \right] \\ & + C_F^2 T_F N_F \left[-\frac{4P_{75}}{(N-1)N^3(1+N)^3(2+N)} + \left(\frac{32(N-1)(2+N)S_2}{N^2(1+N)^2} \right. \right. \\ & - \frac{16P_{20}}{N^4(1+N)^4} \left. \right) S_1 + \frac{8(N-1)(2+N)(2+3N+3N^2)}{N^3(1+N)^3} S_1^2 - \frac{32(N-1)(2+N)}{3N^2(1+N)^2} \\ & \times S_1^3 - \frac{8(2+N)(2-11N-16N^2+9N^3)}{N^3(1+N)^3} S_2 + \frac{32(10+7N+7N^2)}{3N^2(1+N)^2} S_3 \\ & + \left(-\frac{64(10+N+N^2)}{(N-1)N(1+N)(2+N)} + \frac{512}{N^2(1+N)^2} S_1 \right) S_{-2} + \frac{256}{N^2(1+N)^2} S_{-3} \\ & - \frac{64(N-1)(2+N)}{N^2(1+N)^2} S_{2,1} - \frac{512}{N^2(1+N)^2} S_{-2,1} + \frac{192(-2-N-N^2)}{N^2(1+N)^2} C_3 \left. \right] \\ & + C_A^2 T_F N_F \left[\frac{32P_4}{9N^2(1+N)^2} S_2 + \frac{32P_{11}}{9N^2(1+N)^2} S_{-3} - \frac{64P_{11}}{9N^2(1+N)^2} S_{-2,1} \right. \\ & + \frac{16P_{13}}{9N^2(1+N)^2} S_3 + \frac{2P_{70}}{27(N-1)N^3(1+N)^3(2+N)} + \left(\frac{1280}{9} S_2 - \frac{64}{3} S_3 \right. \\ & - \frac{8P_{38}}{27(-1+N)N^4(1+N)^4(2+N)} - 128C_3 \left. \right) S_1 + \frac{64}{3} S_{-2}^2 \\ & + \left(\frac{64P_{45}}{9(N-1)N^2(1+N)^2(2+N)} S_1 - \frac{32P_{30}}{9(N-1)N^3(1+N)^3(2+N)} \right) S_{-2} \\ & + \frac{128(-3+2N+2N^2)}{N^2(1+N)^2} C_3 \left. \right] \end{aligned}$$



- Our results agree in the 3-loop non-singlet vector cases with [Moch, Vermaseren, Vogt, 2004a, 2015](#).
- In the transversity case the formerly guessed expressions in [Velizhanin, 2012](#) agree with the results of our direct calculation.
- We agree with all our earlier results obtained from massive OMEs at $O(T_F)$ (2014-2019).
- in the polarized singlet case we agree with [Moch, Vermaseren, Vogt, 2014](#) in the M-scheme.
- Our results agree with the predictions by [Gracey & Gracey et al.](#) for the leading N_F terms.
- We also have obtained already the EOM, NGI, WI OMEs to 2-loops at higher order in the dimensional parameter ε .

Small x predictions: non-singlet and polarized singlet cases



- The small x approaches based on infrared evolution equations (IEE) or pomeron exchange never specify the factorization scheme of the derived evolution kernels/anomalous dimensions. This complicates comparisons.
- We agree with the LO non-singlet prediction for $\gamma_{qq}^{(2),NS,\pm}$ prediction by [Kirschner and Lipatov, 1983](#), after correcting errors, cf. [Blümlein and Vogt, 1995](#).
- There were no small x predictions for $(\Delta)\gamma_{qq}^{(2),NS,(s)}$ and the transversity anomalous dimensions.
- The LO polarized singlet small x predictions by [Bartels, Ermolaev and Ryskin, 1996](#) are not in the M-scheme from 3-loop order, which our comparison shows, while there is agreement at 1- and 2-loop order.
- Agreement had been reached by [Moch, Vermaseren, Vogt, 2014](#), however, by considering a physical scheme, defined by their 2x2 Matrix of QCD and gravitational Wilson coefficients.
- Here it essential to prove that the small x Wilson coefficients are of less singular order in the IEE approach, which has been investigated by [Kiyo, Kodaira and Tochimura, 1996](#) in the non-singlet case.



- We have calculated all flavor non-singlet and the polarized singlet three-loop anomalous dimensions by using the method of massless off-shell operator matrix elements. Here the method of arbitrary high Mellin moments is of instrumental importance. The current problem requires less than 2000 moments.
- The unpolarized and polarized transversity anomalous dimensions have been calculated for the first time ab initio.
- We agree with the results of earlier calculations using either the forward Compton amplitude or massive on-shell OMEs and the available predictions in for the large N_F terms.
- The comparison to pomeron-dynamics based equations, like the infrared evolution equations, in small x range are more subtle to perform because the calculation scheme is not specified explicitly. Yet we agree with the leading series predictions given there too in a certain physical scheme.
- The two-loop unpolarized case is completely understood, correcting errors in the literature.
- The calculation of the three-loop unpolarized singlet anomalous dimensions by using the same method is forthcoming.
- The method appears to be useful to calculate 4-loop quantities.