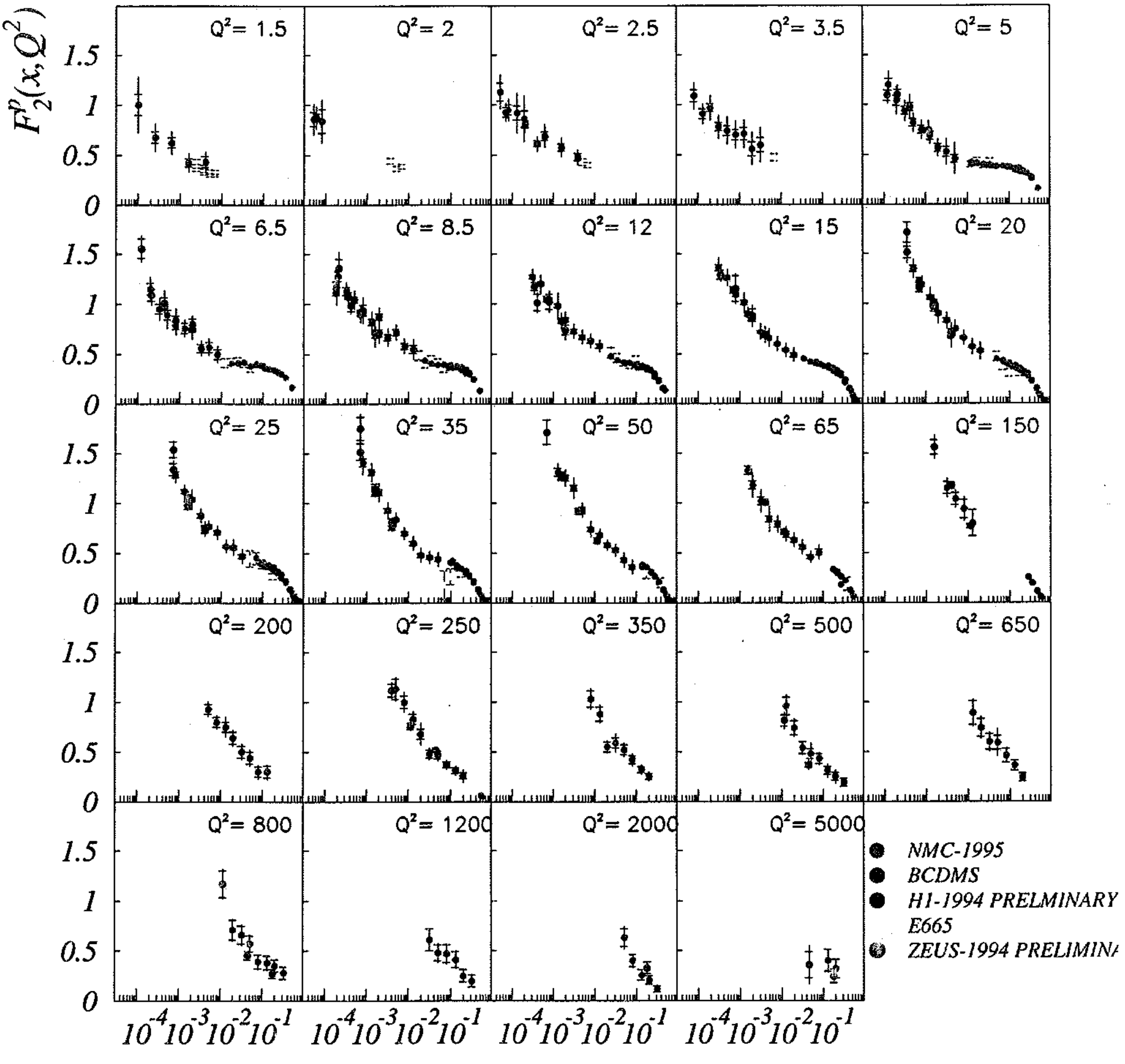
Hamburg
October 1998

QCD–Evolution of Structure Functions at Small x

Johannes Blümlein

DESY



BLUMLEIN@PHOENIX, DEC02/15:20

1. Introduction

- WHY DO STRUCTURE FUNCTIONS GROW AT SMALL x ?

→ DYNAMICAL EFFECT ? (PERT.)

→ INPUT (AT Q_0^2) DRIVEN ? (NON-PERTURBATIVE)

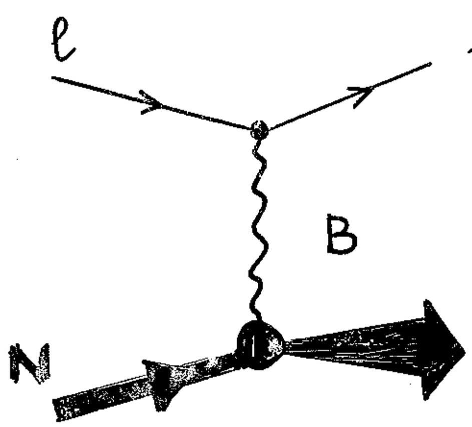
→ BOTH ? • WHICH EFFECT IS STRONGER ?

CAN PT QCD, AT ALL, DESCRIBE THIS BEHAVIOUR ?

→ EFFECTS IN NON-SINGLET AND POLARIZED STRUCTUREFUNCTIONS ?

- HOW TO RESUM ?

DEEP INELASTIC SCATTERING:



WHICH STATE IS SCATTERED OFF?

$$|N\rangle = |\psi\rangle \oplus |\psi\psi\rangle \oplus |\psi\psi\psi\rangle \oplus \cdots$$

- $|\psi\rangle$: SINGLE PARTON STATE
- $|\psi\psi\rangle \oplus |\psi\psi\psi\rangle \oplus \cdots$: MULTI PARTON INITIAL STATE ↖ HIGHER TWIST (OPE)

← SINGLE PARTON (PARTICLE) EVOLUTION EQUATIONS

- LEADING TWIST TERMS (OPERATOR PRODUCT EXPANSION) (OPE)

QCD-IMPROVED PARTON MODEL $\cong$ OPE (LEADING TWIST)

INTUITIVE APPROACH

FORMAL APPROACH.

WHEN IS A (SINGLE) PARTON ?

(INFINITE MOM. FRAME: P)

S. DRELL
R.P. FEYNMAN

$$\tau_{int} \sim \frac{1}{q_0} = \frac{4Px}{Q^2(1-x)}$$

$$\tau_{life} \sim \frac{1}{\sum_i E_i - E} \cong \frac{2x(1-x)P}{k_\perp^2}$$

$$\tau_{int} \ll \tau_{life}$$

$$\searrow \quad \boxed{\frac{\tau_{life}}{\tau_{int}} = \frac{Q^2}{2k_\perp^2}(1-x)^2 \gg 1}$$

BUT BEWARE OF WEE PARTONS:

$$x_{WEE} \lesssim \frac{\mu}{P} \simeq \frac{2(\Lambda \ldots M_p)}{\sqrt{s}} < x_{parton}$$

HERA: $x \gtrsim 10^{-3} \ldots 2 \cdot 10^{-3}$

PURELY KIN. ESTIMATE $\longrightarrow$ DYNAMICS: (x, Q^2).

SINGLE PARTICLE PICTURES GET PROBLEMS AT:

$$Q^2 \lesssim M_p^2 \qquad \text{(ALSO: } \alpha_s \text{ LARGE)}$$

$$x \lesssim \frac{2(\Lambda \dots M_p)}{\sqrt{s}}$$

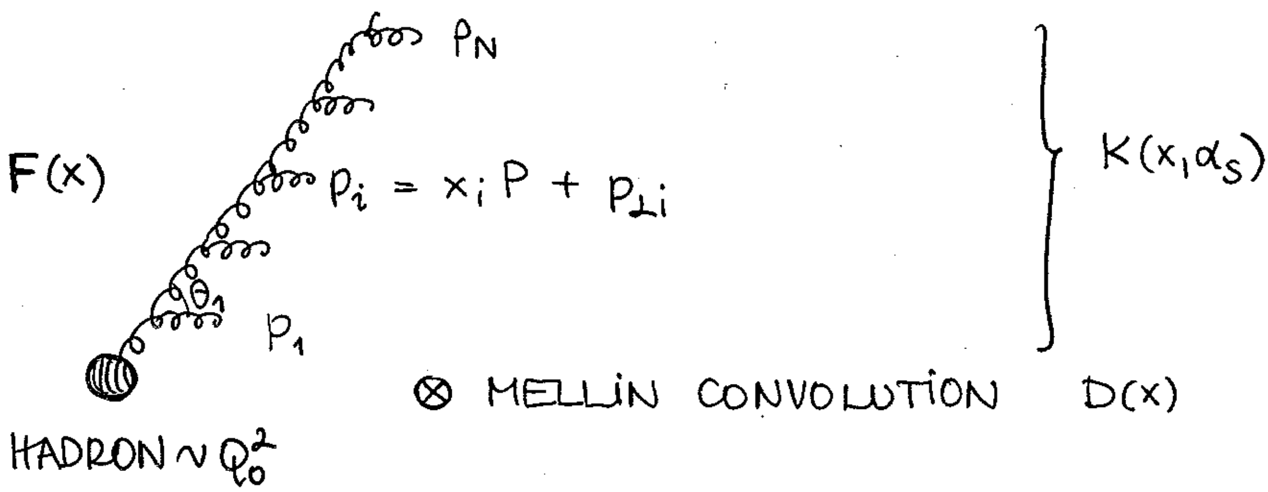
... AT LEAST POTENTIALLY.

⟶ APPLIES TO
- LADDER APPROACHES
- OPE

AND OTHER DESCRIPTIONS.

2. Different LO Resummations

⟶ PARTON LADDERS (PHYSICAL GAUGE)



⊗ MELLIN CONVOLUTION D(x)

$$F(x) \propto \int_0^1 dx_1 \int_0^1 dx_2 \, \delta(x - x_1 \cdot x_2) \; K(x_1, \alpha_s) \cdot D(x_2)$$

A VIEW ON THE CASCADE: (LO)

HOW THE EMISSIONS ARE ORDERED ?

- $p_{1\perp}^2 \ll \ldots \ll p_{2\perp}^2$: GPGWGLAPKSD – EVOLUTION (IN SHORT: ALTARELLI – PARISI) MORE PRECISELY: CALLEN – SYMANZIK EQ.
- $x_1 \ll \ldots \ll x_N$: $S \to \infty$ BFKL – RESUMMATION (LIPATOV)
- ALWAYS TRUE: $\theta_1 < \theta_i \ldots < \theta_N$ CCFM (CIAFALONI et al) ↳ COMPLICATED EQ.

SUCH AN ORDERING FAILS TO EXIST BEYOND LO!

<u>QCD</u>: (MASSLESS PARTONS)

⟶ STRUCTURE FUNCTIONS

⟷ SINGULARITIES: no: SOFT + INFRARED (BLOCH-NORDSIEK)

→ COLLINEAR SINGULARITIES

→ UV SINGULARITIES

RGE_1: RENORMALIZATION OF α_s.

RGE_2: SCALING VIOLATIONS OF PARTON DENSITIES

(ALSO FOR BFKL LADDERS.)

- THE MENTIONED PRESCRIPTIONS ARE

RESUMMATIONS,

AND NOT NECESSARILY 'EVOLUTIONS'.

WHAT IS RESUMMED ?

- BFKL : LO SMALL x CONTRIBUTION TO

  γ_{gg} $\quad O\left(\left(\frac{\alpha_s}{N-1}\right)^{\ell}\right)$

- AP : LO ITERATION OF THE EVOLUTION KERNEL(S).

∴ NS : $\sum_{k=1}^{\infty} \left(\frac{\alpha_s}{2\pi}\right)^{k} \bigotimes_{\ell=1}^{k} P_{qq}(x) = E_{qq}(x,\alpha)$

$$F_{NS}(x,\alpha) = \left[E_{qq}(x,\alpha) + \delta(1-x)\right] F_{NS}(x,\alpha_0)$$

⟶ THESE ARE CLEARLY DIFFERENT TYPE RESUMMATIONS.

- CCFM : LO ACCOUNT FOR A SOLUTION OF THE GLUON EVOLUTION EQUATION INCLUDING BFKL TERMS.

  ⟶ VERY SUITABLE FOR FINAL STATES, MONTE CARLO'S etc.

HOW TO PROCEED IN HIGHER ORDERS ?

- USE RESUMMED ANOMALOUS DIMENSIONS & COEFFICIENT FUNCTIONS

⟶ DO THE RESUMMED TERMS COVER ALL THE SINGULAR TERMS KNOWN IN FIXED ORDER PT ? (SO FAR: YES)

- CONSERVATION LAWS:
  - 4 MOMENTUM : SINGLET
  - FERMION NUMBER : $P^{-}_{NS}(x) = \sum_{\ell=1}^{\infty} \left(\frac{\alpha_s}{4\pi}\right)^{\ell} P^{-}_{NS,\ell}(x)$

$\forall \ell, \quad \int_0^1 dx \; P^{-}_{NS,\ell}(x) = 0$ NOT AUTOMATICALLY FULFILLED.

⟶ LARGE EFFECTS.

⟺ RÔLE OF 'LESS' SINGULAR TERMS !

3. Basic Approach: The Renormalization Group Equation

GOAL : • EVOLUTION EQUATIONS, WHICH ACCOUNT FOR SMALL X EFFECTS.

⟶ α_s MEASUREMENT

⟶ DETERMINATION OF INPUT PARTON DENSITIES.

SIMILAR TO CONVENTIONAL APPROACH, BUT NOW EXTENDED TO ALL ORDERS IN α_s.

$$F(x, Q^2) = C_q(x, \frac{Q^2}{\mu^2}) \otimes f_q(\frac{\mu^2}{M^2}) + C_g(x, \frac{Q^2}{\mu^2}) \otimes f_g(\frac{\mu^2}{M^2})$$

$$\frac{\partial f_i}{\partial \log \mu^2} = \mathbb{P}_{ij}(x, \alpha) \otimes f_j$$

$$\mathbb{P}_{ij} = \sum_{k=1}^{\infty} \left(\frac{\alpha}{4\pi}\right)^k P_{ij}^k(x)$$

$$C_i = \sum_{k=1}^{\infty} \left(\frac{\alpha}{4\pi}\right)^k C_i^k(x)$$

LOW ORDERS: COMPLETE EXPRESSIONS

HIGHER ORDERS: SINGULAR TERMS + CONS. LAWS.

2. LO and NLO Small x Resummation and Evolution Equations

$$F_i(x,Q^2) = \sum_{r=1}^{2N_f} a_{ir}\, c_{i,r}(x,Q^2) \otimes q_r(x,Q^2) + a_{ig}\, c_{i,g}(x,Q^2) \otimes g(x,Q^2)\,,$$

EVOLUTION EQUATIONS:

$$\frac{\partial\, q_{\rm NS}^{\pm}(x,Q^2)}{\partial \ln Q^2} = P_{\rm NS}^{\pm}(x,\alpha_s) \otimes q_{\rm NS}^{\pm}(x,Q^2)\,,$$

$$\frac{\partial\, \boldsymbol{q}_{\rm S}(x,Q^2)}{\partial \ln Q^2} = \boldsymbol{P}_{\rm S}(x,\alpha_s) \otimes \boldsymbol{q}_{\rm S}(x,Q^2)\,.$$

$$\frac{da_s}{d\ln Q^2} = -\sum_{k=0}^{\infty} a_s^{k+2}\beta_k\,.$$

ALL ORDER RESUMMATION:

$$P^{\pm}(x,a_s) = \sum_{l=0}^{\infty} a_s^{l+1} P_l^{\pm}(x)\,,$$

$$\boldsymbol{P}(x,a_s) \equiv \begin{pmatrix} P_{qq}(x,a_s) & P_{qg}(x,a_s) \\ P_{gq}(x,a_s) & P_{gg}(x,a_s) \end{pmatrix} = \sum_{l=0}^{\infty} a_s^{l+1} \boldsymbol{P}_l(x)\,,$$

$$c_{i,j}(x,Q^2) = \delta(1-x)\delta_{jq} + \sum_{l=1}^{\infty} a_s^l c_{ij,l}(x)\,.$$

- LO + NLO exact
- BEYOND: Lx, NLx RESUMMATION
- LO, NLO MOTIVATED: MODEL STUDIES FOR LESS SINGULAR TERMS

4. $\alpha \ln^2 x$ Resummations

- NON-SINGLET $qq \pm q\bar{q}$
  - UNPOLARIZED
    KIRSCHNER, LIPATOV 83
    ERMOLAEV et al. 95
    JB, A. VOGT 95
  - POLARIZED
    BARTELS et al. 95
    JB, A. VOGT 95
- POLARIZED SINGLET — BARTELS et al '96; JB, A. VOGT 96
- QED $e \to e\gamma$ (MULTIPLE EMISSION NS)
  JB, S. RIEMERSMA, A. VOGT '96

FIXED ORDER PT:

$$P^{+\,QCD}_{x\to 0}(x,a) = 2a\,C_F + 2a^2 C_F^2 \log^2 x + \cdots$$

$$P^{-\,QCD}_{x\to 0}(x,a) = 2a\,C_F + 2a^2\left[-3C_F^2 + 2C_F C_A\right]\log^2 x$$

$$\uparrow$$

$$\lim_{x\to 0} 2a\,\frac{1+x^2}{1-x}\,C_F$$

$$O\left(a^k \log^{2(k-1)}(x)\right)$$

$$\mathcal{M}\left[K^{\pm}_{x\to 0}(a)\right](N) \equiv \int_0^1 dx\, x^{N-1} K^{\pm}_{x\to 0}(x,a) \equiv -\frac{1}{2}\Gamma^{\pm}_{x\to 0}(N,a) = \frac{1}{8\pi^2} f_0^{\pm}(N,a)\,.$$

KIRSCHNER, LIPATOV 1983

$$f_0^+(N,a) = 16\pi^2 a_0 \frac{a}{N} + \frac{1}{8\pi^2}\frac{1}{N}\left[f_0^+(N,a)\right]^2,$$

$$f_0^-(N,a) = 16\pi^2 a_0 \frac{a}{N} + 8b_0^- \frac{a}{N^2} f_V^+(N,a) + \frac{1}{8\pi^2}\frac{1}{N}\left[f_0^-(N,a)\right]^2$$

$$f_V^+(N,a) = 16\pi^2 a_V \frac{a}{N} + 2b_V \frac{a}{N}\frac{d}{dN} f_V^+(N,a) + \frac{1}{8\pi^2}\frac{1}{N}\left[f_V^+(N,a)\right]^2$$

$$a_0 = C_F, \quad b_0^- = C_F, \quad a_V = -\frac{1}{2N_c}, \quad b_V = N_c.$$

ANOMALOUS DIM.

$$\Gamma^{+,\mathrm{QCD}}_{x\to 0}(N,a) = -N\left\{1-\sqrt{1-\frac{8aC_F}{N^2}}\right\}$$

$$\Gamma^{-,\mathrm{QCD}}_{x\to 0}(N,a) = -N\left\{1-\sqrt{1-\frac{8aC_F}{N^2}\left[1-\frac{8aN_c}{N}\frac{d}{dN}\ln\left(e^{z^2/4}D_{-1/[2N_c^2]}(z)\right)\right]}\right\},$$

where $z = N/\sqrt{2N_c a}$, and $D_p(z)$ denotes the function of the parabolic cylinder

PREDICTION FOR THE SING. PART OF THE 3 LOOP NS - ANOM. DIM.

$$P^{+,\mathrm{QED}}_{2,x\to 0,\overline{\mathrm{MS}}}(x,a) = \frac{2}{3}a^3 \ln^4 x$$

$$P^{-,\mathrm{QED}}_{2,x\to 0,\overline{\mathrm{MS}}}(x,a) = -\frac{10}{3}a^3 \ln^4 x\,,$$

$$P^{+,\mathrm{QCD}}_{2,x\to 0,\overline{\mathrm{MS}}}(x,a) = \frac{2}{3}C_F^3 a^3 \ln^4 x$$

$$P^{-,\mathrm{QCD}}_{2,x\to 0,\overline{\mathrm{MS}}}(x,a) = \left(-\frac{10}{3}C_F^3 + 4C_F^2 C_G - C_F C_G^2\right) a^3 \ln^4 x\,.$$

l	K_l^+	K_l^-
0	2.667E0	2.667E0
1	3.556E0	5.333E0
2	1.580E0	1.432E0
3	3.512E-1	9.964E-1
4	4.682E-2	-2.078E-1
5	4.162E-3	1.448E-1
6	2.643E-4	-5.777E-2
7	1.258E-5	2.168E-2
8	4.661E-7	-7.173E-3
9	1.381E-8	2.143E-3
10	3.348E-10	-5.827E-4

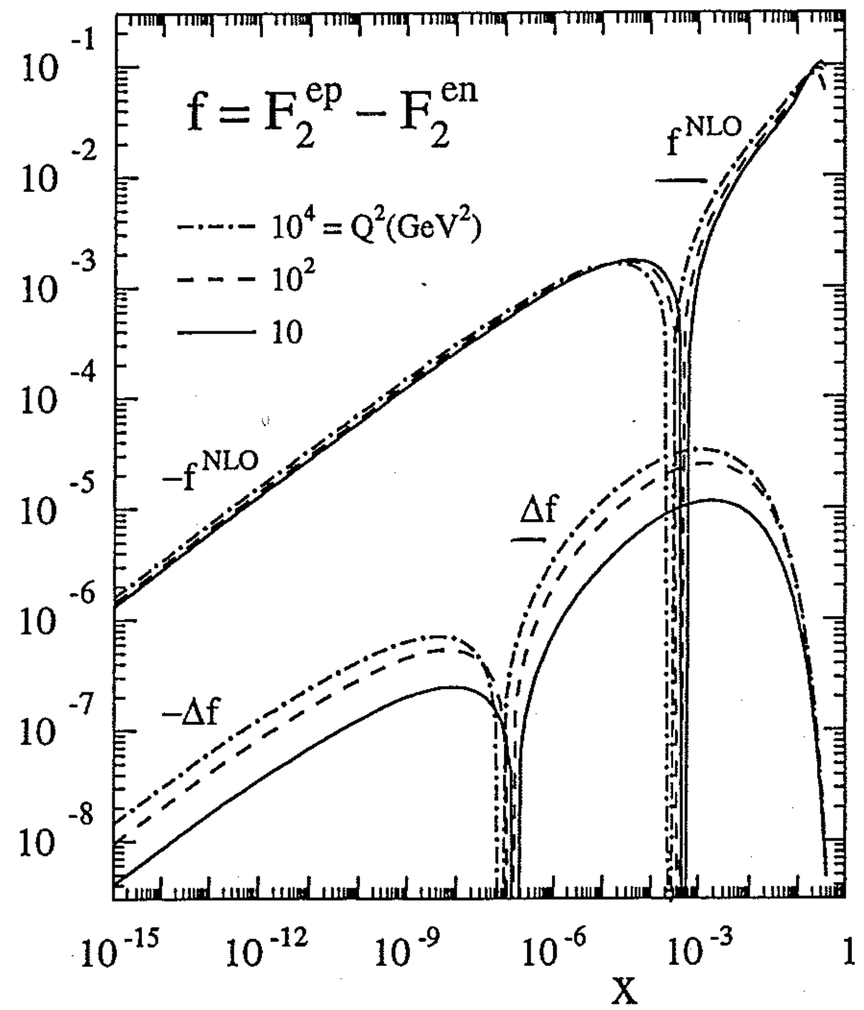
Table 1: The coefficients $K_l^{\pm}$ of the expansion of $K^{\pm}_{x\to 0}(x, a_s)$ in terms of $a_s(a_s \ln^2 x)^l$ as obtained from the resummations in eqs. (17) and (18).

IMPORTANCE OF SUBLEADING TERMS:

e.g. NLO:

$$\gamma_1^+(N)_{x\to 0} = -\frac{128}{9N^3} + \frac{400-32N_f}{9N^2} \overset{N_f=4}{=} -\frac{14.22}{N^3} + \frac{30.22}{N^2},$$

$$\gamma_1^-(N)_{x\to 0} = -\frac{64}{3N^3} + \frac{464-32N_f}{9N^2} \overset{N_f=4}{=} -\frac{21.33}{N^3} + \frac{37.33}{N^2}.$$



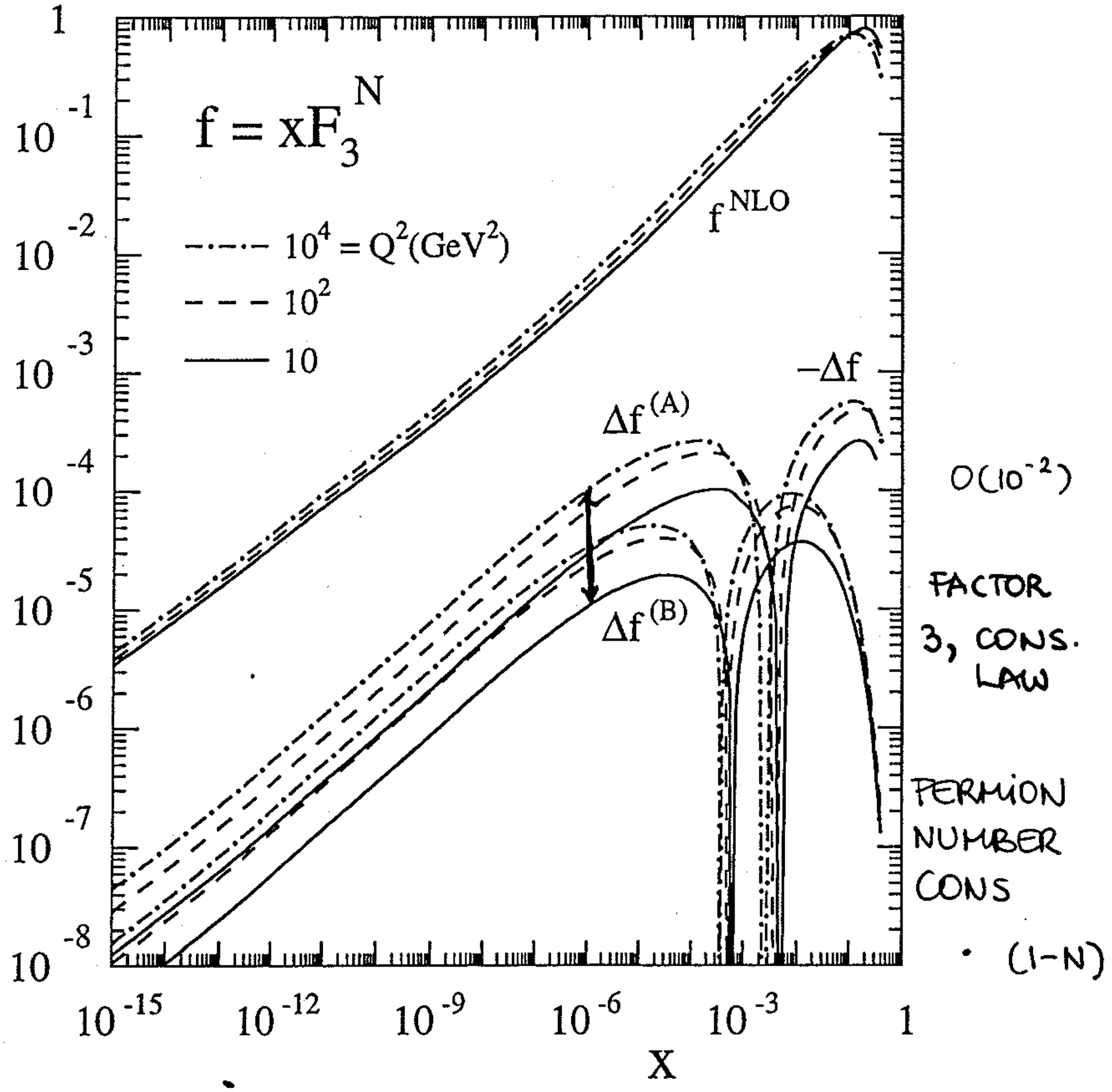
O(10^{-2})

NOT O(10)

(RYSKIN et al.)

Figure 1: The small-x Q^2–evolution of the unpolarized non–singlet structure function combination $F_2^{ep} - F_2^{en}$ in NLO and the absolute corrections to these results due to the resummed kernel derived from ref. [3]. The initial distributions at $Q_0^2 = 4$ GeV^2 have been adopted from [16].

JB, A.VOGT 96



JB, A. VOGT 1996

$$P(x,a_s)_{x\to 0} \equiv \sum_{l=0}^{\infty} P^{(l)}_{x\to 0}\, a_s^{l+1} \ln^{2l} x = \frac{1}{8\pi^2} \mathcal{M}^{-1}\left[F_0(N,a_s)\right](x)\,.$$

The matrix–valued function $F_0(N,a_s)$ is subject to the relation

$$F_0(N,a_s) = 16\pi^2 \frac{a_s}{N} M_0 - \frac{8a_s}{N^2} F_8(N,a_s)\, G_0 + \frac{1}{8\pi^2}\frac{1}{N} F_0^2(N,a_s)$$

derived in ref. [1], where $F_8(N,a_s)$ is the solution of

$$F_8(N,a_s) = 16\pi^2 \frac{a_s}{N} M_8 + \frac{2a_s}{N} C_G \frac{d}{dN} F_8(N,a_s) + \frac{1}{8\pi^2}\frac{1}{N} F_8^2(N,a_s)\,.$$

The basic colour factor matrices are given by

$$M_0 = \begin{pmatrix} C_F & -2T_F N_f \\ 2C_F & 4C_A \end{pmatrix}, \quad G_0 = \begin{pmatrix} C_F & 0 \\ 0 & C_A \end{pmatrix}, \quad M_8 = \begin{pmatrix} C_F - C_A/2 & -T_F N_f \\ C_A & 2C_A \end{pmatrix}.$$

FRAMEWORK:

BARTELS, ERMOLAEV, RYSKIN 1996.

$$P^{(0)}_{x\to 0} = 2\begin{pmatrix} C_F & -2\,T_F N_f \\ 2\,C_F & 4\,C_A \end{pmatrix},$$

$$P^{(1)}_{x\to 0} = 2\begin{pmatrix} C_F(2C_A - 3C_F - 4T_F N_f) & -2T_F N_f(2C_A + C_F) \\ 2C_F(2C_A + C_F) & 8\,C_A^2 - 4\,T_F N_f\, C_F \end{pmatrix}.$$

$$\boxed{P^{(l)}_{qg}/(T_F N_f) = -P^{(l)}_{gq}/C_F\,.}$$

LO resummed exact : $x \to 0$

N=1 SUSY- RELATIONS:

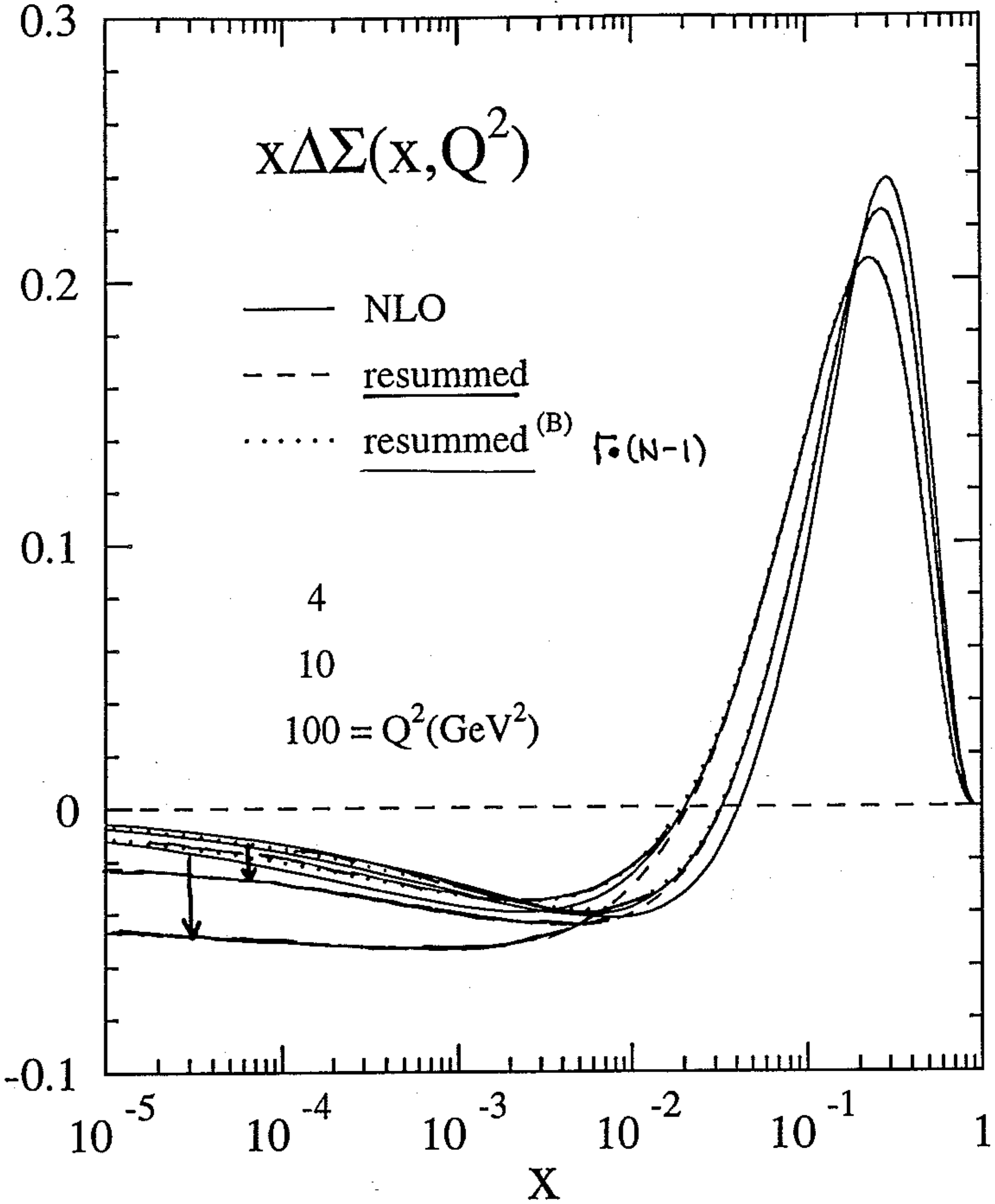
$$P^{(l)}_{qq}(x) + P^{(l)}_{gq}(x) - P^{(l)}_{qg}(x) - P^{(l)}_{gg}(x) = 0$$ ✓

IEVEN:

$$P^{(l)}_{qq} - P^{(l)}_{qg} = 0\,, \quad P^{(l)}_{gq} - P^{(l)}_{gg} = 0$$

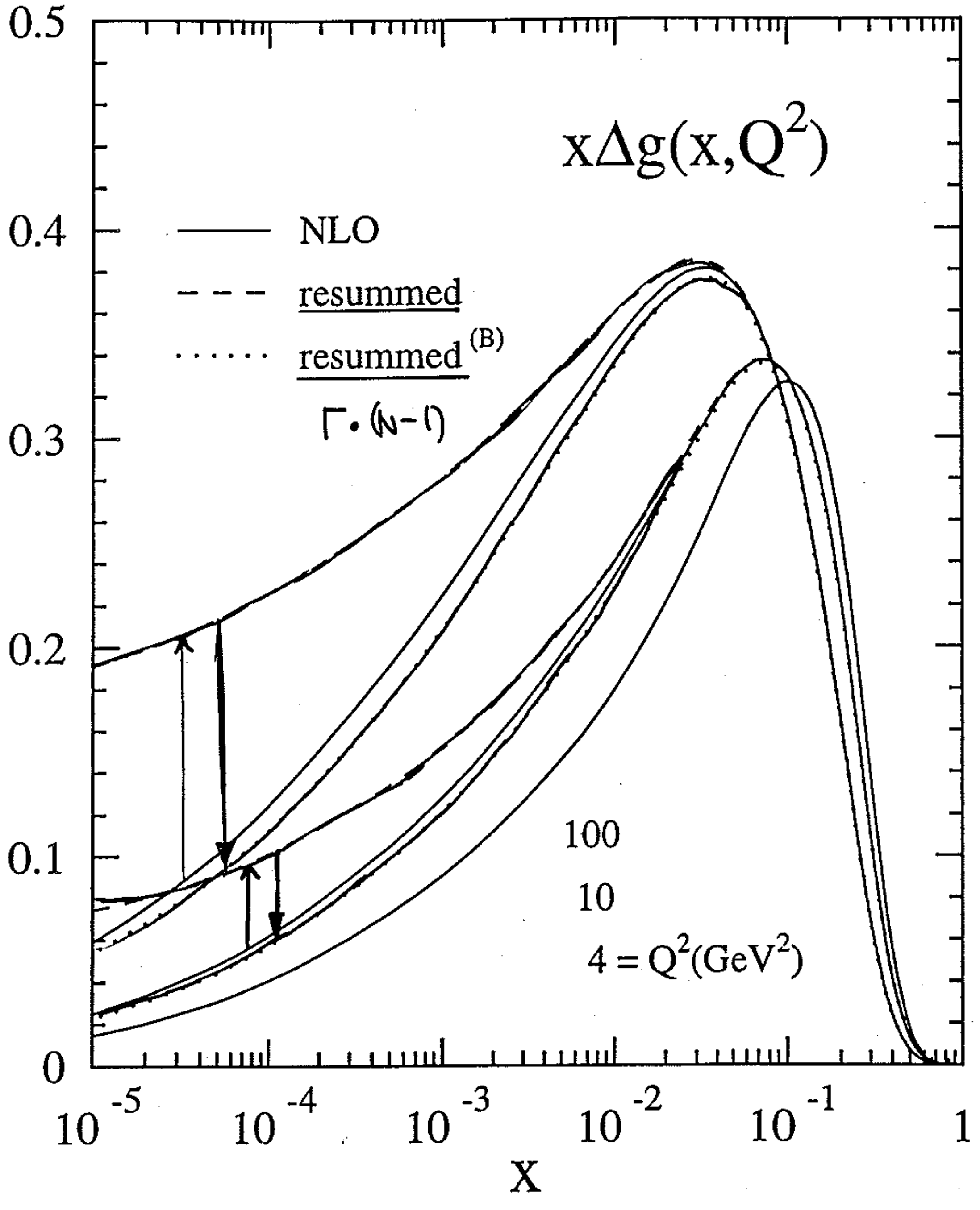
$$P^{\rm SUSY}_{x\to 0} = 2\,a_s M_1 + \sum_{l=1}^{\infty} a_s^{l+1} \ln^{2l} x\, p_l\, M_2$$

$$M_1 \equiv M_0^{\rm SUSY} = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}, \quad M_2 = \begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix}.$$



0.3
0.2
0.1
0
-0.1
$x\Delta\Sigma(x,Q^2)$
NLO
resummed
resummed (B)
Γ•(N-1)
4
10
100 = Q^2(GeV^2)
10 -5
10 -4
10 -3
10 -2
10 -1
1
X

JB , A.VOGT '96



0.5
0.4
0.3
0.2
0.1
0
$x\Delta g(x,Q^2)$
NLO
resummed
resummed$^{(B)}$
Γ·(N-1)
100
10
$4 = Q^2(\mathrm{GeV}^2)$
10^{-5}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
1
x

5. Unpolarized Singlet Structure Functions
5.1. Leading Order Resummation

BALITZKII - FADIN - KURAEV - LIPATOV RESUMMATION
(BFKL) 1976-78, 1986

JAROSZEWICZ 1980/82

RESUMMATION OF ALL TERMS $\propto \left(\frac{\alpha_s}{N-1}\right)^k$
IN $\gamma_{gg}(N, a_s)$ VIA A BETHE - SALPETER eq.

- INFRARED SAFE; KERNEL IS SCALEINVARIANT

$\longrightarrow$ EIGENVALUE $\leftrightarrow$ CONF. ANOM. DIMENSION

$$(N-1) = \frac{\alpha_s}{\pi} N_c \left[2\psi(1) - \psi(\gamma) - \psi(1-\gamma) \right]$$

PERT. SOLUTION: $\gamma \rightarrow \frac{\bar{\alpha}_s}{N-1} \Big|_{N\rightarrow\infty}$ $\quad \bar{\alpha}_s =: \frac{\alpha_s N_c}{\pi}$

$$\gamma_{gg} = \frac{\bar{\alpha}_s}{N-1} \left\{ 1 + 2 \sum_{l=1}^{\infty} b_{2l+1} \gamma_{gg}^{2l+1} \right\}$$

$$\gamma = \frac{\bar{\alpha}_s}{N-1} + 2 b_3 \left(\frac{\bar{\alpha}_s}{N-1}\right)^4 + 2 b_5 \left(\frac{\bar{\alpha}_s}{N-1}\right)^6 + 12 b_3^2 \left(\frac{\bar{\alpha}_s}{N-1}\right)^7 + \ldots$$

/

Conformal Limit and the Anomalous Dimension

$$[M_{\mu\nu}, D] \quad = \quad 0$$

Asymptotic scale and conformal invariance :
K. SYMANZIK, 1971
G. PARISI, 1972

$$\left[\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g} + \gamma_m\, m\frac{\partial}{\partial m} + \gamma_{O_k} - n\gamma_\Phi\right] E_k^n = 0$$

$$m \quad = \quad 0, \qquad \beta \quad = \quad 0$$

$$\Rightarrow \qquad \left[\mu\frac{\partial}{\partial\mu} + \gamma_{O_k} - n\gamma_\Phi\right] E_k^n = 0$$

$$E_k^n(\mu^2) \quad = \quad E_k^n(\mu_0^2)\left(\frac{\mu^2}{\mu_0^2}\right)^{\frac{1}{2}\left(\gamma_{O_k} - n\gamma_\Phi\right)}$$

$$\gamma_{O_k} - n\gamma_\Phi \quad \equiv \quad \Gamma_k^n \quad = \quad \sum_{l=1}^{\infty} a^l \gamma_l^{k,n}$$

ALL ORDERS.

$a =$ fixed coupling constant.

THE CONFORMAL PART EXPONENTIATES TO ALL ORDERS.

NO QUARKONIC ENTRIES AT $O\left(\left(\frac{\alpha_s}{N-1}\right)^k\right)$.

$$\gamma_{qq}^{(0)} = \frac{C_F}{C_A}\,\gamma_{gg}^{(0)}$$

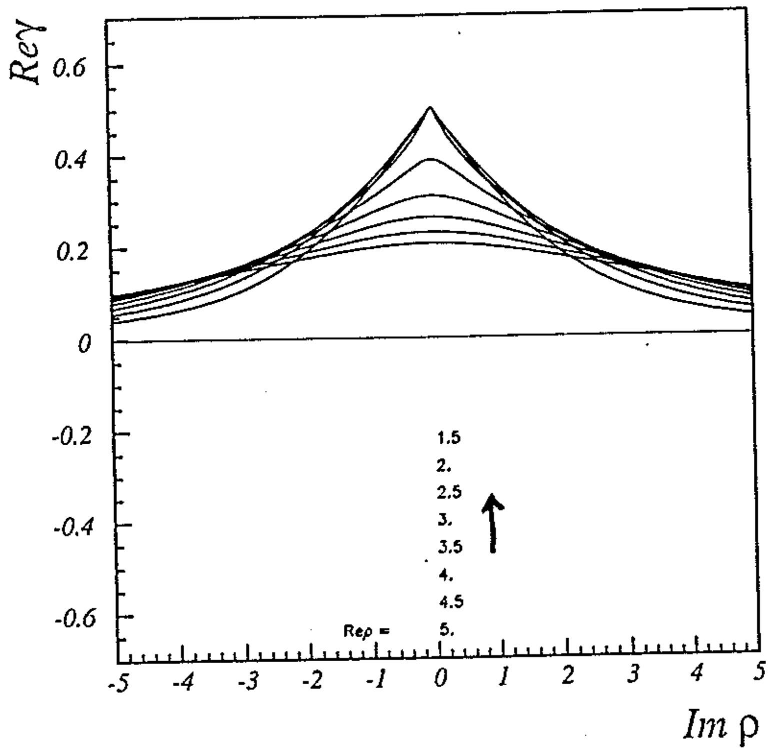
$$\hat{\gamma}_L(N, a_s) = \begin{pmatrix} 0 & 0 \\ \frac{C_F}{C_A} & 1 \end{pmatrix} \gamma_L(N, a_s).$$

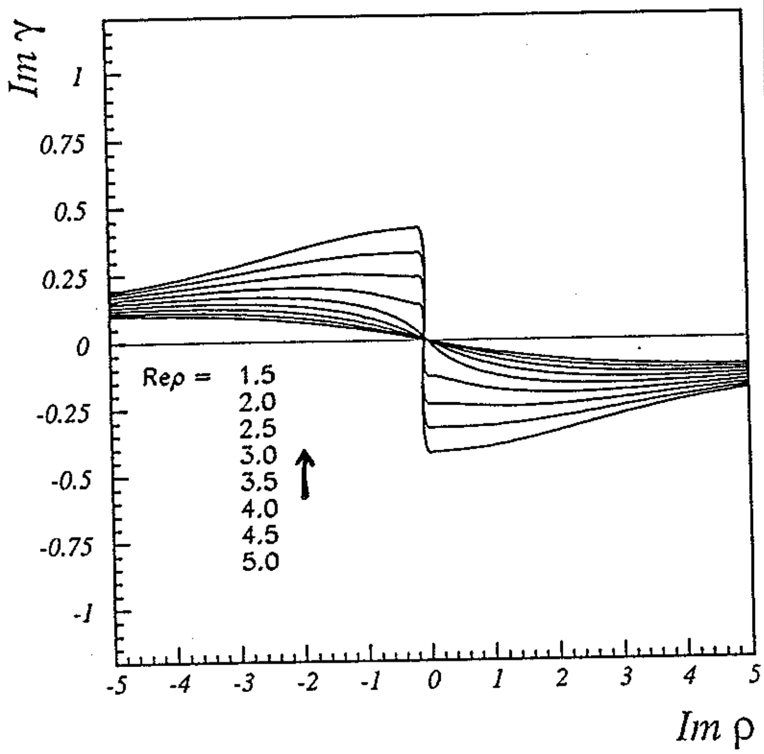
WHAT DRIVES THE QUARKS $\longrightarrow$ NLO.

& LO <u>FINITE</u> x

The behaviour of $\gamma_c(\rho)$ for $\rho \in C$

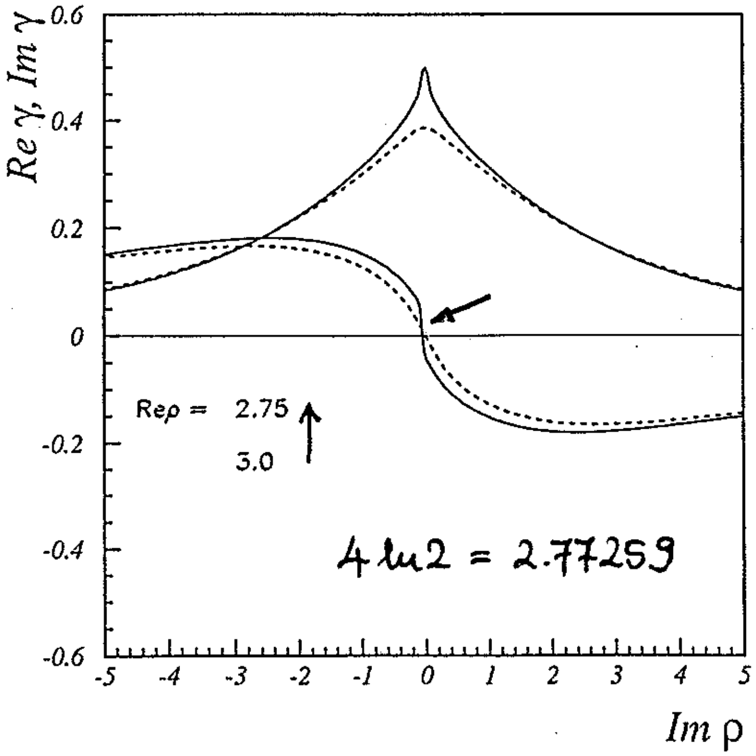
$Re\rho \geq 1.5$:

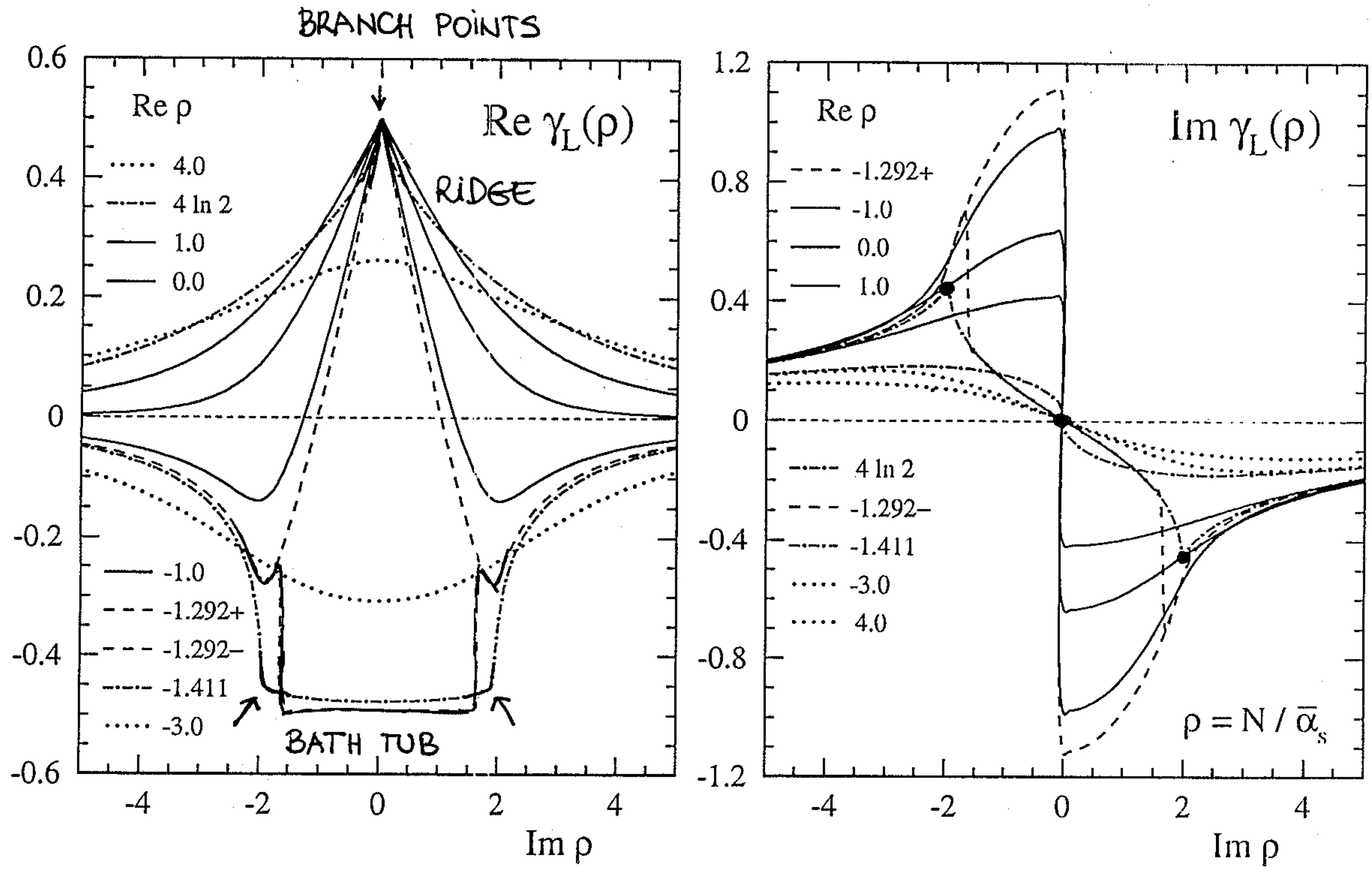




(USE: ADAPTIVE NEWTON ALGORITHM).

$$\rho = \frac{\bar{N}\pi}{\alpha_s N_c}$$




BRANCH POINTS
Re ρ
4.0
4 ln 2
1.0
0.0
Re γ_L(ρ)
RIDGE
-1.0
-1.292+
-1.292–
-1.411
-3.0
BATH TUB
Im ρ
Re ρ
-1.292+
-1.0
0.0
1.0
Im γ_L(ρ)
4 ln 2
-1.292–
-1.411
-3.0
4.0
ρ = N / ᾱ_s
Im ρ

LOCATION OF THE BRANCH POINTS

$$\rho = \frac{\ell - 1}{\bar{\alpha}_s} = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma).$$

$$1 = \left[-\psi'(\gamma) + \psi'(1-\gamma)\right] \frac{\partial \gamma}{\partial \rho}$$

$$\frac{1}{\partial\gamma / \partial\rho} = \psi'(1-\gamma) - \psi'(\gamma) = 0$$

$$\boxed{\psi'(z) - \frac{\pi^2}{2} \frac{1}{\sin^2 \pi z} = 0}$$

$\gamma_1 = \frac{1}{2} + 0i$ $\qquad\qquad$ $\rho_1 = 4 \ln 2$

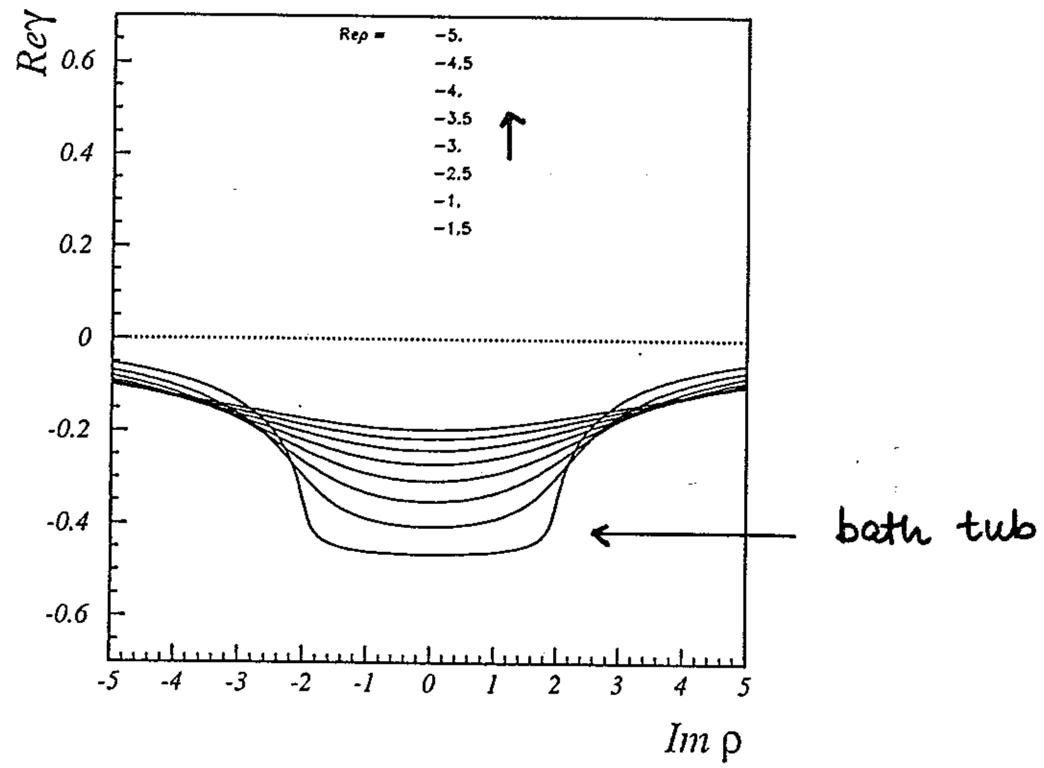
$\gamma_{2,3} = -0.425214 \pm i 0.473898$

$\rho_{2,3} = -1.4105 \pm i\, 1.9721.$

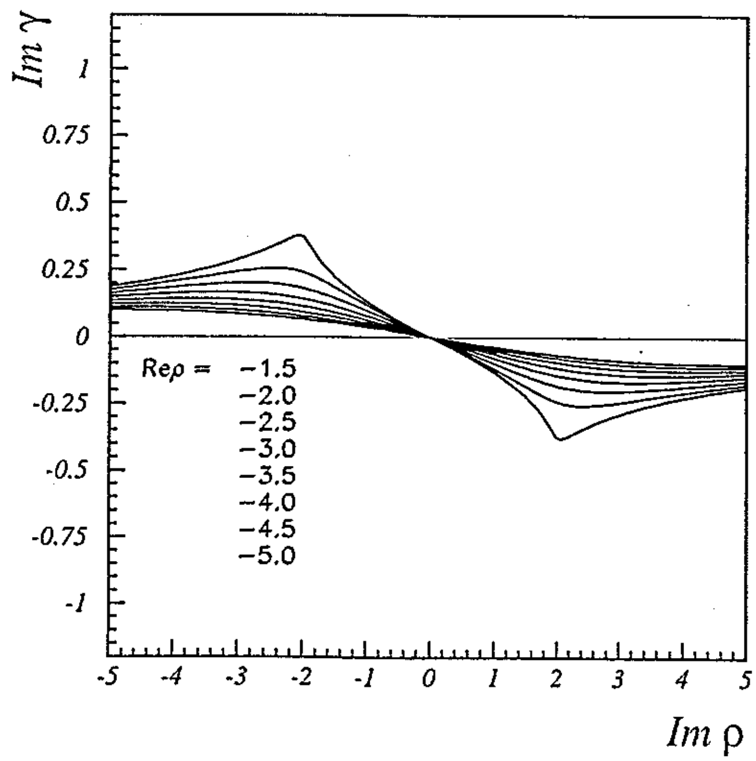
K. ELLIS, HAUTMANN, WEBBER '95

JB '95

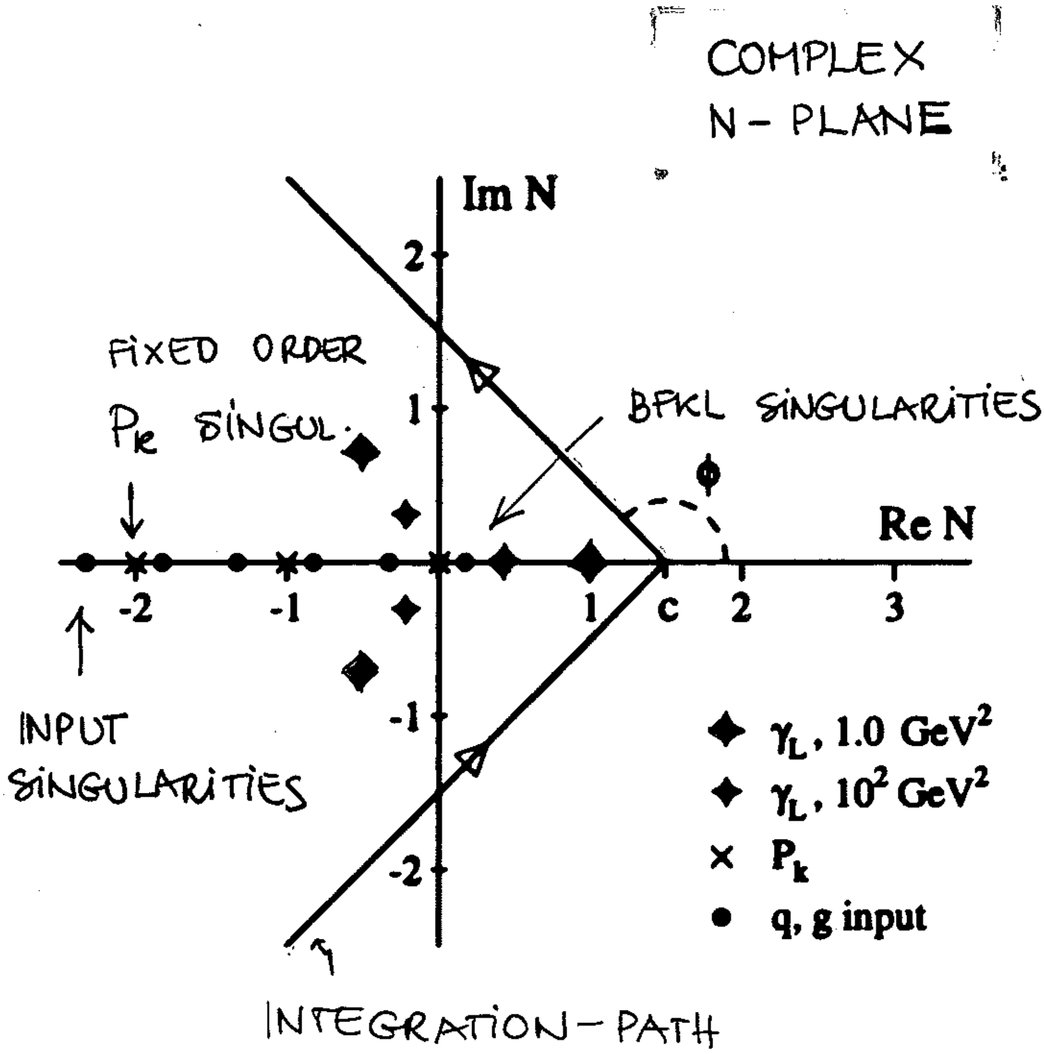
$Re\rho \leq -1.5$



Reγ
Reρ =
−5.
−4.5
−4.
−3.5
−3.
−2.5
−1.
−1.5
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-5
-4
-3
-2
-1
0
1
2
3
4
5
Im ρ
bath tub



Im γ
1
0.75
0.5
0.25
0
-0.25
-0.5
-0.75
-1
Reρ =
−1.5
−2.0
−2.5
−3.0
−3.5
−4.0
−4.5
−5.0
-5
-4
-3
-2
-1
0
1
2
3
4
5
Im ρ



- SOLUTION OF THE RGE'S IN MELLIN SPACE
- EXACT ACCOUNT FOR ALL COMMUTATION RELATIONS: $[P_{ij}^{\ell}, P_{ij}^{m}] \neq 0$ FOR $m \neq \ell$.

5.2. Next-to-Leading Order Resummation

$$\hat{\gamma}_{NL}(N, \alpha_s) = -2 \begin{pmatrix} \frac{C_F}{C_A}\left[\gamma_{NL}^{qg} - \frac{8}{3}\alpha_s T_F\right] & \gamma_{NL}^{qg} \\ \gamma_{gq\,NL} & \gamma_{gg,NL} \end{pmatrix}$$

$\gamma_{gq\,NL}$ ↑ STILL UNKNOWN

$\gamma_{gg,NL}$ ↑ FL '98, CIAFALONI CAMICI '98

QUARKS: γ_{NL}^{qg} CATANI, HAUTMANN 94.

$$\gamma_{NL}^{qg,DIS}(N, \alpha_s) = T_F \frac{\alpha_s}{3\pi} \frac{2 + 3\gamma - 3\gamma^2}{3 - 2\gamma} \frac{[B(1-\gamma, 1+\gamma)]^3}{B(2+2\gamma, 2-2\gamma)} R(\gamma)$$

$$R(\gamma) = \left[\frac{\Gamma(1-\gamma)\,\chi(\gamma)}{\Gamma(1+\gamma)\{-\gamma\,\chi'(\gamma)\}}\right]^{1/2} \exp\left[\gamma\psi(1) + \int_0^\gamma dz\, \frac{\psi'(1) - \psi'(1-z)}{\chi(z)}\right]$$

$$\chi(z) = 2\psi(1) - \psi(z) - \psi(1-z).$$

γ_{NL}^{qg} IS A ANALYTIC FUNCTION OF γ_L ONLY.

⟶ SAME SINGULARITY STRUCTURE.

3. Lx and NLx Anomalous Dimensions in the Conformal Limit and Fixed Order Results

γ_{gg}^{NL} :

The Bethe-Salpeter Equation

BFKL

CIAFALONI, CAMICI
FADIN, LIPATOV
1997/98

$$(N-1)G_N(q_1,q_2) \;=\; \delta^{D-2}(q_1-q_2)+\int d^{D-2}q_3\; K(q_1,q_3)\; G_N(q_3,q_2)$$

with

$$K(q_1,q_2) \;=\; \delta^{D-2}(q_1-q_2)2\omega(q_1)+K_{\rm real}(q_1,q_2)+K_{\rm virt}(q_1,q_2)$$

This equation is infrared finite.

ALSO IN NLO.

The Kernel and its Eigenvalue

($\gamma_+ \to \gamma_{gg}$)

HOW TO EXTRACT THE ANOM. DIMENSION ?

DIS : $q_1^2 \gg q_2^2$

CONF. INV. ↓ NO CONF. INV. ↓

$$\int d^{D-2}q_2\; K(q_1,q_2)\;\left(q_2^2\right)^{\gamma-1}=\overline{\alpha}_s\left[\chi_0(\gamma)-\frac{\overline{\alpha}_s}{4}\delta(\gamma,q_1^2,\mu^2)\right]\;\left(q_1^2\right)^{\gamma-1}$$

with

$$\overline{\alpha}_s=\frac{N_c}{\pi}\alpha_s(\mu^2)$$

and

$$\chi_0(\gamma) \;=\; 2\,\psi(1)-\psi(\gamma)-\psi(1-\gamma)$$

LX: $\gamma \leftrightarrow 1-\gamma$ SYMMETRY

$$\delta(\gamma,q_1^2,\mu^2)=\frac{\beta_0}{3}\left\{\chi_0(\gamma)\log\left(\frac{q_1^2}{\mu^2}\right)+\frac{1}{2}\left[\chi_0^2(\gamma)+\chi_0'(\gamma)\right]\right\}+\hat{\chi}_1^{\rm symm}(\gamma)$$

↑ SCALE DEP. ↑ ASYM.

V. FADIN, L. LIPATOV, 1998;
G. CAMICI, M. CIAFALONI 1997,1998 :

$$
\begin{aligned}
\chi_1(\gamma_L) &= \frac{\beta_0}{6}\left[\chi_0^2(\gamma_L)+\chi_0'(\gamma_L)\right]-\left(\frac{67}{9}-2\zeta(2)-\frac{10}{27}N_f\right)\chi_0(\gamma_L) \\
&\quad -6\zeta(3)+\left[\chi_0^2(\gamma_L)+\chi_0'(\gamma_L)\right]'+4\Phi(\gamma_L)-\frac{\pi^3}{\sin^2(\pi\gamma_L)} \\
&\quad +\frac{\pi^2}{\sin^2(\pi\gamma_L)}\frac{\cos(\pi\gamma_L)}{1-2\gamma_L} \\
&\qquad \times\left[(22-\beta_0)+\frac{\gamma_L(1-\gamma_L)}{(1+2\gamma_L)(3-2\gamma_L)}\left(1+\frac{N_f}{3}\right)\right]
\end{aligned}
$$

with

$$
\begin{aligned}
\Phi(\gamma) &= \int_0^1 dz\frac{1}{1+z}\left[z^{\gamma-1}+z^{\gamma}\right]\left[\mathrm{Li}_2(1)-\mathrm{Li}_2(z)\right] \\
&= \frac{1}{\gamma^2}\left[\psi(\gamma+1)-\psi(1)\right] \\
&+ \sum_{n=1}^{\infty}(-1)^n\left[\frac{\psi(n+1+\gamma)-\psi(1)}{(n+\gamma)^2}-\frac{\psi(n+1-\gamma)-\psi(1)}{(n-\gamma)^2}\right] \\
&= \frac{1}{\gamma}\sum_{l=2}^{\infty}(-1)^l\zeta(l)\gamma^{l-2}+\sum_{k=0}^{\infty}\left[\frac{2\pi^2}{3}\eta(2k+2)+\underline{c_{2k+1}}\right]\gamma^{2k+1}
\end{aligned}
$$

$$
\eta(k)=\zeta(k)\left[1-2^{1-k}\right]
$$

The coefficients c_{2k+1} belong to a <u>NEW CLASS of transcendentals</u> since their corresponding Mellin-sum for $k \in \mathbf{N}$ is not reducible, e.g. J. BLÜMLEIN, S. KURTH, 1997.

$$
c_k=-\frac{2}{k!}\int_0^1 dz\log^k\left(\frac{1}{z}\right)\frac{\mathrm{Li}_2(\mathrm{z})}{1+z}
$$

The structure of $\chi_1(\gamma)$

$$\gamma_L \longleftrightarrow 1-\gamma_L$$

$$\chi_1(\gamma_L) = \left[\frac{\beta_0}{6} + \frac{d}{d\gamma_L}\right]\left[\chi_0^2(\gamma_L) + \chi_0'(\gamma_L)\right] + \hat{\chi}_1^{\rm symm}(\gamma_L)$$

FROM: RUNNING α

universal terms

$$\chi_1(\gamma_L) = \frac{\beta_0}{6}\left[\chi_0^2(\gamma_L) + \chi_0'(\gamma_L)\right] - \left(\frac{67}{9} - 2\zeta(2) - \frac{10}{27}N_f\right)\chi_0(\gamma_L)$$

3LOOP !

$$- 6\zeta(3) + \left[\chi_0^2(\gamma_L) + \chi_0'(\gamma_L)\right]' + 4\Phi(\gamma_L) - \frac{\pi^3}{\sin^2(\pi\gamma_L)}$$

$$+ \frac{\pi^2}{\sin^2(\pi\gamma_L)}\frac{\cos(\pi\gamma_L)}{1-2\gamma_L} \times \left[(22-\beta_0) + \frac{\gamma_L(1-\gamma_L)}{(1+2\gamma_L)(3-2\gamma_L)}\left(1+\frac{N_f}{3}\right)\right]$$

no "running" β – function ! ! (G – SELFENERGY)

- KORCHEMSKY: q-Regge-trajectory: above term $6\zeta(3)$
  g-Regge-trajectory: Different result. ← JB, RAVINDRAN, VAN NEERVEN '98
  3-Loop Term affected ! - Important to clarify.

FIRST AT:

-----	1 LOOP	$1/\gamma_L^2$
-----	2 LOOP	$1/\gamma_L$
-----	3 LOOP	1

$$1 = \frac{\bar{\alpha}}{N-1} \left[\chi_0(\gamma_+) + \alpha \left[\chi_1(\gamma_+) - 2[\chi_0 \chi_0'](\gamma_+) \right] \right]$$

$$\Delta\gamma_+ = -\alpha \frac{\chi_1(\gamma) - 2\chi_0\chi_0'}{\chi_0'}$$

$$\gamma_\pm \simeq \begin{cases} \gamma_{gg} + \frac{C_F}{C_A}\gamma_{qg} + \dots \\ \gamma_{qq} - \frac{C_F}{C_A}\gamma_{qg} \quad \text{JUST FIXED ORDER} + \dots \end{cases}$$

DIS SCHEME, RUNNING COUPLING:

$$\gamma_{gg}^{DIS} = \hat{\gamma}_{gg}^{Q_0} + \frac{\beta_0}{4\pi}\alpha^2 \frac{d\log R(\gamma)}{d\alpha_s} + \frac{C_F}{C_A}(1-R(\gamma))\gamma_{qg}^{Q_0}$$

$$+ \frac{\beta_0}{4\pi}\alpha^2 \frac{d\log[\gamma\sqrt{-\chi_0'(\gamma)}]}{d\alpha_s}.$$

3 LOOP:

$$\gamma^{gg}_{NLX,3} = \left(\frac{3\alpha}{\pi}\right)^3 \frac{1}{(N-1)^2}\ \frac{1}{4}\left[\frac{395}{27} + \frac{71}{81}N_f - \frac{\pi^2}{18}\left(11+\frac{2}{3}N_f\right) - 2\zeta(3)\right]$$

$\uparrow$

NO: $C_A^3\left(\frac{\alpha}{\pi}\right)^3\frac{1}{(N-1)^3}$ TERM.

HERE FOR THE 1ST TIME

RUNNING α - EFFECTS CONTRIBUTE !

STARTING WITH 4-LOOP:

- $\bar{\alpha}\left(\frac{\bar{\alpha}}{N-1}\right)^3\left[3.21\left(\frac{\beta_0}{4} - \frac{2}{9}\frac{C_F}{C_A}T_F\right) + \ldots\ \right]$ $\quad R(\gamma)$ $\quad Q_0 \to$ DIS SCHEME
- $G_+^g(t) = \exp\int_0^t dt\left[\gamma_+(t) - \frac{d}{dt}\log\left(\gamma_+\sqrt{-\chi_0'(\gamma_+)}\right)\right]$

$$= \gamma_+ - \frac{\beta_0}{12}\frac{\bar{\alpha}^4}{(N-1)^3}\left[6\zeta_3 - 2\zeta_3\frac{\bar{\alpha}}{N-1} + 20\,\zeta_5\left(\frac{\bar{\alpha}}{N-1}\right)^2 + \ldots\right]$$

$\uparrow$

VERY BIG TERM!

k	$g_{k,gg}^{(0)}$	$g_{k,qg}^{(1)}$ (Q_0)	$g_{k,qg}^{(1)}$ (DIS)	r_k	c_k^L
0	1.00000 E+0	1.00000 E+0	1.00000 E+0	1.00000 E+0	1.00000 E+0
1	0.00000 E+0	2.16667 E+0	2.16667 E+0	0.00000 E+0	−3.33333 E−1
2	0.00000 E+0	2.29951 E+0	2.29951 E+0	0.00000 E+0	2.13284 E+0
3	2.40411 E+1	5.06561 E+0	8.27109 E+0	3.20549 E+0	2.27231 E+0
4	0.00000 E+0	8.79145 E+0	1.49249 E+1	−8.11742 E−1	4.34344 E−1
5	2.07386 E+1	1.90521 E+1	2.92268 E+1	4.56248 E+1	2.02643 E+1
6	1.73393 E+1	4.58482 E+1	1.02812 E+2	3.27070 E+1	2.30315 E+1
7	2.01670 E+0	9.24159 E+1	1.94887 E+2	−2.95476 E+1	3.46449 E+1
8	3.98863 E+1	2.31063 E+2	4.85100 E+2	1.08183 E+2	2.65004 E+2
9	1.68747 E+2	5.59958 E+2	1.52444 E+3	3.99588 E+2	3.30038 E+2
10	6.99881 E+1	1.24822 E+3	3.11451 E+3	1.33228 E+2	8.50371 E+2
11	6.61253 E+2	3.25381 E+3	8.58375 E+3	2.10243 E+3	3.90849 E+3
12	1.94531 E+3	7.93653 E+3	2.47571 E+4	5.51142 E+3	5.67433 E+3
13	1.71768 E+3	1.89275 E+4	5.47435 E+4	5.30316 E+3	1.77680 E+4
14	1.06433 E+4	4.98520 E+4	1.56195 E+5	3.85296 E+4	6.21982 E+4
15	2.55668 E+4	1.23011 E+5	4.26980 E+5	8.49086 E+4	1.07028 E+5
16	3.67813 E+4	3.06504 E+5	1.01111 E+6	1.40384 E+5	3.51475 E+5
17	1.71685 E+5	8.07771 E+5	2.89398 E+6	6.94998 E+5	1.05058 E+6
18	3.75379 E+5	2.02210 E+6	7.69042 E+6	1.44307 E+6	2.10341 E+6
19	7.36025 E+5	5.17873 E+6	1.91919 E+7	3.22738 E+6	6.80747 E+6

k	$g_{k,gg}^{q\bar{q}\,(a)}$ (Q_0)	$g_{k,gg}^{q\bar{q}\,(b)}$ (Q_0)	$g_{k,gg}^{q\bar{q}\,(a)}$ (DIS)	$g_{k,gg}^{q\bar{q}\,(b)}$ (DIS)	$\Delta g_{k,gg}^{gg}$
0	−1.00000 E+0	0.00000 E+0	−1.00000 E+0	0.00000 E+0	−1.65000 E+1
1	−3.83333 E+0	0.00000 E+0	−3.83333 E+0	0.00000 E+0	0.00000 E+0
2	−2.29951 E+0	0.00000 E+0	−2.29951 E+0	0.00000 E+0	−2.78734 E+1
3	6.42072 E+0	−1.19004 E+2	−6.04506 E+0	3.96679 E+1	−2.25279 E+2
4	−2.59764 E+1	0.00000 E+0	−2.81814 E+1	−5.35750 E+1	−1.65583 E+2
5	5.75787 E+0	−3.42186 E+2	−2.60988 E+1	3.42186 E+1	−7.24788 E+2
6	1.21690 E+2	−2.28879 E+3	−9.43607 E+1	4.40583 E+2	−3.14501 E+3
7	−2.66365 E+2	−6.98786 E+2	−3.54981 E+2	−7.39527 E+2	−3.49585 E+3
8	5.43807 E+2	−1.11881 E+4	−4.27828 E+2	1.11801 E+3	−1.51028 E+4
9	1.96852 E+3	−4.10835 E+4	−1.67366 E+3	4.86665 E+3	−4.91970 E+4
10	−2.04998 E+3	−3.39345 E+4	−5.21390 E+3	−9.10195 E+3	−7.46877 E+4
11	1.49302 E+4	−2.75933 E+5	−7.99079 E+3	2.40902 E+4	−2.99245 E+5
12	3.33837 E+4	−7.55104 E+5	−3.05607 E+4	5.32758 E+4	−8.31843 E+5
13	9.19579 E+3	−1.10387 E+6	−8.37332 E+4	−9.58437 E+4	−1.59528 E+6
14	3.35804 E+5	−6.12763 E+6	−1.57171 E+5	4.46747 E+5	−5.82155 E+6
15	6.26484 E+5	−1.45966 E+7	−5.64262 E+5	5.92510 E+5	−1.49497 E+7
16	9.72892 E+5	−3.01102 E+7	−1.43675 E+6	−6.85258 E+5	−3.37088 E+7
17	7.05626 E+6	−1.30018 E+8	−3.14592 E+6	7.71985 E+6	−1.12828 E+8
18	1.29507 E+7	−2.96814 E+8	−1.05144 E+7	7.22515 E+6	−2.81522 E+8
19	3.18568 E+7	−7.45406 E+8	−2.59548 E+7	2.95797 E+6	−7.03719 E+8

Table 1: The numerical expansion coefficients for the anomalous dimensions and coefficient functions

THE RÔLE OF MEDIUM x TERMS

→ N-MODIFICATION OF:
- ANOMALOUS DIMENSIONS
- COEFFICIENT FUNCTIONS

→ IMPLIED BY CONSERVATION LAWS:

$$\int_0^1 dx \; P_{qq}^{-}(x) = 0$$

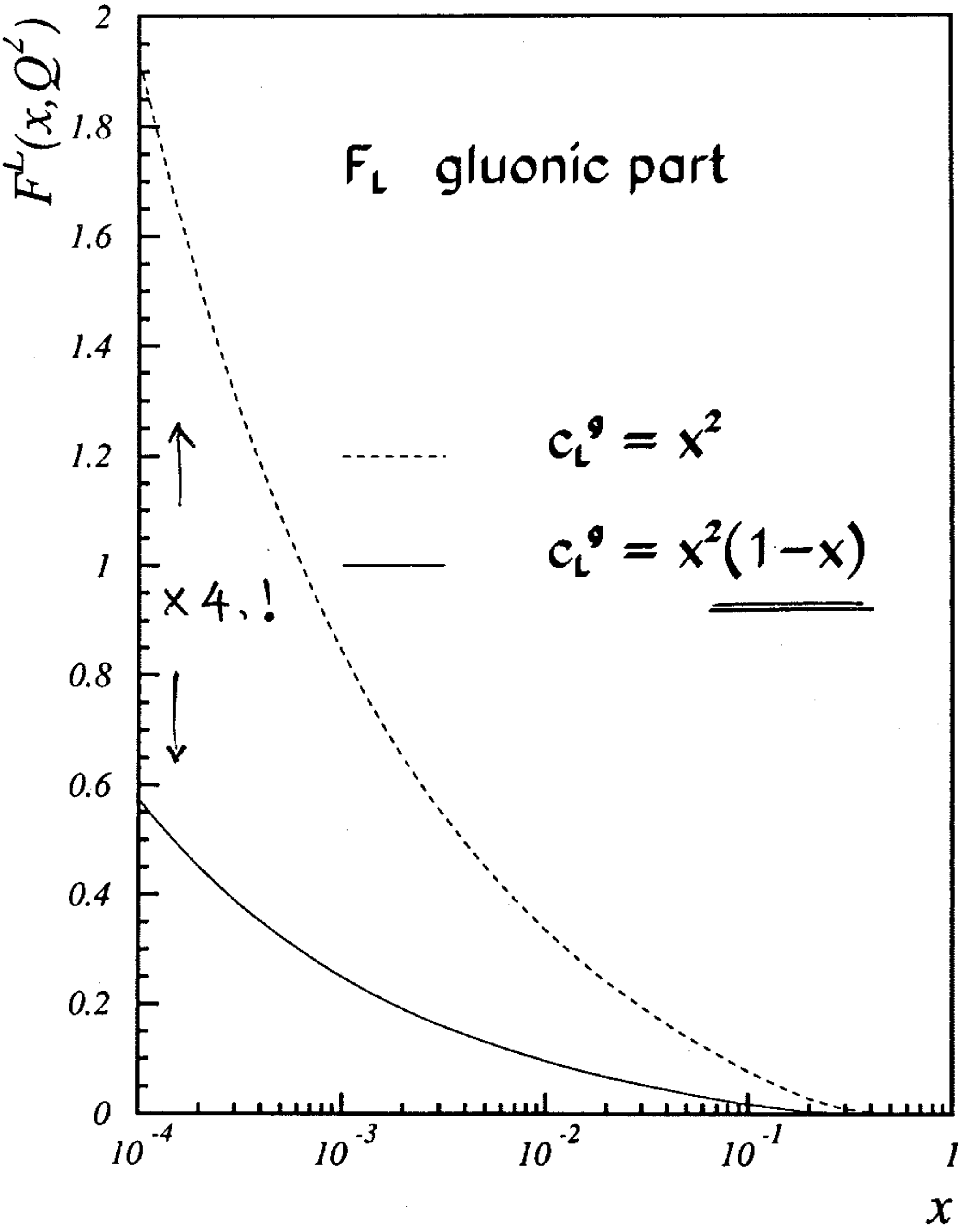
$$\int_0^1 dx \; x \left[2N_f \, P_{qg}(x) + P_{gg}(z) \right] = 0$$

$$\int_0^1 dx \; x \left[P_{qq}(z) + P_{gq}(z) \right] = 0.$$

→ MELLIN CONVOLUTIONS BETWEEN EVOLUTION KERNELS AND INPUT DISTRIBUTIONS.

THE RÔLE OF 'MEDIUM' x TERMS:

EXAMPLE:



- JB, MAR '93 (DURHAM)

- VAN NEERVEN '

$$F \propto K(x) \otimes G(x)$$

$G(x)$: peak @ small x, small @ large x

$K(x)$: MEDIUM x TERMS ARE IMPORTANT.

FIXED ORDER REVISITED: SUBLEADING TERMS

$$\gamma_{qq,\mathrm{LO}} = +10.8793\,N - 6.82222\,N^2 + O(N^3)\,,$$
$$\gamma_{qg,\mathrm{LO}} = -10.6667 + 11.5556\,N - 13.1852\,N^2 + O(N^3)\,,$$
$$\gamma_{gq,\mathrm{LO}} = -\frac{10.6667}{N} + 8.00000 - 9.3333\,N + 10.0000\,N^2 + O(N^3)\,,$$
$$\gamma_{gg,\mathrm{LO}} = -\frac{24.0000}{N} + 27.3333 - 5.1883\,N + 17.0395\,N^2 + O(N^3)\,.$$

$$\gamma^{\mathrm{DIS}}_{qq,\mathrm{NLO}} = -\frac{123.259}{N} + 405.863 - 684.836\,N + 1197.52N^2 + O(N^3)\,,$$
$$\gamma^{\mathrm{DIS}}_{qg,\mathrm{NLO}} = -\frac{277.333}{N} + 846.222 - 1706.18\,N + 2622.76\,N^2 + O(N^3)\,,$$
$$\gamma^{\mathrm{DIS}}_{gq,\mathrm{NLO}} = +\frac{91.2593}{N} - 453.512 + 809.030\,N - 1344.89\,N^2 + O(N^3)\,,$$
$$\gamma^{\mathrm{DIS}}_{gg,\mathrm{NLO}} = +\frac{245.333}{N} - 988.210 + 2093.25\,N - 3109.08\,N^2 + O(N^3)\,.$$

$$\gamma^{\overline{\mathrm{MS}}}_{qq,\mathrm{NLO}} = -\frac{94.8148}{N} + 253.026 - 337.185\,N + 623.259\,N^2 + O(N^3)\,,$$
$$\gamma^{\overline{\mathrm{MS}}}_{qg,\mathrm{NLO}} = -\frac{213.333}{N} + 461.449 - 889.687\,N + 1501.16\,N^2 + O(N^3)\,,$$
$$\gamma^{\overline{\mathrm{MS}}}_{gq,\mathrm{NLO}} = +\frac{62.8148}{N} - 361.805 + 658.108\,N - 1048.43\,N^2 + O(N^3)\,,$$
$$\gamma^{\overline{\mathrm{MS}}}_{gq,\mathrm{NLO}} = +\frac{216.889}{N} - 790.928 + 1616.55\,N - 2423.77\,N^2 + O(N^3)\,.$$

SUBL. TERMS: qq, qg, gq 'MODEL'

$$\gamma_{ij} \to \gamma_{ij}\,(1 - 2N + N^\alpha) \qquad \begin{cases} C: & \alpha = 2 \\ D: & \alpha = 3 \end{cases}$$

(CONSERVATIVE).

gg: NLx, ONLY N^α TERM ADDED.

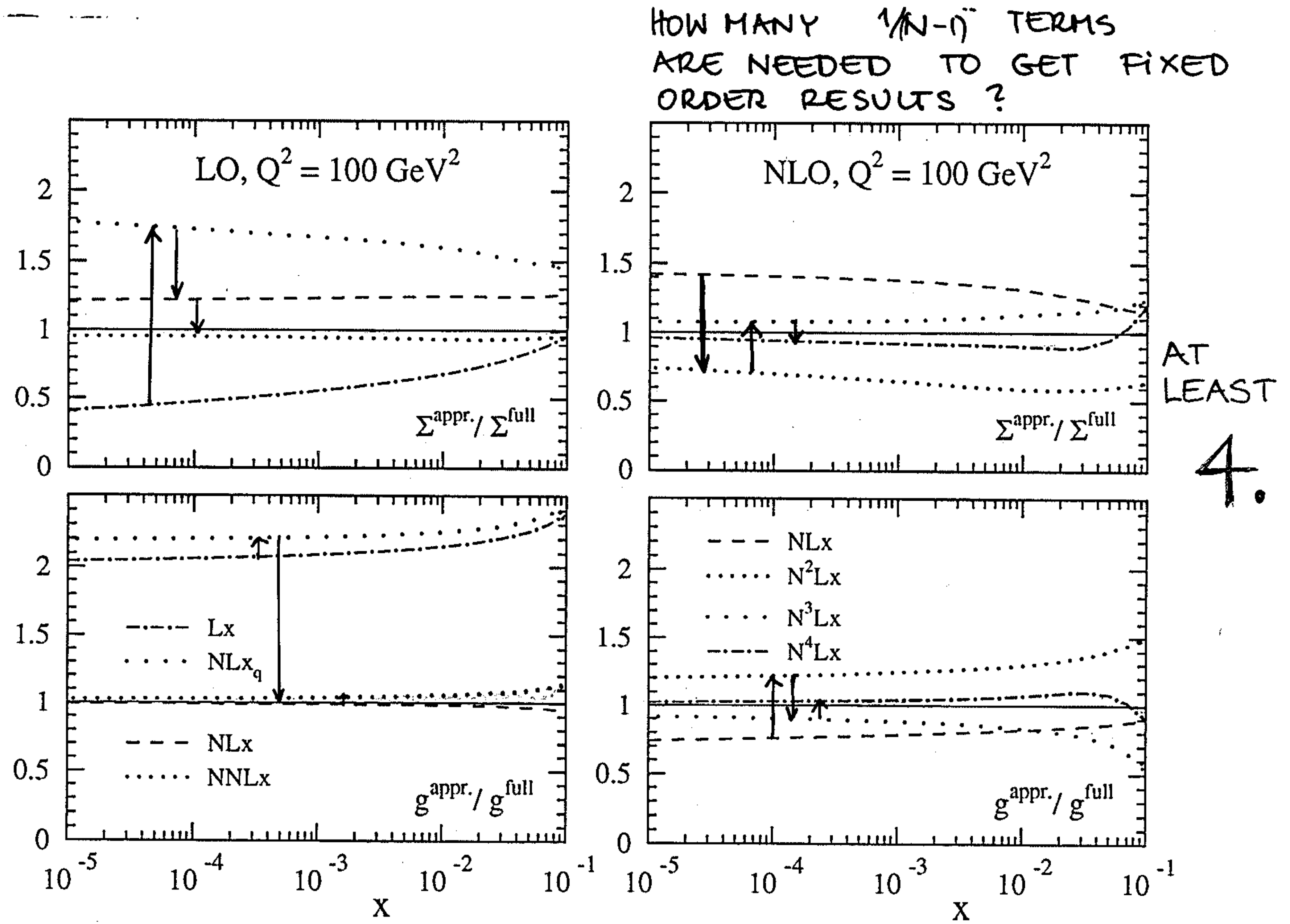
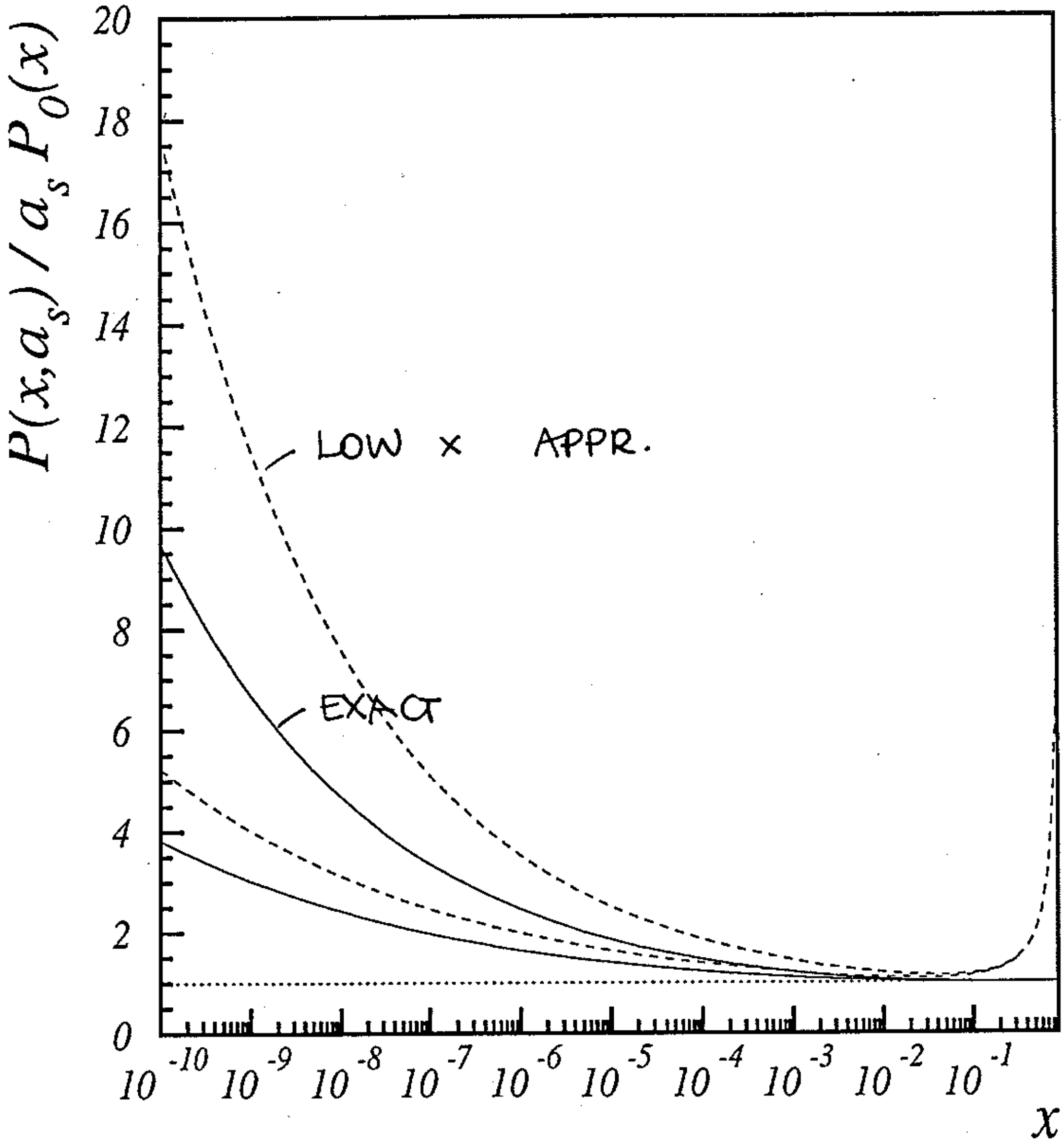

HOW MANY 1/(N-1)ⁿ TERMS ARE NEEDED TO GET FIXED ORDER RESULTS ?
LO, $Q^2 = 100$ GeV^2
$\Sigma^{appr.}/\Sigma^{full}$
NLO, $Q^2 = 100$ GeV^2
AT LEAST 4.
Lx
NLx_q
NLx
NNLx
$g^{appr.}/g^{full}$
NLx
N^2Lx
N^3Lx
N^4Lx
x
x

Fig. 6

AN EXACTLY SOLUBLE CASE : $\phi^3_{D=6}$

LX - RESUMMATION: LADDER

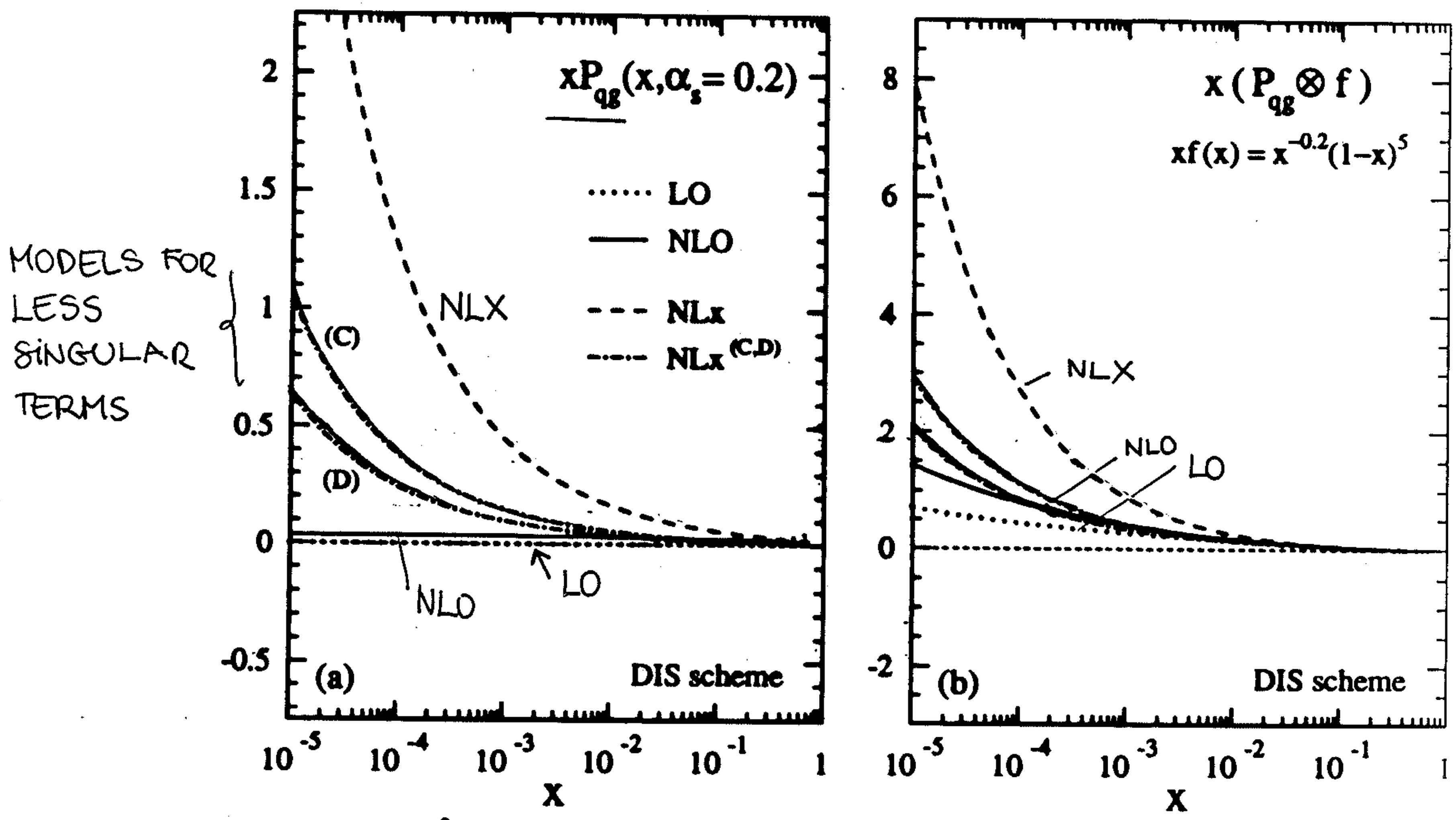


LOVELACE '75

JB, VAN NEERVEN '98

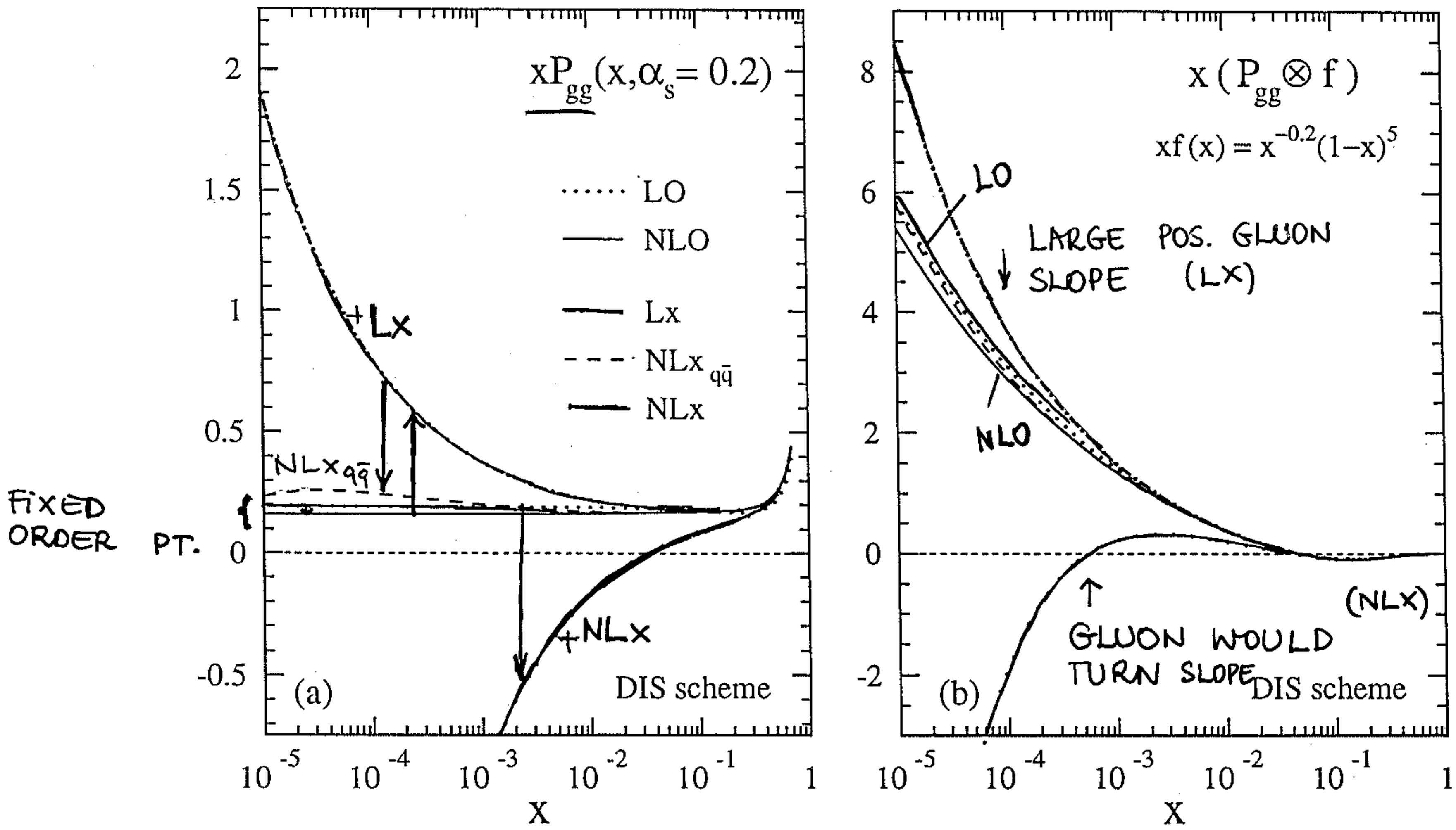
} NO SMALL x DOMINANCE !

EXACT ALL x SOLUTION
DIFFICULT TO GET IN QCD (NO METHOD KNOWN YET)

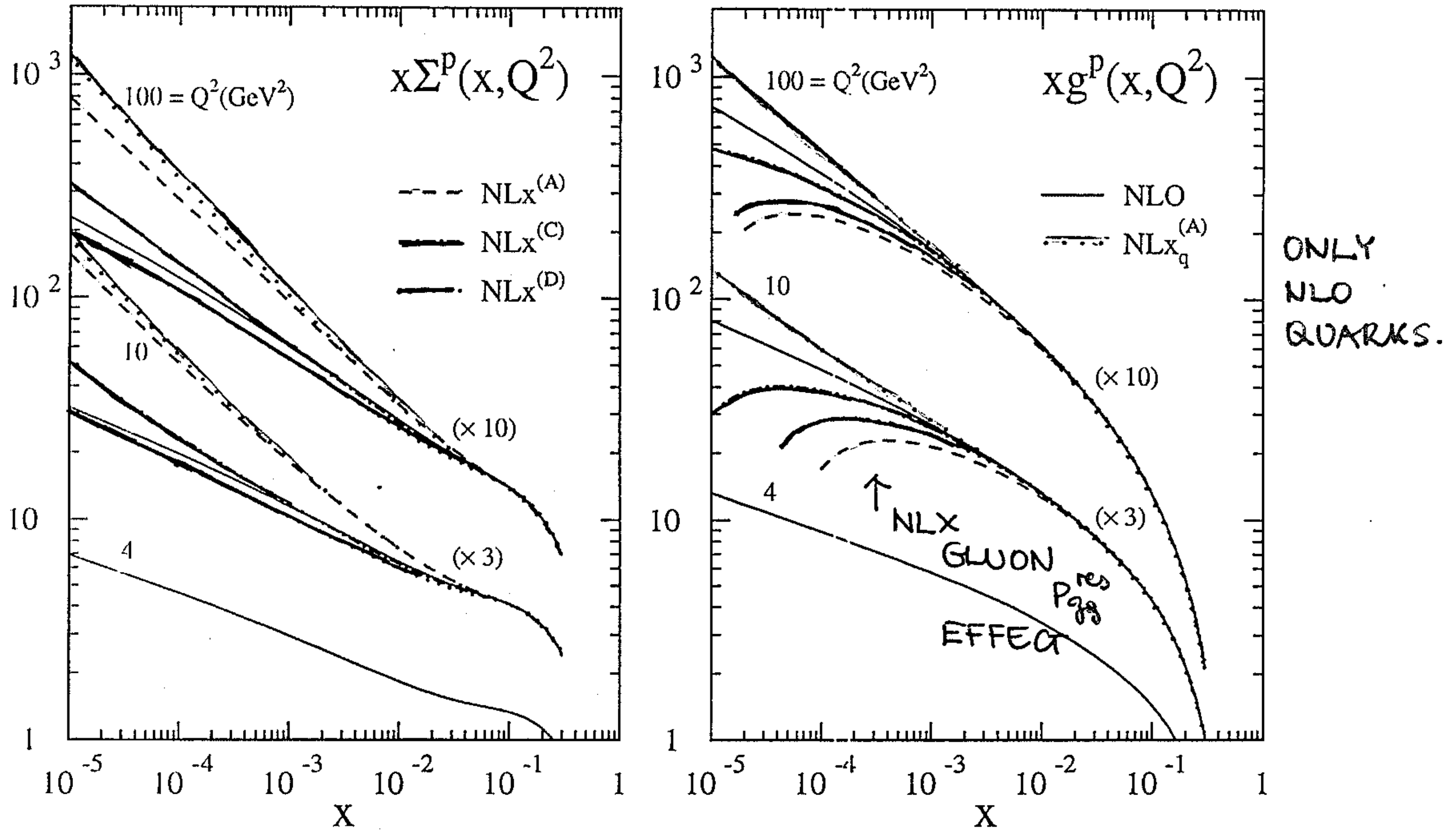


C: $\Gamma \to \Gamma \cdot (1-N)^2$

D: $\Gamma \to \Gamma \cdot (1-2N+N^3)$



NO LESS SINGULAR TERMS CONSIDERED.



A: $\Gamma \rightarrow \Gamma(N) - \Gamma(1)$

C: $\Gamma \rightarrow \Gamma(N) \cdot (1-N)^2$

D: $\Gamma \rightarrow \Gamma(N) \cdot (1 - 2N + N^3)$

4. Numerical results: ii) F_2 and F_L

EFFECT OF POSSIBLE SUB-LEADING TERMS.

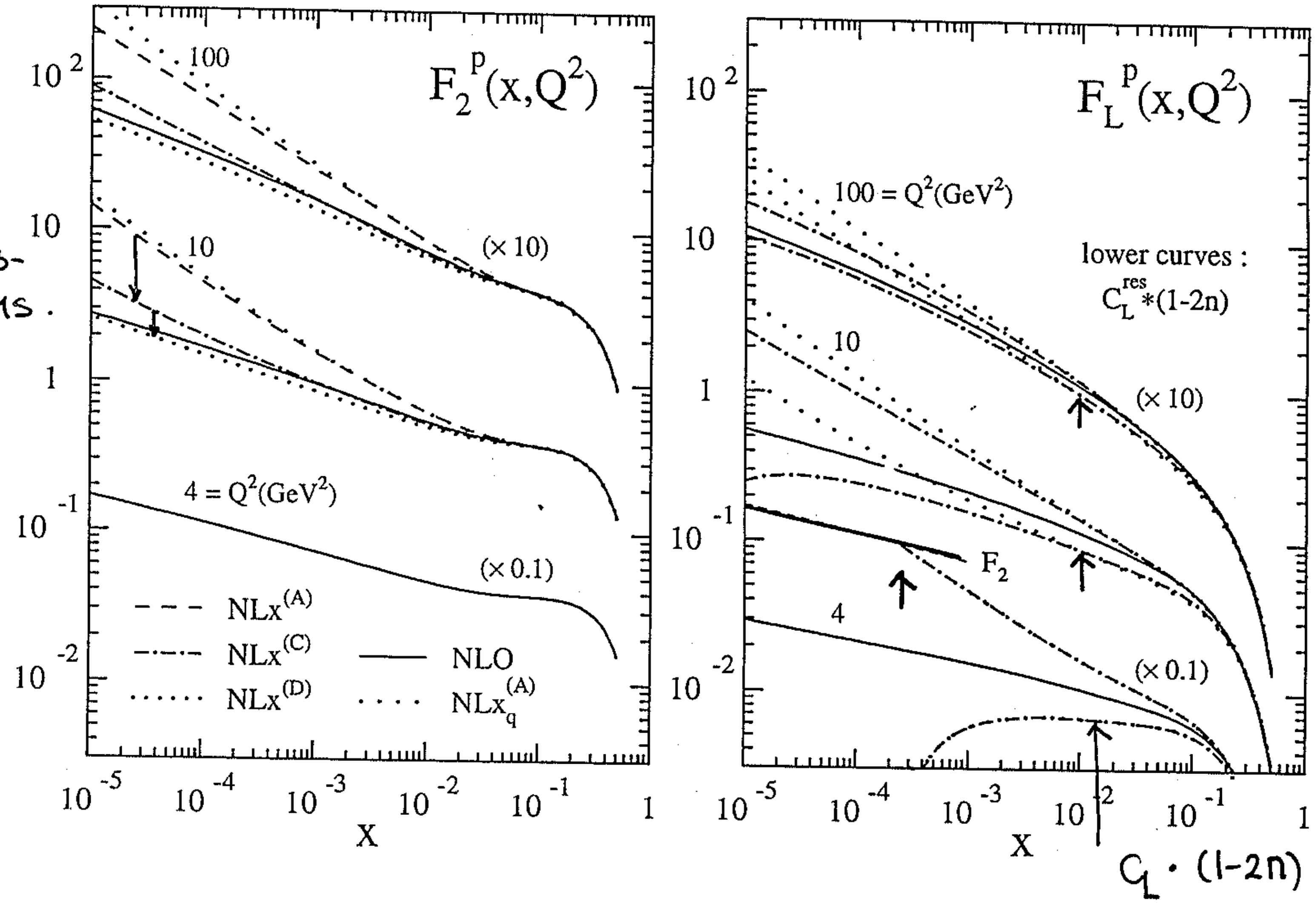
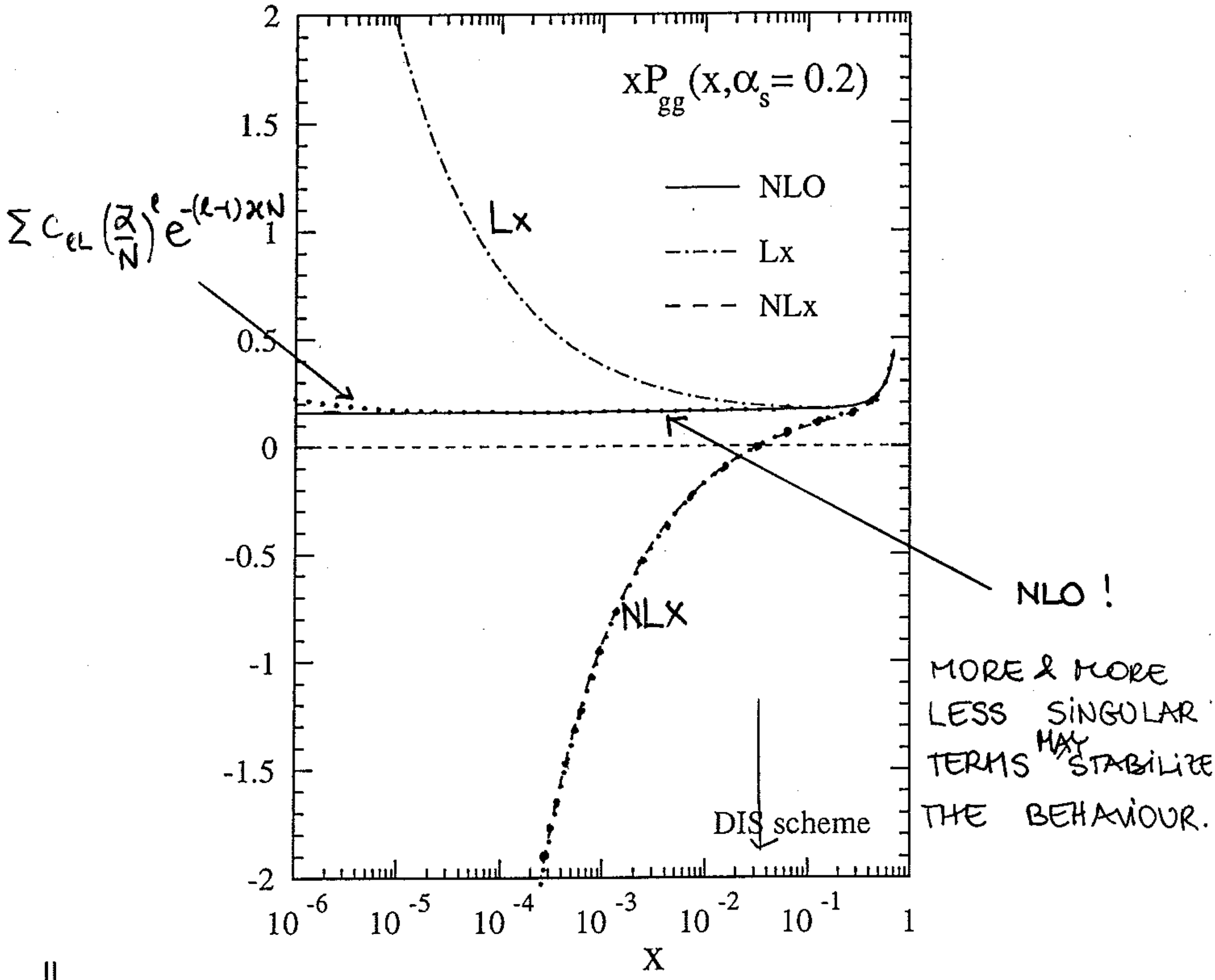


Fig. 10

Blümlein, A.V.



WHAT IFF THIS IS THE TOP OF AN ICEBERG ?

$$Lx = \sum_{\ell=1}^{\infty} C_L^{\ell} \left(\frac{\bar{\alpha}}{N}\right)^{\ell}$$

$$NLx \simeq \sum_{\ell=1}^{\infty} C_L^{\ell} \left(\frac{\bar{\alpha}}{N}\right)^{\ell} (1-(\ell-1)\varkappa N) \quad \cdots\cdots \text{ LINE.}$$

$\uparrow$ $\varkappa(\alpha=0.2) \simeq 3$

LINEAR TERM OUT OF AN EXPONENTIAL ?

5.3. The Conformal Part and s^ω

$$\chi_1(\gamma) = \chi_1^{scal}(\gamma) + \chi_1^{conf}(\gamma)$$

$$\chi_1^{scal}(\gamma) = \left(\frac{\beta_0}{6} + \frac{d}{d\gamma}\right)\left[\chi_0^2(\gamma) + \chi_0'(\gamma)\right]$$

S^ω - BEHAVIOUR : $\gamma \to \frac{1}{2}$

$$\omega_{(1)} = \omega_{(0)}\left[1 + \frac{\alpha_s^2}{\omega_{(0)}}\left[\chi_1^{scal}(\tfrac{1}{2}) + \chi_1^{conf}(\tfrac{1}{2})\right]\right]$$

$$\omega_{(0)} = 12 N_c \frac{\alpha_s}{3\pi} \log 2$$

$$\omega_1 = 2.65\,\alpha_s\left[1 - 6.36\,\alpha_s\right]$$

$\longrightarrow$ $\omega_1^{max} \sim 0.1 \approx 0.808$ @ $Q^2 = 8.7 \times 10^6\ GeV^2$.

$\omega < 0$ for $Q^2 < 600\ GeV^2$.

$$\omega_1^{conf} = 2.64\,\alpha_s\left[1 - 2.55\,\alpha_s\right]$$

$\longrightarrow$ $\omega_1^{max} = 0.26$ @ $Q^2 = 87\ GeV^2$

$\omega > 0$ for $Q^2 > 2\ GeV^2$.

THIS PART, AT LEAST, BEHAVES REASONABLE.
HOWEVER, IT IS NOT CLEAR WHAT IT MEANS.

6. Conclusions

1. THE RENORMALIZATION GROUP EQUATION FORMS, AS IN FIXED ORDER PERTURBATION THEORY, THE BASIS FOR THE EVOLUTION EQS. ALSO IF SMALL x TERMS ARE RESUMMED (LIGHTCONE DOMAIN).

2. DUE TO THE MELLIN-TYPE CONNECTION BETWEEN THE NON $\delta(1-x)$ - LIKE INPUT DENSITIES & EVOL. KERNELS MEDIUM x - TERMS ARE IMPORTANT.

3. LESS SINGULAR TERMS, e.g. $\Gamma_{sing} \cdot N^{1 \ldots 3}$ MAY BE AS IMPORTANT AS THE MOST SINGULAR ONES. THEY THEREFORE NEED TO BE CALCULATED. THIS IS CONFIRMED BY THE BEHAVIOUR FOUND IN φ_6^3.

4. CONSERVATION LAWS NEED TO BE RESTORED. $\longleftrightarrow$ INTEGRAL CONNECTIONS TO LARGE x.

5. THE KNOWN RESUMMATIONS BEHAVE PERFECT, AS FAR AS THEIR CONFORMAL PART IS CONCERNED.

6. HIGHER ORDER FIXED ORDER RESULT WOULD YIELD FURTHER IMPORTANT CROSS CHECKS: 3 LOOP (b_3 PROBLEM e.g.); DYNAMICAL EFFECTS: 4 LOOP.. ... SCHEMES.

7. THE 3 LOOP SPLITTING FUNCTIONS ARE NEEDED TO STABILIZE THE PICTURE, ALSO AT SMALL x.

8. THE TREMENDOUS EFFORT SPENT TO GET THERE, WHERE SMALL x RESUMMATIONS ARE TODAY LED TO A MUCH DEEPER INSIGHT INTO QCD-STRUCTURES.

THE PROBLEM IS STILL CHALLENGING.