QCD precision calculations: precision determinations of the fundamental parameters α_s and m_c

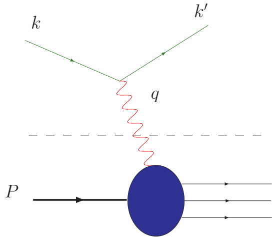
Kolloquium TU Dortmund, December 10, 2024

Johannes Blümlein | November 29, 2024

DESY AND TU DORTMUND

- J. Ablinger et al., The unpolarized and polarized single-mass three-loop heavy flavor operator matrix elements $A_{gg,Q}$ and $\Delta A_{gg,Q}$, JHEP **12** (2022) 134.
- A. Behring, J.B., and K. Schönwald, The inverse Mellin transform via analytic continuation, JHEP **06** (2023) 62.
- J. Ablinger et al., The first-order factorizable contributions to the three-loop massive operator matrix elements $A_{Qg}^{(3)}$ and $\Delta A_{Qg}^{(3)}$, Nucl. Phys.B 999 (2024) 116427.
- J. Ablinger et al., The non-first-order-factorizable contributions to the three-loop single-mass operator matrix elements $A_{Qg}^{(3)}$ and $\Delta A_{Qg}^{(3)}$, 2403.00513 [hep-ph].

Deep–Inelastic Scattering (DIS):



$\longrightarrow \quad L_{\mu\nu}$

$\longrightarrow \quad W_{\mu\nu}$

$$Q^2 := -q^2, \quad x := \frac{Q^2}{2P.q}$$ Bjorken–x

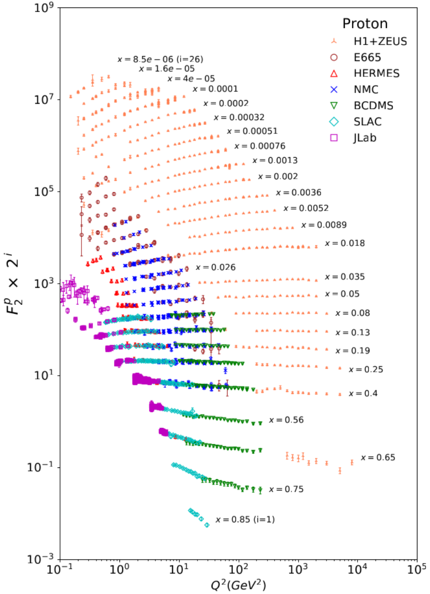
$$\frac{d\sigma}{dQ^2\, dx} \sim W_{\mu\nu} L^{\mu\nu}$$

$$W_{\mu\nu}(q,P,s) = \frac{1}{4\pi}\int d^4\xi \exp(iq\xi)\langle P,s \mid [J_\mu^{em}(\xi), J_\nu^{em}(0)] \mid P,s\rangle =$$

$$\frac{1}{2x}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) F_L(x,Q^2) + \frac{2x}{Q^2}\left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu}\right) F_2(x,Q^2)$$

$$+i\varepsilon_{\mu\nu\lambda\sigma}\frac{q^\lambda S^\sigma}{P.q} g_1(x,Q^2) + i\varepsilon_{\mu\nu\lambda\sigma}\frac{q^\lambda (P.qS^\sigma - S.qP^\sigma)}{(P.q)^2} g_2(x,Q^2)\ .$$

The structure functions $F_{2,L}$ and $g_{1,2}$ contain light and heavy quark contributions.
At 3-loop order also graphs with two heavy quarks of different mass contribute.
$\Longrightarrow$ Single and 2-mass contributions: c and b quarks in one graph.



The current DIS world data for $F_2^p(x, Q^2)$.

Aim of precision data analyses in particle physics:

- fix or constrain the fundamental parameters of the Standard Model
  - coupling constants α, G_F, α_s
  - fundamental particle masses $m_e, m_\mu, m_\tau, m_{\nu_i}, m_c, m_b, m_t, m_u, m_d, m_s$
- determine non-perturbative structures
  - twist-2 parton distribution functions of nucleons [large Q^2 domain]
  - higher twist contributions of nucleons [low Q^2 domain, large x domain]
  - unpolarized case
  - polarized case $\Longrightarrow$ nucleon spin problem
- DIS: $\alpha_s(M_Z^2), m_c$ [m_b to a lesser extent]
- needed high enough massless and massive QCD corrections
- current status: next-to-next-to-leading order (NNLO) corrections completed.

- Calculate all massless corrections to NNLO
- Calculate the single mass corrections to $O(\alpha_s^3)$
- Calculate the two-mass corrections [m_c & m_b] to $O(\alpha_s^3)$

Analysis Levels of Experimental Data

Unpolarized F_2:

- Leading order: $>$ 1976
- Next-to-Leading order: $>$ 1992
- Next-to-Next-to-Leading order: $>$ 2024

Massless Case only:

- Leading order: $>$ 1974
- Next-to-Leading order: $>$ 1982
- Next-to-Next-to-Leading order: $>$ 2004

Polarized g_1:

- Leading order: $>$ 1982
- Next-to-Leading order: $>$ 1996
- Next-to-Next-to-Leading order: $>$ 2024

Massless Case only:

- Leading order: $>$ 1976
- Next-to-Leading order: $>$ 1996
- Next-to-Next-to-Leading order: $>$ 2014

Announced "aN^{3}LO" analyses are heavily incomplete.

Factorization of the Structure Functions

At leading twist the structure functions factorize in terms of a Mellin convolution

$$F_{(2,L)}(x,Q^2) = \sum_j \underbrace{\mathbb{C}_{j,(2,L)}\left(x,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)}_{perturbative} \otimes \underbrace{f_j(x,\mu^2)}_{nonpert.}$$

into (pert.) Wilson coefficients and (nonpert.) parton distribution functions (PDFs).
$\otimes$ denotes the Mellin convolution

$$f(x) \otimes g(x) \equiv \int_0^1 dy \int_0^1 dz \; \delta(x - yz) f(y) g(z) \; .$$

Many of the subsequent calculations are performed in Mellin space, where $\otimes$ reduces to a multiplication, due to the Mellin transformation

$$\hat{f}(N) = \int_0^1 dx \; x^{N-1} f(x) \; .$$

Wilson coefficients:

$$\mathbb{C}_{j,(2,L)}\left(N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) = C_{j,(2,L)}\left(N, \frac{Q^2}{\mu^2}\right) + H_{j,(2,L)}\left(N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) .$$

At $Q^2 \gg m^2$ the heavy flavor part

$$H_{j,(2,L)}\left(N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) = \sum_i C_{i,(2,L)}\left(N, \frac{Q^2}{\mu^2}\right) A_{ij}\left(\frac{m^2}{\mu^2}, N\right)$$

[Buza, Matiounine, Smith, van Neerven 1996]

factorizes into the light flavor Wilson coefficients C and the massive operator matrix elements (OMEs) of local operators O_i between partonic states j

$$A_{ij}\left(\frac{m^2}{\mu^2}, N\right) = \langle j \mid O_i \mid j \rangle .$$

$\rightarrow$ additional Feynman rules with local operator insertions for partonic matrix elements.

The unpolarized light flavor Wilson coefficients are known up to NNLO

[Vermaseren, Moch, Vogt, 2005; JB, Marquard, Schneider, Schönwald, 2022].

For $F_2(x, Q^2)$: at $Q^2 \gtrsim 10m^2$ the asymptotic representation holds at the 1% level.

Why take large projects rather long ?

Examples:

- unpolarized anomalous dimensions and massless DIS Wilson coefficients
  [Vermaseren, Larin, Nogueira, van Ritbergen, Moch, Vogt] 1990-2005: 15 years
  function space: harmonic sums
- unpolarized and polarized massive OMEs and asymptotic Wilson coefficients
  [DESY-Linz collaboration] 2009 - 2024: 15 years
  function spaces: harmonic sums, generalized harmonic sums, finite cyclotomic sums, finite binomial sums, elliptic integrals, higher transcendental ${}_pF_q$ structures

- Initially the function spaces contributing were unknown.
- How to solve the systems of difference equations for the contributing topologies ?
- How to process the differential equations of the master integrals to provide large numbers of moments ?
- How to deal with all first order factorizing difference equations ?
- How to solve the elliptic-affected part ?
- How to tackle 2-mass problems analytically ?
- Are N-space solutions providing the right framework ? [Non-first order factorizable recurrences.]
- Do the computer resources suffice [in space and time] to establish all contributing recurrences ?

Why take large projects rather long ?

- At present, massless 3-loop problems are no problem anymore.
  - Typical computation times $O(1 year)$; pole-terms: $O(month)$.
  - Basically all technologies needed are available in (private) codes.
  - Example: Polarized massless 3-loop Wilson coefficients for DIS.

These calculations are modern adventures.

One enters a terra incognita with rough ideas but insufficient means and one has to develop new technologies all the way along to get through. In this way one lifts the whole field to new levels, which allows to perform many more calculations.

One has to pass many intermediate stops (no-goes) to arrive at the final complete solution: the strategic goal.

The main time-line for the 3-loop corrections

- 2005 F_L [no massive 3-loop OMEs needed] – Valid at $Q^2 > 800$ GeV^2 only.
- 2010 All unpolarized N_F terms and $A_{qg,Q}^{(3)}, A_{qq,Q}^{(3).PS}$
- 2014 unpolarized logarithmic 3-loop contributions and $A_{gq,Q}^{(3)}, (\Delta)A_{qq,Q}^{(3),NS}, A_{Qq}^{(3),PS}$
- 2017 two-mass corrections $A_{gq,Q}^{(3)}, (\Delta)A_{qq,Q}^{(3),NS}, A_{Qq}^{(3),PS}$
- 2018 two-mass corrections $A_{gg,Q}^{(3)}$
- 2019 2-loop correction: $(\Delta)A_{Qq}^{(2),PS}$ whole kinematic region and $\Delta A_{Qq}^{(3),PS}$
- 2019 two-mass corrections $\Delta A_{Qq}^{(3),PS}$
- 2020 two-mass corrections $\Delta A_{gg,Q}^{(3)}$
- 2021 polarized logarithmic 3-loop contributions and $\Delta A_{qg,Q}^{(3)}, \Delta A_{qq,Q}^{(3).PS}, \Delta A_{gq}^{(3)}$
- 2022 3-loop polarized massless Wilson coefficients [JB, Marquard, Schneider, Schönwald]
- 2022 corrected the polarized 2-loop contributions
- 2022 $(\Delta)A_{gg,Q}^{(3)}$
- 2023 $(\Delta)A_{Qg}^{(3)}$: 1st order factorizing parts
- 2024 $(\Delta)A_{Qg}^{(3)}$, [two-mass corrections $(\Delta)A_{Qg}^{(3)}$]

- 45 physics papers (journals)
- 26 mathematical papers
  - 1998 Harmonic sums [Vermaseren; JB]
  - 2000,2005 Analytic continuations of harmonic sums to $N \in \mathbb{C}$ [JB; JB, S. Moch]
  - 2003 Concrete shuffle algebras [JB]
  - 2009 Guessing large recurrences [JB, M. Kauers, S. Klein, C. Schneider]
  - 2009 Structural relations of harmonic sums [JB]
  - 2009 MZV Data mine [JB, D. Broadhurst, J. Vermaseren]
  - 2011 Cyclotomic harmonic sums and harmonic polylogarithms [Ablinger, JB, Schneider]
  - 2013 Generalized harmonic sums and harmonic polylogarithms [Ablinger, JB, Schneider]; 2001 [Moch, Uwer, Weinzierl]
  - 2014 Finite binomial sums and root-valued iterated integrals [Ablinger, JB, Raab, Schneider]
  - 2017 ${}_2F_1$ solutions (iterated non-iterative integrals) [J. Ablinger, JB, A. De Freitas, M. van Hoeij, E. Imamoglu, C. Raab, C.S. Radu, C. Schneider]
  - 2017 Methods of arbitrary high moments [JB, Schneider]
  - 2018 Automated solution of first-order factorizing differential equation systems in an arbitrary basis [J. Ablinger, JB, P. Marquard, N. Rana, C. Schneider]
  - 2023 Analytic continuation form t to x-space [JB, Behring, Schönwald]

Important Computer-Algebra Packages

C. Schneider: `Sigma, EvaluateMultiSums, SumProduction, SolveCoupledSystem`

J. Ablinger: `HarmonicSums`

Function Spaces

Sums

Harmonic Sums

$$\sum_{k=1}^{N} \frac{1}{k} \sum_{l=1}^{k} \frac{(-1)^l}{l^3}$$

gen. Harmonic Sums

$$\sum_{k=1}^{N} \frac{(1/2)^k}{k} \sum_{l=1}^{k} \frac{(-1)^l}{l^3}$$

Cycl. Harmonic Sums

$$\sum_{k=1}^{N} \frac{1}{(2k+1)} \sum_{l=1}^{k} \frac{(-1)^l}{l^3}$$

Binomial Sums

$$\sum_{k=1}^{N} \frac{1}{k^2} \binom{2k}{k} (-1)^k$$

Integrals

Harmonic Polylogarithms

$$\int_0^x \frac{dy}{y} \int_0^y \frac{dz}{1+z}$$

gen. Harmonic Polylogarithms

$$\int_0^x \frac{dy}{y} \int_0^y \frac{dz}{z-3}$$

Cycl. Harmonic Polylogarithms

$$\int_0^x \frac{dy}{1+y^2} \int_0^y \frac{dz}{1-z+z^2}$$

root-valued iterated integrals

$$\int_0^x \frac{dy}{y} \int_0^y \frac{dz}{z\sqrt{1+z}}$$

iterated integrals containing elliptic structures

$$\int_0^z dx \frac{\ln(x)}{1+x} {}_2F_1\left[\begin{matrix}\frac{4}{3},\frac{5}{3}\\ 2\end{matrix}; \frac{x^2(x^2-9)^2}{(x^2+3)^3}\right]$$

Special Numbers

multiple zeta values

$$\int_0^1 dx \frac{\mathrm{Li}_3(x)}{1+x} = -2\mathrm{Li}_4(1/2) + \ldots$$

gen. multiple zeta values

$$\int_0^1 dx \frac{\ln(x+2)}{x-3/2} = \mathrm{Li}_2(1/3) + \ldots$$

cycl. multiple zeta values

$$\mathbf{C} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$$

associated numbers

$$\mathrm{H}_{8,w_3} = 2\mathrm{arccot}(\sqrt{7})^2$$

associated numbers

$$\int_0^1 dx \, {}_2F_1\left[\begin{matrix}\frac{4}{3},\frac{5}{3}\\ 2\end{matrix}; \frac{x^2(x^2-9)^2}{(x^2+3)^3}\right]$$

shuffle, stuffle, and various structural relations $\Longrightarrow$ algebras

The Wilson Coefficients at large Q^2

$$L_{q,(2,L)}^{\rm NS}(N_F+1) = a_s^2\left[A_{qq,Q}^{(2),\rm NS}(N_F+1)\delta_2 + \hat{C}_{q,(2,L)}^{(2),\rm NS}(N_F)\right] + a_s^3\left[A_{qq,Q}^{(3),\rm NS}(N_F+1)\delta_2 + A_{qq,Q}^{(2),\rm NS}(N_F+1)C_{q,(2,L)}^{(1),\rm NS}(N_F+1) + \hat{C}_{q,(2,L)}^{(3),\rm NS}(N_F)\right]$$

$$L_{q,(2,L)}^{\rm PS}(N_F+1) = a_s^3\left[A_{qq,Q}^{(3),\rm PS}(N_F+1)\delta_2 + N_F A_{gq,Q}^{(2),\rm NS}(N_F)\tilde{C}_{g,(2,L)}^{(1),\rm NS}(N_F+1) + N_F\hat{\tilde{C}}_{q,(2,L)}^{(3),\rm PS}(N_F)\right]$$

$$L_{g,(2,L)}^{\rm S}(N_F+1) = a_s^2 A_{gg,Q}^{(1)}(N_F+1)N_F\tilde{C}_{g,(2,L)}^{(2)}(N_F+1) + a_s^3\left[A_{qg,Q}^{(3)}(N_F+1)\delta_2 + A_{gg,Q}^{(1)}(N_F+1)N_F\tilde{C}_{g,(2,L)}^{(2)}(N_F+1)\right.$$
$$\left. + A_{gg,Q}^{(2)}(N_F+1)N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + A_{Qg}^{(1)}(N_F+1)N_F\tilde{C}_{q,(2,L)}^{(2),\rm PS}(N_F+1) + N_F\hat{\tilde{C}}_{g,(2,L)}^{(3)}(N_F)\right]$$

$$H_{q,(2,L)}^{\rm PS}(N_F+1) = a_s^2\left[A_{Qq}^{(2),\rm PS}(N_F+1)\delta_2 + \tilde{C}_{q,(2,L)}^{(2),\rm PS}(N_F+1)\right]$$
$$+ a_s^3\left[A_{Qq}^{(3),\rm PS}(N_F+1)\delta_2 + A_{gq,Q}^{(2)}(N_F+1)\tilde{C}_{g,(1,L)}^{(2)}(N_F+1) + A_{Qq}^{(2),\rm PS}(N_F+1)\tilde{C}_{q,(2,L)}^{(1),\rm NS}(N_F+1) + \tilde{C}_{q,(2,L)}^{(3),\rm PS}(N_F+1)\right]$$

$$H_{g,(2,L)}^{\rm S}(N_F+1) = a_s\left[A_{Qg}^{(1)}(N_F+1)\delta_2 + \tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\right]$$
$$+ a_s^2\left[A_{Qg}^{(2)}(N_F+1)\delta_2 + A_{Qg}^{(1)}(N_F+1)\tilde{C}_{q,(2,L)}^{(1)}(N_F+1) + A_{gg,Q}^{(1)}(N_F+1)\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + \tilde{C}_{g,(2,L)}^{(2)}(N_F+1)\right]$$
$$+ a_s^3\left[A_{Qg}^{(3)}(N_F+1)\delta_2 + A_{Qg}^{(2)}(N_F+1)\tilde{C}_{q,(2,L)}^{(1)}(N_F+1) + A_{gg,Q}^{(2)}(N_F+1)\tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\right.$$
$$\left. + A_{Qg}^{(1)}(N_F+1)\tilde{C}_{q,(2,L)}^{(2),\rm S}(N_F+1) + A_{gg,Q}^{(1)}(N_F+1)\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + \tilde{C}_{g,(2,L)}^{(3)}(N_F+1)\right]$$

- The case for two different masses obeys an analogous representation.
- Note the contributions of the massless Wilson coefficients.

The variable flavor number scheme

- Matching conditions for parton distribution functions:

$$f_k(N_F+2)+f_{\overline{k}}(N_F+2) = A_{qq,Q}^{\rm NS}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\cdot\left[f_k(N_F)+f_{\overline{k}}(N_F)\right]+\frac{1}{N_F}A_{qq,Q}^{\rm PS}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\cdot\Sigma(N_F)$$
$$+\frac{1}{N_F}A_{qg,Q}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\cdot G(N_F)\,,$$

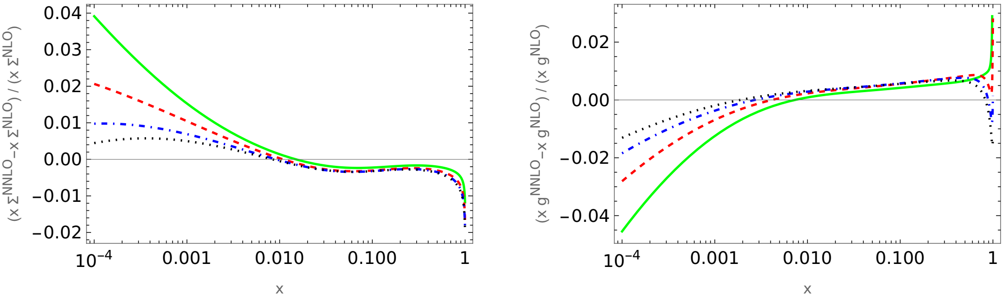
$$f_Q(N_F+2)+f_{\overline{Q}}(N_F+2) = A_{Qq}^{\rm PS}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2},\right)\cdot\Sigma(N_F)+A_{Qg}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\cdot G(N_F)\,,$$

$$\Sigma(N_F+2) = \left[A_{qq,Q}^{\rm NS}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)+A_{qq,Q}^{\rm PS}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)+A_{Qq}^{\rm PS}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\right]\cdot\Sigma(N_F)$$
$$+\left[A_{qg,Q}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)+A_{Qg}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\right]\cdot G(N_F)\,,$$

$$G(N_F+2) = A_{gq,Q}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\cdot\Sigma(N_F)+A_{gg,Q}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\cdot G(N_F)\,.$$

The charm and bottom quark masses are not that much different.

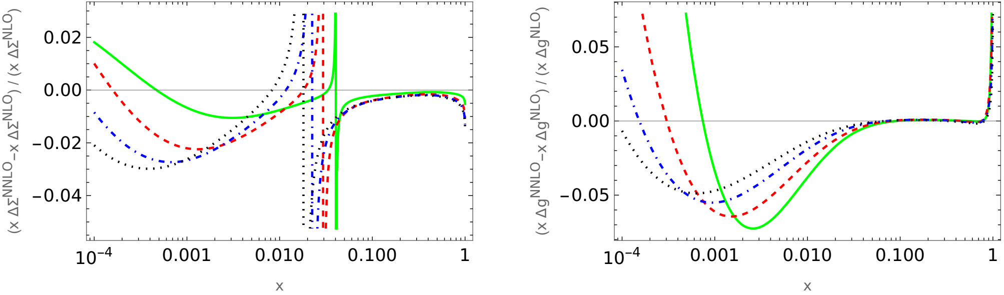
Relative effect in unpolarized NNLO evolution



$Q^2 = 10, 10^2, 10^3, 10^4$ GeV^2 dotted, dash-dotted, dashed, full lines. [M. Saragnese, 2022]

The unpolarized world deep-inelastic data have a precision of $O(1\%)$.

Relative effect in polarized NNLO evolution

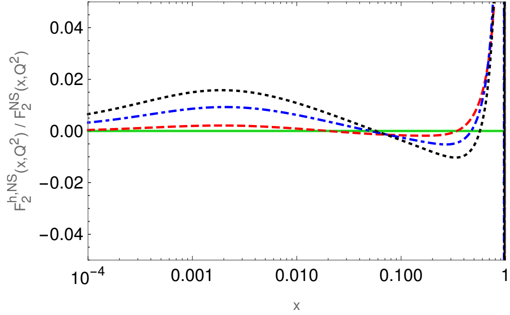


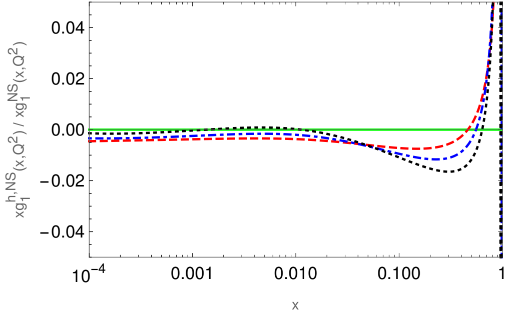
$Q^2 = 10, 10^2, 10^3, 10^4$ GeV^2 dotted, dash-dotted, dashed, full lines. [M. Saragnese, 2022]

The future polarized data at the EIC will reach a precision of $O(1\%)$.

The relative contribution of HQ to non-singlet structure functions at N^{3}LO

Scheme-invariant evolution





Left: The relative contribution of the heavy flavor contributions due to *c* and *b* quarks to the structure function $F_2^{\rm NS}$ at N^{3}LO; dashed lines: 100 GeV^2; dashed-dotted lines: 1000 GeV^2; dotted lines: 10000 GeV^2. Right: The same for the structure function $xg_1^{\rm NS}$ at N^{3}LO. [JB, M. Saragnese, 2021].

Mathematical Background

- massless and massive contributions to two-loops: harmonic sums
- all pole terms to three-loops: harmonic sums
- all massless Wilson coefficients to three-loops: harmonic sums

Single-mass OMEs

- all N_F of the massive OMEs three-loops: harmonic sums
- $(\Delta)A_{qq,Q}^{(3),\mathrm{NS}}, (\Delta)A_{gq,Q}^{(3)}, (\Delta)A_{qg,Q}^{(3)}, (\Delta)A_{qq,Q}^{(3),\mathrm{PS}}$ to three-loops: harmonic sums
- $(\Delta)A_{Qq}^{(3),\mathrm{PS}}$ to three-loops: generalized harmonic sums and also $H_{\vec{a}}(1-2x)$
- $(\Delta)A_{gg,Q}^{(3)}$ to three-loops: finite binomial sums and square-root valued iterated integrals
- $(\Delta)A_{Qg}^{(3)}$ to three-loops:
  - first-order factorizing contributions: finite binomial sums; all iterated integrals in x-space can be rationalized
  - non-first-order factorizing contributions: ${}_2F_1$ letters in iterated integrals in x-space

Two-mass OMEs

- $(\Delta)A_{qq,Q}^{(3),\mathrm{NS}}, (\Delta)A_{gq,Q}^{(3)}$: harmonic sums
- $(\Delta)A_{Qq}^{(3),PS}$: analytic solutions in x-space only; different supports; root-values letters
- $(\Delta)A_{gg,Q}^{(3)}$: root-valued iterated integrals

Integral structure in x space

$$G_{a,\vec{b}}(x) = \int_0^x dx_1 f_a(x_1) G_{\vec{b}}(x_1)$$

Alphabet $\mathfrak{A}$:

$$\mathfrak{A} = \left\{ f_1(x), \dots, f_k(x) \right\}$$

The functions $f_c(x)$ are elementary functions of the type $1/p(x)$ with $p(x)$ polynomials, starting with

$$\left\{ \frac{1}{x}, \frac{1}{1-x}, \frac{1}{1+x} \right\},$$

the cyclotomic polynomials and others and square root valued letters. These case are first-order factorizable.

In the case that also higher transcendental functions emerge in the letters, having only definite integral representations, one speaks about non first-order factorizable iterated integrals.

Iterative non-iterative Integrals

- Master integrals, solving differential equations not factorizing to 1st order
- $_2F_1$ solutions Ablinger et al. [2017]
- Mapping to complete elliptic integrals: duplication of the higher transcendental letters.
- Complete elliptic integrals, modular forms Sabry, Broadhurst, Weinzierl, Remiddi, Tancredi, Duhr, Broedel et al. and many more
- Abel integrals
- K3 surfaces Brown, Schnetz [2012]
- Calabi-Yau motives Klemm, Duhr, Weinzierl et al. [2022]

Refer to as few as possible higher transcendental functions, the properties of which are known in full detail.

- $A_{Qg}^{(3)}$: effectively only one 3×3 system of this kind.
- The system is connected to that occurring in the case of ρ parameter. Ablinger et al. [2017], JB et al. [2018], Abreu et al. [2019]
- Most simple solution: two $_2F_1$ functions.

Massless Corrections

$$\frac{df_i^{\rm NS}(N,\mu^2)}{d\ln(\mu^2)} = -\gamma_{qq}^{\rm NS}(N,a_s) f_i^{\rm NS}(N,\mu^2),$$

$$\frac{d}{d\ln(\mu^2)}\begin{bmatrix}\Sigma(N,\mu^2)\\ G(N,\mu^2)\end{bmatrix} = -\begin{bmatrix}\gamma_{qq}(N,a_s) & \gamma_{qg}(N,a_s)\\ \gamma_{gq}(N,a_s) & \gamma_{gg}(N,a_s)\end{bmatrix}\begin{bmatrix}\Sigma(N,\mu^2)\\ G(N,\mu^2)\end{bmatrix},$$

with $i = 1,2,3$ and $a_s = a_s(\mu^2)$.

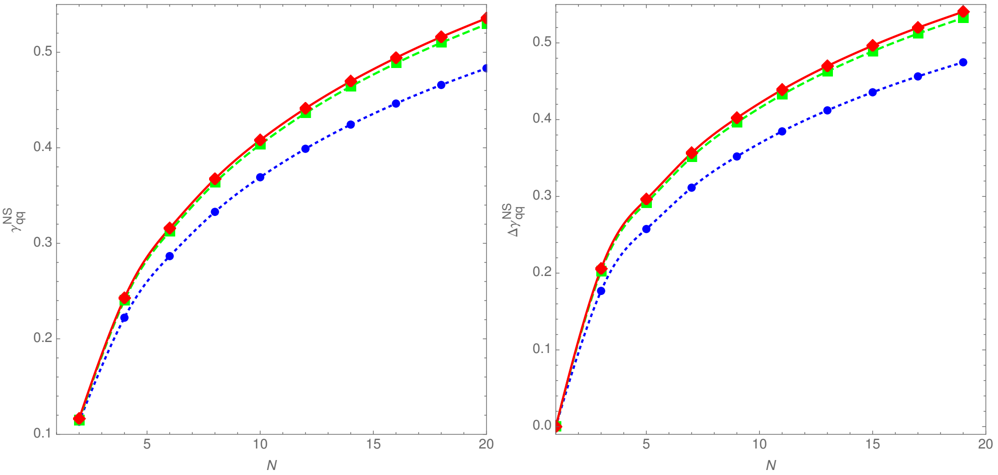
$$\begin{aligned}
C_q^{\rm NS}(a_s,N) &= 1 + a_s C_q^{{\rm NS},(1)}(N) + a_s^2 C_q^{{\rm NS},(2)}(N) + a_s^3\left[C_q^{{\rm NS},(3)}(N) + {\rm w}_2(N_F) C_{q,d_{abc}}^{{\rm NS},(3)}(N)\right]\\
C_q^{\rm PS}(a_s,N) &= a_s^2 C_q^{{\rm PS},(2)}(N) + a_s^3 C_q^{{\rm PS},(3)}(N)\\
C_g^{\rm S}(a_s,N) &= a_s C_g^{{\rm S},(1)}(N) + a_s^2 C_g^{{\rm S},(2)}(N) + a_s^3\left[C_g^{{\rm S},(3)}(N) + {\rm w}_3(N_F) C_{g,d_{abc}}^{{\rm S},(3)}(N)\right],
\end{aligned}$$

with ${\rm w_k}(3) = 0, {\rm w}_2(4) = 1/2, {\rm w}_3(4) = 1/10$.

$$F(x,Q^2) = C_F^{\rm NS}(x,Q^2)\otimes f^{\rm NS}(x,Q^2) + C_{Fq}^{\rm S}(x,Q^2)\otimes\Sigma(x,Q^2) + C_{Fg}^{\rm S}(x,Q^2)\otimes G(x,Q^2),$$

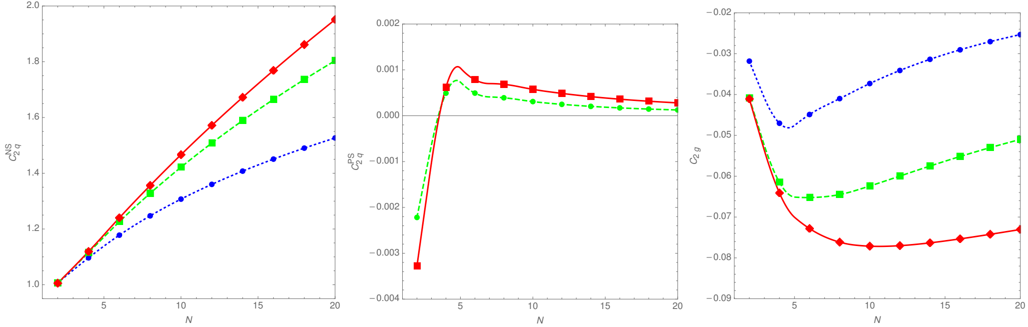
with $F = F_2, g_1$, after Mellin inversion to x-space.

NNLO anomalous dimensions



$\alpha_s = 0.2$, $N_F = 4$ massless quarks.

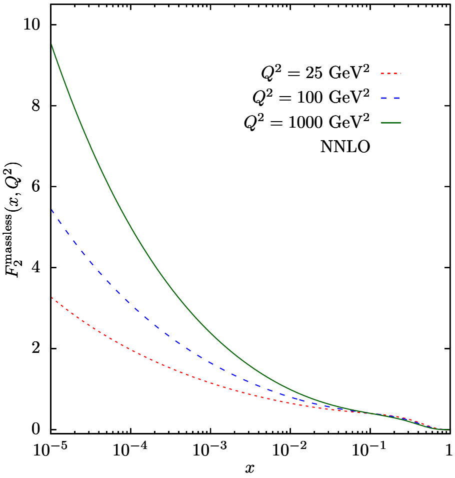
NNLO unpolarized Wilson coefficients for F_2

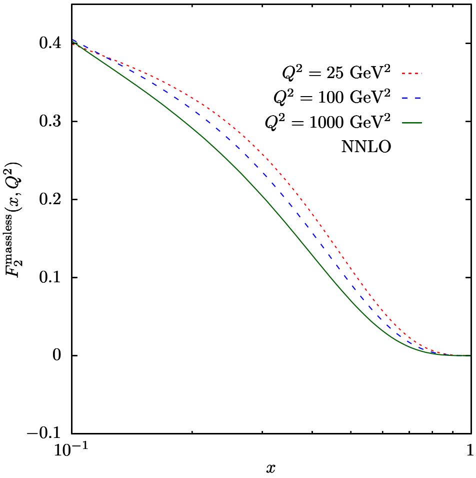


$\alpha_s = 0.2$, $N_F = 4$ massless quarks.

blue: LO, green: NLO, red: NNLO

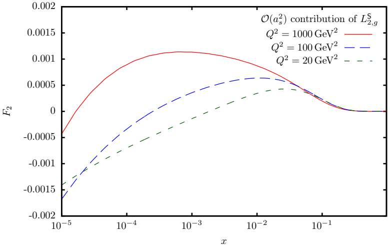
The massless contributions to F_2



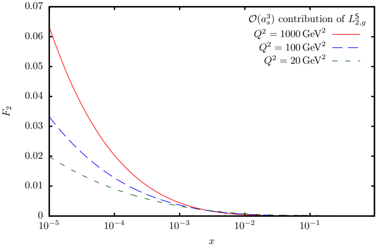


$N_F = 3$ massless quarks.

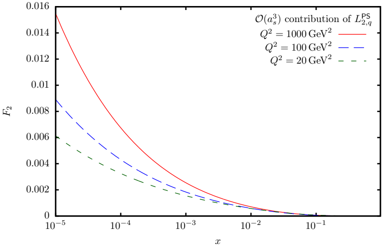
Heavy Quarks: Numerical Results : $L_{g,2}^{\mathrm{S}}$ and $L_{q,2}^{\mathrm{PS}}$



$O(a_s^2)$ $L_{2,g}^{\mathrm{S}}$

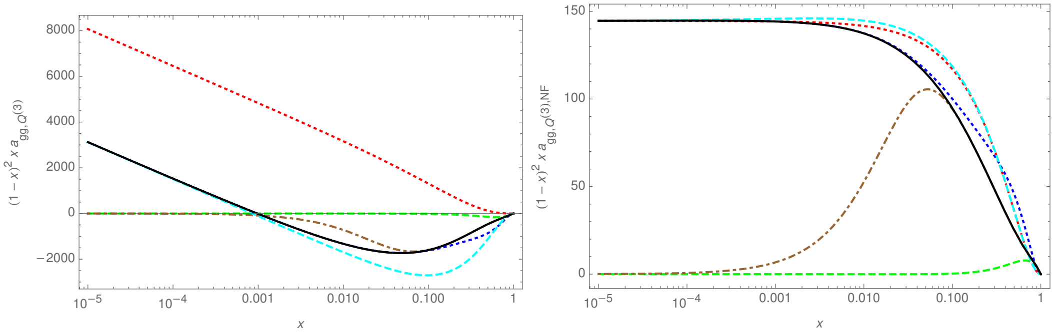


$O(a_s^3)$ $L_{2,g}^{\mathrm{S}}$



$L_{q,2}^{\mathrm{PS}}$

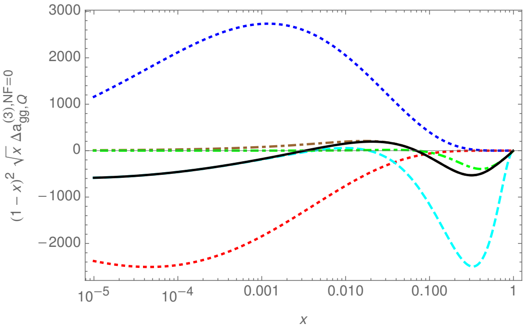
$a_{gg}^{(3)}$



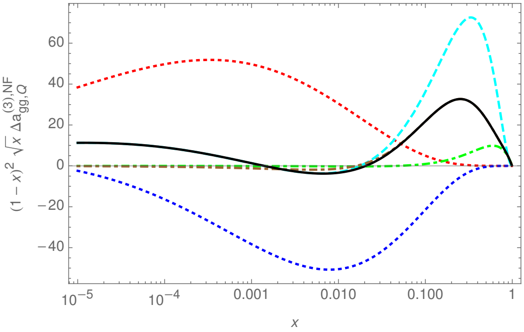
non N_F terms $\qquad$ N_F terms

Left panel: The non–N_F terms of $a_{gg,Q}^{(3)}(N)$ (rescaled) as a function of x. Full line (black): complete result; upper dotted line (red): term $\propto \ln(x)/x$; lower dashed line (cyan): small x terms $\propto 1/x$; lower dotted line (blue): small x terms including all $\ln(x)$ terms up to the constant term; upper dashed line (green): large x contribution up to the constant term; dash-dotted line (brown): complete large x contribution. Right panel: the same for the N_F contribution.

$\Delta a_{gg}^{(3)}$



non N_F terms

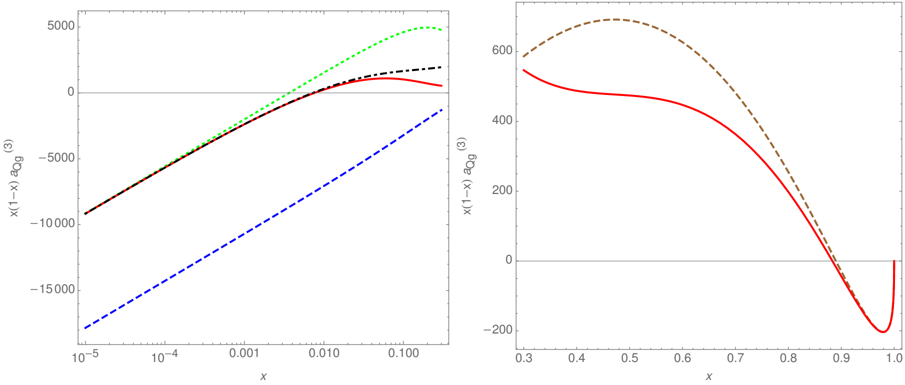


N_F terms

The non–N_F terms of $\Delta a_{gg,Q}^{(3)}(N)$ (rescaled) as a function of x. Full line (black): complete result; lower dotted line (red): term $\ln^5(x)$; upper dotted line (blue): small x terms $\propto \ln^5(x)$ and $\ln^4(x)$; upper dashed line (cyan): small x terms including all $\ln(x)$ terms up to the constant term; lower dash-dotted line (green): large x contribution up to the constant term; dash-dotted line (brown): full large x contribution. Right panel: the same for the N_F contribution.

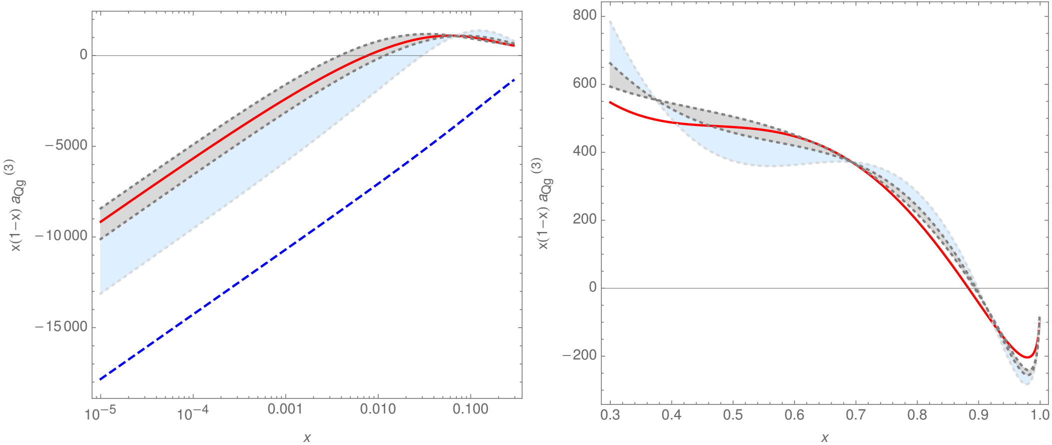
$a_{Qg}^{(3)}$

1009 of the total 1233 Feynman diagrams have first-order factorizing contributions only and are given by G-functions up to root-values letters. The letters for all constants can be rationalized.



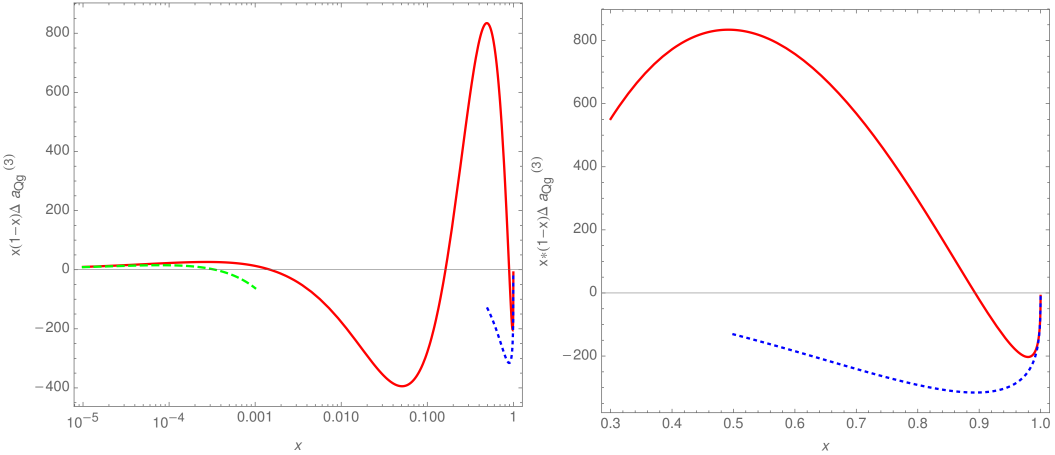
$a_{Qg}^{(3)}(x)$ as a function of x, rescaled by the factor $x(1-x)$. Left panel: smaller x region. Full line (red): $a_{Qg}^{(3)}(x)$; dashed line (blue): leading small-x term $\propto \ln(x)/x$ [Catani, Ciafaloni, Hautmann, 1990]; dotted line (green): $\ln(x)/x$ and $1/x$ term; dash-dotted line (black): all small-x terms, including also $\ln^k(x)$, $k \in \{1,\ldots,5\}$. Right panel: larger x region. Full line (red): $a_{Qg}^{(3)}(x)$; dashed line (brown): leading large-x terms up to the terms $\propto (1-x)$, covering the logarithmic contributions of $O(\ln^k(1-x))$, $k \in \{1,4\}$.

$a_{Qg}^{(3)}$



$a_{Qg}^{(3)}(x)$ as a function of x, rescaled by the factor $x(1-x)$. Left panel: smaller x region. Full line (red): $a_{Qg}^{(3)}(x)$; dashed line (blue): leading small-x term $\propto \ln(x)/x$ [Catani, Ciafaloni, Hautmann, 1990]; light blue region: estimates of [Kawamura et al., 2012]; gray region: estimates of [ABMP 2017]. Right panel: larger x region. Full line (red): $a_{Qg}^{(3)}(x)$; light blue region: estimates of [Kawamura et al., 2012] gray region: estimates of [ABMP 2017].

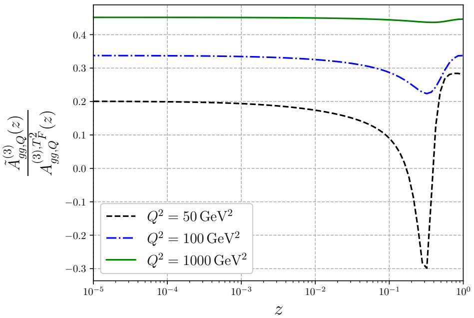
$\Delta a_{Qg}^{(3)}$



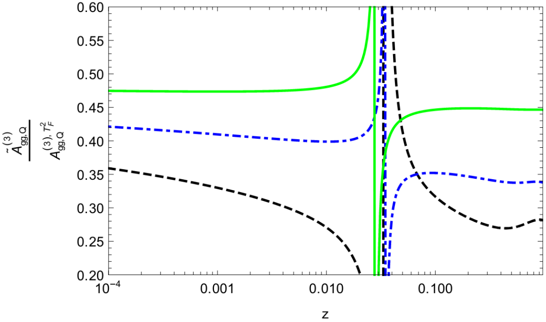
$\Delta a_{Qg}^{(3)}(x)$ as a function of x, rescaled by the factor $x(1-x)$. Left panel: full line (red): $\Delta a_{Qg}^{(3)}(x)$; dashed line (green): the small-x terms $\ln^k(x)$, $k \in \{1, \ldots, 5\}$; dotted line (blue): the large-x terms $\ln^l(1-x)$, $l \in \{1, \ldots, 4\}$. Right panel: larger x region. Full line (red): $\Delta a_{Qg}^{(3)}(x)$; dotted line (blue): the large-x terms $\ln^l(1-x)$, $l \in \{1, \ldots, 4\}$.

Two-mass Results: $\tilde{A}_{gg,Q}^{(3)}$

The two mass contributions over the whole T_F^2-contributions to the OME $\tilde{A}_{gg,Q}^{(3)}$:

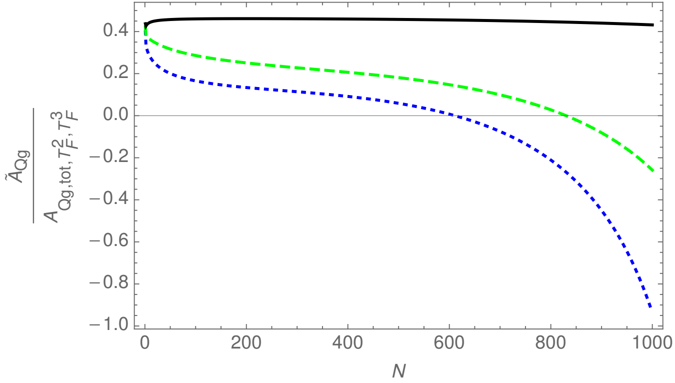


unpolarized



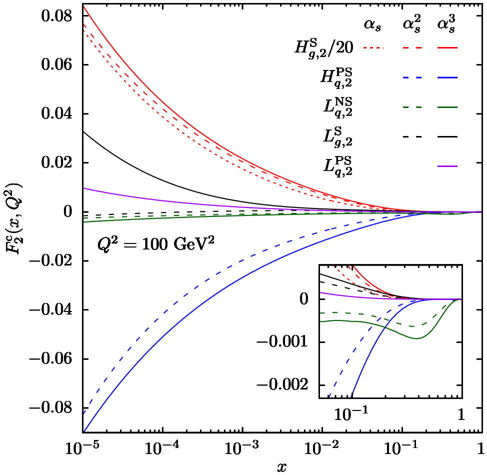
polarized

Relative contribution of $\tilde{A}_{Qg}^{(3)}(N)$

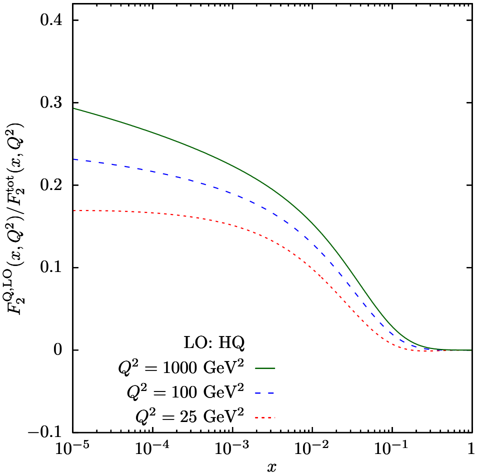


$Q^2 = 30$ GeV^2: dotted line; $Q^2 = 10^2$ GeV^2: dashed line; $Q^2 = 10^4$ GeV^2: full line.

Single-mass contributions to $F_2^{c,b}$



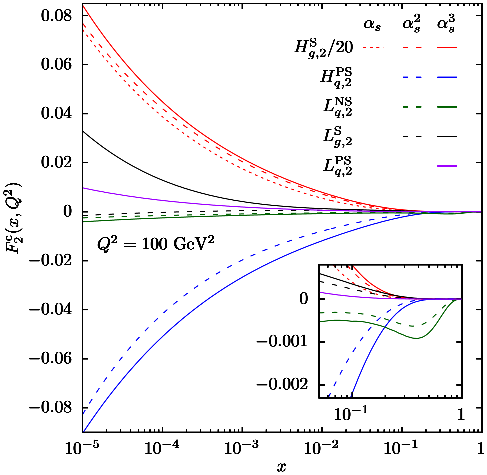
charm for $Q^2 = 100$ GeV^2.



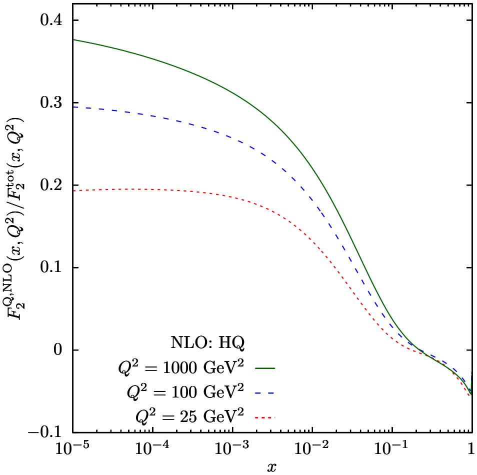
c and *b* single mass contributions

Allows to strongly reduce the current theory error on m_c.

Single-mass contributions to $F_2^{c,b}$



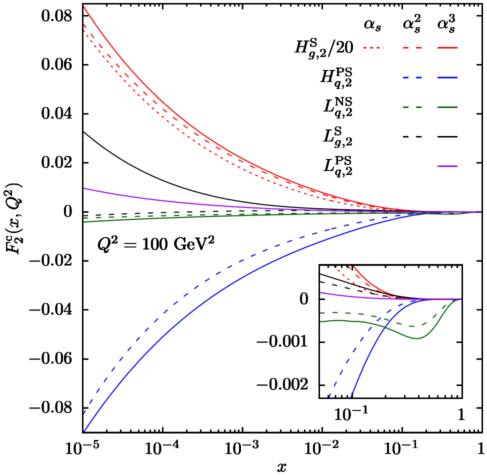
charm for $Q^2 = 100$ GeV^2.

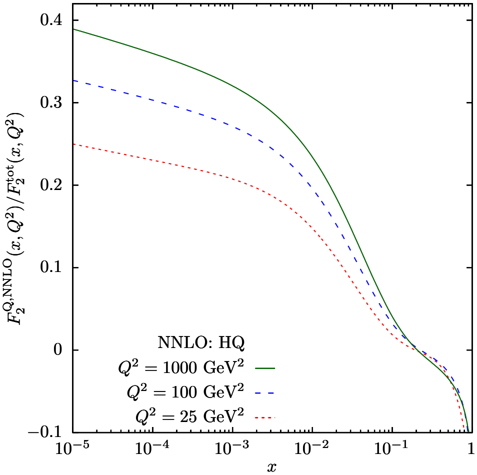


c and *b* single mass contributions

Allows to strongly reduce the current theory error on m_c.

Single-mass contributions to $F_2^{c,b}$



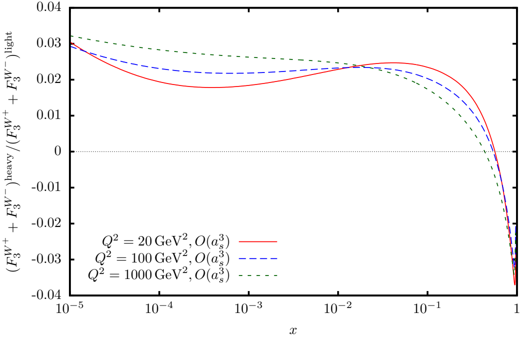


charm for $Q^2 = 100$ GeV^2.

c and *b* single mass contributions

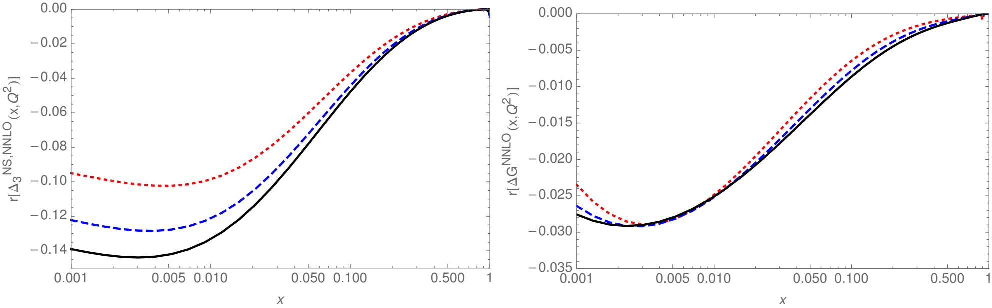
Allows to strongly reduce the current theory error on m_c.

Charged Current Structure Function $xF_3(x, Q^2)$



Non-singlet HQ corrections are usually small in the unpolarized case.

Polarized PDF evolution in the Larin Scheme



[Dotted line: $Q^2 = 100$ GeV^2, dashed line: $Q^2 = 1000$ GeV^2, full line: $Q^2 = 10000$ GeV^2]

$$r(x, Q^2) = \frac{f^L(x, Q^2)}{f^M(x, Q^2)} - 1$$

The pdfs are necessary to match HO Larin-scheme calculations.

Current status of $\alpha_s(M_Z)$ and m_c from DIS

$\alpha_s(M_Z^2)$ from DIS

	$\alpha_s(M_Z^2)$	error	order		Ref.
A02	0.1143	± 0.0014	NNLO	NS	[Alekhin:2002fv]
BBG	0.1134	$+0.0019$ -0.0021	NNLO	NS	[Blumlein:2006be]
BBG	0.1141	$+0.0020$ -0.0022	N^{3}LO	NS	[Blumlein:2006be]
GRS	0.112		NNLO	NS	[Gluck:2006yz]
ABKM	0.1135	± 0.0014	NNLO	FFS	[Alekhin:2009ni]
ABKM	0.1129	± 0.0014	NNLO	BSMN	[Alekhin:2009ni]
JR08	0.1124	± 0.0020	NNLO	dyn. approach	[Jimenez-Delgado:2008orh]
JR08	0.1158	± 0.0035	NNLO	std. approach	[Jimenez-Delgado:2008orh]
JR14	0.1136	± 0.0004	NNLO	dyn. approach	[Jimenez-Delgado:2014twa]
ABM12	0.1134	± 0.0011	NNLO		[Alekhin:2012ig]
ABMP	0.1147	± 0.0024	NNLO		[Alekhin:2017kpj]
Thorne	0.1136		NNLO	cuts:[Alekhin:2009ni]	[Thorne:2013hpa]
BB10	0.1132	$+0.0056$ -0.0095	NLO	polarized	[Blumlein:2010rn]
HERAPDF2.0jets	0.1156	± 0.0031	NNLO	also jets	[H1:2021xxi]
UK20	0.1174	± 0.0013	NNLO	multi data an.	[Cridge:2021qfd]
CT18	0.1164	± 0.0026	NNLO	”	[Hou:2019efy]
NNPDF3.1	0.1185	± 0.0012	NNLO	”	[Ball:2018iqk]

m_c from DIS [$\overline{\rm MS}$ scheme]

Alekhin et al. [2012]

$$m_c(m_c) = 1.24 \pm 0.03(\mathrm{exp.}) \begin{array}{l} +0.03 \\ -0.02 \end{array} (\mathrm{scale}) \begin{array}{l} +0.00 \\ -0.07 \end{array} (\mathrm{theory})\ \mathrm{GeV},$$

HERA [2018]

$$m_c(m_c) = 1.290\ {}^{+0.046}_{-0.041}\ (\mathrm{exp}) \begin{array}{l} +0.062 \\ -0.014 \end{array} (\mathrm{model}) \begin{array}{l} +0.003 \\ -0.031 \end{array} (\mathrm{param.})$$

e^+e^- Chetyrkin et al. [2017]

$$m_c(m_c) = 1.2730 \pm 0.0046\ \mathrm{GeV}$$

CTEQ [2013] by using four different methods.

$$m_c(m_c) = 1.12 \begin{array}{l} +0.05 \\ -0.11 \end{array} \mathrm{GeV};\ 1.18\ {}^{+0.05}_{-0.11}\ \mathrm{GeV};\ 1.19\ {}^{+0.06}_{-0.15}\ \mathrm{GeV};\ 1.24\ {}^{+0.06}_{-0.15}\ \mathrm{GeV}$$

MSTW 2021: global minimum for $m_c(m_c) = 1.35$ GeV

Conclusions

- All unpolarized and polarized single-mass OMEs and the associated massive Wilson coefficients for $Q^2 \gg m_Q^2$ have been calculated. The unpolarized and polarized massless three-loop Wilson coefficients were calculated and contribute to the present results.
- The calculation of all unpolarized and polarized two-mass OMEs, except for $(\Delta)A_{Qg}^{(3)}$, are finished and the remaining OMEs will be available very soon.
- Various new mathematical and technological methods were developed during the present project. They are available for use in further single- and two-mass calculations in other QFT projects.
- Very soon new precision analyses of the world DIS-data to measure $\alpha_s(M_Z)$ and m_c at higher precision can be carried out.
- Both the single- and two-mass VFNS at 3-loop order will be available in form of a numerical program, to be used e.g. in applications at hadron colliders.
- The results in the polarized case prepare the analysis of the precision data, which will be taken at the EIC starting at the end of this decade.
- For all sub-processes it turned out that the small x BFKL approaches fail to present the physical result due to quite a series of missing subleading terms, which substantially correct the LO behaviour. The correct description requires the full calculation.