Algebraic and Structural Relations between Harmonic Sums up to Weight w = 6

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DESY

1. Introduction
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1. Introduction

Consider hard scattering processes in massless field theories:

QCD, QED, $m_i \to 0$

Factorization Theorem Leading Twist:

The cross section σ factorizes as

$$\sigma = \sum_k \sigma_{k,W} \otimes f_k$$

σ_W perturbative Wilson Coefficient

f non-perturbative Parton Density

$\otimes$ Mellin convolution

$$\begin{aligned} [A \otimes B](x) &= \int_0^1 dx_1 \int_0^1 dx_2 \delta(x - x_1 x_2) A(x_1) B(x_2) \\ \mathbf{M}[A \otimes B](N) &= \mathbf{M}[A](N) \cdot \mathbf{M}[B](N) \end{aligned}$$

with the Mellin transform :

$$\mathbf{M}[f(x)](N) = \int_0^1 dx x^{N-1} f(x), \quad Re[N] > c$$

Observation :

Feynman Amplitudes seem to obey the **Mellin Symmetry**

i.e. to significantly simplify in **Mellin Space**

WHICH ARE THE FUNDAMENTAL OBJECTS EXPRESSING SINGLE-SCALE FEYNMAN DIAGRAMS?

$\boxed{\longrightarrow \text{HARMONIC SUMS !}}$

NOT HARMONIC POLYLOGARITHMS IN THE FIRST PLACE.

LIGHT CONE EXPANSION: $Q^2 = -q^2 > 0$
$\to \infty$

$\nu \to \infty$ (ENERGY)

$\frac{Q^2}{\nu}$ = fixed. $\nu = p.q$

MUELLER CUT-VERTEX METHOD: $\infty \leftarrow \begin{cases} Q^2 \equiv q^2 > 0 \\ \nu \end{cases}$

$\frac{\nu}{Q^2}$ = fixed.

$\longrightarrow$ INTEGER MOMENTS OF "EXPECTATION VALUES"

$a_n = \langle p | O_a^n | p \rangle$ $n \in \mathbb{N}$, even <u>or</u> odd.

$\longrightarrow$ (CURRENT) CROSSING RELATIONS

$T(q,p) \longrightarrow T(-q,p)$

- NESTED HARMONIC SUMS, N fixed $\longrightarrow$ PERIODS? or shuffles.

$\longrightarrow$ HARMONIC POLYLOGARITHMS EMERGE AS ANALYTIC CONTINUATIONS UP TO A MELLIN INVERSION.

References

A. Gonzalez-Arroyo, C. Lopez, and F.J. Yndurain, Nucl. Phys. **B153** (1979) 161;
A. Gonzalez-Arroyo and C. Lopez, Nucl. Phys. **B166** (1980) 429.

J.A.M. Vermaseren, Int. J. Mod. Phys. **A14** (1999) 2037.

J. Blümlein and S. Kurth, DESY 97–160, hep-ph/9708388; Phys. Rev. **D60** (1999) 014018.

N. Nielsen, Der Eulersche Dilogarithmus und seine Verallgemeinerungen, Nova Acta Leopoldina, Vol. XC, Nr. 3, Halle, 1909.

J. Blümlein, Comput. Phys. Commun. **159** 92004) 19.

E. Remiddi and J.A.M. Vermaseren , Int. J. Mod. Phys. **A15** (2000) 725.

J. Blümlein, Comput. Phys. Commun. **133** (2000) 76.

S.O. MOCH, J. VERMASEREN, A. VOGT, NUCL. PHYS. B688 (2004) 101; B691 (2004) 129; PHYS. LETT. B606 (2005) 123; hep-ph 0504242 → NPB.

J. BLÜMLEIN, S.O. MOCH, PHYS. LETT. B 614 (2005) 53.

2. x Space Results

Usual Starting Point of Higher Order Calculations :

$\Longrightarrow$ Nielsen type Integrals and their Generalization

$$S_{n,p,q}(x) = \frac{(-1)^{n+p+q-1}}{(n-1)!p!q!} \int_0^1 \frac{dz}{z} \ln^{(n-1)}(z) \ln^p(1-zx) \ln^q(1+zx)$$

Special Cases:

$$\begin{aligned} \mathrm{Li}_n(x) &= S_{n-1,1}(x) && w=n \\ \frac{d\mathrm{Li}_2(\pm x)}{d\ln(x)} &= -\ln(1\mp x) && w=1 \\ \mathrm{Li}_0(x) &= \frac{x}{1-x} && w=0 \end{aligned}$$

PHYSICS:

- ALL WILSON COEFFICIENTS AND ANOMALOUS DIMENSIONS TO 3-LOOP ORDER

- DIS, $e^+e^- \longrightarrow H+X$, DRELL-YAN PROCESS,
- HIGGS PRODUCTION $m_H \to \infty$
- HEAVY QUARK PROD. $m_Q^2/Q^2 \longrightarrow 0$, m_Q FINITE.

$\longrightarrow$ SINGLE SCALE INTEGRALS !

van Neerven, Zijlstra 1992

$$c_{2,-}^{(2)}(x) = C_F\,(C_F - C_A/2) \times$$

$$\left\{ \frac{1+x^2}{1-x} \left[\left[4\ln^2(x) - 16\ln(x)\ln(1+x) - 16\mathrm{Li}_2(-x) - 8\zeta_2 \right] \ln(1-x) \right. \right.$$

$$+ \left[-2\ln^2(x) + 20\ln(x)\ln(1+x) - 8\ln^2(1+x) + 8\mathrm{Li}_2(1-x) + 16\mathrm{Li}_2(-x) - 8 \right] \ln(x)$$

$$-16\ln(1+x)\mathrm{Li}_2(-x) - 8\zeta_2\ln(1+x) - 16\left[\mathrm{Li}_3\left(-\frac{1-x}{1+x}\right) - \mathrm{Li}_3\left(\frac{1-x}{1+x}\right) \right]$$

$$\left. -16\mathrm{Li}_2(1-x) + 8S_{1,2}(1-x) + 8\mathrm{Li}_3(-x) - 16S_{1,2}(-x) + 8\zeta_3 \right]$$

$$+(4+20x)\left[\ln^2(x)\ln(1+x) - 2\ln(x)\ln^2(1+x) - 2\zeta_2\ln(1+x) - 4\ln(1+x)\mathrm{Li}_2(-x) \right.$$

$$\left. +2\mathrm{Li}_3(-x) - 4S_{1,2}(-x) + 2\zeta_3 \right] + \left(32 + 32x + 48x^2 - \frac{72}{5}x^3 + \frac{8}{5x^2} \right)$$

$$\times \left[\mathrm{Li}_2(-x) + ln(x)\ln(1+x) \right] + 8(1+x)\left[\mathrm{Li}_s(1-x) + \ln(x)\ln(1-x) \right] + 16(1-x)\ln(1-x)$$

$$+ \left(-4 - 16x - 24x^2 + \frac{36}{5}x^3 \right) \ln^2(x) + \frac{1}{5}\left(-26 - 106x + 72x^2 - \frac{8}{x} \right) \ln(x)$$

$$\left. + \left(-4 + 20x + 48x^2 - \frac{72}{5}x^3 \right) \zeta_2 + \frac{1}{5}\left(-162 + 82x + 72x^2 + \frac{8}{x} \right) \right\}$$

.... several other pages for $c_2^{(+)}(x), c_2^G(x), c_L^{(q,G)}(x)$

$\Longrightarrow$ 77 Functions @ 2 Loops

- $\Longrightarrow$ partly rather complicated arguments
- $\Longrightarrow$ relations are not directly visible ...

The 77 functions do roughly correspond in number to the number of all possible harmonic sums up to weight w=4: 80.

x Space Results

No.	$f(z)$	$M[f](N) = \int_0^1 dz z^{N-1} f(z)$
1	$\delta(1-z)$	1
2	z^r	$\frac{1}{N+r}$
3	$\left(\frac{1}{1-z}\right)_+$	$-S_1(N-1)$
4	$\frac{1}{1+z}$	$(-1)^{N-1}[\log(2) - S_1(N-1)]$ $+\frac{1+(-1)^{N-1}}{2} S_1\left(\frac{N-1}{2}\right) - \frac{1-(-1)^{N-1}}{2} S_1\left(\frac{N-2}{2}\right)$
5	$z^r \log^n(z)$	$\frac{(-1)^n}{(N+r)^{n+1}} \Gamma(n+1)$
6	$z^r \log(1-z)$	$-\frac{S_1(N+r)}{N+r}$
7	$z^r \log^2(1-z)$	$\frac{S_1^2(N+r) + S_2(N+r)}{N+r}$
8	$z^r \log^3(1-z)$	$-\frac{S_1^3(N+r) + 3S_1(N+r)S_2(N+r) + 2S_3(N+r)}{N+r}$
9	$\left[\frac{\log(1-z)}{1-z}\right]_+$	$\frac{1}{2} S_1^2(N-1) + \frac{1}{2} S_2(N-1)$
10	$\left[\frac{\log^2(1-z)}{1-z}\right]_+$	$-\left[\frac{1}{3} S_1^3(N-1) + S_1(N-1)S_2(N-1) + \frac{2}{3} S_3(N-1)\right]$
11	$\left[\frac{\log^3(1-z)}{1-z}\right]_+$	$\frac{1}{4} S_1^4(N-1) + \frac{3}{2} S_1^2(N-1)S_2(N-1)$ $+\frac{3}{4} S_2^2(N-1) + 2S_1(N-1)S_3(N-1)$ $+\frac{3}{2} S_4(N-1)$
12	$\frac{\log^n(z)}{1-z}$	$(-1)^{n+1}\Gamma(n+1)\left[S_{n+1}(N-1) - \zeta(n+1)\right]$

Only single sums.

No.	$f(z)$	$M[f](N)$
64	$\dfrac{\mathrm{Li}_3(-z)}{1+z}$	$(-1)^{N-1}\Big\{S_{3,-1}(N-1)+\big[S_3(N-1)-S_{-3}(N-1)\big]\log 2$ $+\frac{1}{2}\zeta(2)S_{-2}(N-1)-\frac{3}{4}\zeta(3)S_{-1}(N-1)$ $+\frac{1}{8}\zeta^2(2)-\frac{3}{4}\zeta(3)\log 2\Big\}$
65	$\mathrm{Li}_3(1-z)$	$\frac{1}{N}\big[S_1(N)S_2(N)-\zeta(2)S_1(N)+S_3(N)$ $-S_{2,1}(N)+\zeta(3)\big]$
66	$\dfrac{\mathrm{Li}_3(1-z)}{1-z}$	$-S_{1,1,2}(N-1)+\frac{1}{2}\zeta(2)S_1^2(N-1)+\frac{1}{2}\zeta(2)S_2(N-1)$ $-\zeta(3)S_1(N-1)+\frac{2}{5}\zeta^2(2)$
67	$\dfrac{\mathrm{Li}_3(1-z)}{1+z}$	$(-1)^{N-1}\Big[S_{-1,1,2}(N-1)-\zeta(2)S_{-1,1}(N-1)$ $+\zeta(3)S_{-1}(N-1)+\mathrm{Li}_4\left(\frac{1}{2}\right)-\frac{9}{20}\zeta^2(2)$ $+\frac{7}{8}\zeta(3)\log 2+\frac{1}{2}\zeta(2)\log^2 2+\frac{1}{24}\log^4 2\Big]$
68	$\mathrm{Li}_3\left(\frac{1-z}{1+z}\right)$ $-\mathrm{Li}_3\left(-\frac{1-z}{1+z}\right)$	$\frac{(-1)^N}{N}\Big[-S_{-1,2}(N)-S_{-2,1}(N)+S_1(N)S_{-2}(N)$ $+S_{-3}(N)$ $+\zeta(2)S_{-1}(N)+\frac{1}{2}\zeta(2)S_1(N)-\frac{7}{8}\zeta(3)+\frac{3}{2}\zeta(2)\log 2\Big]$ $+\frac{1}{N}\Big[-S_{-1,-2}(N)-S_{2,1}(N)+S_1(N)S_2(N)+S_3(N)$ $-\frac{1}{2}\zeta(2)S_{-1}(N)-\zeta(2)S_1(N)+\frac{21}{8}\zeta(3)-\frac{3}{2}\zeta(2)\log 2\Big]$
69	$\frac{1}{1+z}\Big[\mathrm{Li}_3\left(\frac{1-z}{1+z}\right)$ $-\mathrm{Li}_3\left(-\frac{1-z}{1+z}\right)\Big]$	$(-1)^{N-1}\Big\{\underline{S_{1,1,-2}(N-1)-S_{1,-1,2}(N-1)}$ $\underline{+S_{-1,1,2}(N-1)-S_{-1,-1,-2}(N-1)}$ $+2\zeta(2)S_{1,-1}(N-1)+\frac{1}{4}\zeta(2)S_1^2(N-1)-\frac{1}{4}\zeta(2)S_{-1}^2(N-1)$ $-\zeta(2)S_1(N-1)S_{-1}(N-1)-\zeta(2)S_{-2}(N-1)$ $-\left[\frac{7}{8}\zeta(3)-\frac{3}{2}\zeta(2)\log 2\right]S_1(N-1)$ $+\left[\frac{21}{8}\zeta(3)-\frac{3}{2}\zeta(2)\log 2\right]S_{-1}(N-1)$ $-2\mathrm{Li}_4\left(\frac{1}{2}\right)+\frac{19}{40}\zeta^2(2)+\frac{1}{2}\zeta(2)\log^2 2-\frac{1}{12}\log^4 2\Big\}$

2 loop coefficient functions $\Longrightarrow$ Nested Harmonic Sums of Weight $w=4$

x Space Results

$$S_{-1,-1,-2}(N) =$$
$$(-1)^{N+1}\mathbf{M}\left\{\frac{1}{1+x}\left[F_1(x)+\log(1-x)\mathrm{Li}_2(-x)\right]\right\}(N)$$
$$+(-1)^{N+1}\mathbf{M}\left\{\frac{1}{1+x}\left[\frac{1}{2}\mathrm{S}_{1,2}(x^2)-\mathrm{S}_{1,2}(x)-\mathrm{S}_{1,2}(-x)\right]\right\}(N)$$
$$+\frac{1}{2}\zeta(2)\left[S_{-1,1}(N)-S_{-1,-1}(N)\right]$$
$$+\left[\frac{9}{8}\zeta(3)-\frac{3}{2}\zeta(2)\log(2)-\frac{1}{6}\log^3(2)\right]S_{-1}(N)$$
$$-\frac{1}{10}\zeta(2)^2+\frac{17}{8}\zeta(3)\log(2)-\frac{7}{4}\zeta(2)\log^2(2)-\frac{1}{6}\log^4(2)$$

with

$$\begin{aligned} F_1(x) \quad = \quad & S_{1,2}\left(\frac{1-x}{2}\right)+S_{1,2}(1-x)-S_{1,2}\left(\frac{1-x}{1+x}\right) \\ & +S_{1,2}\left(\frac{1}{1+x}\right)-\ln(2)\left(\frac{1-x}{2}\right) \\ & +\frac{1}{2}\ln^2(2)\ln\left(\frac{1+x}{2}\right)-\ln(2)\mathrm{Li}_2\left(\frac{1-x}{1+x}\right) \end{aligned}$$

$F_1(x)$, although of complicated structure, it reduces completely via algebraic relations

$\Rightarrow$ Mellin polynomial of simpler objects

These objects can be very complicated integrals. J.B., van Neerven, Ravindran, Kawamura 2000, 2003

3. Multiple Harmonic Sums to Level 6

The simplest example :

$$P_{qq}(x) = \left(\frac{1+x^2}{1-x}\right)_+ = \frac{2}{(1-x)_+} + \dots$$

$$\int_0^1 dx \frac{x^{N-1}}{(1-x)_+} = -\sum_{k=0}^{N-2} \int_0^1 dx x^k = -\sum_{k=1}^{N-1} \frac{1}{k} = -S_1(N-1)$$

Alternating sums :

$$S_{-1}(N-1) = (-1)^{N-1} \mathbf{M}\left[\frac{1}{1+x}\right](N) - \ln(2) = \int_0^1 dx \frac{x^{N-1}}{(1-x)_+} = \sum_{k=1}^{N-1} \frac{(-1)^k}{k}$$

(Finite for $N \to \infty$.)

General case :

$$S_{a_1,\dots,a_l}(N) = \sum_{k_1=1}^{N} \frac{(\mathrm{sign}(a_1))^{k_1}}{k_1^{|a_1|}} \sum_{k_2=1}^{N} \frac{(\mathrm{sign}(a_2))^{k_2}}{k_2^{|a_2|}} \dots$$

Vermaseren, 1997

All Mellin transforms occurring in massless Field Theories for 1-Parameter Quantities can be represented by Harmonic Sums

(at least to 3–loop order).

Algebraic Relations

First relation:

L. Euler, 1775

$$S_{m,n} + S_{n,m} = S_m \cdot S_n + S_{m+n}, \qquad m, n > 0$$

Generalized to alternating sums by

$$\begin{aligned} S_{m,n} + S_{n,m} &= S_m \cdot S_n + S_{m \wedge n}, \\ m \wedge n &= [|m| + |n|] \, \mathrm{sign}(m)\mathrm{sign}(n) \end{aligned}$$

Ternary relations: Sita Ramachandra Rao, 1984,

4-ary relation: J.B., Kurth, 1998.

These & other relations hold widely independent of their **Value** and **Type**.

Determined by :
- Index Structure
- Multiplication Relation

The Formalism applies as well to the Harmonic Polylogarithms. Remiddi, Vermaseren, 1999.

Application to QED: T. Riemann et al., 2004

Linear Representations of Mellin Transform by Harmonic Sums:

$$\mathbf{M}[F_w(x)](N) = S^w_{k_1,\ldots,k_m}(N) + P\left(S^{\tau'}_{k_1,\ldots,k_r}, \sigma^{\tau''}_{k_1,\ldots,k_p}\right)$$

$w = \sum_{i=1}^m |k_i|$ Weight

$\tau', \tau'' < w$ P is a polynomial.

w	#	Σ	
1	2	2	
2	6	8	
3	18	26	2 Loop anom. Dimensions
4	54	80	2 Loop Wilson Coefficients
5	162	242	3 Loop anom. Dimensions
6	486	728	3 Loop Wilson Coefficients
	$2 \cdot 3^{w-1}$	$3^w - 1$	

Shuffle Products

Depth 2:

$$S_{a_1}(N) \sqcup\!\sqcup\, S_{a_2}(N) = S_{a_1,a_2}(N) + S_{a_2,a_1}(N)$$

Depth 3:

$$S_{a_1}(N) \sqcup\!\sqcup\, S_{a_2,a_3}(N) = S_{a_1,a_2,a_3}(N) + S_{a_2,a_1,a_3}(N) + S_{a_2,a_3,a_1}(N)$$

Depth 4:

$$\begin{aligned}
S_{a_1}(N) \sqcup\!\sqcup\, S_{a_2,a_3,a_4}(N) &= S_{a_1,a_2,a_3,a_4}(N) + S_{a_2,a_1,a_3,a_4}(N) + S_{a_2,a_3,a_1,a_4}(N) \\
&+ S_{a_2,a_3,a_4,a_1}(N) \\
S_{a_1,a_2}(N) \sqcup\!\sqcup\, S_{a_3,a_4}(N) &= S_{a_1,a_2,a_3,a_4}(N) + S_{a_1,a_3,a_2,a_4}(N) + S_{a_1,a_3,a_4,a_2}(N) \\
&+ S_{a_3,a_4,a_1,a_2}(N) + S_{a_3,a_1,a_4,a_2}(N) + S_{a_3,a_1,a_2,a_4}(N)
\end{aligned}$$

Depth 5:

$$\begin{aligned}
S_{a_1}(N) \sqcup\!\sqcup\, S_{a_2,a_3,a_4,a_5}(N) &= S_{a_1,a_2,a_3,a_4,a_5}(N) + S_{a_2,a_1,a_3,a_4,a_5}(N) \\
&+ S_{a_2,a_3,a_1,a_4,a_5}(N) + S_{a_2,a_3,a_4,a_1,a_5}(N) \\
&+ S_{a_2,a_3,a_4,a_5,a_1}(N) \\
S_{a_1,a_2}(N) \sqcup\!\sqcup\, S_{a_3,a_4,a_5}(N) &= S_{a_1,a_2,a_3,a_4,a_5}(N) + S_{a_1,a_3,a_2,a_4,a_5}(N) \\
&+ S_{a_1,a_3,a_4,a_2,a_5}(N) + S_{a_1,a_3,a_4,a_5,a_2}(N) \\
&+ S_{a_3,a_1,a_2,a_4,a_5}(N) + S_{a_3,a_1,a_4,a_2,a_5}(N) \\
&+ S_{a_3,a_1,a_4,a_5,a_2}(N) + S_{a_3,a_4,a_5,a_1,a_2}(N) \\
&+ S_{a_3,a_4,a_1,a_5,a_2}(N) + S_{a_3,a_4,a_1,a_2,a_5}(N)
\end{aligned}$$

Depth 6: .....

Algebraic Equations

Depth 2:

$$S_{a_1}(N) \sqcup\!\sqcup\, S_{a_2}(N) - S_{a_1}(N)S_{a_2}(N) - S_{a_1\wedge a_2}(N) = 0$$

Depth 3:

$$S_{a_1}(N) \sqcup\!\sqcup\, S_{a_2,a_3}(N) \quad - \quad S_{a_1}(N)S_{a_2,a_3}(N) - S_{a_1\wedge a_2,a_3}(N) - S_{a_2,a_1\wedge a_3}(N) = 0$$

Depth 4:

$$\begin{aligned}
S_{a_1}(N) \sqcup\!\sqcup\, S_{a_2,a_3,a_4}(N) &- S_{a_1}(N)S_{a_2,a_3,a_4}(N) - S_{a_1\wedge a_2,a_3,a_4}(N) \\
&- S_{a_2,a_1\wedge a_3,a_4}(N) - S_{a_2,a_3,a_1\wedge a_4}(N) = 0 \\
S_{a_1,a_2}(N) \sqcup\!\sqcup\, S_{a_3,a_4}(N) &- S_{a_1,a_2}(N)S_{a_3,a_4}(N) - S_{a_1,a_2\wedge a_3,a_4}(N) \\
&- S_{a_1,a_3,a_2\wedge a_4}(N) - S_{a_3,a_1\wedge a_4,a_2}(N) \\
&- S_{a_3,a_1,a_2\wedge a_4}(N) - S_{a_1\wedge a_3,a_2,a_4}(N) \\
&- S_{a_1\wedge a_3,a_4,a_2}(N) + S_{a_1\wedge a_3,a_2\wedge a_4} = 0
\end{aligned}$$

Depth 5: .....

Basic Sums = # Permutations - # Independent Equations

Some Solution for $d = 6$

$$
\begin{aligned}
&S_{a,a,a,a,b,b} = \\
&-\frac{1}{4}S_a S_{b,a,a,a,b} + \frac{3}{4}S_{a\wedge b,a,a,a,b} - \frac{1}{4}S_{b,a,a,a,a\wedge b} + \frac{1}{12}S_a S_{a,a,b,b,a} + S_{a,a,a,a,b\wedge b} \\
&-\frac{1}{12}S_{a\wedge a,b,b,a,a} - \frac{1}{12}S_{a,b,b,a\wedge a,a} - \frac{1}{12}S_{a,b,b,a,a\wedge a} - \frac{1}{4}S_{b,a\wedge a,a,a,b} - \frac{1}{4}S_{b,a,a\wedge a,a,l} \\
&-\frac{1}{4}S_{b,a,a,a\wedge a,b} - \frac{1}{4}S_{a,a\wedge a,a,b,b} - \frac{1}{4}S_{a,a,a\wedge a,b,b} - \frac{1}{4}S_{a,b,a\wedge a,a,b} - \frac{1}{4}S_{a,b,a,a\wedge a,b} \\
&+\frac{1}{12}S_{a\wedge a,b,a,b,a} + \frac{1}{12}S_{a,b,a\wedge a,b,a} - \frac{1}{4}S_{a,a,a,b,a\wedge b} - \frac{1}{4}S_{a,a,b,a,a\wedge b} + \frac{1}{12}S_{a,a,a\wedge b,b,a} \\
&+\frac{3}{4}S_{a,a,a\wedge b,a,b} - S_{b,b,a,a,a,a} + \frac{1}{4}S_{b,b,a,a\wedge a,a} - \frac{1}{4}S_{a\wedge a,a,a,b,b} + \frac{1}{12}S_{a,b,a,b,a\wedge a} \\
&+\frac{1}{12}S_{b,a\wedge a,a,b,a} + \frac{1}{12}S_{b,a,a\wedge a,b,a} + \frac{1}{12}S_{b,a,a,b,a\wedge a} - \frac{1}{12}S_{b,a\wedge a,b,a,a} - \frac{1}{12}S_{b,a,b,a\wedge c} \\
&+\frac{1}{4}S_{b,b,a\wedge a,a,a} + \frac{1}{4}S_{b,b,a,a,a\wedge a} - \frac{1}{4}S_{a\wedge a,a,b,a,b} - \frac{1}{4}S_{a,a\wedge a,b,a,b} - \frac{1}{4}S_{a,a,b,a\wedge a,b} \\
&+\frac{1}{12}S_{a\wedge a,a,b,b,a} + \frac{1}{12}S_{a,a\wedge a,b,b,a} + \frac{1}{12}S_{a,a,b,b,a\wedge a} - \frac{1}{4}S_{a\wedge a,b,a,a,b} - \frac{1}{12}S_{b,a,b,a,a\wedge} \\
&+\frac{1}{12}S_{a\wedge b,a,a,b,a} + \frac{1}{12}S_{b,a,a,a\wedge b,a} - \frac{1}{12}S_{a\wedge b,a,b,a,a} + \frac{1}{4}S_{a\wedge b,b,a,a,a} + \frac{1}{4}S_{b,a\wedge b,a,a,a} \\
&-\frac{1}{12}S_{b,a,a\wedge b,a,a} + \frac{3}{4}S_{a,a,a,a\wedge b,b} + \frac{1}{12}S_{a,a,b,a\wedge b,a} + \frac{3}{4}S_{a,a\wedge b,a,a,b} - \frac{1}{4}S_{a,b,a,a,a\wedge b} \\
&+\frac{1}{12}S_{a,a\wedge b,a,b,a} + \frac{1}{12}S_{a,b,a,a\wedge b,a} - \frac{1}{12}S_{a,a\wedge b,b,a,a} - \frac{1}{12}S_{a,b,a\wedge b,a,a} - \frac{1}{4}S_a S_{a,a,a,b} \\
&-\frac{1}{12}S_a S_{a,b,b,a,a} + \frac{1}{12}S_a S_{a,b,a,b,a} - \frac{1}{12}S_a S_{b,a,b,a,a} + \frac{1}{12}S_a S_{b,a,a,b,a} - \frac{1}{4}S_a S_{a,b,a,a} \\
&+S_b S_{a,a,a,a,b} + \frac{1}{4}S_a S_{b,b,a,a,a} - \frac{1}{4}S_a S_{a,a,b,a,b}
\end{aligned}
$$

Depth $d = 3$

Index Set	Number	Dep. Sums of Depth 3	min. Weight	Fraction of fund. Sums
$\{a, a, a\}$	1	1	3	0
$\{a, a, b\}$	3	2	3	1/3
$\{a, b, c\}$	6	4	4	1/3

Depth $d = 4$

Index Set	Number	Dep. Sums of Depth 4	min. Weight	Fraction of fund. Sums
$\{a, a, a, a\}$	1	1	4	0
$\{a, a, a, b\}$	4	3	4	1/4
$\{a, a, b, b\}$	6	5	4	1/6
$\{a, a, b, c\}$	12	9	5	1/4
$\{a, b, c, d\}$	24	18	6	1/4

Depth $d = 6$

Index Set	Number	Dep. Sums of Depth 6	min. Weight	Fraction of fund. Sums
$\{a, a, a, a, a, a\}$	1	1	6	0
$\{a, a, a, a, a, b\}$	6	5	6	1/6
$\{a, a, a, a, b, b\}$	15	13	6	2/15
$\{a, a, a, b, b, b\}$	20	17	6	3/20
$\{a, a, a, a, b, c\}$	30	25	7	1/6
$\{a, a, a, b, b, c\}$	60	50	7	1/6
$\{a, a, b, b, c, c\}$	90	76	8	7/45
$\{a, a, a, b, c, d\}$	120	100	8	1/6
$\{a, a, b, b, c, d\}$	180	150	8	1/6
$\{a, a, b, c, d, e\}$	360	300	10	1/6
$\{a, b, c, d, e, f\}$	720	600	12	1/6

Theory of Words

Can we count the Basis in simpler way ? $\Longrightarrow$ YES.

Free Algebras and Elements of the Theory of Codes
$\Longrightarrow$ **Particle Physics**

Only the multiplication relation and the Index structure matters

$\mathfrak{A} = \{a, b, c, d, \ldots\}$ **Alphabet**

$a < b < c < d < \ldots$ **ordered**

$\mathfrak{A}^*(\mathfrak{A})$ **Set of all words W**

$W = a_1 \cdot a_2 \cdot a_{27} \ldots a_{532} \equiv$ **concatenation product (nc)**

$W = p \cdot x \cdot s$ **p** $=$ **prefix; s** $=$ **suffix**

Definition:

A Lyndon word is smaller than any of its suffixes.

Theorem:[Radford, 1979]

The shuffle algebra $K\langle\mathfrak{A}\rangle$ is freely generated by the Lyndon words.
I.e. the number of Lyndon words yields the number of basic elements.

Examples :

$\{a, a, \ldots, a, b\} = aaa\ldots ab$ 1 Lyndon word for these sets

$n \quad a's: \quad n_{basic}/n_{all} = 1/n \qquad n \equiv$ depth of the sums

$\{a, a, a, b, b, b\}$ $aaabbb, aababb, aabbab$ 3 Lyndon words

$n_{basic}/n_{all} = 3/20 < 1/6.$ Symmetries lead to a smaller fraction.

Is there a general Counting Relation ?

E. Witt, 1937

$$l_n(n_1, \ldots, n_q) = \frac{1}{n} \sum_{d \| n_i} \mu(d) \frac{(n/d)!}{(n_1/d)! \ldots (n_q/d)!}, \qquad \sum_i n_i = n$$

$\mu(k)$ Möbius function

2nd Witt formula.

The Length of the Basis is a function mainly of the Depth.

$$\begin{aligned} l_6(\{a, a, a, b, b, b\}) &= \frac{1}{6}\left[\mu(1)\frac{6!}{3!3!} + \mu(3)\frac{2!}{1!1!}\right] = 3 \\ n_6(\{a, a, a, b, b, b\}) &= \frac{6!}{2!3!} = 20 \end{aligned}$$

Weight	# Sums	Cum. # Sums	# Basic Sums	Cum. # Basic Sums	Cum. Fraction
1	2	2	0	0	0.0
2	6	8	1	1	0.1250
3	18	26	6	7	0.2692
4	54	80	16	23	0.2875
5	162	242	46	69	0.2851
6	486	728	114	183	0.2513

$\Uparrow$ 2nd Witt formula

$\Longrightarrow \quad \frac{\mathrm{Li}_2(x)}{x \pm 1} \quad , \quad \frac{\ln^2(1+x)}{x \pm 1}$

OBSERVATION IN QUANTUM FIELD THEORY:

AT LEAST UP TO $w = 6$, $S_{a_1, \dots, -1, \dots a_k}(N)$ DO NEVER OCCUR, IFF ONE USES COMPACT RERESENTATIONS.

$\Longrightarrow$ CONSIDER THIS SUBSET FURTHER ON.

One may show that likewise the number of all harmonic sums of weight w, which do not contain any index $i = -1$ is obtained by expanding the following generating function [3]

$$\frac{1-x}{1-2x-x^2} = \sum_{\mathrm{w}=0}^{\infty} N_{\neg(-1)}(\mathrm{w}) x^{\mathrm{w}} . \tag{14.8}$$

with

$$N_{\neg\{-1\}}(\mathrm{w}) = \mathrm{round}\left[\frac{1}{2}\left(1+\sqrt{2}\right)^{\mathrm{w}}\right] \tag{14.9}$$

$$N_{\neg\{-1\}}(\mathrm{w}) = \frac{1}{2}\left[\left(1-\sqrt{2}\right)^{\mathrm{w}} + \left(1+\sqrt{2}\right)^{\mathrm{w}}\right] = \sum_{k=0}^{[w/2]} \binom{w}{2k} 2^k , \tag{14.10}$$

with the recursion relation

$$N_{\neg\{-1\}}(\mathrm{w}) = 2 \cdot N_{\neg\{-1\}}(\mathrm{w}-1) + N_{\neg\{-1\}}(\mathrm{w}-2) \tag{14.11}$$

and $N_{\neg\{-1\}}(1) = 1$, $N_{\neg\{-1\}}(1) = 3$, which easily follows by induction.

In the following table the number of sums is given in dependence of the weight w. Here the number of a-basic sums is the number of sums obtained using the algebraic relations [5]. The number of s-basic sums counts the basic sums after using structural relations : single sums are not counted; etc etc [11].

Weight	Number of: Sums	a-basic sums	Sums $\neg\{-1\}$	a-basic sums	Sums $i > 0$	a-basic sums
1	2	2	1	1	1	1
2	6	3	3	2	2	1
3	18	8	7	4	4	2
4	54	18	17	7	8	3
5	162	48	41	16	16	6
6	486	116	99	30	32	9
7	1458	312	239	68	64	18
8	4374	810	577	140	128	30
9	13122	2184	1393	308	256	56
10	39366	5880	3363	664	512	99

4. Structural Relations

WEIGHT $w = 1$:

$$\frac{1}{1-x} \quad \& \quad \frac{1}{1+x}$$

$$\frac{1}{1-x^2} = \frac{1}{2}\left[\frac{1}{(1-x)} + \frac{1}{(1+x)}\right]$$

$$M\left[\left(\frac{1}{1-x}\right)_+\right]\left(\frac{N}{2}\right) = M\left[\left(\frac{1}{1-x}\right)_+\right](N) + M\left[\frac{1}{1+x}\right](N) + \ln(2)$$

$$-\psi\left(\frac{N}{2}\right) - \gamma_E = -\psi(N) - \gamma_E + \beta(N) + \ln(2)$$

$$\beta(N) := \frac{1}{2}\left[\psi\left(\frac{N+1}{2}\right) - \psi(N/2)\right].$$

$S_{-1}(N)$ DEPENDS ON $S_{+1}(N)$
FOR $N \in \mathbb{Q}$: $N \rightarrow \frac{N}{2}$.

$N \in \mathbb{R}$:

$$S_2(N) = -\frac{d}{dN} S_1(N) + \zeta_2 \quad \text{etc.}$$

FOR $N \in \mathbb{R}$:

ONLY ONE INDEPENDENT SINGLE SUM:

$$S_1(N) = \sum_{k=1}^{N} \frac{1}{k} = \psi(N+1) + \gamma_E$$

Sums of weight two

$$
\begin{aligned}
\mathbf{M}\left[\frac{\ln(1-x)}{1+x}\right](N) &= -\mathbf{M}\left[\frac{\ln(1+x)}{1+x}\right](N) \\
&\quad +(-1)^N \left[S_1(N-1)S_{-1}(N-1) + S_{-2}(N-1)\right. \\
&\quad \left. + \left[S_1(N-1) - S_{-1}(N-1)\right]\ln(2) - \ln^2(2) + \frac{1}{2}\zeta_2\right] . \\
&= -\mathbf{M}\left[\frac{\ln(1+x)}{1+x}\right](N) - \left[\psi(N) + \gamma_E + \ln(2)\right]\beta(N) + \beta'(N) .
\end{aligned}
$$

Similar to (3.5) one may decompose

$$
\frac{\ln(1-x^2)}{1-x^2} = \frac{1}{2}\left\{\frac{\ln(1-x)}{1-x} + \frac{\ln(1-x)}{1+x} + \frac{\ln(1+x)}{1-x} + \frac{\ln(1+x)}{1+x}\right\} .
$$

which yields

$$
\begin{aligned}
\mathbf{M}\left[\left(\frac{\ln(1-x)}{1-x}\right)_+\right]\left(\frac{N}{2}\right) &= \mathbf{M}\left[\left(\frac{\ln(1-x)}{1-x}\right)_+\right](N) + \mathbf{M}\left[\frac{\ln(1-x)}{1+x}\right](N) \\
&\quad + \mathbf{M}\left[\frac{\ln(1+x)}{1+x}\right](N) + \mathbf{M}\left[\left(\frac{\ln(1+x)}{1-x}\right)_+\right](N) - \frac{1}{2}\zeta_2 + \ln^2(2) .
\end{aligned}
$$

Since the l.h.s. of (4.5) and the first and fourth term in the r.h.s. are polynomials of single sums, the remainder two terms are related. However, (4.4) does not yield a new relation. Therefore

$$
\Longrightarrow \qquad F_1(N) := \mathbf{M}\left[\frac{\ln(1+x)}{1+x}\right](N) \qquad \to S_{1,-1}(N)
$$

is the first new non-trivial MELLIN transform.

As a historical remark we mention that NIELSEN [25] considered these functions a well using the notation $\xi(N), \eta(N), \xi_1(N)$ and $\xi_2(N)$,

$$
\begin{aligned}
\xi(N) &= \mathbf{M}\left[\left(\frac{\ln(1-x)}{x-1}\right)_+\right](N) = \frac{1}{2}\left[\psi'(N) - \zeta_2 - \left(\psi(N) + \gamma_E\right)^2\right] \\
&= -S_{1,1}(N-1) \\
\eta(N) &= \mathbf{M}\left[\left(\frac{\ln(1+x)}{x-1}\right)_+\right](N) = -\frac{1}{2}\left[\beta^2(N) - \psi'(N) + \zeta_2 - \ln^2(2)\right] \\
&\quad + \ln(2)\left[\psi(N) + \gamma_E\right] \\
&= -S_{-1,-1}(N-1) + \ln(2)\left[S_1(N-1) - S_{-1}(N-1)\right] \\
\xi_1(N) &= \mathbf{M}\left[\frac{\ln(1+x)}{x+1}\right](N) \\
-\xi_2(N) &= \mathbf{M}\left[\frac{\ln(1-x)}{x+1}\right](N) .
\end{aligned}
$$

and [25]

$$
\begin{aligned}
\left[\psi(z) + \gamma_E\right]\left[\psi(1-z) + \gamma_E\right] &= 2\zeta_2 - \xi(z) - \xi(1-z) \\
\beta(z)\beta(1-z) &= \eta(z) - \eta(1-z)
\end{aligned}
$$

holds [25] for $z \in]0,1[$.

$\boxed{\text{The Reduction for } \mathrm{Li}_k(-x)/(x \pm 1)}$

In some of the harmonic sums Mellin transforms of the type

$$\frac{\mathrm{Li}_k(-x)}{x \pm 1}.$$

For odd values of $k = 2l + 1$ the harmonic sums $S_{1,-(k-1)}(N), S_{-(k-1),1}(N)$ and $S_{-l,-l}(N)$ allow to substitute the Mellin transforms of these functions in terms of Mellin transforms of basic functions and derivatives thereof.

For even values of k this argument applies to $M[\mathrm{Li}_k(-x)/(1 + x)](N)$ but not for $M[\mathrm{Li}_k(-x)/(1 + x)](N)$. In the latter case one may use the relation

$$\frac{1}{2^{k-2}} \frac{\mathrm{Li}_k(x^2)}{1-x^2} = \frac{\mathrm{Li}_k(x)}{1-x} + \frac{\mathrm{Li}_k(x)}{1+x} + \frac{\mathrm{Li}_k(-x)}{1-x} + \frac{\mathrm{Li}_k(x)}{1+x}.$$

Since in massless quantumfield-theoretic calculations both denominators occur, one may use this decomposition based on the first two cyclotomic polynomials, cf. [7], and the decomposition relation for $\mathrm{Li}_k(x^2)$. The corresponding Mellin transforms also require half–integer arguments. In more general situations other cyclotomic polynomials might emerge. The relation

$$\begin{aligned}
\frac{1}{2^{k-1}} \mathbf{M}\left[\left(\frac{\mathrm{Li}_k(x^2)}{x^2-1}\right)_+\right]\left(\frac{N-1}{2}\right) &= \mathbf{M}\left[\left(\frac{\mathrm{Li}_k(x)}{x-1}\right)_+\right](N) + \mathbf{M}\left[\left(\frac{\mathrm{Li}_k(x)}{x+1}\right)_+\right](N) \\
&+ \mathbf{M}\left[\left(\frac{\mathrm{Li}_k(-x)}{x-1}\right)_+\right](N) + \mathbf{M}\left[\left(\frac{\mathrm{Li}_k(-x)}{x+1}\right)_+\right](N) \\
&- \int_0^1 dx \frac{\mathrm{Li}_k(x^2)}{1+x}
\end{aligned}$$

determines $M[\mathrm{Li}_k(-x)/(1+x)](N)$. For $k = 2, 4$ the last integral in (7.16) is given by

$$\begin{aligned}
\int_0^1 dx \frac{\mathrm{Li}_2(x^2)}{1+x} &= \zeta_2 \ln(2) - \frac{3}{4}\zeta_3 \\
\int_0^1 dx \frac{\mathrm{Li}_4(x^2)}{1+x} &= \frac{2}{5}\ln(2)\zeta_2^2 + 3\zeta_2\zeta_3 - \frac{25}{4}\zeta_5
\end{aligned}$$

The corresponding relations for $\mathbf{M}[\mathrm{Li}_k(-x)/(1+x)](N)$ are :

$$\begin{aligned}
\mathbf{M}\left[\frac{\mathrm{Li}_2(-x)}{x+1}\right](N) &= -\frac{1}{2}\mathbf{M}\left[\left(\frac{\mathrm{Li}_2(x)}{x-1}\right)_+\right]\left(\frac{N-1}{2}\right) + \mathbf{M}\left[\left(\frac{\mathrm{Li}_2(x)}{x-1}\right)_+\right](N) \\
&+ \mathbf{M}\left[\left(\frac{\mathrm{Li}_2(-x)}{x-1}\right)_+\right](N) - \mathbf{M}\left[\frac{\mathrm{Li}_2(x)}{x+1}\right](N) \\
&+ \frac{3}{8}\zeta_3 - \frac{1}{2}\zeta_2 \ln(2)
\end{aligned}$$

$$\begin{aligned}
\mathbf{M}\left[\frac{\mathrm{Li}_4(-x)}{x+1}\right](N) &= -\frac{1}{8}\mathbf{M}\left[\left(\frac{\mathrm{Li}_4(x)}{x-1}\right)_+\right]\left(\frac{N-1}{2}\right) + \mathbf{M}\left[\left(\frac{\mathrm{Li}_4(x)}{x-1}\right)_+\right](N) \\
&+ \mathbf{M}\left[\left(\frac{\mathrm{Li}_4(-x)}{x-1}\right)_+\right](N) - \mathbf{M}\left[\frac{\mathrm{Li}_4(x)}{x+1}\right](N) \\
&- \frac{1}{20}\zeta_2^2 \ln(2) - \frac{3}{8}\zeta_2\zeta_3 + \frac{25}{32}\zeta_5
\end{aligned}$$

Sums of weight three

There are 18 harmonic sums of weight w=3 [3]. 12 of these sums are related to the remaining 6 harmonic sums by shuffle relations [18]. The latter are represented by the MELLIN transforms

$$\mathbf{M}\left[\left(\frac{\ln^2(1+x)}{1-x}\right)_+\right](N) \qquad \mathbf{M}\left[\frac{\ln^2(1+x)}{1+x}\right](N)$$

$$\mathbf{M}\left[\left(\frac{\mathrm{Li}_2(-x)}{1-x}\right)_+\right](N) \qquad F_4 = \mathbf{M}\left[\left(\frac{\mathrm{Li}_2(x)}{1-x}\right)_+\right](N)$$

$$\mathbf{M}\left[\frac{\mathrm{Li}_2(-x)}{1+x}\right](N) \qquad F_5 = \mathbf{M}\left[\frac{\mathrm{Li}_2(x)}{1+x}\right](N)$$

The first two functions in (5.1) correspond to the harmonic sums of the type $S_{-1,1,-1}(N)$ and $S_{1,1,-1}(N)$, which are not related. Using the general relation

$$\mathbf{M}\left[\ln^l(x)f(x)\right](N) = \frac{d^l}{dN^l}\mathbf{M}\left[f(x)\right](N) ,$$

the MELLIN transform $\mathbf{M}\left[\ln(x)\ln(1+x)/(1+x)\right](N)$ can be calculated from $F_1(N)$. By this we can use EULER'S relation for the sums $S_{1,-2}(N)$ and $S_{-2,1}(N)$, [3], to express $\mathbf{M}\left[\mathrm{Li}_2(-x)/(1+x)\right](N)$ in terms of $\mathbf{M}\left[\mathrm{Li}_2(x)/(1+x)\right](N)$,

$$\begin{aligned}\mathbf{M}\left[\frac{\mathrm{Li}_2(-x)}{1+x}\right](N) &= \mathbf{M}\left[\frac{\mathrm{Li}_2(x)}{1+x}\right](N) - \mathbf{M}\left[\ln(x)\frac{\ln(1+x))}{1+x}\right](N)\\ &-\frac{\zeta_2}{2}\beta(N) - \left[\psi(N)+\gamma_E\right]\beta'(N) + (-1)^{N-1}\frac{3}{4}\zeta_3 .\end{aligned}$$

Furthermore, the identity

$$\frac{1}{2^n}\frac{\mathrm{Li}_n(x^2)}{1-x^2} = \frac{\mathrm{Li}_n(x)}{1-x} + \frac{\mathrm{Li}_n(x)}{1+x} + \frac{\mathrm{Li}_n(-x)}{1-x} + \frac{\mathrm{Li}_n(-x)}{1+x}$$

holds and one obtains :

$$\begin{aligned}\frac{1}{2}\mathbf{M}\left[\left(\frac{\mathrm{Li}_2(x)}{1-x}\right)_+\right]\left(\frac{N}{2}\right) &= \mathbf{M}\left[\left(\frac{\mathrm{Li}_2(x)}{1-x}\right)_+\right](N) + \mathbf{M}\left[\frac{\mathrm{Li}_2(x)}{1+x}\right](N)\\ &+\mathbf{M}\left[\left(\frac{\mathrm{Li}_2(-x)}{1-x}\right)_+\right](N) + \mathbf{M}\left[\frac{\mathrm{Li}_2(-x)}{1+x}\right](N)\\ &+\frac{1}{2}\ln(2)\zeta_2 - \frac{3}{8}\zeta_3 ,\end{aligned}$$

(5.5) can be used to express $\mathbf{M}\left[(\mathrm{Li}_2(-x)/(1-x))_+\right](N)$ in terms of the remaining MELLIN transforms.

$$\begin{aligned}\mathbf{M}\left[\left(\frac{\mathrm{Li}_2(-x)}{1-x}\right)_+\right](N) &= \frac{1}{2}F_4\left(\frac{N}{2}\right) - F_4(N) - 2F_5(N) + F_1'(N)\\ &+\frac{\zeta_2}{2}\beta(N) + \left[\psi(N)+\gamma_E\right]\beta'(N) + (-1)^N\frac{3}{4}\zeta_3\end{aligned}$$

WEIGHT $w=4$:

$i \neq -1$ HARMONIC SUMS:

$$\Longrightarrow \quad \frac{\mathrm{Li}_3(x)}{x \pm 1}, \quad \frac{S_{1,2}(x)}{x \pm 1}$$

$$S_{2,2}(N) = \frac{1}{2}\left[S_2^2(N) + S_4(N)\right]$$

$$\propto 2\, M\left[\left(\frac{\mathrm{Li}_3(x)}{x-1}\right)_+\right](N) - M\left[\ln(x)\,\frac{\mathrm{Li}_2(x)}{x-1}\right](N).$$

$\Longrightarrow$ ONLY

$$\frac{\mathrm{Li}_3(x)}{x+1}, \quad \frac{S_{1,2}(x)}{x \pm 1}$$

CONTRIBUTE AS NEW FUNCTIONS.

WEIGHT W = 5 :

The above multiple harmonic sums obey the following representation [4] :

$$S_{4,1}(N) = -\mathrm{M}\left[\left(\frac{\mathrm{Li}_4(x)}{x-1}\right)_+\right](N) + S_1(N)\zeta_4 - S_2(N)\zeta_3 + S_3(N)\zeta_2 \quad (12.5)$$

$$S_{-4,1}(N) = (-1)^{N+1}\mathrm{M}\left[\frac{\mathrm{Li}_4(x)}{x+1}\right](N) + S_{-1}(N)\zeta_4 - S_{-2}(N)\zeta_3 + S_{-3}(N)\zeta_2 + \zeta_4\ln(2) + \frac{3}{4}\zeta_2\zeta_3 - \frac{59}{32}\zeta_5 \quad (12.6)$$

$$S_{1,-4}(N) = (-1)^N\mathrm{M}\left[\frac{\mathrm{Li}_4(-x) - \ln(x)\mathrm{Li}_3(-x) + \ln^2(x)\mathrm{Li}_2(-x)/2 + \ln^3(x)\ln(1+x)/6}{x+1}\right](N) + \frac{7}{8}\zeta_4\left[S_{-1}(N) - S_1(N)\right] - \frac{1}{2}\zeta_2\zeta_3 + \frac{7}{8}\zeta_4\ln(2) + \frac{29}{32}\zeta_5 \quad (12.7)$$

$$S_{3,1,1}(N) = \mathrm{M}\left[\left(\frac{S_{2,2}(x)}{x-1}\right)_+\right](N) + \zeta_3 S_2(N) - \frac{\zeta_4}{4}S_1(N) \quad (12.8)$$

$$S_{-3,1,1}(N) = (-1)^N\mathrm{M}\left[\frac{S_{2,2}(x)}{1+x}\right](N) + \zeta_3 S_{-2}(N) - \frac{\zeta_4}{4}S_{-1}(N) - \left[-\frac{7}{8}\zeta_3\zeta_2 + \frac{1}{4}\zeta_4\ln(2) + \frac{7}{8}\zeta_3\ln^2(2) - \frac{1}{3}\zeta_2\ln^3(2)\right] - \frac{15}{32}\zeta_5 + 2\ln(2)\mathrm{Li}_4\left(\frac{1}{2}\right) + 2\mathrm{Li}_5\left(\frac{1}{2}\right) + \frac{1}{15}\ln^5(2) \quad (12.9)$$

$$S_{2,1,1,1}(N) = -\mathrm{M}\left[\left(\frac{S_{1,3}(x)}{x-1}\right)_+\right](N) + \zeta_4 S_1(N) \quad (12.10)$$

$$S_{-2,1,1,1}(N) = (-1)^{N+1}\mathrm{M}\left[\frac{S_{1,3}(x)}{1+x}\right](N) + \zeta_4 S_{-1}(N) + \zeta_4\ln(2) - \frac{7}{16}\zeta_2\zeta_3 - \frac{1}{6}\zeta_2\ln^3(2) + \frac{7}{16}\zeta_3\ln^2(2) - \frac{27}{32}\zeta_5 + \ln(2)\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{1}{30}\ln^5(2) + \mathrm{Li}_5\left(\frac{1}{2}\right) \quad (12.11)$$

$$S_{2,2,1}(N) = \mathrm{M}\left[\left(\frac{2S_{2,2}(x) - \mathrm{Li}_2^2(x)/2}{x-1}\right)_+\right](N) + \zeta_2 S_{2,1}(N) + \frac{3}{10}\zeta_2^2 S_1(N) \quad (12.12)$$

$$S_{-2,1,-2}(N) = \mathrm{M}\left[\left(\frac{S_{2,2}(-x) - S_{2,2}(x) + \ln(x)S_{1,2}(x) - \mathrm{Li}_2^2(-x)}{x-1}\right)_+\right](N) + \left(\frac{1}{8}\zeta_3 - \frac{1}{2}\ln(2)\zeta_2\right)S_2(N) \left(-\frac{39}{40}\zeta_2^2 + 2\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{7}{4}\zeta_3\ln(2) - \frac{1}{2}\zeta_2\ln^2(2) + \frac{1}{12}\ln^4\right)S_1(N) \quad (12.13)$$

$$S_{-2,2,1}(N) = (-1)^N\mathrm{M}\left[\frac{2S_{2,2}(x) - \mathrm{Li}_2^2(x)/2}{1+x}\right](N) + \zeta_2 S_{-2,1}(N) + \frac{3}{10}\zeta_2^2 S_{-1}(N) + 4\mathrm{Li}_5\left(\frac{1}{2}\right) + 4\mathrm{Li}_5\left(\frac{1}{2}\right)\ln(2) - \frac{89}{64}\zeta_5 - \frac{9}{8}\zeta_2\zeta_3 + \frac{2}{15}\ln^5(2) - \frac{2}{3}\zeta_2\ln^3(2) + \frac{7}{4}\zeta_3\ln^2(2) - \frac{3}{10}\zeta_2^2\ln(2) \quad (12.14)$$

$$S_{2,1,-2}(N) = (-1)^{N+1}\mathrm{M}\left[\frac{S_{2,2}(x) - S_{2,2}(-x) - \ln(x)S_{1,2}(x) + \mathrm{Li}_2^2(-x)/2}{1+x}\right](N) - \int_0^1 dx\frac{S_{2,2}(x) - S_{2,2}(-x) - \ln(x)S_{1,2}(x) + \mathrm{Li}_2^2(-x)/2}{1+x} - \frac{1}{8}\zeta_3 S_{-2}(N) - \left[2\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{7}{4}\zeta_3\ln(2) - \frac{1}{2}\zeta 2\ln^2(2) + \frac{1}{12}\ln^4(2) - \frac{39}{40}\zeta_2^2\right]S_{-1}(N) \quad (12.16)$$

$$
\begin{aligned}
S_{2,3}(N) &= \mathbf{M}\left[\left(\frac{\ln(x)\left[S_{1,2}(1-x)-\zeta_3\right]+3\left[S_{1,3}(1-x)-\zeta_4\right]}{x-1}\right)_+\right](N) \\
&\quad +3\zeta_4 S_1(N) && (12.18) \\
S_{-2,3}(N) &= (-1)^N\,\mathbf{M}\left[\frac{\ln(x)\left[S_{1,2}(1-x)-\zeta_3\right]+3\left[S_{1,3}(1-x)-\zeta_4\right]}{1+x}\right](N) \\
&\quad +3\zeta_4 S_{-1}(N)+\frac{21}{32}\zeta_5+3\zeta_4\ln(2)-\frac{3}{4}\zeta_2\zeta_3 \\
& && (12.19) \\
S_{2,-3}(N) &= (-1)^{N+1}\,\mathbf{M}\left[\frac{1}{1+x}\left[\frac{1}{2}\ln^2(x)\mathrm{Li}_2(-x)-2\ln(x)\mathrm{Li}_3(-x)+3\mathrm{Li}_4(-x)\right]\right](N) \\
&\quad +\frac{3}{4}\left[S_{-2}(N)-S_2(N)\right]-\frac{21}{8}\zeta_4 S_{-1}(N)-\frac{41}{32}\zeta_5-\frac{21}{8}\zeta_4\ln(2)+\zeta_2\zeta_3 && (12.20) \\
S_{-2,-3}(N) &= \mathbf{M}\left\{\left[\frac{1}{1-x}\left(\frac{1}{2}\ln^2(x)\mathrm{Li}_2(-x)-2\ln(x)\mathrm{Li}_3(-x)+3\mathrm{Li}_4(-x)\right)\right]_+\right\}(N) \\
&\quad +\frac{3}{4}\zeta_3\left[S_2(N)-S_{-2}(N)\right]-\frac{21}{8}\zeta_4 S_1(N)\,. && (12.21)
\end{aligned}
$$

In the above integrands we use the relations

$$
\begin{aligned}
S_{1,2}(1-x) &= -\mathrm{Li}_3(x)+\ln(x)\mathrm{Li}_2(x)+\frac{1}{2}\ln(1-x)\ln^2(x)+\zeta_3 \\
S_{1,3}(1-x) &= -\mathrm{Li}_4(x)+\ln(x)\mathrm{Li}_3(x)-\frac{1}{2}\ln^2(x)\mathrm{Li}_2(x)-\frac{1}{6}\ln^3(x)\ln(1-x)+\zeta_4 \\
S_{2,2}(1-x) &= -S_{2,2}(x)+\ln(x)S_{1,2}(x)+\frac{\zeta_4}{4} \\
&\quad -\left[\mathrm{Li}_3(x)-\ln(x)\mathrm{Li}_2(x)-\zeta_3\right]\ln(1-x)+\frac{1}{4}\ln^2(x)\ln^2(1-x)\,,
\end{aligned}
$$

$\Longrightarrow \qquad \dfrac{\mathrm{Li}_4(x)}{x \pm 1}$

W=5 BASIC FUNCTIONS:

$$\frac{\mathrm{Li}_4(x)}{x\pm 1} \qquad \frac{S_{1,3}(x)}{x+1} \qquad \frac{S_{2,2}(x)}{x\pm 1} \qquad \frac{\mathrm{Li}_2^2(x)}{x+1}$$

$$\frac{S_{2,2}(-x)-\mathrm{Li}_2^2(-x)/2}{x\pm 1}.$$

Sums of Weight $w = 6$

Twofold Sums

The following sums occur : $S_{\pm 5,1}(N), S_{\pm 4,\pm 2}(N), S_{-3,3}(N)$ and $S_{3,3}(N), S_{-3,-3}(N)$. The latter sums are related to single harmonic sums through Euler's relation. In the case of the former sums we only consider the algebraically irreducible cases. We show that all sums can be related to $S_{\pm 5,1}(N)$. Their representation is :

$$\begin{aligned}
S_{5,1}(N) &= \mathbf{M}\left[\left(\frac{\mathrm{Li}_5(x)}{x-1}\right)_+\right](N) - S_1(N)\zeta_5 + S_2(N)\zeta_2 - S_3(N)\zeta_3 + S_4\zeta_2 \\
S_{-5,1}(N) &= (-1)^N \mathbf{M}\left[\frac{\mathrm{Li}_5(x)}{1+x}\right](N) - C_{-5,1} - S_{-1}(N)\zeta_5 + S_{-2}(N)\zeta_4 \\
&\quad -S_{-3}(N)\zeta_3 + S_{-4}\zeta_2 \;,
\end{aligned}$$

with

$$C_{-5,1} = s_6 - \frac{15}{16}\ln(2)\zeta_5 \;.$$

The other two-fold sums are

$$\begin{aligned}
S_{-4,-2}(N) &= -\mathbf{M}\left[\frac{4\mathrm{Li}_5(-x) - \ln(x)\mathrm{Li}_4(-x)}{x-1}\right](N) \\
&\quad +\frac{1}{2}\zeta_2\left[S_4(N) - S_{-4}(N)\right] - \frac{3}{2}\zeta_3 S_3(N) + \frac{21}{8}\zeta_4 S_2(N) - \frac{15}{4}\zeta_5 S_1(N) \\
S_{-4,2}(N) &= (-1)^{N+1}\mathbf{M}\left[\frac{4\mathrm{Li}_5(x) - \mathrm{Li}_4(x)\ln(x)}{1+x}\right](N) \\
&\quad +2\zeta_3 S_{-3}(N) - 3\zeta_4 S_{-2}(N) + 4\zeta_5 S_{-1}(N) + \frac{239}{840}\zeta_2^3 - \frac{3}{4}\zeta_3^2 - \frac{15}{4}\zeta_5\ln(2) + 4s_6
\end{aligned}$$

$$\begin{aligned}
S_{4,-2}(N) &= \frac{1}{2}\zeta_2\left[S_{-4}(N) - S_4(N)\right] - \frac{3}{2}\zeta_3 S_{-3}(N) + \frac{21}{8}\zeta_4 S_{-2}(N) - \frac{15}{4}\zeta_5 S_{-1}(N) \\
&\quad +(-1)^{N+1}\mathbf{M}\left[\frac{4\mathrm{Li}_5(-x) - \ln(x)\mathrm{Li}_4(-x)}{1+x}\right](N) \\
&\quad -\frac{313}{840}\zeta_2^3 + \frac{15}{16}\zeta_3^2 + 4\zeta_5\ln(2) - 4s_6 \\
S_{4,2}(N) &= -\mathbf{M}\left[\left(\frac{4\mathrm{Li}_5(x) - \ln(x)\mathrm{Li}_4(x)}{x-1}\right)_+\right](N) + 2\zeta_3 S_3(N) - 3\zeta_4 S_2(N) + 4\zeta_5 S_1(N)
\end{aligned}$$

$$\begin{aligned}
S_{-3,-3}(N) &= \frac{1}{2}\left[S_{-3}^2(N) + S_6(N)\right] \\
S_{-3,3}(N) &= 3\zeta_4 S_{-2}(N) - 6\zeta_5 S_{-1}(N) + (-1)^{N+1} 6\,\mathbf{M}\left[\left(\frac{S_{1,4}(1-x) - \zeta_5}{1+x}\right)_+\right](N) \\
&\quad +(-1)^{N+1}\mathbf{M}\left[\left(\frac{3\ln(x)\left[S_{1,3}(1-x) - \zeta_5\right] + \ln^2(x)\left[S_{1,2}(1-x) - \zeta_3\right]/2}{1+x}\right)_+\right](N) \\
&\quad -\frac{271}{280}\zeta_2^3 + \frac{81}{32}\zeta_3^2 + \frac{45}{8}\zeta_5\ln(2) - 6s_6
\end{aligned}$$

$$
\begin{aligned}
S_{3,3}(N) &= \frac{1}{2}\left[S_3^2(N)+S_6(N)\right] \\
&= 3\zeta_4 S_2(N) - 6\zeta_5 S_1(N) - 6\,\mathbf{M}\left[\left(\frac{S_{1,4}(1-x)-\zeta_5}{x-1}\right)_+\right](N) \\
&\quad - \mathbf{M}\left[\left(\frac{\ln^2(x)\left[S_{1,2}(1-x)-\zeta_3\right]/2 + 3\ln(x)\left[S_{1,3}(1-x)-\zeta_4\right]}{x-1}\right)_+\right](N)\,,
\end{aligned}
$$

where

$$
\begin{aligned}
S_{1,4}(1-x) &= -\mathrm{Li}_5(x) + \ln(x)\mathrm{Li}_4(x) - \frac{1}{2}\ln^2(x)\mathrm{Li}_3(x) + \frac{1}{6}\ln(x)\mathrm{Li}_2(x) \\
&\quad + \frac{1}{24}\ln(x)\ln(1-x) + \zeta_5
\end{aligned}
$$

The algebraic relations for $S_{3,3}(N)$ can be used to express $\mathbf{M}[(\mathrm{Li}_5(x)/(x-1))_+](N)$. $S_{4,2}(N)$ and $S_{-3,3}(N)$ do not contain new Mellin transforms. Similarly, the $S_{-3,-3}(N)$ determines the new Mellin-transform in $S_{-4,-2}(N)$. The two representations for $S_{-5,1}(N)$ relate $\mathbf{M}[\mathrm{Li}_5(-x)/(1+x)](N)$ and $\mathbf{M}[\mathrm{Li}_5(x)/(1+x)](N)$. Therefore contributing the double sums at w=6 are determined by only one new baswic function $\mathbf{M}[\mathrm{Li}_5(x)/(1+x)](N)$.

$$
\begin{aligned}
S_{-5,1}(N) &= S_{-5}(N)S_1(N) + S_{-6}(N) \\
&\quad - \left\{(-1)^N\,\mathbf{M}\left[\frac{\mathrm{Li}_5(-x) - \ln(x)\mathrm{Li}_4(-x) + \ln^2(x)\mathrm{Li}_3(-x)/2}{x+1}\right](N)\right. \\
&\quad + (-1)^N\,\mathbf{M}\left[\frac{-\ln^3(x)\mathrm{Li}_2(-x)/6 - \ln^4(x)\ln(1+x)/24}{x+1}\right](N) \\
&\quad \left. + \frac{15}{16}\zeta_5\left[S_{-1}(N) - S_1(N)\right] + \frac{23}{70}\zeta_2^3 - \frac{3}{4}\zeta_3^2 - \zeta_5\ln(2) + s_6\right\}.
\end{aligned}
$$

$\Longrightarrow \quad \dfrac{\mathrm{Li}_5(x)}{1+x}$

Threefold Sums

$$
\begin{aligned}
S_{4,1,1}(N) &= -\mathbf{M}\left[\left(\frac{S_{3,2}(x)}{x-1}\right)_+\right](N) + S_1(N)(2\zeta_5-\zeta_2\zeta_3) - \frac{\zeta_4}{4}S_2(N) + \zeta_3 S_3(N)\\
S_{-4,1,1}(N) &= (-1)^{N+1}\mathbf{M}\left[\frac{S_{3,2}(x)}{1+x}\right](N) + (2\zeta_5-\zeta_2\zeta_3)S_{-1}(N) - \frac{\zeta_4}{4}S_{-2}(N)\\
&\quad +\zeta_3 S_{-3}(N) + C_{-4,1,1}\\
C_{-4,1,1} &= \int_0^1 dx \frac{S_{3,2}(x)}{1+x} = \frac{71}{840}\zeta_2^3 + \frac{1}{8}\zeta_3^2 + \frac{29}{32}\zeta_5\ln(2) - \zeta_2\zeta_3\ln(2) + \frac{3}{2}s_6
\end{aligned}
$$

+ some more to come

The quadruple index sums are :

$$
S_{-3,1,1,1}(N) = (-1)^N\mathbf{M}\left[\frac{S_{2,3}(x)}{x+1}\right](N) - C_{-3,1,1,1} + \zeta_4 S_{-2}(N) - (2\zeta_5-\zeta_2\zeta_3)S_{-1}(N)
$$

$$
\begin{aligned}
S_{3,1,1,1}(N) &= \mathbf{M}\left[\left(\frac{S_{2,3}(x)}{x-1}\right)_+\right](N) + \zeta_4 S_2(N) - (2\zeta_5-\zeta_2\zeta_3)S_1(N)\\
C_{-3,1,1,1} &= \int_0^1 dx\frac{S_{2,3}(x)}{1+x}\\
&= -\frac{1}{8}\zeta_2\zeta_3\ln(2) + \frac{1}{6}\zeta_2\ln(2) + \zeta_2\mathrm{Li}_4\left(\frac{1}{2}\right) - \frac{1}{4}\zeta_2^2\ln^2(2) + \frac{257}{840}\zeta_3\ln^3(2)\\
&\quad -\frac{41}{64}\zeta_3^2 + \frac{33}{32}\zeta_5\ln(2) - 2\ln(2)\mathrm{Li}_5\left(\frac{1}{2}\right) - \ln^2(2)\mathrm{Li}_4\left(\frac{1}{2}\right) - \frac{1}{36}\ln^6(2)\\
&\quad -2\mathrm{Li}_6\left(\frac{1}{2}\right) + \frac{1}{2}s_6
\end{aligned}
$$

+ some more to come.

Finally, two 5–fold sums contribute.

$$
\begin{aligned}
S_{2,1,1,1,1}(N) &= -\mathbf{M}\left[\left(\frac{S_{1,4}(x)}{x-1}\right)_+\right](N)+\zeta_5 S_1(N)\\
S_{-2,1,1,1,1}(N) &= (-1)^{N+1}\mathbf{M}\left[\frac{S_{1,4}(x)}{1+x}\right](N)+C_{-2,1,1,1,1}+\zeta_5 S_{-1}(N)\,,
\end{aligned}
$$

where

$$
\begin{aligned}
C_{-2,1,1,1,1} &= \int_0^1 dx\frac{S_{1,4}(x)}{1+x} = \ln(2)\zeta_5+\frac{7}{16}\zeta_2\zeta_3\ln(2)+\frac{1}{12}\zeta_2\ln^4(2)+\frac{1}{2}\zeta_2\mathrm{Li}_4\left(\frac{1}{2}\right)\\
&-\frac{1}{8}\zeta_2^2\ln^2(2)-\frac{7}{48}\zeta_3\ln^3(2)-\frac{7}{48}\zeta_3\ln^3(2)-\frac{49}{128}\zeta_3^2\\
&-\ln(2)\mathrm{Li}_5\left(\frac{1}{2}\right)-\frac{1}{2}\ln^2(2)\mathrm{Li}_4\left(\frac{1}{2}\right)-\frac{1}{72}\ln^6(2)-\mathrm{Li}_6\left(\frac{1}{2}\right)\,.
\end{aligned}
$$

ARE ALL NUMERATORS NIELSEN INTEGRALS?

TO BE CLARIFIED, BUT MAY BE NOT:

$$\int_0^x \frac{dy}{y}\,\mathrm{Li}_2^2(\pm y)$$

COULD OCCUR.

$$\longleftrightarrow \qquad S_{1,2}(\pm x) = \frac{1}{2}\int_0^x \frac{dy}{y}\ln^2(1-y)\,.$$

5. Factorial Series

CONSIDER:

$$\Omega(z) = \int_0^1 dt\, t^{z-1} \varphi(t)$$

$$\varphi(1-t) = \sum_{k=0}^{\infty} a_k t^k \qquad (*)$$

$\mathrm{Re}(z) > 0$

$$\Omega(z) = \sum_{k=0}^{\infty} a_{k+1} \frac{k!}{z(z+1)\dots(z+k)}$$

THIS REPRESENTATION CAN BE OBTAINED FOR ALL BASIC FUNCTIONS.

SOME "SOFT" TERMS HAVE TO BE SUBTRACTED TO MEET COND. (*) ; THEY CAN BE TREATED EVEN ALGEBRAICALLY.

$$\Omega(z) \sim \sum_{k=1}^{\infty} \frac{\mathfrak{a}_k}{z^k} \quad , \quad |z| \to \infty$$

$$\mathfrak{a}_1 = a_1 \ ; \quad \mathfrak{a}_k = \sum_{\ell=0}^{k-2} (-1)^{\ell} \mathcal{S}_{k-\ell}^{\ell}\, a_{k-\ell}$$

↑ Stirling-2 numbers.

$\Omega(z)$ @ FINITE z:

APPLY RECURSION: $z \to z+1$

$\longrightarrow$ OTHER BASIC FUNCTIONS OF LOWER WEIGHT.

$\Omega(z)$ IS MEROMORPHIC.

$\longrightarrow$ DIFFERENTIABLE.

EXAMPLES:

- $F_1(z) = \mathbf{M}\left[\frac{\ln(1+\hat{z})}{1+\hat{z}}\right](N)_{|z=N} \underset{\text{Nielsen}}{\equiv} \xi_1(z)$.

$$F_1(z) = F_1(z+1) - \frac{1}{z}\left[\ln(2) - \beta(z+1)\right]$$

$$F_1(z) = \sum_{k=1}^{\infty} (-1)^{k-1} S_1(k) \frac{1}{k+z}$$

$$F_1(z) \underset{|z|\to\infty}{\approx} \frac{1}{2}\frac{1}{N}\ln(2) - \frac{1}{4}(1-\ln(2))\frac{1}{N^2} - \frac{1}{8N^3} + \frac{1}{16}(3-2\ln(2))\frac{1}{N^4} + \dots \quad |_{N\equiv z}.$$

- $F_5(z) = M\left[\frac{\mathrm{Li}_2(x)}{1+x}\right](z).$

$$F_5(z+1) = -F_5(z) + \frac{1}{z}\left[\zeta_2 - \frac{\psi(z+1)+\gamma_E}{z}\right]$$

Asymptotic expansion: $\mathrm{Li}_2(z) \longrightarrow \mathrm{Li}_2(1-z).$

$$M\left[\frac{\mathrm{Li}_2(1-z)}{1+z}\right](N) \underset{|N|\to\infty}{=} \frac{1}{2N^2} + \frac{1}{4N^3} - \frac{7}{24}\frac{1}{N^4} - \frac{1}{3}\frac{1}{N^5} + \frac{73}{120}\frac{1}{N^6} + \ldots$$

etc.

- $S_1(N) = \psi(N+1) + \gamma_E$

$$\psi(z) = \psi(z-1) + \frac{1}{z-1}$$

$$S_1(N) \underset{|N|\to\infty}{\propto} \ln(z) + \frac{1}{2z} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2k\, z^{2k}}$$

$$\psi(1+z) = -\gamma_E + \sum_{k=1}^{\infty} \frac{z}{k(k+z)}.$$

6. A Basis

$i \neq -1.$

1 $W = 1$: $\dfrac{1}{(1-x)_+}$

$W = 2$: SUBSIDIARY : $\dfrac{\ln(1+x)}{1+x}$ HELPS A LOT FOR $W \geq 4$.

2 $W = 3$: $\dfrac{\mathrm{Li}_2(x)}{x \pm 1}$

4 $W = 4$: $\dfrac{\mathrm{Li}_3(x)}{x+1}$, $\dfrac{S_{1,2}(x)}{x \pm 1}$

8 $W = 5$: $\dfrac{\mathrm{Li}_4(x)}{x \pm 1}$, $\dfrac{S_{1,3}(x)}{x+1}$, $\dfrac{S_{2,2}(x)}{x \pm 1}$, $\dfrac{\mathrm{Li}_2^2(x)}{x+1}$

$$\frac{S_{2,2}(-x) - \mathrm{Li}_2^2(-x)/2}{x \pm 1}$$

$W = 6$: $\dfrac{\mathrm{Li}_5(x)}{x+1}$, $\dfrac{S_{1,4}(x)}{x \pm 1}$, $\dfrac{S_{2,3}(x)}{x \pm 1}$, $\dfrac{S_{3,2}(x)}{x \pm 1}$

AND MORE TO COME.

(A FEW MORE).

7. Conclusions

1) THE SINGLE SCALE QUANTITIES IN QFT'S (MASSLESS OR SINGLE MASS / $Q^2 \to 0$) TO 3 LOOP ORDER $\Leftrightarrow$ $W = 6$ HARMONIC SUMS ARE $\in$ POLYNOMIAL RING SPANNED BY A FEW BASIC MELLIN TRANSFORMS ONLY.

2)

	CUMUL. #
$W = 1$:	1
$W = 2$:	1
$W = 3$:	3
$W = 4$:	$6 + 1 = 7$
$W = 5$:	15
$W = 6$:	$\dots 15 + 7 + \dots$

3) THE BASIC FUNCTIONS ARE MEROMORPHIC + "SOFT COMPONENTS" $\propto \ln^k N$ $\quad |N| \to \infty$.

4) THE TOTAL AMOUNT OF HARMONIC SUMS REDUCES BY ALGEBRAIC RELATIONS [INDICES OF H-SUMS].

STRUCTURAL RELATIONS : $N \in \mathbb{Q}$

$N \in \mathbb{R}$

5) ANALYTIC CONTINUATION TO $N \in \mathbb{C}$ IS POSSIBLE BY EXPRESSING THE BASIC SUMS IN TERMS OF FACTORIAL SERIES UP TO THEIR 'SOFT COMPONENT'. THE LATTER IS USUALLY OF LOWER WEIGHT AND/OR EXPRESSED BY SINGLE HARMONIC SUMS IN PART.

6) AS WE OBSERVE THE SAME SET OF BASIC FUNCTIONS IN A WIDE CLASS OF PHYSICAL PROCESSES, THEY SEEM TO BE OF SOME GENERAL NATURE.

7) UP TO $w=6$, PHYSICAL QUANTITIES OF THE ABOVE TYPE APPEAR TO BE <u>FREE</u> OF INDECES $\{-1\}$ IN THE HARMONIC SUMS CONTRIBUTING.

UP TO WEIGHT $w=5$ ALL NUMERATORS CAN BE EXPRESSED BY NIELSEN INTEGRALS