From Moments to Functions in Higher Order QCD

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- Introduction
- Single Scale Feynman Integrals as Recurrent Quantities
- Establishing and Solving Recurrences
- Application to 3-Loop Anomalous Dimensions and Wilson coefficients
- Conclusions

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1. Introduction

• Higher order calculations in Quantum Field Theories easily become tedious due to the larger number of terms and the sophistication of the Feynman parameter integrals.

• This even applies to Zero Scale and Single Scale Quantities.

• Even more this is the case for higher scale problems.

• While in the latter case the mathematical structure of the solution for the Feynman Integrals is widely unknown, it is explored to a certain extent for Zero Scale and Single Scale quantities.

• Zero Scale quantities are the expansion coefficients of the running couplings and masses, fixed moments of splitting functions etc.

• They can be expressed by rational numbers and certain special numbers as multiple $\zeta$-values and related quantities.
Single Scale quantities depend on a scale $z \in [0, 1]$, with $z$ a ratio of Lorentz invariants. One may perform a Mellin Transform over $z$

$$\int_0^1 dzz^{N-1} f(z) = M[f](N)$$

Here one assumes $N \in \mathbb{N}, N > 0$. Due to this the problem on hand becomes discrete.

One may seek a description in terms of difference equations.

Zero Scale problems are obtained from Single Scale problems treating $N$ as a fixed integer or considering the limit $N \to \infty$. 
Some Remarks about MZV’s

- General question on the bases of MZV’s: length in the non-alternating and alternating cases.
- Do Zero Scale Feynman integrals always lead to MZV’s?
  - No! e.g. Y. Andre, 2008.
- At lower orders in perturbation theory one has just MZV’s even in single-mass problems.
- J.B., Broadhurst, Vermaseren, 2008: explicit calculation of bases and all relations of alternating MZV’s to $w=12$ and non-alternating MZV’s to $w=22$. [World Record.]; Verification to $w=26$.
- Broadhurst 1996 conjecture is proven. shuffles, stuffles, doubling, gen. doubling relations However, we did not find further reductions - which still may exist.
Introduction

- Can one reconstruct the general formula for Single Scale quantities out of a finite number of fixed moments?
- This is possible for recurrent quantities.
- At least up to 3-loop order, presumably to higher orders, single scale quantities belong to this class.
- Goal: design a general formalism to solve the problem.
2. Single Scale Feynman Integrals as Recurrent Quantities

- Can one reconstruct the general formula for Single Scale quantities out of a finite number of fixed moments?
- Polynomials and Nested Harmonic Sums obey recurrence relations, so do their polynomials.
- Example: Harmonic Sums or linear combinations thereof:

\[ F(N + 1) - F(N) = \frac{\text{sign}(a)^{N+1}}{(N + 1)^{|a|}} \]

is solved by \( S_a(N) \); and similarly for deeper nested sums

\[ S_{a,\vec{b}}(N) = \sum_{k=1}^{N} \frac{(\text{sign}(a))^k}{k^{|a|}} S_{\vec{b}}(k) \]
Feynman integrals have often a form like
\[
\int_0^1 dz \frac{z^{N-1} - 1}{1 - z} H_\alpha(z), \quad \int_0^1 dz \frac{(-z)^{N-1} - 1}{1 + z} H_\alpha(z)
\]

This structure leads to recurrences.

It is very likely that single scale Feynman diagrams do always obey difference equations.
3. Establishing and Solving Recurrences

- One seeks the relation
  \[
  \sum_{k=0}^{l} \left[ \sum_{i=0}^{d} c_{i,k} N^i \right] F(N + k) = 0.
  \]

- The corresponding linear system is dense.

- Rational number arithmetics is not applicable for the large systems to be determined; \( C_{2,q,C^3_F}^{(3)} \) would require 11 Tb of memory.

- Use arithmetic in finite fields together with Chinese remaindering \( \implies \) few Gb of memory

- The linear system approximately minimizes for \( l \approx d \).

- Join different recurrences found to reduce \( l \) to a minimal value.
Establishing and Solving Recurrences

- For the solution of the recurrence low degrees are clearly preferred.
- The linear difference equation of order $l$ with polynomial coefficients is equivalent to a linear system in $l$ variables.
- It is solved in $\Pi - \Sigma$ fields.
- Apply advanced symbolic summation methods: telescoping, creative telescoping and its refinement. Code: sigma.
- The solutions are found as linear combinations of rational terms in $N$ combined with functions, which cannot be further reduced in the $\Pi - \Sigma$ fields. In the present application they turn out all to be harmonic sums $S_{\bar{b}}(N)$.
- Other or higher order applications may consist of other sums too, which are uniquely found by the algorithm.
4. Application to 3-Loop Anomalous Dimensions and Wilson coefficients

- We apply the method for the unfolding of the unpolarized anomalous dimensions and Wilson coefficients up to 3-loop order.

- \( \Rightarrow \) analyze for individual color factors; 141 contributions from 1 – 3 loops

- Input: Moch, Vermaseren, Vogt, 2004/05. The expressions are given in terms of harmonic sums.

- Calculate the moments (rational numbers) recursively through recursions for the harmonic sums; MAPLE code.

- Establish the corresponding difference equation by a recurrency finder; build a difference equation of minimal order possible; test the recurrency.

- Solve the difference equation order by order with the summation package \texttt{sigma} C. Schneider.; most complicated cases: 4 weeks @ \( \leq 10 \)Gb, 2 GHz Proc.
C2qq3CF^3

N=3:
#11 digits / #10 digits

-98268084191 / 1166400000

N=500:
#1262 digits / #1256 digits

164184070424196780953020619176376506284305344481262083057197600746507008493793994422411032344159163031114822220582876889422095708591511216773075853139951009783631792518952817622034037186132846974627021672678012913675099511203807811938593043910803504434592021869605258833203635532508999836135422688236732214903763105373671643487725403810874264968729520075619227285741802419403727207822473765999900236383740315299205053601633484348249454757553446642108141111400654753911367986891674100650767493578709478683390573977410013520894494463909291327425815766566386397276158317387748594547139264608970087515744507507319232854289096546200480571199874814441437938609399373617980290444257899537261336751997905237704272985005100634640619858400662960713372543015648919155964069606994597363886301185067827291937065300754786947063672848938208192687107860032862813193676605747597045089655666762216336589580877342811972153527921310890635770450696939622130611988940570336060686956071232719672698106005601158460943602399862339178722607227732369045013237683625354915213011664567056504596669459201645860239580602071746606798898861360772330880307417756055465187887933272264368297071217405654474375844238250889238538974548421298170425909521742559494728720178770039473965622616598603666839154407853462338171648227013134266795320251847 /
3057444614247225372882570514367358697278130741348282122206492932828352440850471902
74910469621053366455636564873675690796713906565688820365601907263710863954826386081
3227580037879361869941003802807590860358894142891046776447162895908787986423254678
5776778283337231702130612499429819559798501074020676282769289102955679421885795867
1982932998601320344971927374905889934059987271939760212836368619501189238215442366
3805773701929509268157747992859384837403751183019423692868569168206789710047557452
5131217382272060267681480496298975522467614707848639773185909858278799786637303834
1017166676276847525704755493166263297079720470719813623901545811953853986456533543
9994182050551827959988760121168490745476969259468454613431624179198860751513076481
0304734205926703138519418575751315944374873897873646706993620825697218523316375559
40682220047659627159242085261060081019740402380126260947524640509361283802755722132
4856690051525724685919792641506082307567956962328560073471086799287131287564668441
62566980835042338974364847020024713143308342146773925541151273924985946178771189
231243716221343813770389606473498715702080141315355435311326719739117599044341913
59226935873856609594245948237469293148702516714038297077639382323251255360181047
4965862324750911265976299767973752788271111167745930035200000000000000000

N=5114:
#13388 digits / #13381 digits
Table 1: Run parameters for the unfolding of the non-singlet anomalous dimensions

<table>
<thead>
<tr>
<th>$P_{NS1}$</th>
<th>number of terms needed</th>
<th>order of recurrence</th>
<th>degree of recurrence</th>
<th>total time [sec]</th>
<th>length of recurrence [kbyte]</th>
<th>number of harm. sums, a b</th>
<th>solution time [sec]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_{NS1,C}$</td>
<td>14</td>
<td>2</td>
<td>3</td>
<td>0.05</td>
<td>0.087</td>
<td>1 [1]</td>
<td>0.55</td>
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<tr>
<td>$P_{NS1,C,F}$</td>
<td>142</td>
<td>5</td>
<td>31</td>
<td>3.32</td>
<td>4.666</td>
<td>6 [10]</td>
<td>7.45</td>
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<tr>
<td>$P_{NS1,C,F}$</td>
<td>109</td>
<td>4</td>
<td>24</td>
<td>1.91</td>
<td>2.834</td>
<td>6 [7]</td>
<td>6.28</td>
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<tr>
<td>$P_{NS1,C,F}$</td>
<td>24</td>
<td>2</td>
<td>7</td>
<td>0.13</td>
<td>0.271</td>
<td>2 [2]</td>
<td>0.92</td>
</tr>
</tbody>
</table>

| $P_{NS2,C}$ | 142                    | 5                   | 31                  | 3.35            | 4.707                       | 6 [10]                  | 7.45              |
| $P_{NS2,C}$ | 109                    | 4                   | 23                  | 1.88            | 2.703                       | 6 [7]                   | 6.23              |
| $P_{NS2,C}$ | 24                     | 2                   | 7                   | 0.09            | 0.271                       | 2 [2]                   | 0.89              |

| $P_{NS2,C,F}$ | 1079                   | 16                  | 192                 | 3152.19 | 529.802                     | 25 [68]                 | 1194.41           |
| $P_{NS2,C,F}$ | 48                     | 3                   | 11                  | 0.49     | 0.643                       | 1 [1]                   | 1.56              |
| $P_{NS2,C,F}$ | 974                    | 15                  | 181                 | 1736.08 | 450.919                     | 25 [62]                 | 1194.41           |
| $P_{NS2,C,F}$ | 48                     | 3                   | 11                  | 0.53     | 0.643                       | 1 [1]                   | 1.53              |
| $P_{NS2,C,F}$ | 48                     | 3                   | 11                  | 0.56     | 0.643                       | 1 [1]                   | 1.56              |
| $P_{NS2,C,F}$ | 39                     | 2                   | 11                  | 0.31     | 0.369                       | 3 [3]                   | 1.20              |
| $P_{NS2,C,F}$ | 377                    | 8                   | 68                  | 76.34    | 33.946                      | 12 [24]                 | 72.22             |
| $P_{NS2,C,F}$ | 14                     | 2                   | 3                   | 0.12     | 0.101                       | 1 [1]                   | 0.53              |
| $P_{NS2,C,F}$ | 356                    | 7                   | 62                  | 65.25    | 23.830                      | 12 [20]                 | 52.67             |
| $P_{NS2,C,F}$ | 14                     | 2                   | 3                   | 0.12     | 0.101                       | 1 [1]                   | 0.55              |

| $P_{NS2,C,F}$ | 1079                   | 16                  | 192                 | 4713.27 | 527.094                     | 25 [68]                 | 1165.22           |
| $P_{NS2,C,F}$ | 48                     | 3                   | 11                  | 0.55     | 0.643                       | 1 [1]                   | 1.562             |
| $P_{NS2,C,F}$ | 974                    | 15                  | 178                 | 1715.03 | 442.031                     | 25 [62]                 | 889.047           |
| $P_{NS2,C,F}$ | 48                     | 3                   | 11                  | 0.61     | 0.643                       | 1 [1]                   | 1.531             |
| $P_{NS2,C,F}$ | 749                    | 12                  | 146                 | 991.22   | 240.325                     | 25 [50]                 | 516.812           |
| $P_{NS2,C,F}$ | 48                     | 3                   | 11                  | 0.61     | 0.643                       | 1 [1]                   | 1.593             |
| $P_{NS2,C,F}$ | 377                    | 8                   | 69                  | 111.38   | 33.872                      | 12 [24]                 | 71.235            |
| $P_{NS2,C,F}$ | 14                     | 2                   | 3                   | 0.15     | 0.101                       | 1 [1]                   | 0.531             |
| $P_{NS2,C,F}$ | 14                     | 2                   | 3                   | 0.15     | 0.101                       | 1 [1]                   | 0.547             |
| $P_{NS2,C,F}$ | 39                     | 2                   | 11                  | 0.40     | 0.369                       | 3 [3]                   | 1.172             |
| $P_{NS2,C,F}$ | 39                     | 2                   | 11                  | 0.55     | 0.369                       | 3 [3]                   | 1.19              |
Table 2: Run parameters for the unfolding of the unpolarized quarkonic Wilson Coefficients for the structure function $F_2(x, Q^2)$.

<table>
<thead>
<tr>
<th>$c^{(i)}_{\Delta g}$</th>
<th>number of terms needed</th>
<th>order of recurrence</th>
<th>degree of recurrence</th>
<th>total time [sec]</th>
<th>length of recurrence [kbyte]</th>
<th>number of harm. sums</th>
<th>solution time [sec]</th>
</tr>
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<td>$c^{(i)}_{\Delta g}$</td>
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<td>0.26</td>
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<td>137</td>
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<td>10</td>
<td>121</td>
<td>413.33</td>
<td>127.893</td>
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<td>178.73</td>
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<tr>
<td>$c^{(i)}_{\Delta g}$</td>
<td>15</td>
<td>2</td>
<td>3</td>
<td>0.27</td>
<td>0.100</td>
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<td>0.112</td>
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<td>71</td>
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<td>16</td>
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<td>$c^{(i)}_{\Delta g}$</td>
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<td>5</td>
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<td>63</td>
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<td>30.46</td>
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<td>11</td>
<td>0.32</td>
<td>0.643</td>
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<td>1.01</td>
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<td>5</td>
<td>0.08</td>
<td>0.175</td>
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<td>348</td>
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<td>21</td>
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<td>2.83</td>
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<td>30</td>
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<td>$c^{(i)}_{\Delta g}$</td>
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<td>3</td>
<td>0.06</td>
<td>0.101</td>
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<td>14</td>
<td>242</td>
<td>6583.27</td>
<td>738.498</td>
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<td>841.24</td>
</tr>
<tr>
<td>$c^{(i)}_{\Delta g}$</td>
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<td>25</td>
<td>2.33</td>
<td>3.164</td>
<td>2[7]</td>
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<td>$c^{(i)}_{\Delta g}$</td>
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<td>1</td>
<td>2</td>
<td>0.03</td>
<td>0.041</td>
<td>0[0]</td>
<td>0.10</td>
</tr>
</tbody>
</table>
A complicated example

\[ C_{2,q} \propto C_F^3 \] :

- 5114 moments needed. Use a clever way to calculate the input.
- Largest moment: fraction: numerator 13388 digits; denominator 13381 digits.
- CPU time to determine the recurrence: 20.7 days.
  - modular prediction of the dimension: 4 h; modular LEQ’s: 5.8 days; modular operator GCDs: 11 days; Chinese Remainder + Rat. Reconstruction: 3.8 days. 140 large primes needed.
  - 31 MB recurrence is established; largest integer: 1227 digits; order: 35; degree: 938
- Solved by \texttt{sigma} within about one week.
- 3 loop anomalous dimensions: much smaller recurrences & shorter computation times.

\[ \implies \] In practice no method does yet exists to calculate such a high number of moments.

\[ \implies \] Existence proof of a quite general and powerful automatic difference-equation solver, standing rather demanding tests.
Structure of the Results

- We carry out all algebraic reductions, J.B. 2003.
- Different color factor contributions lead to the same or nearly the same amount of sums at a given quantity.
- This points to the fact that the amount of harmonic sums is governed by topology rather than the fields and color.
- The linear harmonic sum representations by Vermaseren et al. 2004/05 require many more sums than our representation.
- There are reductions in the number of sums as 264 $\rightarrow$ 29.
- Further use of structural relations will lead to maximally 35 sums for the 3-loop Wilson coefficients; J.B. 2008.
\[ \nu_{q_0,0}(N) = S_1(N) - \frac{3N^2 + 3N + 2}{N(N + 1)} \quad (1) \]

\[ \nu_{q_0,1}^{(+)}(N) = C_FN_F \left[ S_1(N) - \frac{3N^2 + 3N + 2 \times 3N^4 + 6N^3 + 47N^2 + 20N - 12}{9N^2(N + 1)^2} - \frac{40}{9}S_1(N) + \frac{8}{3}S_2(N) \right] \]

\[ \quad + C_AC_F \left[ \frac{-51N^5 - 153N^4 - 757N^3 - 144(-1)^N N^2 - 851N^2 - 208N + 132}{18N^3(N + 1)^3} + 8S_{-3}(N) + \frac{268}{9}S_1(N) + S_{-2}(N) \left( 16S_1(N) - \frac{8}{N(N + 1)} \right) - \frac{44}{3}S_2(N) + 8S_3(N) + 16S_{-2,1}(N) \right] \quad (2) \]

\[ \nu_{q_0,1}^{(-)}(N) = C_FN_F \left[ \frac{3N^4 + 6N^3 + 47N^2 + 20N - 12}{9N^2(N + 1)^2} - \frac{40}{9}S_1(N) + \frac{8}{3}S_2(N) \right] \]

\[ \quad + C_AN_F \left[ \frac{-144(-1)^N N^3 - (N + 1)(51N^5 + 102N^4 + 655N^3 + 484N^2 + 12N + 144)}{18N^3(N + 1)^3} + 8S_{-3}(N) + \frac{268}{9}S_1(N) + S_{-2}(N) \left( 16S_1(N) - \frac{8}{N(N + 1)} \right) - \frac{44}{3}S_2(N) + 8S_3(N) - 16S_{-2,1}(N) \right] \quad (3) \]
\[
\gamma_{qq2}(N) = C_F^2 N_F C_3 \left[ 32 S_1(N) - \frac{8(3N^2 + 3N + 2)}{N(N+1)} \right] \\
+ C_A C_F N_F C_3 \left[ \frac{8(3N^2 + 3N + 2)}{N(N+1)} - 32 S_1(N) \right] \\
+ C_F N_F^2 \left[ \frac{51N^6 + 153N^5 + 57N^4 + 35N^3 + 96N^2 + 16N - 24}{27N^3(N+1)^3} - \frac{16}{27} S_1(N) - \frac{80}{27} S_2(N) \right] \\
+ \frac{16}{9} S_3(N) \\
+ C_F^3 C_3 \left[ \frac{24(5N^4 + 10N^3 + 9N^2 + 4N + 4)}{N^2(N+1)^2} - 192 S_{-2}(N) \right] \\
+ C_A C_F^2 C_3 \left[ \frac{36(5N^4 + 10N^3 + 9N^2 + 4N + 4)}{N^2(N+1)^2} + 288 S_{-2}(N) \right] \\
+ C_A^2 C_F C_3 \left[ \frac{-12(5N^4 + 10N^3 + 9N^2 + 4N + 4)}{N^2(N+1)^2} - 96 S_{-2}(N) \right] \\
+ C_A C_F N_F \left[ -\frac{2(N+1)(270N^7 + 810N^6 - 463N^5 - 1392N^4 - 211N^3 - 206N^2 - 156N + 144)}{27N^4(N+1)^4} \right] \\
- \frac{296N^4(4N+1)}{27N^4(N+1)^4} + \frac{64}{3} S_4(N) + S_{-3}(N) \left( \frac{32}{3} S_1(N) - \frac{16}{9N(N+1)} \right) \\
+ \frac{1336}{27} S_2(N) + S_{-2}(N) \left( \frac{16(16N^2 + 10N - 3)}{9N^2(N+1)^2} - \frac{329}{9} S_1(N) + \frac{64}{3} S_2(N) \right) \\
- \frac{8(14N^2 + 14N + 3)}{3N(N+1)} + \frac{80}{3} S_4(N) + \frac{32(10N^2 + 10N - 3)}{9N(N+1)} S_{-2,1}(N) \\
+ S_1(N) \left( \frac{-4(209N^6 + 627N^5 + 627N^4 + 72N^3 + 281N^2 + 36N^2 + 36N + 18)}{27N^3(N+1)^3} \right) \\
+ 16 S_3(N) + \frac{64}{3} S_{-2,1}(N) \right) - \frac{32}{3} S_{2,-2}(N) - \frac{64}{9} S_{3,1}(N) - \frac{128}{3} S_{-2,1,1}(N) \\
+ \text{various pages more.}
\]

\[
\gamma_{qq1}(N) = \frac{d_{abc}}{N_c} N_F \left[ \frac{51N^6 + 153N^5 + 57N^4 + 35N^3 + 96N^2 + 16N - 24}{27N^3(N+1)^3} - \frac{16}{27} S_1(N) - \frac{80}{27} S_2(N) \right] \\
+ 16 S_3(N) \left( \frac{8}{9} \right)
\]

(1)
Other Processes

- The present method can be applied irrespectively of the loop order to all single scale processes.

- As has been found before J.B. & Ravindran 2004/05, J.B. & Moch 2005, J.B. & S. Klein 2007 representing a large number of 2- and 3-loop processes in terms of harmonic sums, the basis elements emerging are always the same.
  \{anomalous dimensions, Wilson coefficients, space- and time-like, polarized/unpolarized, Drell-Yan process, hadronic Higgs Boson production in the heavy mass limit, HO QED corrections in $e^+e^-$ annihilation, soft+virtual corrections to Bhabha scattering\}.

- The formalism also applies to Heavy Flavor Wilson Coefficients at $Q^2 \gg m^2$, c.f. Bierenbaum, J.B., Klein 2007/08.

- Basis to $w = 6$, c.f. J.B. 2008.
5. Conclusions

- We established a general algorithm to calculate the exact expression for single scale quantities from a finite (suitably large) number of moments (zero scale quantities).

- The latter ones are much more easily calculable.

- We applied the method to the anomalous dimensions and Wilson coefficients up to 3-loop order.

- To solve 3-loop problems this way is not possible at present, since the number of required moments is too large for the methods available.

- We attempted to solve the quantities for all color projections at once. This problem is too voluminous.

- Yet we showed that giant difference equations [order 35; degree \(\sim 1000\)] can be reliably and fast established and solved unconditionally for advanced problems in Quantum Field Theory.