.

Two-loop Heavy Fermion Corrections to Bhabha Scattering

Tord Riemann, DESY, Zeuthen

based on work with:

S. Actis (DESY), M. Czakon (U. Würzburg), Janusz Gluza (Katowice)

LCWS, electroweak session, 31 May 2007, DESY, Hamburg

See also: • http://www-zeuthen.desy.de/theory/research/bhabha/

and • hep-ph/0412164 (List of all masters)

and • hep-ph/0604101 (planar box masters), hep-ph/0609051 (Nf=2 masters)

- **Introduction: Two-Loop corrections to Bhabha Scattering**
- **The Heavy Fermion Contributions [arXiv:0704.2400, hep-ph], → NPB**
- **Results**
- **Summary**

The Physics Needs

For more details see e.g.:
K. Mönig, "Bhabha scattering at the ILC"
talk at Mini-WS on Bhabha scattering, Univ. Karlsruhe, April 2005
/afs/ifh.de/user/m/moenig/public/www/bhabha_ilc.pdf

ILC – Need Bhabha cross-sections with 3–4 significant digits.

Why?

- **ILC**: $e^+e^- \to W^+W^-, f\bar{f}$ **with** $O(10^6)$ **events** $\to 10^{-3}$
- **GigaZ: relevant physics derived from** $Z \to$ **hadrons**, l^+l^-, **the latter with** $O(10^8)$ **events** $\to 10^{-4}$, **the systematic errors (luminosity!) influence this**
- **ILC**:$e^+e^- \to e^+e^-$, **a probe for New Physics with** $O(10^5)$ **events/year** $\to 10^{-3}$

Conclude: will need $\Delta\mathcal{L}/\mathcal{L} \approx 2 \times 10^{-4}$

The luminosity comes from very forward Bhabha scattering.

Some Kinematics

Need a cross-section prediction with 5 significant digits.

Perturbative orders:

$$\left(\frac{\alpha}{\pi}\right) = 2 \times 10^{-3}$$

$$\left(\frac{\alpha}{\pi}\right)^2 = 0.6 \times 10^{-5}$$

Kinematics:

$$\begin{aligned}
\sqrt{s} &= 90 \ldots 1000 \text{ GeV} \\
\vartheta &= 26 \ldots 82 \text{ mrad} \\
\cos\vartheta &\sim 0.999\,66 \ldots 0.996\,64 \\
T &= \frac{s}{2}(1-\beta^2\cos\vartheta) > 1.36 \text{ GeV}|_{GigaZ}, \qquad 42.2 \text{ GeV}|_{ILC500}
\end{aligned}$$

Conclude:

- t-**channel exchange of** γ **dominates everything else**
- $m_e^2/s \;<\; m_e^2/T \;\leq\; 10^{-5} \ldots 10^{-7}$
- **Calculate: 1-loop EWRC + 2-loop QED + corresp. bremsstrahlung**

Status by end of 2004

Established: 10^{-3} MC programs for LEP, ILC

Introduction to NLLBHA by Trentadue and to BHLUMI by Jadach in: Proc. of Loops and Legs, Rheinsberg, Germany, 1996

Recent mini-review: Jadach, "Theoretical error of luminosity cross section at LEP", hep-ph/0306083 [1]

- **BHLUMI v.4.04: Jadach, Placzek, Richter-Was, Was: CPC 1997**
- **see also: Jadach, Melles, Ward, Yost: PLB 1996, thesis Melles 1996 [2]**
- **NLLBHA: Arbuzov, Fadin, Kuraev, Lipatov, Merenkov, Trentadue: NPB 1997, CERN 96-01**
- **SAMBHA: Arbuzov, Haidt, Matteuzzi, Paganoni, Trentadue: hep-ph/0402211**

See e.g.: Table 1 of [1] and Figure 3.1 of [2] $\rightarrow$ Conclude:
The nonlogarithmic $O(\alpha^2)$ terms, originating from pure QED radiative 1-loop and from 2-loop diagrams are not completely covered.

They have to be calculated and integrated into the MC programs.

Beware:

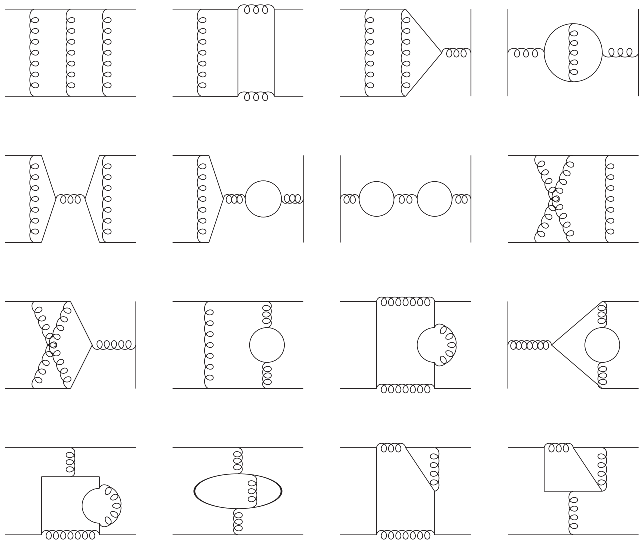
$$m_e, m_\gamma, (d-4), E_\gamma$$

$m = 0$

Two Loop Bhabha Scattering

To calculate Bhabha scattering it is best to first compute $e^+e^- \to \mu^+\mu^-$, since it's closely related but has less diagrams.

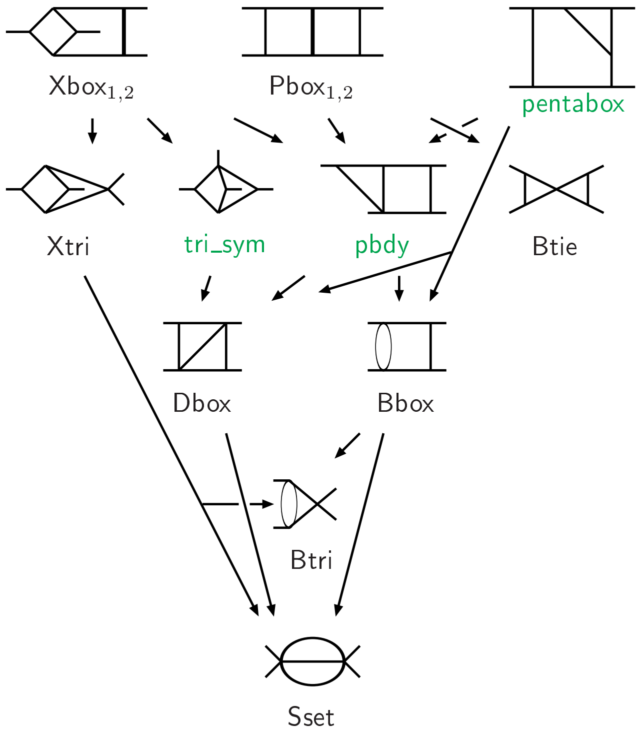
There are 47 QED diagrams contributing to $e^+e^- \to \mu^+\mu^-$.



In this calculation all particles massless.

The Bhabha scattering amplitude can be obtained from $e^+e^- \to \mu^+\mu^-$ simply by summing it with the crossed amplitude (including fermi minus sign).

Two-loop integral inheritance chart



From Talk of Zvi Bern, LoopFest 2002

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Status 2005

Know the constant term ($m_e = 0$) of 2-loop photonic corrections

A. Penin, Two-Loop Corrections to Bhabha Scattering, hep-ph/0501120 v.3, $\rightarrow$ PRL

Transform the massless 2-loop results of Bern, Dixon, Ghinculov (2002) with InfraRed (IR) regulation by $D = 4 - 2\epsilon$ into the on-mass-shell renormalization with $m_e \rightarrow 0$ and IR regulation by $\lambda = m_\gamma \neq 0$

Use IR-properties of amplitudes (see Penin):

[A] Exponentiation of the IR logarithms (Sudakov 1956,...)

[B] Factorization of the collinear logarithms into expernal legs (Frenkel, Taylor 1976)

[C] Non-renormalization of the IR exponents (YFS 1961, ...)

Isolate the closed fermion loop contribution (does not fulfil [C]) and add it separately (Burgers 1985, Bonciani, v.d.Bij et al. 2005, Penin)

After that the small mass limit in m_e is known – but the radiative one-loops with 5-point functions and the $nf = 2$ corrections.

From Talk of Zvi Bern, LoopFest 2002

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Status 2006

Determination of master integrals from 2-loop Bhabha scattering

[A] All planar box masters for $m_e^2 << s, t, u$

[B] All masters for $N_f = 2$ and $m_e^2 << m_f^2 << s, t, u$

We had to develop for this

Technique of semi-automatized derivation of Mellin-Barnes integrals ($\rightarrow$ AMBRE package, 2007)

Automatized small-mass expansion for Mellin-Barnes integrals

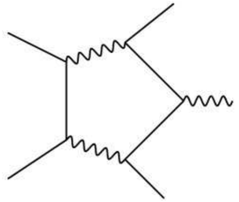
We – and all the others – failed with a determination of non-planar 2-loop boxes. Little is known due to Smirnov,Heinrich.

T. Riemann, 31 May 2007- LCWS, electroweak, Hamburg

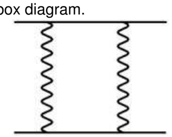
AMBRE - Automatic Mellin-Barnes REpresentation (arXiv:0704.2423)

To download 'right click' and 'save target as'.

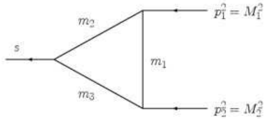
- The package AMBRE.m
- Kinematics generator for 4- 5- and 6- point functions with any external legs KinematicsGen.m
- Tarball with examples given below examples.tar.gz
  - example1.nb, example2.nb - Massive QED pentagon diagram.



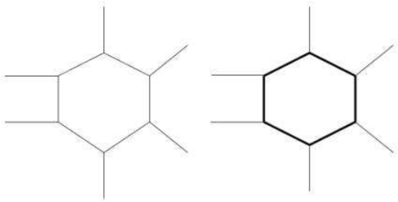
  - example3.nb - Massive QED one-loop box diagram.



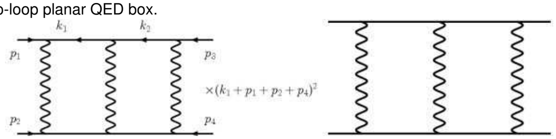
  - example4.nb - General one-loop vertex.



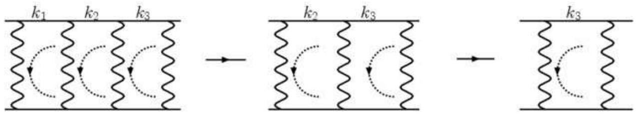
  - example5.nb - Six-point scalar functions;
    left: massless case,
    right: massive case.



  - example6.nb - **left**, example7.nb - **right**
    Massive two-loop planar QED box.



  - example8.nb - The loop-by-loop iterative procedure.



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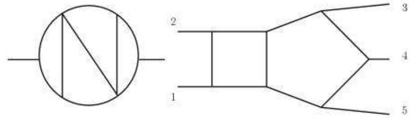

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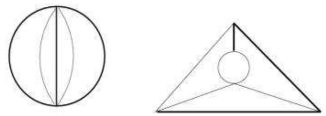

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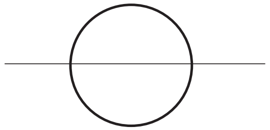
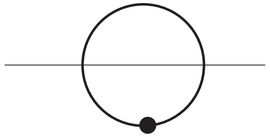
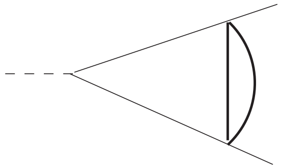
From Talk of Zvi Bern, LoopFest 2002

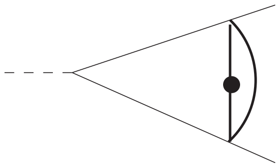
The master integrals for the $N_f = 2$ contributions

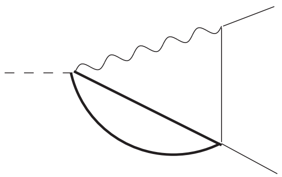
S. Actis, M. Czakon, J. Gluza, TR, 2006(publ.) / 2007(box master expansion corrected)

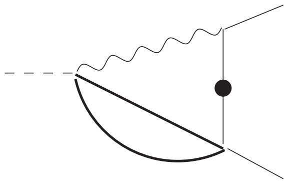

SE3l2M1m

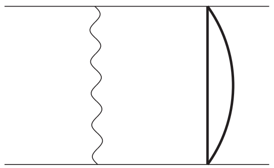

SE3l2M1md

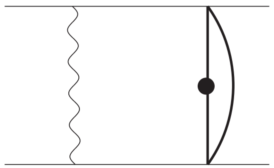

V4l2M2m


V4l2M2md


V4l2M1m


V4l2M1md


B5l2M2md


B5l2M2m

There are eight additional master integrals with two different mass scales.

The 2-box-diagrams represent a three-scale problem: $s/m_e^2, t/m_e^2, M^2/m_e^2$

Self-energy master integrals:

Actis,Czakon,Gluza,TR, NPB(PS) 160 (2006) 91, hep-ph/0609051

$$L(R) \quad = \quad \ln\left(\frac{m_e^2}{M^2}\right)$$

$$\begin{aligned}
\texttt{SE3l2M1m[on shell]} \quad &= \quad M^2\, m^{-4\epsilon}\Big\{R\Big[\frac{1}{2\epsilon^2}+\frac{5}{4\epsilon}-\frac{3}{8}+\frac{\zeta_2}{2}+\frac{3}{2}L(R)-\frac{1}{2}L^2(R)\Big] \\
&+ \quad R^2\Big[\frac{11}{18}-\frac{1}{3}L(R)\Big]+\epsilon\Big[R\Big(\frac{45}{16}+\frac{5}{4}\zeta_2-\frac{\zeta_3}{3}-\frac{7}{4}L(R)+L^2(R) \\
&- \quad \frac{1}{2}L^3(R)\Big)+R^2\Big(-\frac{3}{4}+\frac{8}{9}L(R)-\frac{1}{2}L^2(R)\Big)\Big]\Big\},
\end{aligned}$$

$$\begin{aligned}
\texttt{SE3l2M1md[on shell]} \quad &= \quad m^{-4\epsilon}\Big\{\frac{1}{2\epsilon^2}+\frac{1}{2\epsilon}\Big[1+2L\left(R\right)\Big]+\frac{1}{2}\left(1+\zeta_2\right)+L\left(R\right)+L^2\left(R\right) \\
&+ \quad \epsilon\Big[\frac{1}{6}\left(3+3\zeta_2-2\zeta_3\right)+\left(1+\zeta_2\right)L\left(R\right)+L^2\left(R\right)+\frac{2}{3}L^3\left(R\right)\Big] \\
&+ \quad R\Big[-\frac{3}{4}+\frac{1}{2}L(R)+\epsilon\left(\frac{7}{8}-L(R)+\frac{3}{4}L^2(R)\right)\Big] \\
&+ \quad R^2\Big[-\frac{5}{36}+\frac{1}{6}L(R)+\epsilon\Big(-\frac{5}{72}+\frac{1}{18}L(R)+\frac{1}{4}L^2(R)\Big)\Big]\Big\}.
\end{aligned}$$

Vertex master integrals:

Actis,Czakon,Gluza,TR, NPB(PS) 160 (2006) 91, hep-ph/0609051

$L_m(x) = \ln(-m^2/x)$ and $L_M(x) = \ln(-M^2/x)$,

$$
\begin{aligned}
\texttt{V412M1m[x]} &= m^{-4\epsilon}\Big\{\frac{1}{2\epsilon^2} + \frac{5}{2\epsilon} + \frac{1}{2}\Big[19 - 3\zeta_2 - L_m^2(x)\Big] \\
&+ \frac{M^2}{x}\Big[-2 + 4\zeta_2 - 4\zeta_3 - 2L_m(x) + 2L_M(x) - 4\zeta_2 L_M(x) \\
&+ 2L_m(x)L_M(x) - L_M^2(x) - L_m(x)L_M^2(x) + \frac{1}{3}L_M^3(x)\Big]\Big\},
\end{aligned}
$$

$$
\begin{aligned}
\texttt{V412M1md[x]} &= \frac{m^{-4\epsilon}}{m^2}\Big\{\frac{1}{2\epsilon^2} + \frac{1}{\epsilon}\Big[1 + \frac{1}{2}L_m(x)\Big] + 2 - \zeta_2 + L_m(x) + \frac{1}{4}L_m^2(x) \\
&+ \frac{M^2}{x}\Big[\frac{1}{\epsilon} - \frac{1}{\epsilon}L_M(x) - 1 + 3\zeta_2 + L_m(x) + L_M(x) \\
&- L_m(x)L_M(x) - \frac{1}{2}L_M^2(x)\Big]\Big\},
\end{aligned}
$$

$$
\texttt{V412M2m[x]} = m^{-4\epsilon}\Big\{\frac{1}{2\epsilon^2} + \frac{1}{\epsilon}\Big[\frac{5}{2} + L_m(x)\Big] + \frac{1}{2}(19 + \zeta_2) + 5\ L_m(x) + L_m^2(x)\Big\},
$$

$$
\texttt{V412M2md[x]} = \frac{m^{-4\epsilon}}{6x}\Big[12\zeta_3 - 6\zeta_2 L_M(x) - L_M^3(x)\Big],
$$

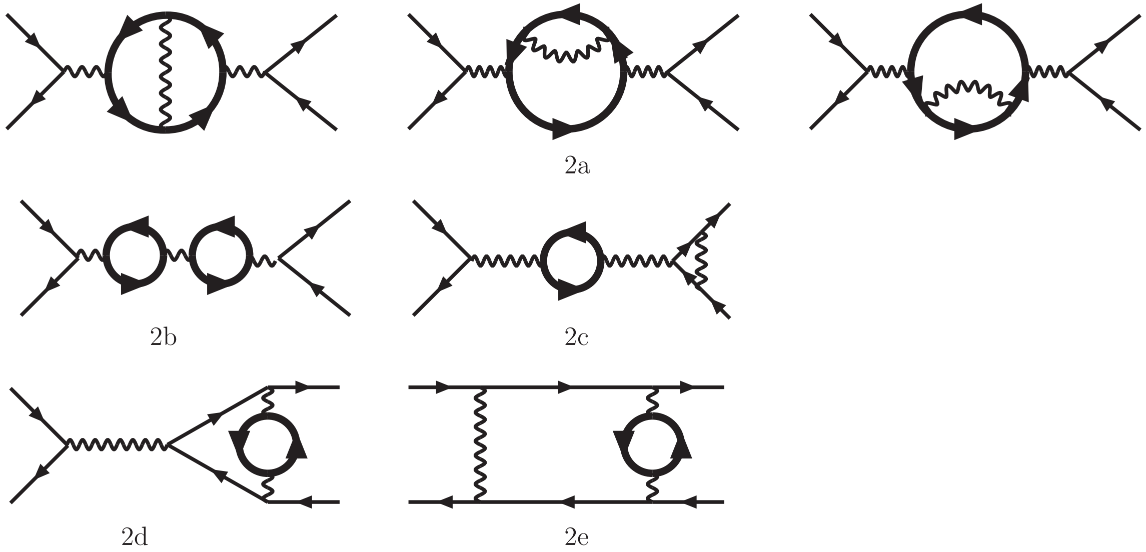
Box master integrals:

Correct Mellin-Barnes representations in Actis et al., NPB(PS) 160 (2006) 91, hep-ph/0609051

But wrong mass expansion there! Correct results are:

$$
\begin{aligned}
\texttt{B5l2M2m[x,y]} &= \frac{m^{-4\epsilon}}{x}\Bigg\{\frac{1}{\epsilon^2}L_m(x)+\frac{1}{\epsilon}\Big(-\zeta_2+2L_m(x)+\frac{1}{2}L_m^2(x)+L_m(x)L_m(y)\Big)\\
&- 2\zeta_2-2\zeta_3+4L_m(x)+L_m^2(x)+\frac{1}{3}L_m^3(x)-4\zeta_2L_m(y)\\
&+ 2L_m(x)L_m(y)+L_m(x)L_m^2(y)-\frac{1}{6}L_m^3(y)\\
&- \Big(3\zeta_2+\frac{1}{2}L_m^2(x)-L_m(x)L_m(y)+\frac{1}{2}L_m^2(y)\Big)\ln\left(1+\frac{y}{x}\right)\\
&- \Big(L_m(x)-L_m(y)\Big)\mathrm{Li}_2\left(-\frac{y}{x}\right)+\mathrm{Li}_3\left(-\frac{y}{x}\right)\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
\texttt{B5l2M2md[x,y]} &= \frac{m^{-4\epsilon}}{xy}\Bigg\{\frac{1}{\epsilon}\Big[-L_m(x)L_m(y)+L_m(x)L(R)\Big]-2\zeta_3+\zeta_2L_m(x)+4\zeta_2L_m(y)\\
&- 2L_m(x)L_m^2(y)+\frac{1}{6}L_m^3(y)-2\zeta_2L(R)+2L_m(x)L_m(y)L(R)-\frac{1}{6}L^3(R)\\
&+ \Big(3\zeta_2+\frac{1}{2}L_m^2(x)-L_m(x)L_m(y)+\frac{1}{2}L_m^2(y)\Big)\ln\left(1+\frac{y}{x}\right)\\
&+ \Big(L_m(x)-L_m(y)\Big)\mathrm{Li}_2\left(-\frac{y}{x}\right)-\mathrm{Li}_3\left(-\frac{y}{x}\right)\Bigg\}.
\end{aligned}
$$



Classes of Bhabha-scattering -loop diagrams containing at least one fermion loop.

After combining the 2-loop terms with the loop-by-loop terms and with soft real corrections:

$$\frac{d\sigma^{\mathrm{NNLO}}}{d\Omega} + \frac{d\sigma_{\gamma}^{\mathrm{NLO}}}{d\Omega} = \frac{d\sigma^{\mathrm{NNLO,e}}}{d\Omega} + \sum_{f \neq e} Q_f^2 \frac{d\sigma^{\mathrm{NNLO,f^2}}}{d\Omega} + \sum_{f \neq e} Q_f^4 \frac{d\sigma^{\mathrm{NNLO,f^4}}}{d\Omega}$$

$$+ \sum_{f_1, f_2 \neq e} Q_{f_1}^2 Q_{f_2}^2 \frac{d\sigma^{\mathrm{NNLO,2f}}}{d\Omega}.$$

The Box Corrections

The contribution of the renormalized two-loop box diagrams of class 2e is given by

$$\frac{d\sigma^{2\mathrm{e}\times\mathrm{tree}}}{d\Omega} = \frac{\alpha^2}{2\,s}\left[\frac{1}{s}\,A_1^{2\mathrm{e}\times\mathrm{tree}}(s,t) + \frac{1}{t}\,A_2^{2\mathrm{e}\times\mathrm{tree}}(s,t)\right]$$

Here the auxiliary functions can be conveniently expressed through three independent form factors $B^{(2)}_{\mathrm{I,f}}(x,y)$, where $\mathrm{i} = \mathrm{A,B,C}$,

$$\begin{aligned} A_1^{2\mathrm{e}\times\mathrm{tree}}(s,t) &= F_\epsilon^2 \sum_f Q_f^2\, \mathrm{Re}\left[B^{(2)}_{\mathrm{A},f}(s,t) + B^{(2)}_{\mathrm{B},f}(t,s) + B^{(2)}_{\mathrm{C},f}(u,t) - B^{(2)}_{\mathrm{B},f}(u,s)\right], \\ A_2^{2\mathrm{e}\times\mathrm{tree}}(s,t) &= F_\epsilon^2 \sum_f Q_f^2\, \mathrm{Re}\left[B^{(2)}_{\mathrm{B},f}(s,t) + B^{(2)}_{\mathrm{A},f}(t,s) - B^{(2)}_{\mathrm{B},f}(u,t) + B^{(2)}_{\mathrm{C},f}(u,s)\right]. \end{aligned}$$

The normalization factor is

$$F_\epsilon = \left(\frac{m_e^2\,\pi\,e^{\gamma_E}}{\mu^2}\right)^{-\epsilon}$$

Look e.g. at $B^{(2)}_{\mathrm{A},f}(t,s)$

The interference of the box diagram of class 2e with the s-channel tree-level amplitude,

$$B_{2\mathrm{e},f} = \frac{\alpha^2}{4\,s^2}\mathrm{Re}\left[B^{(2)}_{A,f}(s,t)\right]$$

$$
\begin{aligned}
B^{(2)}_{\mathrm{A},f}(x,y) \;=\;& \frac{1}{\epsilon}\,\frac{2}{3}\left(\frac{x^2}{y}+2x+y\right)\left[\frac{5}{3}-L(R_f)+L_e(y)\right]L_e(x)\\
+\;& \frac{1}{3}\,\frac{x^2}{y}\Big\{2\left(\frac{131}{27}-10\,\zeta_2-2\,\zeta_3\right)-2\left(\frac{25}{9}-6\,\zeta_2\right)L(R_f)+\frac{7}{6}\,L^2(R_f)\\
-\;& \frac{1}{3}\,L^3(R_f)+\left[\frac{82}{9}-2\,\zeta_2-\frac{4}{3}\,L(R_f)\right]L_e(x)-2\left[\frac{1}{3}+8\,\zeta_2-\frac{1}{2}\,L(R_f)\right]L_e(y)\\
-\;& \left[\frac{23}{6}-2\,L(R_f)\right]L_e^2(y)+4\left[2-L(R_f)\right]L_e(x)\,L_e(y)-4\Big[\frac{5}{12}\,L_e^3(y)\\
-\;& L_e(x)\,L_e^2(y)\Big]-\left[6\,\zeta_2+\ln^2\left(\frac{y}{x}\right)\right]\ln\left(1+\frac{y}{x}\right)-2\,\ln\left(\frac{y}{x}\right)\mathrm{Li}_2\left(-\frac{y}{x}\right)\\
+\;& 2\,\mathrm{Li}_3\left(-\frac{y}{x}\right)\Big\}+\frac{x}{3}\Big\{2\left(\frac{262}{27}-9\,\zeta_2-4\,\zeta_3\right)-4\left(\frac{25}{9}-3\,\zeta_2\right)L(R_f)\\
+\;& \frac{7}{3}\,L^2(R_f)-\frac{2}{3}\,L^3(R_f)+2\left[\frac{121}{9}-\frac{10}{3}\,L(R_f)\right]L_e(x)-2\Big[\frac{10}{3}+12\,\zeta_2\\
-\;& 2\,L(R_f)\Big]L_e(y)+\left[\frac{13}{3}-2\,L(R_f)\right]L_e^2(x)-\left[\frac{16}{3}-2\,L(R_f)\right]L_e^2(y)\\
+\;& 2\left[\frac{17}{3}-2\,L(R_f)\right]L_e(x)\,L_e(y)+\frac{2}{3}\,L_e^3(x)\\
+\;& 6\,L_e(x)\,L_e^2(y)-2\,L_e^3(y)-2\left[6\,\zeta_2+\ln^2\left(\frac{y}{x}\right)\right]\ln\left(1+\frac{y}{x}\right)\\
-\;& 4\,\ln\left(\frac{y}{x}\right)\mathrm{Li}_2\left(-\frac{y}{x}\right)+4\,\mathrm{Li}_3\left(-\frac{y}{x}\right)\Big\}+\frac{y}{3}\Big\{2\left(\frac{131}{27}-7\zeta_2-2\zeta_3\right)\\
-\;& 2\left(\frac{25}{9}-3\zeta_2\right)L(R_f)+\frac{7}{6}L^2(R_f)-\frac{1}{3}L^3(R_f)+\left[\frac{130}{9}-\frac{10}{3}L(R_f)\right]L_e(x)\\
-\;& \left[6+12\zeta_2-3L(R_f)\right]L_e(y)+\left[\frac{5}{3}-L(R_f)\right]L_e^2(x)-\left[\frac{25}{6}-L(R_f)\right]L_e^2(y)\\
+\;& 2\left[\frac{10}{3}-L(R_f)\right]L_e(x)L_e(y)+\frac{1}{3}L_e^3(x)-L_e^3(y)+3L_e(x)L_e^2(y)\\
-\;& \left[6\,\zeta_2+\ln^2\left(\frac{y}{x}\right)\right]\ln\left(1+\frac{y}{x}\right)-2\ln\left(\frac{y}{x}\right)\mathrm{Li}_2\left(-\frac{y}{x}\right)+2\,\mathrm{Li}_3\left(-\frac{y}{x}\right)\Big\}
\end{aligned}
$$

(1)

(2)

$B_{2\mathrm{e},f}$ [nb] / $\sqrt{s}$ [GeV]	10	91	500
e	188758	5200.08	284.711
μ	1635.62	1686.88	130.579
τ			39.5554

Table 1: Finite part of $B_{2\mathrm{e},f}$ in nanobarns at a scattering angle $\theta = 3^\circ$.

$B_{2\mathrm{e},f}$ [nb] / $\sqrt{s}$ [GeV]	10	91	500
e	143.162	3.23102	0.160582
μ	61.3875	1.79381	0.0995184
τ	10.0105	0.935319	0.0639576
t			-0.00256757

Table 2: Finite part of $B_{2\mathrm{e},f}$ in nanobarns at a scattering angle $\theta = 90^\circ$.

$\sqrt{s}$ [GeV]	10	91	500
e	-124.237	-254.293	-400.574
μ	-4.8036	-29.1057	-70.1032
τ		-2.08719	-13.4901

Table 3: Real part for the vertex form factor.

$$\frac{d\sigma^{\mathrm{NNLO,f^2}}}{d\Omega} = \frac{\alpha^2}{s}\left\{\sigma_1^{\mathrm{NNLO,f^2}} + \sigma_2^{\mathrm{NNLO,f^2}} \ln\left(\frac{2\omega}{\sqrt{s}}\right)\right\}$$

The $\sigma_1^{\mathrm{NNLO,f^2}}$ is the main result of this study:

$$
\begin{aligned}
\sigma_1^{\mathrm{NNLO,f^2}} \quad = \quad & \frac{\left(1-x+x^2\right)^2}{3x^2}\Bigg\{-\frac{1}{3}\left[\ln^3\left(\frac{s}{m_e^2}\right)+\ln^3\left(R_f\right)\right]+\ln^2\left(\frac{s}{m_e^2}\right)\left[\frac{55}{6}-\ln\left(R_f\right)\right.\\
+ \quad & \left.\ln\left(1-x\right)-\ln\left(x\right)\right]+\ln\left(\frac{s}{m_e^2}\right)\left[-\frac{589}{18}+\frac{37}{3}\ln\left(R_f\right)-\ln^2\left(R_f\right)\right.\\
- \quad & \left.2\ln\left(R_f\right)\Big(\ln\left(x\right)-\ln\left(1-x\right)\Big)-8\mathrm{Li}_2\left(x\right)\right]+\frac{4795}{108}-\frac{409}{18}\ln\left(R_f\right)+\frac{19}{6}\ln^2\left(R_f\right)\\
- \quad & \ln^2\left(R_f\right)\Big(\ln\left(x\right)-\ln\left(1-x\right)\Big)-8\ln\left(R_f\right)\mathrm{Li}_2\left(x\right)+\frac{40}{3}\mathrm{Li}_2\left(x\right)\Bigg\}\\
+ \quad & \ln\left(\frac{s}{m_e^2}\right)\left[\zeta_2\left(-\frac{2}{3x^2}+\frac{4}{3x}+\frac{11}{2}-\frac{23}{3}x+\frac{16}{3}x^2\right)+\ln^2\left(x\right)\left(-\frac{1}{3x^2}+\frac{17}{12x}\right.\right.\\
- \quad & \left.\frac{5}{4}-\frac{x}{12}+\frac{2}{3}x^2\right)+\ln^2\left(1-x\right)\left(-\frac{2}{3x^2}+\frac{11}{6x}-\frac{5}{2}+\frac{11}{6}x-\frac{2}{3}x^2\right)\\
+ \quad & \ln\left(x\right)\ln\left(1-x\right)\left(\frac{2}{3x^2}-\frac{4}{3x}-\frac{1}{2}+\frac{5}{3}x-\frac{4}{3}x^2\right)+\ln\left(x\right)\left(\frac{55}{9x^2}-\frac{83}{9x}+\frac{65}{6}\right.\\
- \quad & \left.\left.\frac{85}{18}x+\frac{10}{9}x^2\right)+\frac{1}{3}\ln\left(1-x\right)\left(-\frac{10}{3x^2}+\frac{31}{6x}-10+\frac{31}{6}x-\frac{10}{3}x^2\right)\right]\\
+ \quad & \frac{1}{3}\ln^3\left(x\right)\left(-\frac{1}{3x^2}+\frac{31}{12x}-\frac{11}{6}-\frac{x}{6}+\frac{x^2}{3}\right)+\frac{1}{3}\ln^3\left(1-x\right)\left(-\frac{1}{3x^2}+\frac{1}{x}\right.\\
- \quad & \left.\frac{4}{3}+x-\frac{x^2}{3}\right)+\ln^2\left(x\right)\ln\left(1-x\right)\left(-\frac{1}{3x^2}+\frac{1}{3x}-\frac{4}{3}+x-\frac{x^2}{3}\right)\\
+ \quad & \frac{1}{3}\ln\left(x\right)\ln^2\left(1-x\right)\left(-\frac{1}{x^2}+\frac{2}{x}-\frac{7}{4}+\frac{x}{2}\right)+\ln^2\left(x\right)\left[\frac{55}{18x^2}-\frac{46}{9x}+\cdots\cdots\cdots\right.
\end{aligned}
$$

$$
\begin{aligned}
\cdots\cdots\cdots\cdots + \quad & \frac{14}{3} - \frac{4}{9}x - \frac{10}{9}x^2 + \ln(R_f)\Big(-\frac{1}{3x^2} + \frac{17}{12x} - \frac{5}{4} - \frac{x}{12} + \frac{2}{3}x^2\Big)\Big] \\
+ \quad & \ln^2(1-x)\Big[\frac{10}{9x^2} - \frac{29}{9x} + \frac{9}{2} - \frac{29}{9}x + \frac{10}{9}x^2 + \ln(R_f)\Big(-\frac{2}{3x^2} + \frac{11}{6x} \\
- \quad & \frac{5}{2} + \frac{11}{6}x - \frac{2}{3}x^2\Big)\Big] + \ln(x)\ln(1-x)\Big[-\frac{10}{9x^2} + \frac{37}{18x} + \frac{1}{2} - \frac{25}{9}x \\
+ \quad & \frac{20}{9}x^2 + \ln(R_f)\Big(\frac{2}{3x^2} - \frac{4}{3x} - \frac{1}{2} + \frac{5}{3}x - \frac{4}{3}x^2\Big)\Big] + \ln(x)\Big[-\frac{589}{54x^2} + \frac{1753}{108x} \\
- \quad & \frac{701}{36} + \frac{925}{108}x - \frac{56}{27}x^2 + \mathrm{Li}_2(x)\Big(-\frac{4}{x^2} + \frac{19}{3x} - 7 + 3x - \frac{2}{3}x^2\Big) \\
+ \quad & \ln(R_f)\Big(\frac{37}{9x^2} - \frac{56}{9x} + \frac{47}{6} - \frac{67}{18}x + \frac{10}{9}x^2\Big) + \zeta_2\Big(-\frac{2}{3x^2} + \frac{4}{x} - \frac{1}{6} \\
- \quad & \frac{10}{3}x + 2x^2\Big)\Big] + \ln(1-x)\Big[\frac{56}{27x^2} - \frac{161}{54x} + \frac{56}{9} - \frac{161}{54}x + \frac{56}{27}x^2 \\
+ \quad & \ln(R_f)\Big(-\frac{10}{9x^2} + \frac{31}{18x} - \frac{10}{3} + \frac{31}{18}x - \frac{10}{9}x^2\Big) + \zeta_2\Big(-\frac{2}{x^2} + \frac{20}{3x} - \frac{32}{3} + \frac{20}{3}x \\
- \quad & 2x^2\Big)\Big] + \mathrm{Li}_3(x)\Big(\frac{4}{3x^2} - \frac{7}{3x} + 3 - \frac{5}{3}x + \frac{2}{3}x^2\Big) + \frac{2}{3}S_{1,2}(x)\Big(-\frac{1}{x^2} + \frac{1}{x} \\
- \quad & x + x^2\Big) + \zeta_2\Big[\frac{19}{9x^2} - \frac{13}{18x} - \frac{43}{3} + \frac{311}{18}x - \frac{98}{9}x^2 + \ln(R_f)\Big(-\frac{2}{3x^2} + \frac{4}{3x} \\
+ \quad & \frac{11}{2} - \frac{23}{3}x + \frac{16}{3}x^2\Big)\Big] + \zeta_3\Big(-\frac{4}{3x^2} + \frac{3}{x} - 5 + \frac{11}{3}x - 2x^2\Big)
\end{aligned}
$$

$d\sigma$ / $d\Omega$ [nb] | $\sqrt{s}$ [GeV]	10	91	500
LO QED	440873	5323.91	176.349
LO Zfitter	440875	5331.5	176.283
NNLO (e)	-1397.35	-35.8374	-1.88151
NNLO ($e + \mu$)	-1394.74	-43.1888	-2.41643
NNLO ($e + \mu + \tau$)			-2.55179
NNLO photonic	9564.09	251.661	12.7943

$d\sigma$ / $d\Omega$ [nb] | $\sqrt{s}$ [GeV]	10	91	500
LO QED [Eq. (**??**)]	0.466409	0.00563228	0.000186564
LO Zfitter	0.468499	0.127292	0.0000854731
NNLO (e)	-0.00453987	-0.0000919387	-4.28105 $\cdot 10^{-6}$
NNLO ($e + \mu$)	-0.00570942	-0.000122796	-5.90469 $\cdot 10^{-6}$
NNLO ($e + \mu + \tau$)	-0.00586082	-0.000135449	-6.7059 $\cdot 10^{-6}$
NNLO ($e + \mu + \tau + t$)			-6.6927 $\cdot 10^{-6}$
NNLO photonic	0.0358755	0.000655126	0.0000284063

Table 4: Numerical values for the NNLO corrections to the differential cross section respect to the solid angle. Results are expressed in nanobarns for a scattering angle $\theta = 3^\circ$ and $\theta = 90^\circ$. Empty entries are related to cases where the high-energy approximation cannot be applied.

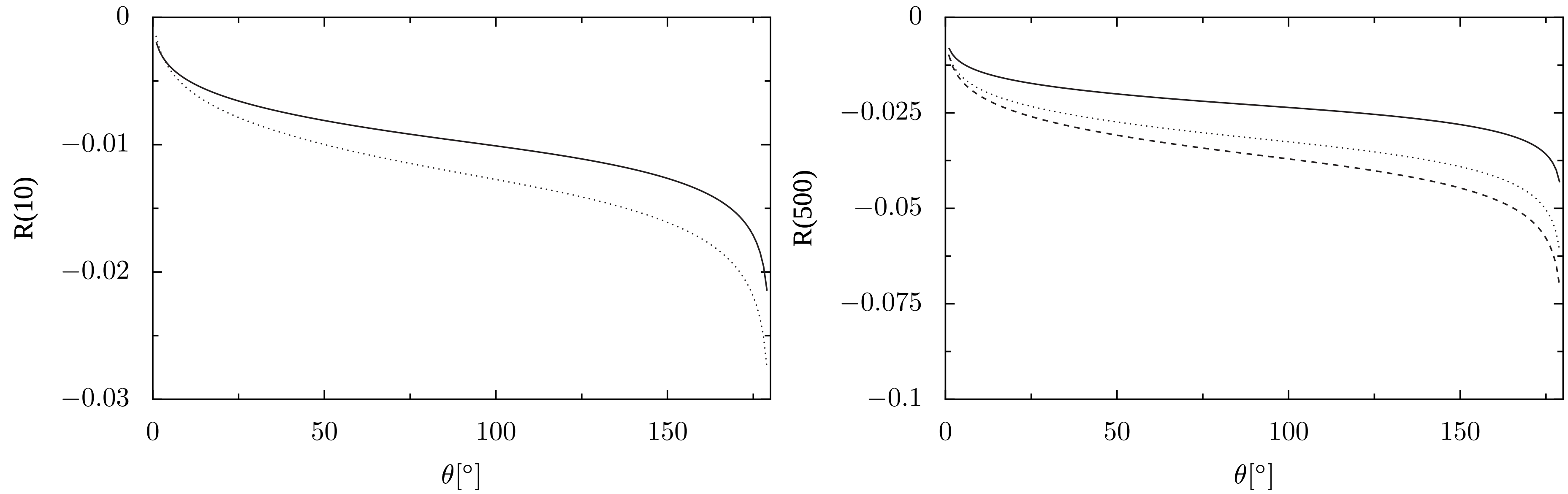


Figure 1: Ratio of the fermionic NNLO corrections to the differential cross section respect to the tree-level result for $\sqrt{s} = 10$ GeV and $\sqrt{s} = 500$ GeV. Solid line: electron-loop contributions, a dotted one the sum of electron- and muon-loop ones, and a dashed one includes also τ leptons.

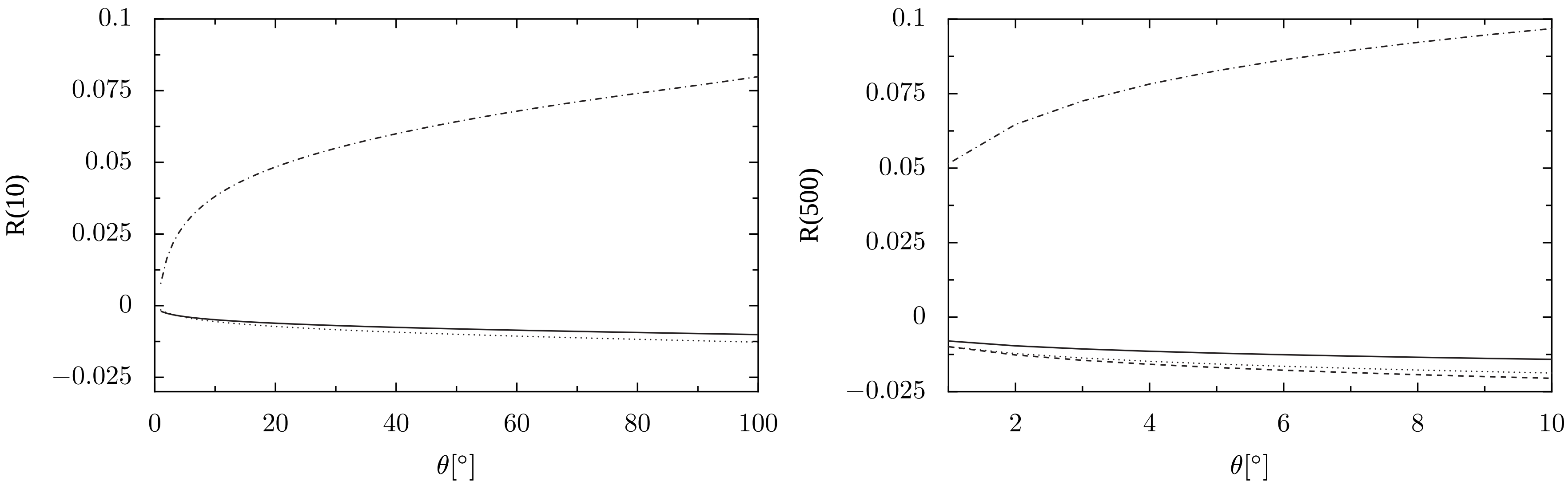


Figure 2: Here also with the photonic contributions of Arbuzov et al., Glover et al., Penin (dash-dotted lines).

Summary

- We determined the $N_f = 2$ contributions to 2-loop Bhabha scattering
- The contribution is small, but non-negligible at the scale 10^{-4} ($\rightarrow$ No LEP influencing)
- Agreement with:
  "Two-loop QED corrections to Bhabha scattering"
  Thomas Becher (Fermilab) , Kirill Melnikov (Hawaii U.), arXiv:0704.3582 [hep-ph], subm. to JHEP
  They pointed out to us an error which we had to find then, unfortunately ...
- Status: Now a nearly complete knowledge of the NNLO corrections to Bhabha scattering
  To be determined yet:
  $\longrightarrow$ Quarkonic $N_f = 2$ contributions (non-perturbative)
  $\longrightarrow$ 1-loop diagrams with real photon emission, interfering with real (Born) radiation, including 5-point functions
  The latter was studied already by Arbuzov, Kuraev, Shaitchatdenov (1998, massless case, small photon mass)