Symbolic Summation

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Introduction

Task

Given expression \( g(n) \) (depending on \( n \)) find expression \( f(n) \), such that

\[
f(n) = \sum_{i=0}^{n} g(i)
\]
Introduction

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- Summation algorithms
  - Polynomial summation
  - Hypergeometric summation
  - Harmonic summation
  - Beyond
Introduction

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$$f(n) = \sum_{i=0}^{n} g(i)$$

Summation algorithms

- Polynomial summation
- Hypergeometric summation
- Harmonic summation
- Beyond

Examples in particle physics
Introduction

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Given expression $g(n)$ (depending on $n$) find expression $f(n)$, such that

$$f(n) = \sum_{i=0}^{n} g(i)$$

- Summation algorithms
  - Polynomial summation
  - Hypergeometric summation
  - Harmonic summation
  - Beyond
- Examples in particle physics
- Summary
Polynomial summation

Examples

Polynomials

\[
\sum_{i=0}^{n-1} i = \frac{1}{2} n(n - 1)
\]

\[
\sum_{i=0}^{n-1} i^2 = \frac{1}{6} n(n - 1)(2n - 1)
\]

\[
\sum_{i=0}^{n-1} i^3 = \frac{1}{4} n^2 (n - 1)^2
\]

\[
\sum_{i=0}^{n-1} i^4 = \frac{1}{30} n(n - 1)(2n - 1)(3n^2 - 3n - 1)
\]
**Difference operator**

- Introduce operator $\Delta$ with $(\Delta f)(n) = f(n + 1) - f(n)$
- If $g = (\Delta f)$, then (for $a, b \in \mathbb{N}, a \leq b$)

\[
\sum_{i=a}^{b-1} g(i) = \sum_{i=a}^{b-1} (f(i + 1) - f(i))
\]
**Difference operator**

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- If $g = (\Delta f)$, then (for $a, b \in \mathbb{N}, a \leq b$)

\[
\sum_{i=a}^{b-1} g(i) = \sum_{i=a}^{b-1} (f(i + 1) - f(i)) = \sum_{i=a}^{b-1} f(i + 1) - \sum_{i=a}^{b-1} f(i)
\]
Difference operator

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$$

$$
= \sum_{i=a+1}^{b} f(i) - \sum_{i=a}^{b-1} f(i)
$$
Difference operator

- Introduce operator $\Delta$ with $(\Delta f)(n) = f(n + 1) - f(n)$

- If $g = (\Delta f)$, then (for $a, b \in \mathbb{N}, a \leq b$)

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\sum_{i=a}^{b-1} g(i) = \sum_{i=a}^{b-1} (f(i + 1) - f(i)) = \sum_{i=a}^{b-1} f(i + 1) - \sum_{i=a}^{b-1} f(i)
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\[
= \sum_{i=a+1}^{b} f(i) - \sum_{i=a}^{b-1} f(i) = f(b) - f(a)
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Difference operator

- Introduce operator $\Delta$ with $(\Delta f)(n) = f(n + 1) - f(n)$
- If $g = (\Delta f)$, then (for $a, b \in \mathbb{N}, a \leq b$)

\[
\sum_{i=a}^{b-1} g(i) = \sum_{i=a}^{b-1} (f(i + 1) - f(i)) = \sum_{i=a}^{b-1} f(i + 1) - \sum_{i=a}^{b-1} f(i) = \sum_{i=a+1}^{b} f(i) - \sum_{i=a}^{b-1} f(i) = f(b) - f(a)
\]

- Consecutive cancellation of summands: telescoping
- Symbolic summation problem
  $g = (\Delta f)$ with $f = (\sum g)$, operator $\Delta$ is left inverse $\Delta(\sum f) = f$
- Cf. symbolic integration (differential operator $D$)

\[
g = Df = \frac{d}{dx}f \quad \rightarrow \quad \int_{a}^{b} dx g(x) = f(b) - f(a)
\]
Difference operator (cont’d)

Differential operator $D$ acts in continuum as $D(x^m) = mx^{m-1}$
Difference operator (cont’d)

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Difference operator (cont’d)

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- Action of discrete analog $\Delta$ on polynomials?
  - Example: $\Delta(n^3) = 3n^2 + 3n - 1$

Rising and falling factorials

- Define rising factorials as $f^{\overline{m}} = f(x)f(x + 1)\ldots f(x + m - 1)$
  (also known as Pochhammer symbols $(x)_m$)
Difference operator (cont’d)

- Differential operator $D$ acts in continuum as $D(x^m) = mx^{m-1}$
- Action of discrete analog $\Delta$ on polynomials?
  - Example: $\Delta(n^3) = 3n^2 + 3n - 1$

Rising and falling factorials

- Define falling factorials as $f[n^m] = f(x)f(x-1) \ldots f(x-m+1)$
Difference operator (cont’d)

- Differential operator $D$ acts in continuum as $D(x^m) = mx^{m-1}$
- Action of discrete analog $\Delta$ on polynomials?
  - Example: $\Delta(n^3) = 3n^2 + 3n - 1$

Rising and falling factorials

- Define falling factorials as $f_m^n = f(x)f(x-1)\ldots f(x-m+1)$
- Then, with falling factorials
  $$\Delta(x^m) = mx^{m-1}$$
  $$\sum_{i=0}^{n-1} im = \frac{1}{m+1}n^{m+1}$$

- Conversion of polynomial powers $x^m$
  (decomposition with Stirling numbers of second kind $\{ \binom{m}{i} \}$)
  $$x^m = \sum_{i=0}^{m} \left\{ \binom{m}{i} \right\} x^i$$

- Stirling numbers of second kind denote # of ways to partition $n$ things in $k$ non-empty sets
Examples

Polynomials

\[\sum_{i=0}^{n-1} i = \sum_{i=0}^{n-1} i^{\frac{1}{1}} = \frac{1}{2} n^2 = \frac{1}{2} n(n-1)\]

\[\sum_{i=0}^{n-1} i^2 = \sum_{i=0}^{n-1} (i^2 + i^{\frac{1}{1}}) = \frac{1}{3} n^3 + \frac{1}{2} n^2 = \frac{1}{6} n(n+1)(2n+1)\]

\[\sum_{i=0}^{n-1} i^3 = \sum_{i=0}^{n-1} (i^3 + 3i^2 + i^{\frac{1}{1}}) = \frac{1}{4} n^4 + n^3 + \frac{1}{2} n^2 = \frac{1}{4} n^2 (n+1)^2\]
Hypergeometric summation

**Definition**

Hypergeometric function $mF_n$

$$mF_n \left( \begin{array}{c} a_1, \ldots, a_m \\ b_1, \ldots, b_n \end{array} \vline \right) z = \sum_{i \geq 0} \frac{a_1^i \ldots a_m^i}{b_1^i \ldots b_n^i} \frac{z^i}{i!}$$
Hypergeometric summation

Definition

Hypergeometric function $mF_n$

$$mF_n \left( \begin{array}{c} a_1, \ldots, a_m \\ b_1, \ldots, b_n \end{array} \middle| z \right) = \sum_{i \geq 0} \frac{a_1^i \cdots a_m^i}{b_1^i \cdots b_n^i} \frac{z^i}{i!}$$

Examples

$$0F_0 \left( \middle| z \right) = \sum_{i \geq 0} \frac{z^i}{i!} = e^z$$

$$2F_1 \left( \begin{array}{c} a, 1 \\ 1 \end{array} \middle| z \right) = \sum_{i \geq 0} a \frac{z^i}{i!} = \frac{1}{(1 - z)^a}$$

$$2F_1 \left( \begin{array}{c} 1, 1 \\ 2 \end{array} \middle| z \right) = z \sum_{i \geq 0} \frac{i^i 1^i}{2^i} \frac{z^i}{i!} = -\ln(1 - z)$$
Ratios

- A term $g_n$ is hypergeometric, if the ratio $r(n)$ of two consecutive terms is a rational function of $n$.

$$r(n) = \frac{g_{n+1}}{g_n}$$
Ratios

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$$r(n) = \frac{g_{n+1}}{g_n}$$

Example: binomial coefficient

$$\binom{m}{n+1} \binom{n}{m} = \frac{\Gamma(m+1)\Gamma(n+1)\Gamma(m-n+1)}{\Gamma(n+2)\Gamma(m-n)\Gamma(m)} = \frac{-n+m}{n+1}$$
Ratios

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$$r(n) = \frac{g_{n+1}}{g_n}$$

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Given a hypergeometric term $g$, is there hypergeometric term $f$ such that $\Delta f = g$?

$$f_{n+1} - f_n = g_n$$
**Gospers algorithm**

Gospers algorithm for indefinite hypergeometric summation determines $f_n$ from a given recursion

$$f_n = f_{n-1} + g_{n-1} = f_{n-2} + g_{n-1} + g_{n-2} = \cdots = f_0 + \sum_{k=0}^{n-1} g_k$$

Idea: recursive algorithm; telescoping
**Gospers algorithm**

- Gospers algorithm for indefinite hypergeometric summation determines $f_n$ from a given recursion

\[
f_n = f_{n-1} + g_{n-1} = f_{n-2} + g_{n-1} + g_{n-2} = \cdots = f_0 + \sum_{k=0}^{n-1} g_k
\]

- Idea: recursive algorithm; telescoping

- Ratio \( \frac{f_n}{g_n} = \frac{f_n}{f_{n+1} - f_n} = \frac{1}{\frac{f_{n+1}}{f_n} - 1} \) is rational function of \( n \)
Gospers algorithm

Gospers algorithm for indefinite hypergeometric summation determines $f_n$ from a given recursion

$$f_n = f_{n-1} + g_{n-1} = f_{n-2} + g_{n-1} + g_{n-2} = \cdots = f_0 + \sum_{k=0}^{n-1} g_k$$

- Idea: recursive algorithm; telescoping
- Ratio $\frac{f_n}{g_n} = \frac{f_n}{f_{n+1} - f_n} = \frac{1}{\frac{f_{n+1}}{f_n} - 1}$ is rational function of $n$
- Solve recursion with ansatz $f_n = y(n)g_n$ and (unknown) rational function $y(n)$

$$f_{n+1} - f_n = g_n \quad \longrightarrow \quad r(n)y(n+1) - y(n) = 1$$
Gospers algorithm

- Gospers algorithm for indefinite hypergeometric summation determines $f_n$ from a given recursion

$$f_n = f_{n-1} + g_{n-1} = f_{n-2} + g_{n-1} + g_{n-2} = \cdots = f_0 + \sum_{k=0}^{n-1} g_k$$

- Idea: recursive algorithm; telescoping

- Ratio $\frac{f_n}{g_n} = \frac{f_n}{f_{n+1} - f_n} = \frac{1}{f_n} - 1$ is rational function of $n$

- Solve recursion with ansatz $f_n = y(n) g_n$ and (unknown) rational function $y(n)$

$$f_{n+1} - f_n = g_n \quad \rightarrow \quad r(n) y(n+1) - y(n) = 1$$

- Upshot
  - Solve first-order linear recursion for $y(n)$
Gospers algorithm (cont’d)

Given \( f_{n+1} - f_n = g_n \) and ansatz \( f_n = y(n)g_n \) with rational function \( y(n) \), then \( r(n)y(n+1) - y(n) = 1 \)
Gospers algorithm (cont’d)

Given $f_{n+1} - f_n = g_n$ and ansatz $f_n = y(n)g_n$ with rational function $y(n)$, then $r(n)y(n + 1) - y(n) = 1$

Let $r(n) = \frac{a(n)}{b(n)} \frac{c(n + 1)}{c(n)}$ with polynomials $a(n), b(n), c(n)$ and $\gcd(a(n), b(n + k)) = 1$
Gospers algorithm (cont’d)

- Given \( f_{n+1} - f_n = g_n \) and ansatz \( f_n = y(n)g_n \) with rational function \( y(n) \), then \( r(n)y(n + 1) - y(n) = 1 \)

- Let \( r(n) = \frac{a(n)}{b(n)} \frac{c(n + 1)}{c(n)} \) with polynomials \( a(n), b(n), c(n) \) and \( \gcd(a(n), b(n + k)) = 1 \)

- Ansatz for \( y(n) \) becomes \( y(n) = \frac{b(n - 1)}{c(n)}x(n) \) with (unknown) polynomial \( x(n) \)
Gospers algorithm (cont’d)

Given \( f_{n+1} - f_n = g_n \) and ansatz \( f_n = y(n)g_n \) with rational function \( y(n) \), then \( r(n)y(n + 1) - y(n) = 1 \)

Let \( r(n) = \frac{a(n) c(n + 1)}{b(n) c(n)} \) with polynomials \( a(n), b(n), c(n) \) and \( \gcd(a(n), b(n + k)) = 1 \)

Ansatz for \( y(n) \) becomes \( y(n) = \frac{b(n - 1)}{c(n)} x(n) \) with (unknown) polynomial \( x(n) \)

Solve for non-zero \( x(n) \)

\[ a(n)x(n + 1) - b(n - 1)x(n) = c(n) \]

If non-zero \( x(n) \) exists, hypergeometric recursion is summable.
Wilf-Zeilberger algorithm

- WZ algorithm
  - definite hypergeometric summation
  - telescoping
Wilf-Zeilberger algorithm

- WZ algorithm
  - definite hypergeometric summation
  - telescoping

Examples

- Definite vs. indefinite summation

\[
\sum_{k} \binom{n}{k} = \sum_{k} \binom{n}{k}
\]
\[
\sum_{k=0}^{n} \binom{n}{k} = 2^n
\]
Harmonic summation

- Harmonic sums $S_{m_1,\ldots,m_k}(n)$ [see lecture by Blümlein]

- Recursive definition

$$S_{\pm m_1,\ldots,m_k}(n) = \sum_{i=1}^{n} \frac{(\pm 1)^i}{i m_1} S_{m_2,\ldots,m_k}(i)$$
Harmonic summation

- Harmonic sums $S_{m_1,\ldots,m_k}(n)$ [see lecture by Blümlein]

- recursive definition

$$S_{\pm m_1,\ldots,m_k}(n) = \sum_{i=1}^{n} \frac{(\pm 1)^i}{i^{m_1}} S_{m_2,\ldots,m_k}(i)$$

- Particle physics

  - dimensional regularization $D = 4 - 2\epsilon$ requires expansion of the Gamma-function around positive integers values ($n \geq 0$)

$$\frac{\Gamma(n + 1 + \epsilon)}{\Gamma(1 + \epsilon)} = \Gamma(n + 1) \exp \left( - \sum_{k=1}^{\infty} \epsilon^k \frac{(-1)^k}{k} S_k(n) \right)$$
Algorithms for harmonic sums

- Multiplication (Hopf algebra)
  - basic formula (recursion)

\[
S_{m_1,\ldots,m_k}(n) \times S'_{m'_1,\ldots,m'_l}(n) = \sum_{j_1=1}^{n} \frac{1}{m_1} S_{m_2,\ldots,m_k(j_1)}S'_{m'_1,\ldots,m'_l(j_1)} \\
+ \sum_{j_2=1}^{n} \frac{1}{m'_1} S_{m_1,\ldots,m_k(j_2)}S'_{m'_2,\ldots,m'_l(j_2)} \\
- \sum_{j=1}^{n} \frac{1}{j m_1 + m'_1} S_{m_2,\ldots,m_k(j)}S'_{m'_2,\ldots,m'_l(j)}
\]

- Proof uses decomposition

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{i} a_{ij} + \sum_{j=1}^{n} \sum_{i=1}^{j} a_{ij} - \sum_{i=1}^{n} a_{ii}
\]
Algorithms for harmonic sums (cont’d)

- Convolution (sum over \( n - j \) and \( j \))

\[
\sum_{j=1}^{n-1} \frac{1}{j^{m_1}} S_{m_2, \ldots, m_k}(j) \frac{1}{(n - j)^{n_1}} S_{n_2, \ldots, n_l}(n - j)
\]

- Conjugation

\[- \sum_{j=1}^{n} \binom{n}{j} (-1)^j \frac{1}{j^{m_1}} S_{m_2, \ldots, m_k}(j) \]

- Binomial convolution (sum over binomial, \( n - j \) and \( j \))

\[- \sum_{j=1}^{n-1} \binom{n}{j} (-1)^j \frac{1}{j^{m_1}} S_{m_2, \ldots, m_k}(j) \frac{1}{(n - j)^{n_1}} S_{n_2, \ldots, n_l}(n - j)\]
Beyond

- Generalized sums \( S(n; m_1, \ldots, m_k; x_1, \ldots, x_k) \)
  - recursive definition
    \[
    S(n; m_1, \ldots, m_k; x_1, \ldots, x_k) = \sum_{i=1}^{n} \frac{x_i^{m_1}}{i^{m_1}} S(i; m_2, \ldots, m_k; x_2, \ldots, x_k)
    \]
  - multiple scales \( x_1, \ldots, x_k \)
  - depth \( k \), weight \( w = m_1 + \ldots + m_k \)

Example

- Powers of logarithm \( \ln(1 - x) \)
  \[
  \sum_{j=1}^{\infty} \frac{x^j}{j!} \Gamma(j - \epsilon) = \sum_{j=1}^{\infty} \frac{x^j}{j} - \epsilon \sum_{j=1}^{\infty} \frac{x^j}{j} S_1(j - 1) + \epsilon^2 \ldots
  \]
  \[
  = -\ln(1 - x) - \epsilon \frac{1}{2} \ln(1 - x)^2 + \epsilon^2 \ldots
  \]
Algorithms for nested sums

- Same structures as for harmonic sums, in particular
  - multiplication
    \[ S(n; m_1, ...; x_1, ...) \times S(n; m'_1, ...; x'_1, ...) \]
  - convolution
  - conjugation
  - binomial convolution

- Recursive algorithms analogous to harmonic sums solve multiple nested sums
Higher transcendental functions

- Expansion of higher transcendental functions in small parameter
  - Expansion parameter $\epsilon$ occurs in the rising factorials (Pochhammer symbols)

- Hypergeometric function

\[
\begin{align*}
2F_1(a, b; c, x_0) &= \sum_{i=0}^{\infty} \frac{a^i b^i x_0^i}{c^i i!}
\end{align*}
\]

- First Appell function

\[
\begin{align*}
F_1(a, b_1, b_2; c; x_1, x_2) &= \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{a^{m_1+m_2} b_1^{m_1} b_2^{m_2} x_1^{m_1} x_2^{m_2}}{c^{m_1+m_2} m_1! m_2!}
\end{align*}
\]

- Second Appell function

\[
\begin{align*}
F_2(a, b_1, b_2; c_1, c_2; x_1, x_2) &= \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{a^{m_1+m_2} b_1^{m_1} b_2^{m_2} x_1^{m_1} x_2^{m_2}}{c_1^{m_1} c_2^{m_2} m_1! m_2!}
\end{align*}
\]
Examples in particle physics

Feynman integrals

Scalar diagram with external momenta $P$ and $Q$

Four-point function with underlying ladder topology

$[\text{see lecture by Vermaseren}]$
Examples in particle physics

**Feynman integrals**

- Scalar diagram with external momenta $P$ and $Q$
- Four-point function with underlying ladder topology
  [see lecture by Vermaseren]

\[ \int \prod_{n} d^D l_n \frac{1}{(P - l_1)^2} \frac{1}{l_1^2 \ldots l_8^2} \]

- $N$-th moment: coefficient of $(2P \cdot Q)^N$

\[ \frac{(2P \cdot Q)^N}{(Q^2)^{N+\alpha}} C_N \]
Examples in particle physics

Feynman integrals

Scalar diagram with external momenta $P$ and $Q$

Four-point function with underlying ladder topology

[see lecture by Vermaseren]

$N$-th moment:
Coefficient of $(2P \cdot Q)^N$

Taylor expansion

$\frac{1}{(P - l_1)^2} = \sum_i \frac{(2P \cdot l_1)^i}{(l_1^2)^{i+1}} \quad \rightarrow \quad \frac{(2P \cdot l_1)^N}{(l_1^2)^N}$
Difference equations

- Single-step difference equation in $N$
- extremely simple example

$$I(N) = \frac{N+3+3\varepsilon}{N+2} \frac{2p \cdot q}{q^2} + \frac{2}{N+2}$$
Difference equations

- Single-step difference equation in \( N \)
  - extremely simple example

\[
I(N) = -\frac{N+3+3\epsilon}{N+2} \cdot \frac{2p \cdot q}{q^2} I(N-1) + \frac{2}{N+2} G(N)
\]

- Formal equation
  - automatic build-up of nested sums
  - efficient implementation in FORM

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Difference equations

- Single-step difference equation in $N$
  - extremely simple example

\[
    -\frac{N+3+3\varepsilon}{N+2} \frac{2p \cdot q}{q^2} = 2 \frac{N+2}{N+2}
\]

- Formal equation, formal solution

\[
    I(N) = (-1)^N \frac{\prod_{j=1}^{N} (j+3+3\varepsilon)}{\prod_{j=1}^{N} (j+2)} I(0) + (-1)^N \sum_{i=1}^{N} (-1)^j \frac{\prod_{j=i+1}^{N} (j+3+3\varepsilon)}{\prod_{j=i}^{N} (j+2)} G(i)
\]
Difference equations

- Single-step difference equation in $N$
- extremely simple example

\[
\frac{N+3+3\epsilon}{N+2} = -\frac{2p \cdot q}{q^2} \quad + \quad \frac{2}{N+2}
\]

- Formal equation, formal solution, input to solution

\[
I(N) = (-1)^N \frac{\prod_{j=1}^{N}(j+3+3\epsilon)}{\prod_{j=1}^{N}(j+2)} I(0) + (-1)^N \sum_{i=1}^{N} (-1)^j \frac{\prod_{j=i+1}^{N}(j+3+3\epsilon)}{\prod_{j=i}^{N}(j+2)} G(i)
\]

\[
I(0) = -\frac{2}{3} \frac{1}{\epsilon^2} + \frac{23}{3} \frac{1}{\epsilon} - 42
\]

\[
G(i) = \frac{(-1)^i}{\epsilon^2} \frac{2}{3} \left( \frac{S_1(i+2)}{i+2} - \frac{S_1,2(i)}{2} - \frac{S_2(i+1)}{2(i+1)} - S_2(i) - \frac{1}{(i+1)^2} - \frac{1}{(i+2)^2} \right) + \ldots
\]
Difference equations

- Single-step difference equation in $N$
  - extremely simple example

\[
I(N) = -\frac{N + 3 + 3\varepsilon}{N + 2} \frac{2p \cdot q}{q^2} I(0) + \frac{2}{N + 2} \quad + \quad G(i)
\]

- Formal equation, formal solution, input to solution

\[
I(N) = (-1)^N \frac{\prod_{j=1}^{N} (j + 3 + 3\varepsilon)}{\prod_{j=1}^{N} (j + 2)} I(0) + (-1)^N \sum_{i=1}^{N} (-1)^j \frac{\prod_{j=i+1}^{N} (j + 3 + 3\varepsilon)}{\prod_{j=i}^{N} (j + 2)} G(i)
\]

\[
I(0) = -\frac{2}{3} \frac{1}{\varepsilon^2} + \frac{23}{3} \frac{1}{\varepsilon} - 42
\]

\[
G(i) = \frac{(-1)^i}{\varepsilon^2} \frac{2}{3} \left( \frac{S_1(i + 2)}{i + 2} - \frac{S_1,2(i)}{2} - \frac{S_2(i + 1)}{2(i + 1)} - S_2(i) - \frac{1}{(i + 1)^2} - \frac{1}{(i + 2)^2} \right) + \ldots
\]

- Upshot
  - automatic build-up of nested sums
  - efficient implementation in FORM

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Symbolic Summation – p.19
\[
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\]

Symbolic Summation – p.20
\[
\begin{align*}
\text{Result for } I(N) \text{ in the G-scheme}
\end{align*}
\]
Summary

Symbolic summation

- Polynomial summation → solved in Mathematica, Maple, ...

Examples in particle physics, Feynman diagram calculations and more...

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Summary

Symbolic summation

- Polynomial summation $\longrightarrow$ solved in *Mathematica*, *Maple*, ...
- Hypergeometric summation $\longrightarrow$ algorithms of Gosper, Wilf-Zeilberger
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- Beyond → generalizations ...
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- Beyond $\rightarrow$ generalizations ...

Examples in particle physics

- Feynman diagram calculations and more ...
Literature

Text books

- *Modern Computer Algebra*, J. von zur Gathen, J. Gerhard
- *Concrete Mathematics*, R. L Graham, D. E. Knuth, O. Pataschnik
- *A=B*, M. Petkovsek, H. S. Wilf, D. Zeilberger

Research articles

- *Harmonic sums, Mellin transforms and integrals*, J. Vermaseren; hep-ph/9806280
Software

Commercial programs

Mathematica
Maple

Freeware/Add-on packages

Mathematica, Maple
Several packages for hypergeometric summation
[see for instance www.cis.upenn.edu/~wilf/AeqB.html]

GiNaC
nestedsums, S. Weinzierl

FORM
Summer6, J. Vermaseren
XSummer, S. Moch, P. Uwer to be published
Exercises 1

- Use *Mathematica* or *Maple* for polynomial summation.
- Check some of the examples for hypergeometric summation with *Mathematica* or *Maple* like

$$
\sum_{i \geq 0} \frac{a^i z^i}{i!} = \frac{1}{(1 - z)^a}
$$

$$
-z \sum_{i \geq 0} \frac{1^i 1^i z^i}{2^i i!} = \ln(1 - z)
$$

- Try to evaluate the sum

$$
\sum_{j_1 = 1}^{N} \frac{1}{j_1} S_1(j_1)
$$

in *Mathematica* or *Maple*.

What happens?
Exercises 2

- Use the FORM package summer6.h for harmonic summation.

- Evaluate the product of harmonic sums $S_2(N)S_1(N)^2$. Use the procedure basis.prc.

```c
#include summer6.h
.global
L exampleproduct = S(R(2),N)*S(R(1),N)^2;
#call basis(S)
Print;
.end
```

- Check your result with the following sequence of calls.
  Multiply, replace $(N, <some_number>)$;
  #call subesses(S)
Exercises 3

- Use the FORM package summer6.h for harmonic summation.

- Evaluate the sum $\sum_{j_1=1}^{N} \frac{1}{j_1} S_1(j_1)$ Use the procedure summer.prc.

```c
#include summer6.h
.global
.left
examplesum = sum1(j1,1,N)*den(j1)*S(R(1),j1);
#call summer(1)
Print;
.end
```

- Compute examples for the convolution and conjugation of harmonic sums. Use the notation $\text{fac}(N), \text{invfac}(N)$ and $\text{sign}(N)$ with the (obvious) meaning $N!, \frac{1}{N!}$ and $(-1)^N$
Exercises 4

- Program the expansion of the Gamma-function around any integer value in FORM. To do so, use the result for expansions around positive integers ($n \geq 0$)

$$\frac{\Gamma(n+1+\epsilon)}{\Gamma(1+\epsilon)} = \Gamma(n+1) \exp \left( - \sum_{k=1}^{\infty} \epsilon^k \frac{(-1)^k}{k} S_k(n) \right)$$

For expansions around negative integers ($n \leq 0$) use the well-known relation

$$\frac{\Gamma(-n+1+\epsilon)}{\Gamma(1+\epsilon)} = (-1)^n \frac{\Gamma(-\epsilon)}{\Gamma(n-\epsilon)}$$

- Use the FORM package summer6.h and your Gamma-function expansion to solve the difference equation

$$I(N) = -\frac{N+3+3\epsilon}{N+2} I(N-1) + \frac{2}{N+2} G(N)$$

with the boundary conditions for $I(0)$ and $G(N)$ given in the lecture.