Asymptotic expansion of Feynman diagrams

- Outline of the problem:

  complexity of Feynman integral

  ~ # loops

  ~ # scales

- simplest cases:
  - * no scale : $\int d^dk \, \frac{1}{k^\alpha} = 0$ (massless tadpole)
  - * 1 scale:

→ MINCER (3 loops)

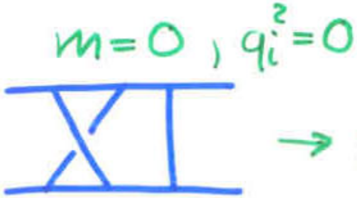
4 loops in progress

→ MATAD (3 loops)

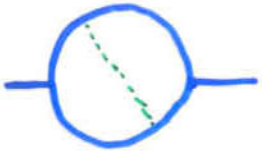
4 loops in progress

→ on-shell (3 loops)

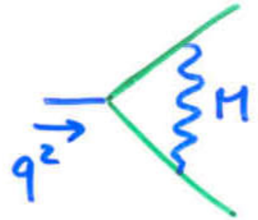
* 2 scales:



→ s,t Laporta approach



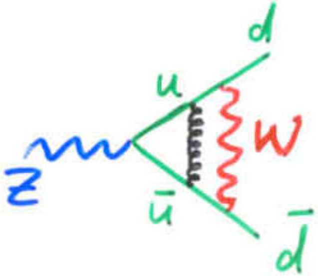
→ q^2, m^2 2-loop



2-loop in progress

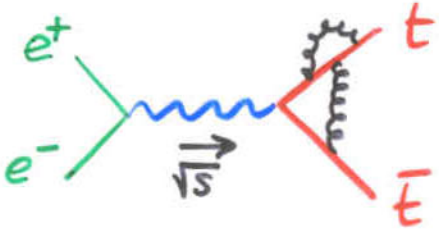
But: very important!

e.g. • EW radiative corrections:



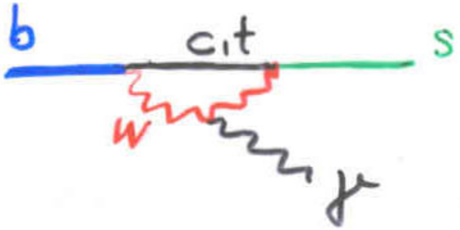
M_Z, M_W

- $e^+e^- \to t\bar{t}$:



$\sqrt{s}, m_t$

- B-physics:



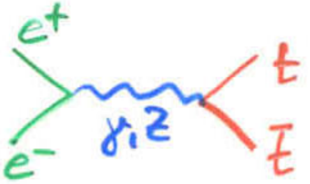
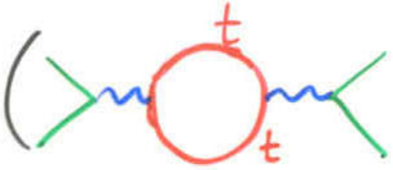
- extended theories: (e.g. Supersymmetry)

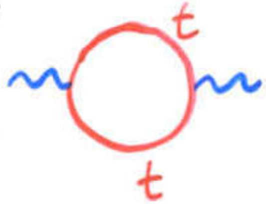
  even more scales ($\tilde{q}, \tilde{g}, h^\circ, H^\circ, \dots$)

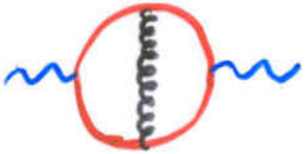
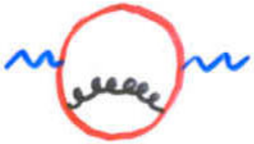
  ↑ even QCD!

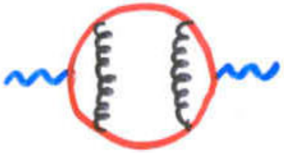
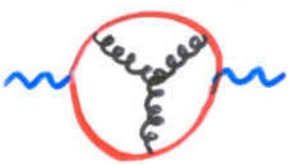
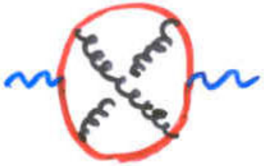
- IBP identities more complex
- master integrals?

Example: $e^+e^- \to t\bar{t}\ (+X)$

$\sigma \sim \Big|$  $\Big|^2 \sim \mathrm{Im}\Big($  $\Big) \sim$

$\sim \mathrm{Im}\Big($  $\Big) = \mathrm{Im}\,\Pi(q^2, m^2)$

higher orders:   → Im: virtual + real ✓

$\mathcal{O}(\alpha_s^2)$:    ....

relevant scales: $\sqrt{s} \sim Q^2 \gtrsim 500$ GeV

$2m_t \approx 350$ GeV

$$x = \frac{2m_t}{\sqrt{s}} \approx 0.7$$

(?) → approximate result for $x \ll 1$ $(\Leftrightarrow \sqrt{s} \to \infty \Leftrightarrow m_t \to 0)$

Example: scalar 1-loop

- consider unphysical regime: $q^2 < 0$

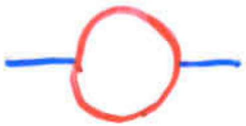
  define Euclidean $Q^2 = -q^2 > 0, \quad Q = \underbrace{(q_0, \vec{q})_E}_{= q_E}$

- loop momenta Wick-rotated: $k \to (ik_0, \vec{k})$

  $\Rightarrow k^2 \to -k_E^2$

  $(k+q)^2 \to -(k_E + Q)^2$

  ↑ drop index in the following.

- 

$$= \int d^D \overset{\text{Euclidean}}{k} \frac{1}{(k^2+m^2)[(k+Q)^2+m^2]}$$

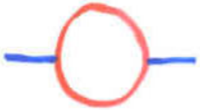
$$= \frac{1}{\epsilon} + 2 - \ln m^2 + \sqrt{1+\frac{4m^2}{Q^2}} \ln \frac{\sqrt{1+\frac{4m^2}{Q^2}} - 1}{\sqrt{1+\frac{4}{Q^2}} + 1} + \mathcal{O}(\epsilon)$$

$$= \frac{1}{\epsilon} + 2 - \ln m^2 - \ln \frac{Q^2}{m^2} + \frac{m^2}{Q^2}\left[2 + 2\ln\frac{Q^2}{m^2}\right] + \mathcal{O}\left(\frac{m^4}{Q^4}\right)$$

$$= \frac{1}{\epsilon} + 2 - \ln Q^2 + \frac{m^2}{Q^2}\left[2 + 2\ln\frac{Q^2}{m^2}\right] + \mathcal{O}\left(\frac{m^4}{Q^4}\right)$$

Naive attempt: expand integrand

$\mathcal{J}_m$



$$\equiv \mathcal{J}_m \int d^Dk \, \frac{1}{(k^2+m^2)[(k+Q)^2+m^2]} =$$

$$= \int d^Dk \, \frac{1}{k^2(k+Q)^2}\left[1-\frac{m^2}{k^2}+\left(\frac{m^2}{k^2}\right)^2 \pm \cdots\right]\left[1-\frac{m^2}{(k+Q)^2}+\frac{m^4}{(k+Q)^4}\pm\cdots\right]$$

integrals are of the form:

$$(m^2)^n \int d^Dk \, \frac{1}{(k^2)^r[(k+Q)^2]^{n-r+2}} \sim \left(\frac{m^2}{Q^2}\right)^n (Q^2)^{-\epsilon}\, \hat{I}(r, n-r+2)$$

$(r \geq 1)$

$n = 0, 1, 2, \ldots.$

- $n=0$: $(Q^2)^{-\epsilon}\hat{I}(1,1) = \frac{1}{\epsilon} + 2 - \ln Q^2 + \mathcal{O}(\epsilon)$ ✓
- $n=1$: $\frac{m^2}{Q^2}(Q^2)^{-\epsilon}\underbrace{\left[\hat{I}(1,2)+\hat{I}(2,1)\right]}_{=-\frac{2}{\epsilon}} = \frac{m^2}{Q^2}\left[-\frac{2}{\epsilon} + 2\ln Q^2\right]$

$$\neq \frac{m^2}{Q^2}\left[2 + 2\ln\frac{Q^2}{m^2}\right]$$ ↯

More careful: expand by regions

$$= \int d^Dk \, \frac{1}{(k^2+m^2)[(k+Q)^2+m^2]}$$

[all momenta Euclidean!]

$Q^2 \gg m^2$:

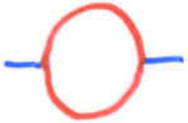
- region (i): $k^2 \gg m^2$ and $(k+Q)^2 \gg m^2$

  → expand integrand in $\frac{m^2}{k^2}, \frac{m^2}{(k+Q)^2}$

  → result above

- region (ii): $k^2 \sim m^2$ $[\Rightarrow (k+Q)^2 \gg m^2]$

  → expand in $\frac{m}{Q}, \frac{k}{Q}$

$\mathcal{T}_{k,m}$ 

$$= \mathcal{T}_{km} \int_{(ii)} d^Dk \, \frac{1}{(k^2+m^2)} \frac{1}{[Q^2+2Q\cdot k+k^2+m^2]} =$$

$$= \int_{(ii)} d^Dk \, \frac{1}{k^2+m} \frac{1}{Q^2} \left[1 - \frac{2Q\cdot k+k^2+m^2}{Q^2} + \cdots \right]$$

typical term: $\frac{1}{(Q^2)^{n+1}} \int_{(ii)} d^Dk \, \frac{(2Q\cdot k)^n}{k^2+m^2}$

- region (iii): $(k+Q)^2 \sim m^2 \quad [\Rightarrow k^2 \gg m^2]$

  → substitute $k \to -k-Q$

  → same as region (ii)

- crucial step:

$$\int_{(i)} d^Dk \to \int d^Dk \,, \quad \int_{(ii)} d^Dk \to \int d^Dk \,, \quad \int_{(iii)} d^Dk \to \int d^Dk$$

allowed ?

$$\int \frac{1}{k^2+m^2}\,\frac{1}{(k+Q)^2+m^2} \xrightarrow{Q^2\gg m^2} \int_{(i)} \mathcal{T}_m \frac{1}{k^2+m^2}\,\frac{1}{(k+Q)^2+m^2} + 2\int_{(ii)} \frac{1}{k^2+m^2}\,\mathcal{T}_{km}\frac{1}{(k+Q)^2+m^2}$$

$$= \int \mathcal{T}_m \frac{1}{k^2+m^2}\,\frac{1}{(k+Q)^2+m^2} + 2\int \frac{1}{k^2+m^2}\,\mathcal{T}_{km}\frac{1}{(k+Q)^2+m^2}$$

$k' = -k-Q$

$$- \int_{(ii)\cup(iii)} \mathcal{T}_m \frac{1}{k^2+m^2}\,\frac{1}{(k+Q)^2+m^2} - \int_{(i)\cup(iii)} \frac{1}{k^2+m^2}\,\mathcal{T}_{km}\frac{1}{(k+Q)^2+m^2} - \int_{(i)\cup(ii)} \frac{1}{k'^2+m^2}\,\mathcal{T}_{k'm}\frac{1}{(k'+Q)^2+m^2}$$

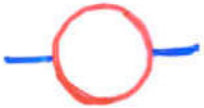
expand again, e.g.: $\int_{(ii)} \mathcal{T}_m \frac{1}{k^2+m^2}\,\frac{1}{(k+Q)^2+m^2} \longrightarrow \int_{(ii)} \mathcal{T}_m \frac{1}{k^2+m^2}\,\mathcal{T}_{km}\frac{1}{(k+Q)^2+m^2}$

typical term: $\left(\frac{m^2}{Q^2}\right)^n \int_{(ii)} \frac{1}{(k^2)^n}$

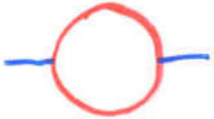
$$\Rightarrow \quad \int_{(i)} dk + \int_{(ii)} dk + \int_{(iii)} dk =$$

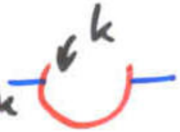
$$= \int dk + \int dk + \int dk \quad - C$$

$$C = 2\int_{(i)\cup(ii)\cup(iii)} \mathcal{T}_m \frac{1}{k^2+m^2}\, \mathcal{T}_{km} \frac{1}{(k+Q)^2+m^2} \sim \int d^D k \, \frac{1}{(k^2)^n} = 0 \; !$$

$\Rightarrow$ 

$$= \underbrace{\int \mathcal{T}_m \frac{1}{k^2+m^2}\, \frac{1}{(k+Q)^2+m^2}}_{\sim \int m^{2n} \frac{1}{(k^2)^r} \frac{1}{[(k+Q)^2]^{2-r+n}}} + 2 \underbrace{\int \frac{1}{k^2+m^2}\, \mathcal{T}_{km} \frac{1}{(k+Q)^2+m^2}}_{\sim \left(\frac{m^2}{Q^2}\right)^n \frac{1}{Q^2} \int \frac{1}{k^2+m^2}}$$

$\mathcal{T}_m$  $+ \; 2$ 

$* \; \mathcal{T}_{mk}$ 

check:

$\mathcal{J}_m$  $= \frac{1}{\epsilon} + 2 - \ln Q^2 + \frac{m^2}{Q^2}\left[-\frac{2}{\epsilon} + 2\ln Q^2\right] + \mathcal{O}\left(\frac{m^4}{Q^4}\right)$

2  $* \; \mathcal{J}_{km}$  $= 2\,\frac{1}{Q^2}\int\frac{1}{k^2+m^2} + \mathcal{O}(m^4) = -\frac{2}{Q}\,A(1) = \frac{m^2}{Q^2}\left[\frac{2}{\epsilon} + 2 - 2\ln m^2\right]$

$$\overset{\uparrow}{=}\left(-\frac{1}{\epsilon} - 1 + \ln m^2\right)\cdot m^2$$

$\Rightarrow \; \mathcal{J}_m$  $+ \; 2$  $* \; \mathcal{J}_{km}$ 

$$= \frac{1}{\epsilon} + 2 - \ln Q^2 + \frac{m^2}{Q^2}\left[2 + 2\ln\frac{Q^2}{m^2}\right] + \mathcal{O}\left(\frac{m^4}{Q^4}\right)$$

✓

Diagammatic approach

[for Euclidean cases]

- Large momentum procedure

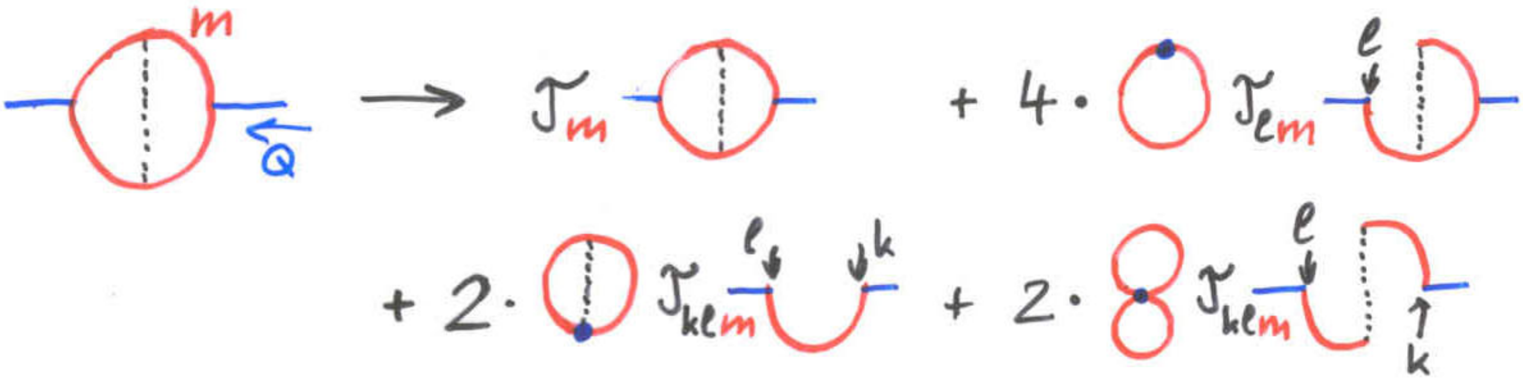
$Q_i \gg q_j , m_j$

- expand all subdiagrams in $\frac{q}{Q} , \frac{m}{Q} , \frac{k}{Q}$ which
  - * contain all vertices with Q_i
  - * are 1-PI when identifying these vertices

loop momentum of full diagram

ext. mom. of subgraph

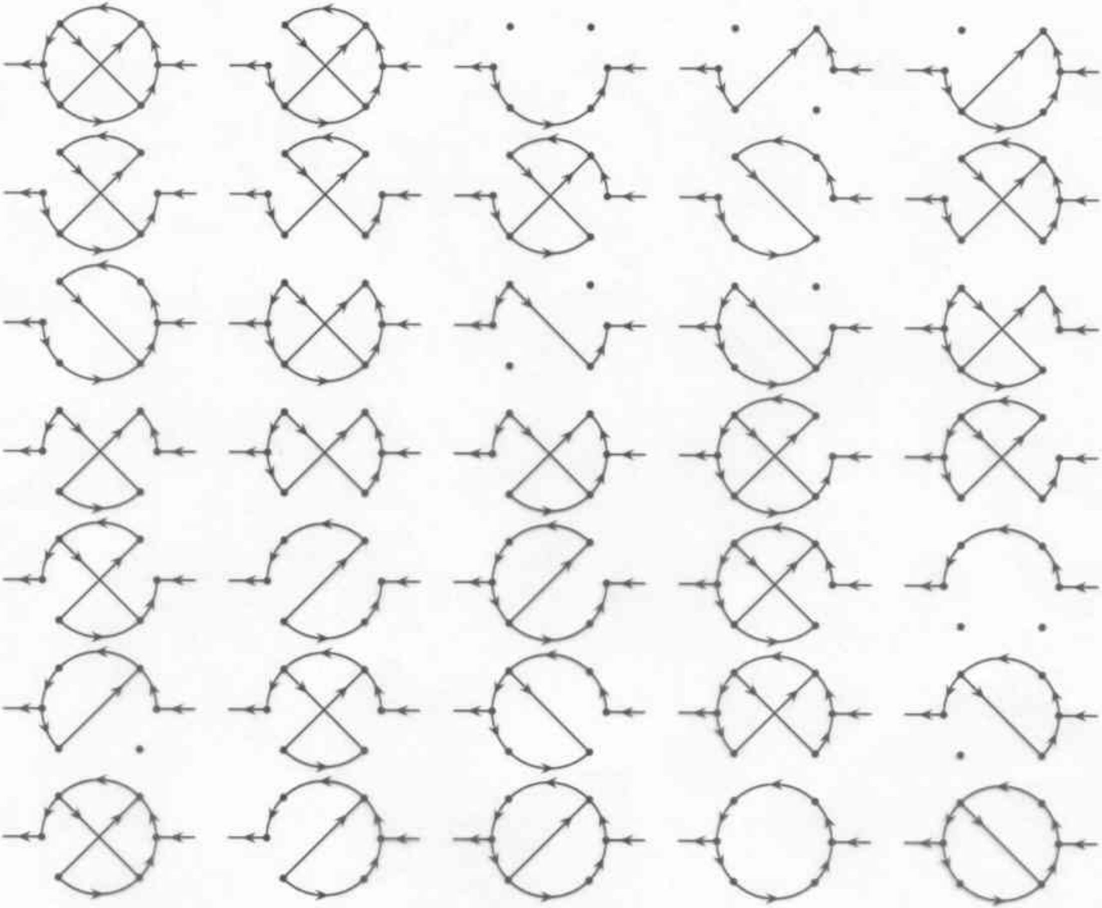
example:



"identifying vertices":

1-PI → valid

→ not valid

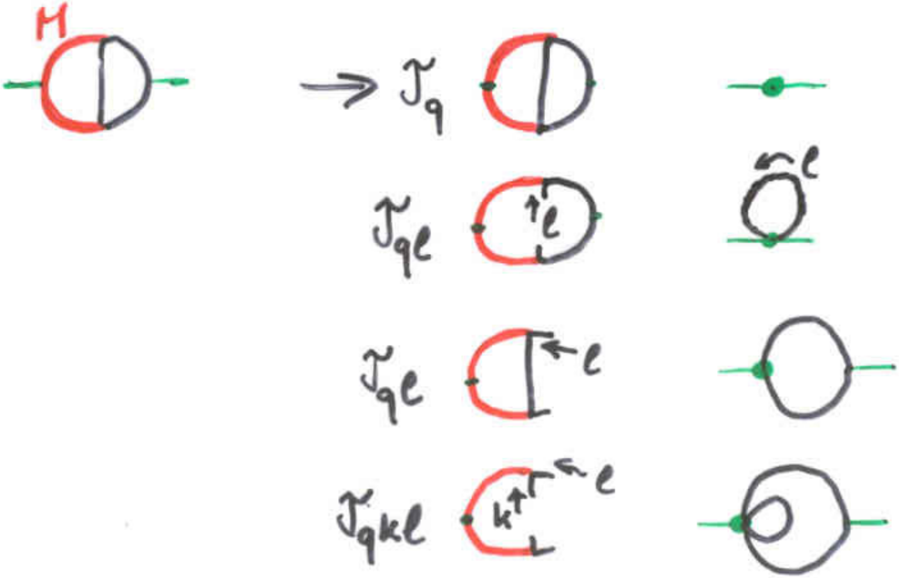


- **Large mass procedure**

$$M_i \gg q_j, m_j$$

- expand all subdiagrams in $\frac{q}{M}, \frac{m}{M}, \frac{k}{M}$ which
  - * contain all heavy lines
  - * are 1-PI in connected pieces when shrinking heavy lines to points

example:



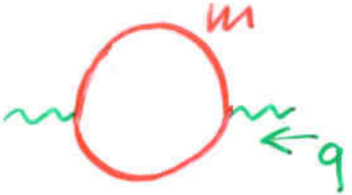
But: Caution!

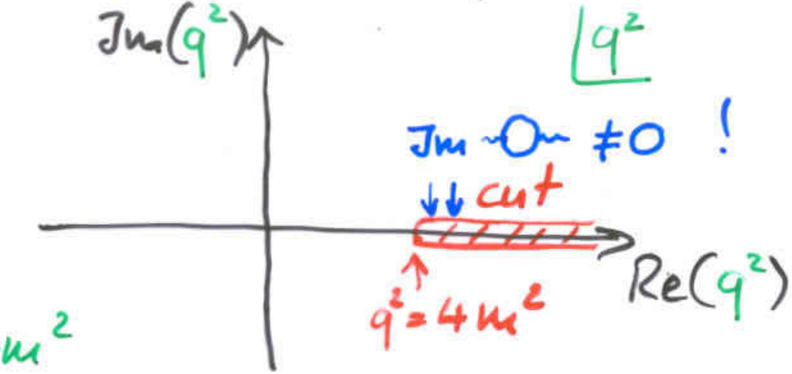
Expansions work only within radius of convergence!

For Feynman diagrams: cuts! [Branching points]

e.g.

$\Pi(q^2) =$



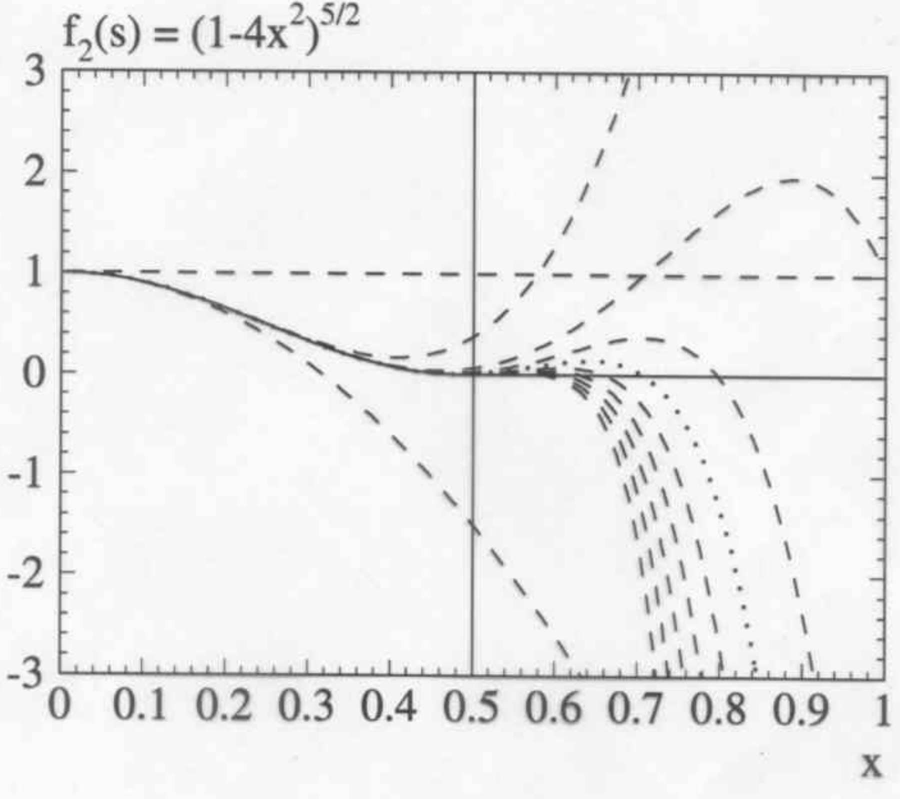


$\text{Im}\,\Pi(q^2) = 0$ for $q^2 \leq 4m^2$

$\neq 0$ for $q^2 > 4m^2$

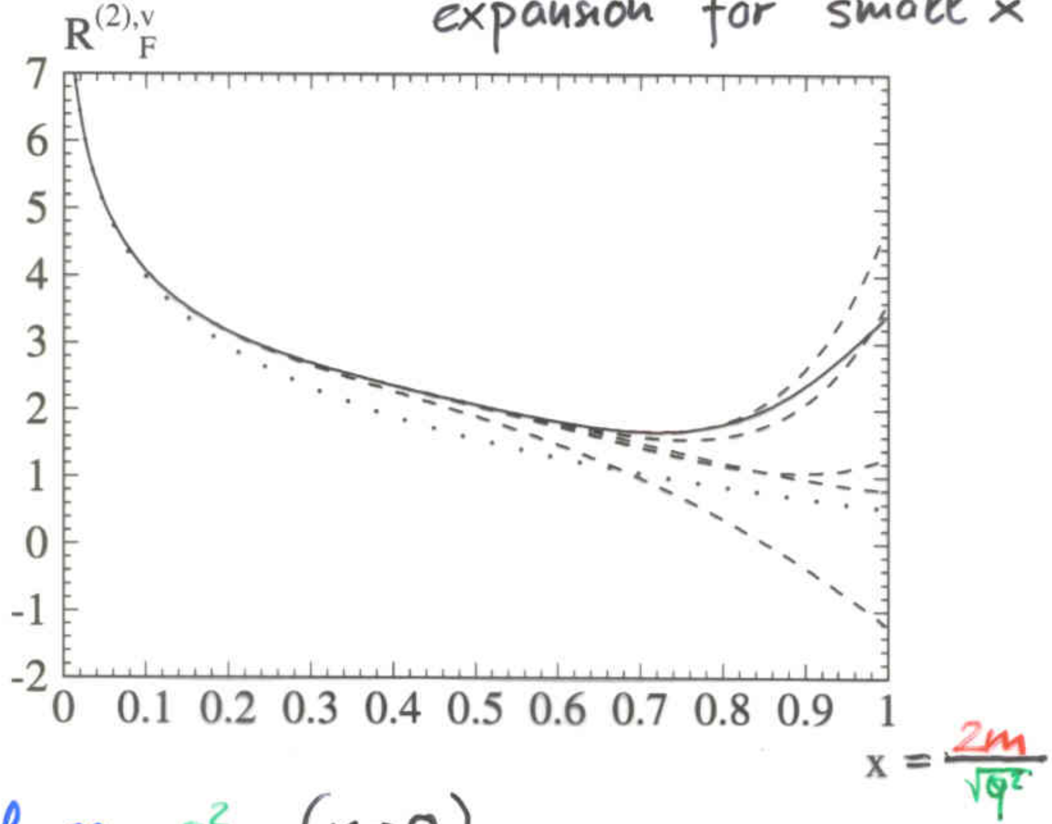
physical reason: optical theorem

Im


$f_2(s) = (1-4x^2)^{5/2}$
3
2
1
0
-1
-2
-3
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
x

$R_F \sim \mathrm{Im}$ 

expansion for small x



at large q^2 $(x \to 0)$

R_F : 

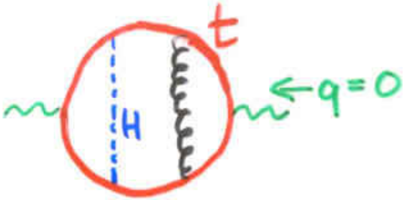
but: if $\sqrt{q^2} < 4m$ / absent!

→ branch point at $x = 1/2$!

- possible way out:

  combine exp's in various limits

e.g.



$\rightarrow \quad \delta\rho = \rho - 1$

where $\rho^{LO} = \frac{M_W^2}{M_Z^2 \cdot \sin^2\Theta} = 1$

function of $\frac{m_t^2}{M_H^2} \approx \left(\frac{175}{M_H}\right)^2$

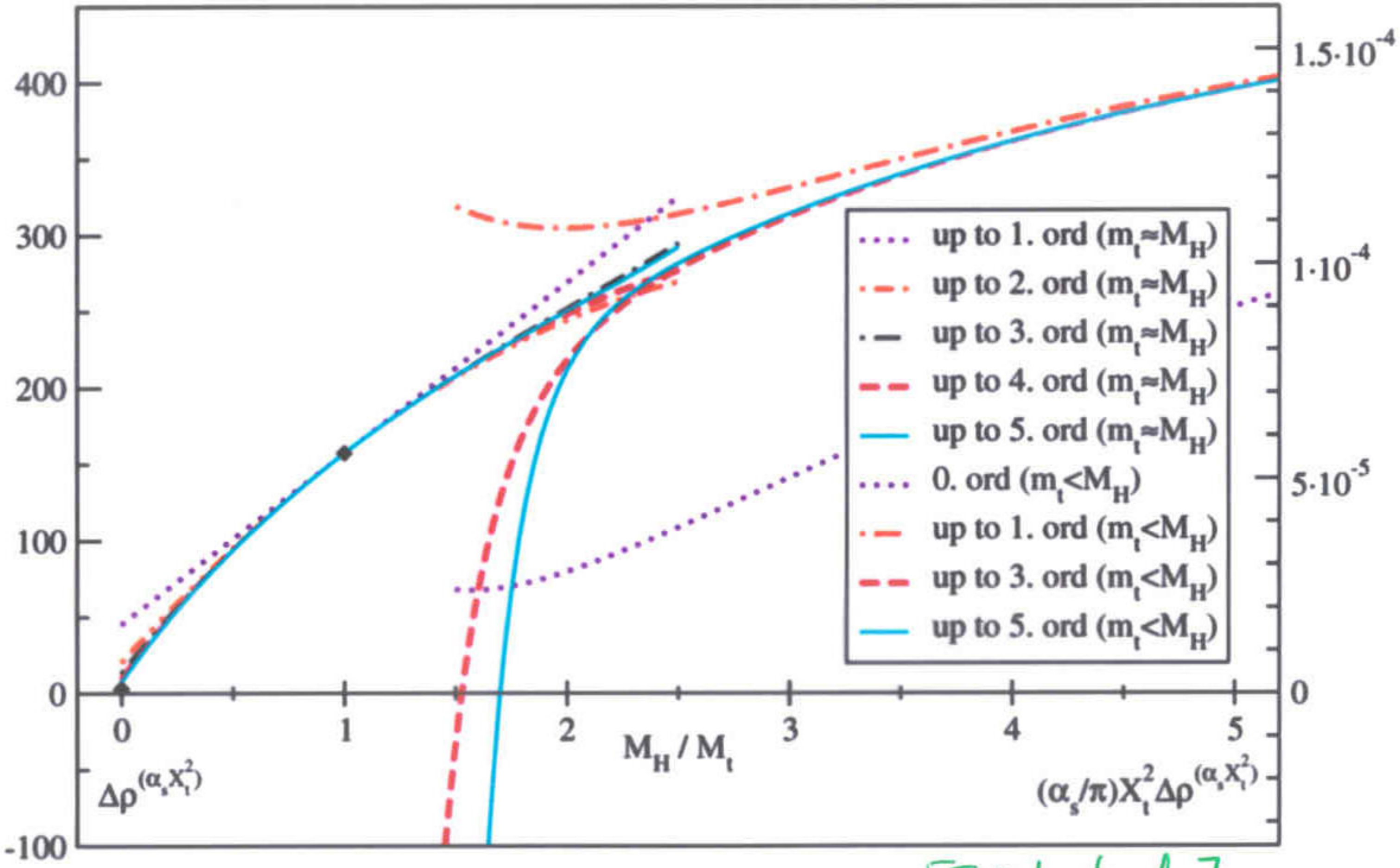
- calculation at

  $M_H = 0$

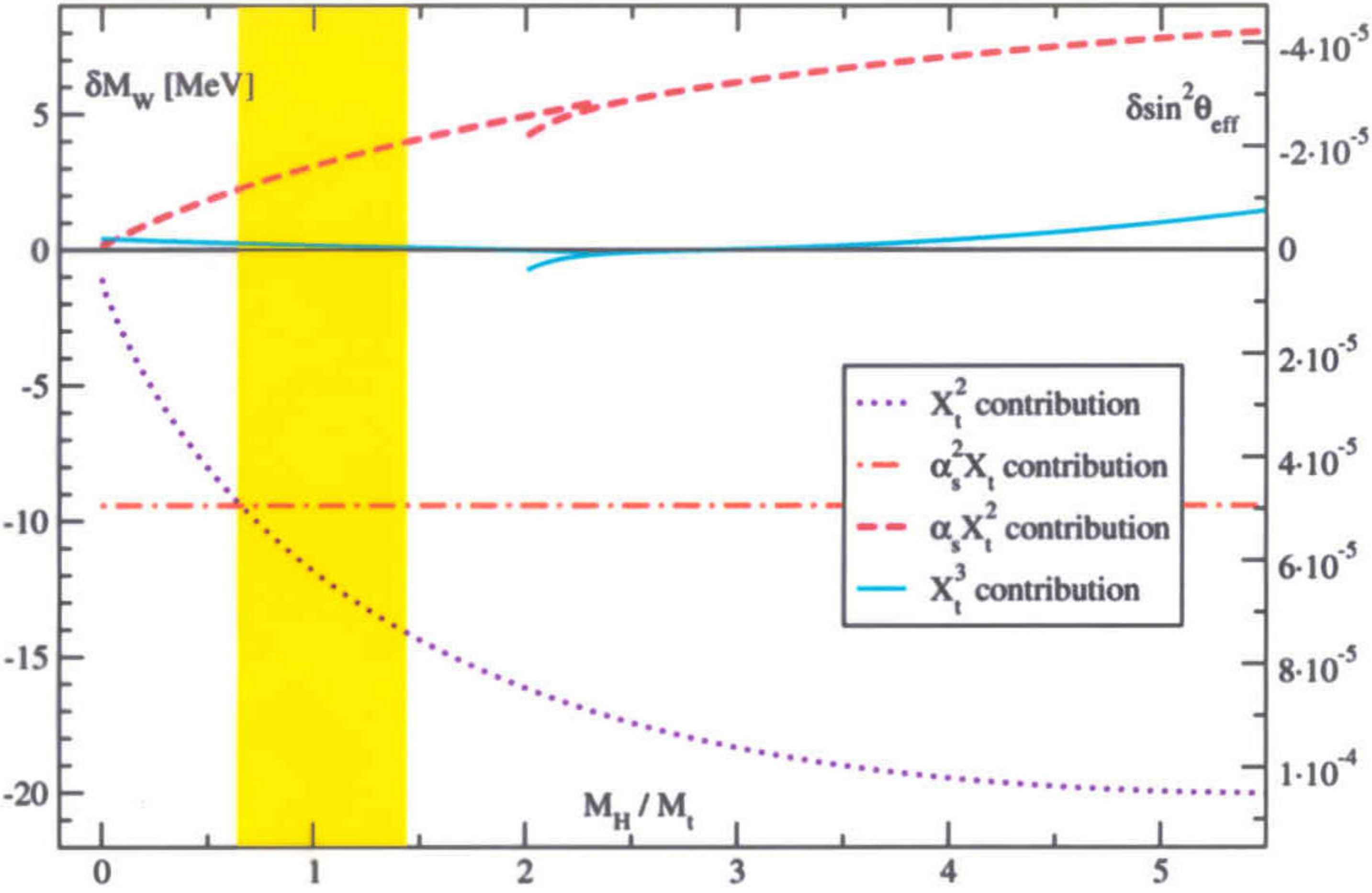
  $M_H \approx m_t$ (5 terms)

  $M_H \gg m_t$ (- " -)

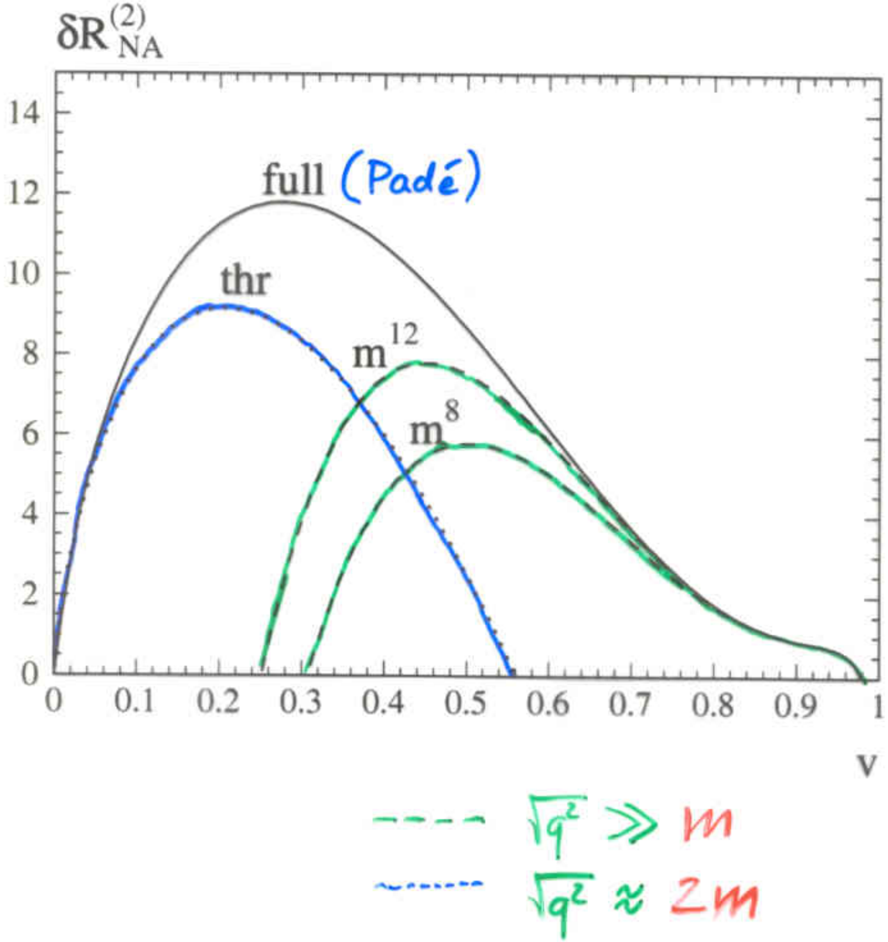
Contributions of order $\alpha_s X_t^2$ to $\Delta\rho$



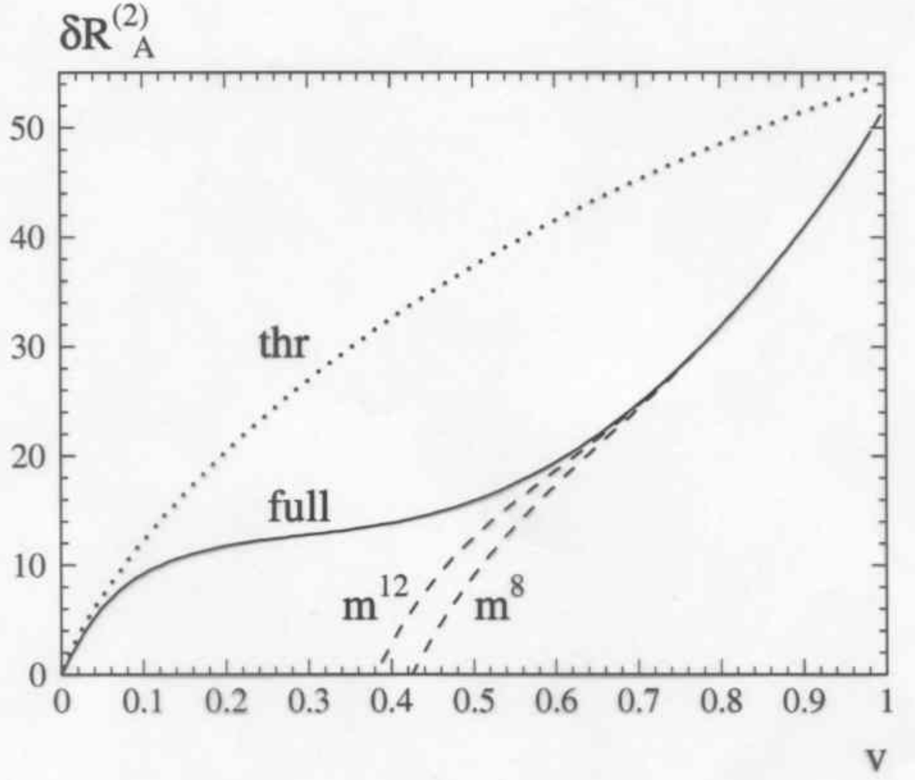
[Faisst et al.]



δM_W [MeV]
$\delta \sin^2\theta_{eff}$
5
0
-5
-10
-15
-20
$-4\cdot10^{-5}$
$-2\cdot10^{-5}$
0
$2\cdot10^{-5}$
$4\cdot10^{-5}$
$6\cdot10^{-5}$
$8\cdot10^{-5}$
$1\cdot10^{-4}$
X_t^2 contribution
$\alpha_s^2 X_t$ contribution
$\alpha_s X_t^2$ contribution
X_t^3 contribution
M_H / M_t
0
1
2
3
4
5

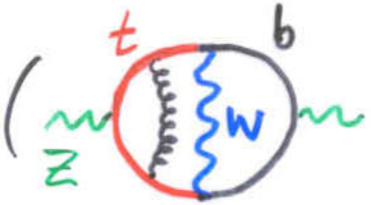


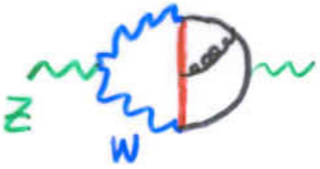
Padé : [Avdeev, Fleischer]
[Baikov, Broadhurst]
[Chetyrkin, Kühn, Steinhauser]
[RH, — "]


$\delta R^{(2)}_A$
50
40
30
20
10
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
v
thr
full
m^{12}
m^8

Iterative expansions

- for multi-scale multi-loop calculations
- e.g

Im() $\rightarrow \Gamma(Z \rightarrow b\bar{b})$ @ $\mathcal{O}(G_F^2 \alpha_s)$



scales: M_Z, m_t, M_W, $(\xi_W \cdot M_W)$

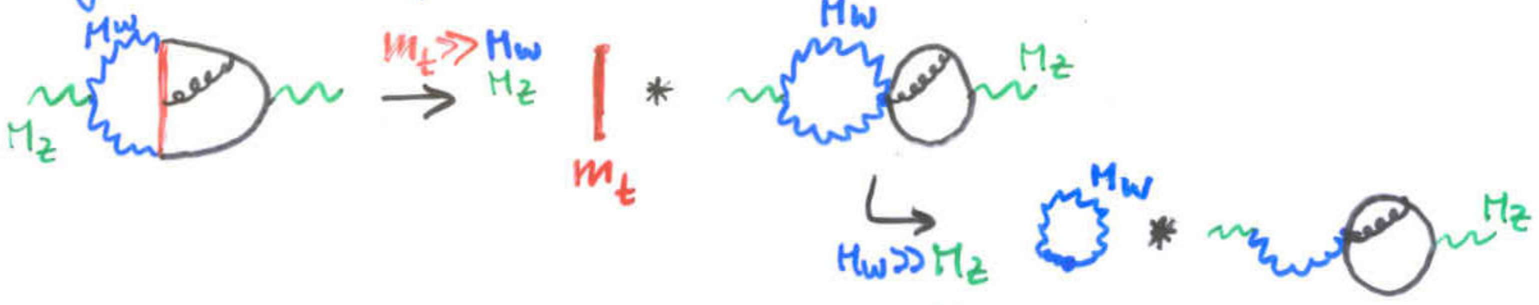
define hierarchy: how?

note: if we assume $M_Z \gg M_W$
then result contains  ↯

⇒ assume $\boxed{m_t \gg M_W \gg M_Z}$ → works, because actually:

$\frac{2m_t}{M_Z}$, $\frac{m_t+M_W}{M_Z}$, $\frac{2M_W}{M_Z}$

- *diagrammatically*



result:

$$\Gamma(Z \to b\bar{b}) = \Gamma^{LO} + \frac{\alpha_s}{\pi}\Gamma^{QCD} + \frac{\alpha_s}{\pi}\cdot G_F \cdot \Gamma^{EW+QCD} + \cdots$$

$$\begin{aligned}\Gamma^{EW+QCD} = \Gamma^{0}\cdot\Bigg[& \frac{m_t^2}{M_Z^2}\left(c_{00} + \frac{M_Z^2}{4M_W^2}\cdot c_{10} + \left(\frac{M_Z^2}{4M_W^2}\right)^2 c_{20} + \cdots \right) \\ & + \left(c_{01} + \left(M_Z^2 \right) c_{11} + \left(M_Z^2 \right) c_{21} + \right) \\ & + \left(\frac{M_Z^2}{m_t^2}\right)\left(\cdots \right) \\ & + \left(\frac{M_Z^2}{m_t^2}\right)^2\left(\cdots \right) \\ & + \cdots \Bigg]\end{aligned}$$

Automatic Setup :

QGRAF → **Q2E** → **EXP** → **MATAD/MINCER/....**

QGRAF: generation of diagrams

↑

- process (e.g. $Z \to b\bar{b}$)

Q2E: translation to MINCER-notation

EXP: asymptotic expansions

↑

- scales hierarchy $m_t > M_W > M_Z$

MATAD/MINCER/....: reduction to MASTER INTEGRALS, evaluation

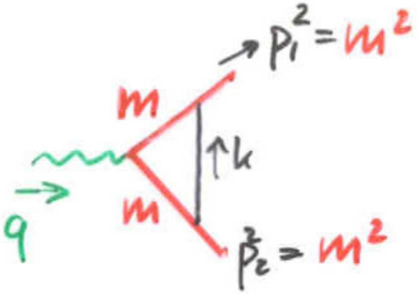
↓

RESULT

Expansions in non-Euklidean case

- more regions

example:



$q = p_1 + p_2$

$$\int d^Dk \frac{1}{k^2((k-p_1)^2-m^2)((k+p_2)^2-m^2)} = \int \frac{1}{k^2(k^2-2k\cdot p_1)(k^2+2k\cdot p_2)}$$

let $q = (Q, \vec{0})$, $p_1 = \frac{Q}{2}(1, v)$, $p_2 = \frac{Q}{2}(1, -v)$

$v = \sqrt{1-\frac{4m^2}{Q^2}}$

typical:

(a) hard: $k \sim Q$

(b) potential: $k_0 \sim v^2 Q$ $\vec{k} \sim vQ$

(c) soft: $k_0 \sim vQ$ $\vec{k} \sim vQ$

(d) ultra-soft: $k_0 \sim v^2 Q$ $\vec{k} \sim v^2 Q$

Conclusions: Asymptotic Expansions

- Purpose: reducing # scales in loop diagram
- increasing importance:
  - extended theories ?
  - Field Theory connection (e.g. SCET)
- validity often extends way beyond nominal limit (e.g. "$M_W \gg M_Z$" for $Z \to b\bar{b}$, ...)
- possible: reconstruction of full result ?

e.g.

$$\int_0^1 \frac{dx}{1-\tau x} \approx \int_0^1 dx \left\{ 1 + \tau x + \tau^2 x^2 + \tau^3 x^3 + \cdots \right\} =$$

$$= 1 + \frac{1}{2}\tau + \frac{1}{3}\tau^2 + \cdots = -\frac{\ln(1-\tau)}{\tau}$$