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Algebraic Methods for Multi-Loop Integrals

- Integration by Parts
  → Reduction to master integrals

- Asymptotic Expansion
  → systematic approximations

Integration by Parts

- Motivation / Physics $\rightarrow$ later!
- preliminaries: typical Feynman integral:

$$I(n_1, \ldots, n_N) = \int d^Dk_1 \cdots d^Dk_\ell \frac{1}{(p_1^2 - m_1^2)^{n_1} \cdots (p_N^2 - m_N^2)^{n_N}}$$

N : # of propagators
n_i : indices
ℓ : # of loops
p_i : linear combinations of
loop momenta (k_i) and
external momenta (q_i)

$D = 4 - 2\epsilon$

... a word about normalization:

* usually: $\frac{i}{(2\pi)^D}$ for each loop

  angular integration $\rightarrow$ $\pi^{D/2}$

  $\rightarrow \left[\frac{i}{(4\pi)^{D/2}}\right]^{\ell}$ in final result

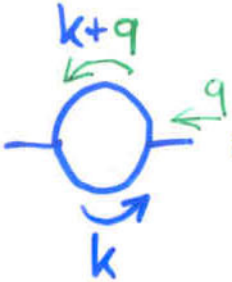
  $\rightarrow$ will be dropped

* typical $\gamma_E = 0.577\ldots$ is cancelled by multiplication with $e^{\epsilon\gamma_E}$ for each loop.

effectively: replace $\frac{1}{\epsilon} - \gamma_E + \ln 4\pi$ by $\frac{1}{\epsilon}$

and drop $\frac{1}{16\pi^2}$

Example: 1-loop self-energy-type



$$= I(n_1, n_2) = \int d^q k \, \frac{1}{(k^2 - m_1^2)^{n_1}((k+q)^2 - m_2^2)^{n_2}}$$

• special cases:

(i) $m_1 = m_2 = 0$: $(a = n_1, \, b = n_2)$

$$I_q(a,b) = (-1)^{a+b}(-q^2)^{D/2-a-b} \cdot \hat{I}(a,b)$$

$$\hat{I}(a,b) = \frac{\Gamma(a+b-D/2)\,\Gamma(D/2-b)\,\Gamma(D/2-a)}{\Gamma(a)\,\Gamma(b)\,\Gamma(D-a-b)}$$

remarks: * valid for arbitrary a, b, D (almost...)

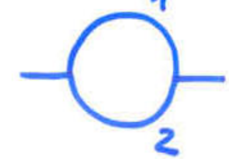
$[-a, -b \notin \mathbb{N}_0]$ ....

* expansion in ϵ through

$$\Gamma(1+\epsilon) = 1 - \epsilon\gamma_E + \frac{\epsilon^2}{2}\left(\gamma_E^2 + \frac{\pi^2}{6}\right) + \dots$$

[γ_E is cancelled by our normalization]

- special cases at 1 loop (cont'd):



(ii) $n_2 = 0$ [equivalently: $q=0$, $m_1 = m_2$]

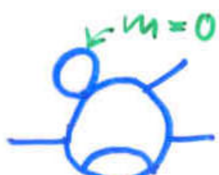
$$= A(a) = \int d^D k \, \frac{1}{(k^2 - m^2)^a}$$

$$= (-1)^a (m^2)^{D/2 - a} \, \frac{\Gamma(a - D/2)}{\Gamma(a)}$$

note: massless tadpoles $= 0$

(scale-less integrals)

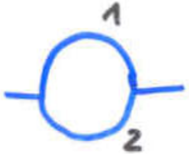
e.g.



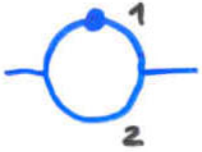
$= 0$

more notation

- "dots on lines":

e.g. if 

$$= \int d^D k \, \frac{1}{(p_1^2 - m_1^2)^{n_1} (p_2^2 - m_2^2)^{n_2}} = I(n_1, n_2)$$

then 

$$= \int d^D k \, \frac{1}{(p_1^2 - m_1^2)^{n_1+1} (p_2^2 - m_2^2)^{n_2}} = I(n_1 + 1, n_2)$$

useful notation:

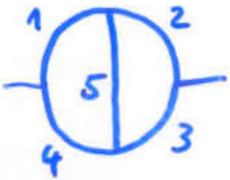
$$1^{\pm} I(n_1, n_2) = I(n_1 \pm 1, n_2)$$

$$2^{\pm} I(n_1, n_2) = I(n_1, n_2 \pm 1)$$

more notation

- "shrinking a line to a point (pinching)":

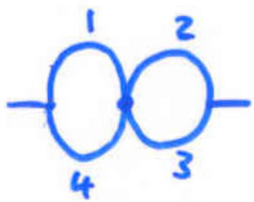
e.g.



"line 2 shrinks to a point" $:\Leftrightarrow$ $n_2 = 0$

graphically:

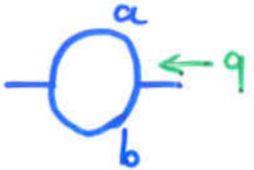
line 5 shrinks ...



how do indices change?

e.g. : derivatives

exercise:

$q_\mu \frac{\partial}{\partial q_\mu}$ 

$$= -a\left[1 - 1^+2^- + (m_1^2 - m_2^2 + q^2)\,1^+\right] I(a,b)$$

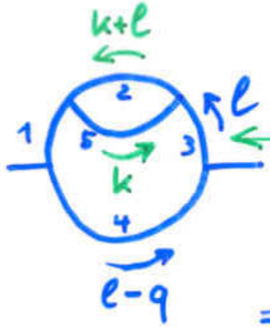
$$\left(= I(a,b) = \int d^Dk \, \frac{1}{[(k+q)^2 - m_1^2]^a \, [k^2 - m_2^2]^b} \right)$$

$$= -a\Big[I(a,b) - I(a+1, b-1)$$
$$+ (m_1^2 - m_2^2 + q^2)\, I(a+1, b) \Big]$$

Products and Convolutions of 1-loop

- 

  → separate integrations ✓

- 

  ← q , $m_i = 0 \ \forall i$:

$$= \int d^D\ell \, \frac{1}{(\ell^2)^{n_1+n_3}\left[(\ell-q)^2\right]^{n_4}} \underbrace{\int d^D k \, \frac{1}{(k^2)^{n_5}\left[(k+\ell)^2\right]^{n_2}}}_{\sim (\ell^2)^{D/2 - n_5 - n_2} \, \hat{I}(n_2, n_5)}$$

$a := n_1 + n_3 + n_2 + n_5 - D/2$

$$\sim \int d^D\ell \, \frac{1}{(\ell^2)^{a}\left[(\ell-q)^2\right]^{n_4}} \cdot \hat{I}(n_2, n_5)$$

$$\sim (q^2)^{D - \sum n_i} \, \hat{I}(n_1+n_3+n_2+n_5 - D/2, \, n_4) \cdot \hat{I}(n_2, n_5)$$

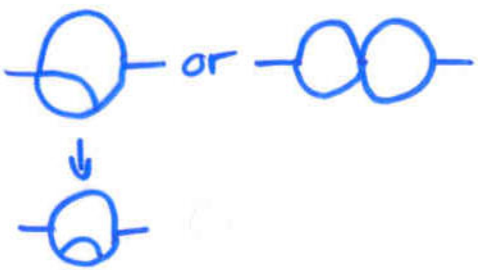
analoguously: ($m_i = 0$)

convolutions of 1-loop self-energy diagrams.

- <u>however</u>:

genuine 2-loop:  "T_1" - topology

remark: shrinking any line $\Rightarrow$ convolution of 1-loop!



Integration-by-Parts identities

[Chetyrkin, Tkachov '81]

observation: (in dimensional regularization)

- integrals are finite
- integration region is infinite

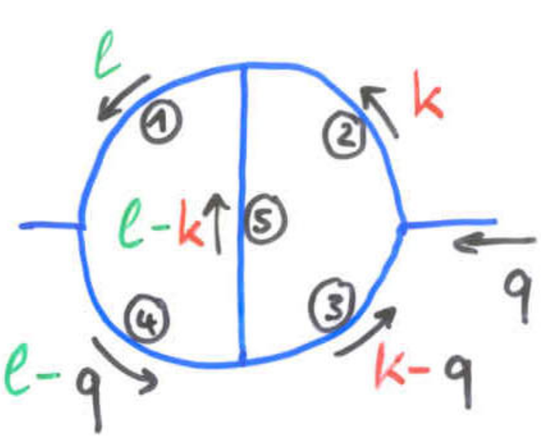
$$\Rightarrow \int d^D k \, \frac{\partial}{\partial k_\mu} \cdot p_\mu \, f(k, p, \cdots) = 0$$

"Surface terms at ∞"

- if $p = k + q_1 + \cdots$ (q_i external)

$$\Rightarrow \quad D \cdot \int d^D k \, f(k, p, \cdots) = \int d^D k \, p_\mu \cdot \frac{\partial}{\partial k_\mu} f(k, p, \cdots)$$

Topology T_1 (massless)



$$= \int d^D\ell \int d^Dk \, \frac{1}{[\ell^2]^{n_1}[k^2]^{n_2}[(k-q)^2]^{n_3}[(\ell-q)^2]^{n_4}[(\ell-k)^2]^{n_5}}$$

act on integrand with

$$\frac{\partial}{\partial \ell_\mu}\cdot \ell_\mu \;,\quad \frac{\partial}{\partial \ell_\mu} k_\mu \;,\quad \frac{\partial}{\partial \ell_\mu} q_\mu$$

$$\frac{\partial}{\partial k_\mu} \ell_\mu \;,\quad \frac{\partial}{\partial k_\mu} k_\mu \;,\quad \frac{\partial}{\partial k_\mu} q_\mu$$

example: $\frac{\partial}{\partial \ell_\mu}\cdot \ell_\mu = D + \ell_\mu \frac{\partial}{\partial \ell_\mu}$

$$\ell_\mu \frac{\partial}{\partial \ell_\mu} \frac{1}{[\ell^2]^{n_1}} = -2n_1 \frac{1}{[\ell^2]^{n_1}}$$

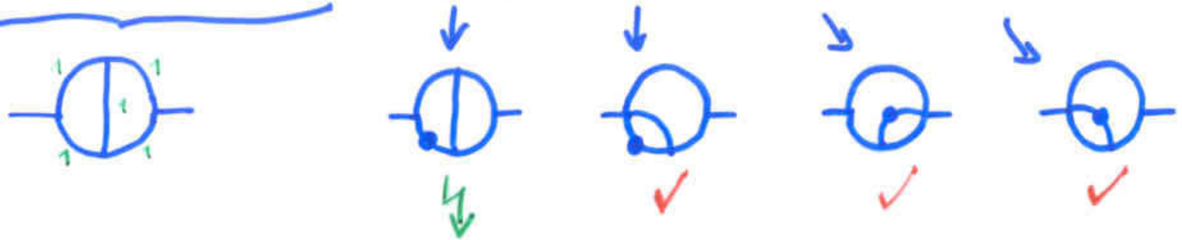
$$\ell_\mu \frac{\partial}{\partial \ell_\mu} \frac{1}{[(\ell-q)^2]^{n_4}} = -2n_4 \underbrace{\frac{1}{[(\ell-q)^2]^{n_4+1}}}_{4^+} \Big\{ \underbrace{\ell^2}_{1^-} - \ell\cdot q \Big\} =$$

$$\ell\cdot q = \frac{1}{2}\Big[\underbrace{(\ell-q)^2}_{4^-} - \underbrace{\ell^2}_{1^-} - q^2\Big]$$

$$\Rightarrow \Big[D - 2n_1 - n_4 - n_5 + n_4 (q^2 - 1^-)\, 4^+ + n_5 (2^- - 1^-)\, 5^+ \Big] T_1 = 0$$

IBP identities for T_1

$$\ell_\mu \frac{\partial}{\partial \ell_\mu}: \quad \left[D - 2n_1 - n_4 - n_5 + n_4\left(q^2 - 1^-\right)4^+ + n_5\left(2^- - 1^-\right)5^+\right] T_1(n_1, \dots, n_5) = 0$$



Q: Combine <u>all</u> IBP identities such that

$$T(n_1, n_2, n_3, n_4, n_5) \rightarrow$$ "simpler" integrals,

i.e.
- conv. of 1-loop
- low values of n_i

IBP for T_1

$$D - 2\,n_1 + n_4\,(-1 + (q^2 - \mathbf{1}^-)\,\mathbf{4}^+) + n_5\,(-1 + (-\mathbf{1}^- + \mathbf{2}^-)\,\mathbf{5}^+) = 0 \qquad (1)$$

$$n_1\,(-1 + (-q^2 + \mathbf{4}^-)\,\mathbf{1}^+) + n_4\,(1 + (q^2 - \mathbf{1}^-)\,\mathbf{4}^+) + (-\mathbf{1}^- + \mathbf{2}^- - \mathbf{3}^- + \mathbf{4}^-)\,n_5\,\mathbf{5}^+ = 0 \qquad (2)$$

$$n_1\,(-1 + (-\mathbf{2}^- + \mathbf{5}^-)\,\mathbf{1}^+) + (q^2 - \mathbf{1}^- - \mathbf{3}^- + \mathbf{5}^-)\,n_4\,\mathbf{4}^+ + n_5\,(1 + (-\mathbf{1}^- + \mathbf{2}^-)\,\mathbf{5}^+) = 0 \qquad (3)$$

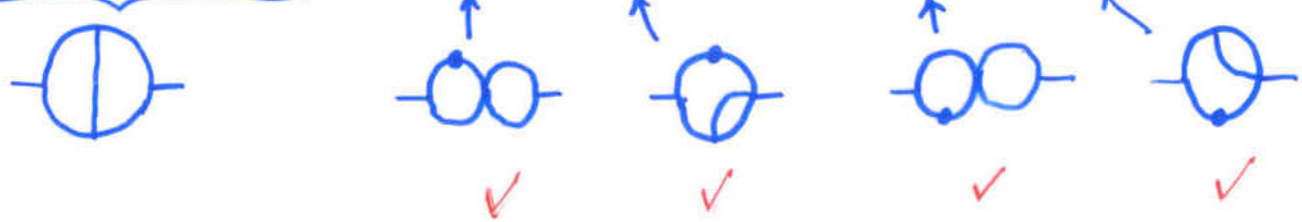
$$n_2\,(-1 + (-\mathbf{1}^- + \mathbf{5}^-)\,\mathbf{2}^+) + (q^2 - \mathbf{2}^- - \mathbf{4}^- + \mathbf{5}^-)\,n_3\,\mathbf{3}^+ + n_5\,(1 + (\mathbf{1}^- - \mathbf{2}^-)\,\mathbf{5}^+) = 0 \qquad (4)$$

$$n_2\,(-1 + (-q^2 + \mathbf{3}^-)\,\mathbf{2}^+) + n_3\,(1 + (q^2 - \mathbf{2}^-)\,\mathbf{3}^+) + (\mathbf{1}^- - \mathbf{2}^- + \mathbf{3}^- - \mathbf{4}^-)\,n_5\,\mathbf{5}^+ = 0 \qquad (5)$$

$$D - 2\,n_2 + n_3\,(-1 + (q^2 - \mathbf{2}^-)\,\mathbf{3}^+) + n_5\,(-1 + (\mathbf{1}^- - \mathbf{2}^-)\,\mathbf{5}^+) = 0 \qquad (6)$$

(1) − (3):

$$\underbrace{D - 2n_5 - n_1 - n_4} - n_1(5^- - 2^-)\,1^+ - n_4(5^- - 3^-)\,4^+ = 0$$



Triangle rule (2)

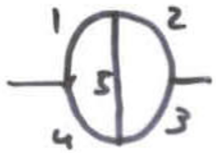
$$T_1(n_1, n_2, n_3, n_4, n_5) =$$

$$= \frac{1}{D - 2n_5 - n_1 - n_4}\left[n_1 (5^- - 2^-) 1^+ + n_4 (5^- - 3^-) 4^+ \right] T_1(n_1, n_2 \ldots, n_5)$$

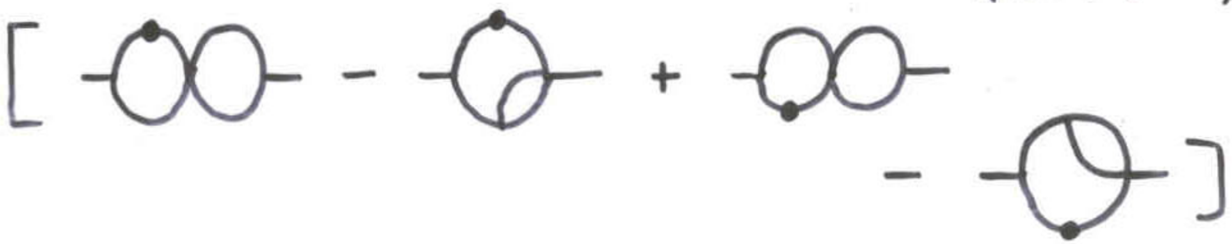
recurrence relation

example: $n_1 = \cdots = n_5 = 1$

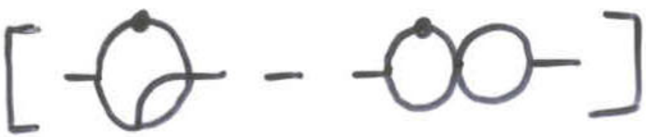
$$\Rightarrow T_1(1,1,1,1,1) = \frac{1}{D-4}\left[T_1(2,1,1,1,0) - T_1(2,0,1,1,1) + T_1(1,1,1,2,0) + T_1(1,1,0,2,1) \right]$$



$= -\frac{1}{2\epsilon}$



$= \frac{1}{\epsilon}$



$= 6\zeta(3) + \mathcal{O}(\epsilon)$

Triangle rule (3)

indices > 1 $\longrightarrow$ apply rec. rel. repeatedly

$T_1(1,3,1,1,1) \to$ ✓ $T_1(2,3,1,1,\underline{0})$ 

$T_1(2,2,1,1,1) \longrightarrow T_1(3,2,1,1,\underline{0})$ ✓

✓ $T_1(1,3,1,2,\underline{0})$ 

$T_1(3,1,1,1,1) \longrightarrow$ ✓ $T_1(4,1,1,1,\underline{0})$

✓ $T_1(1,3,\underline{0},2,1)$ 

$T_1(2,2,1,2,\underline{0})$ ✓

✓ $T_1(4,\underline{0},1,1,1)$

$T_1(2,2,\underline{0},2,1)$ ✓

✓ $T_1(3,1,1,2,\underline{0})$

✓ $T_1(3,1,\underline{0},2,1)$

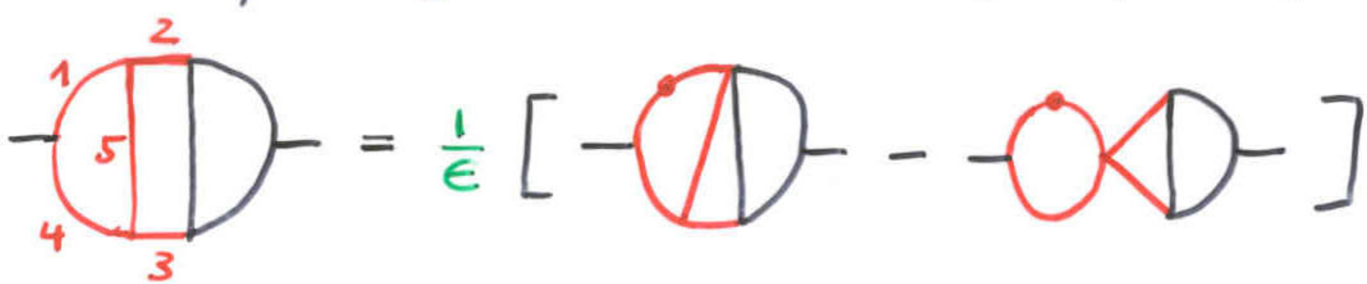
$\longrightarrow$ generates many terms

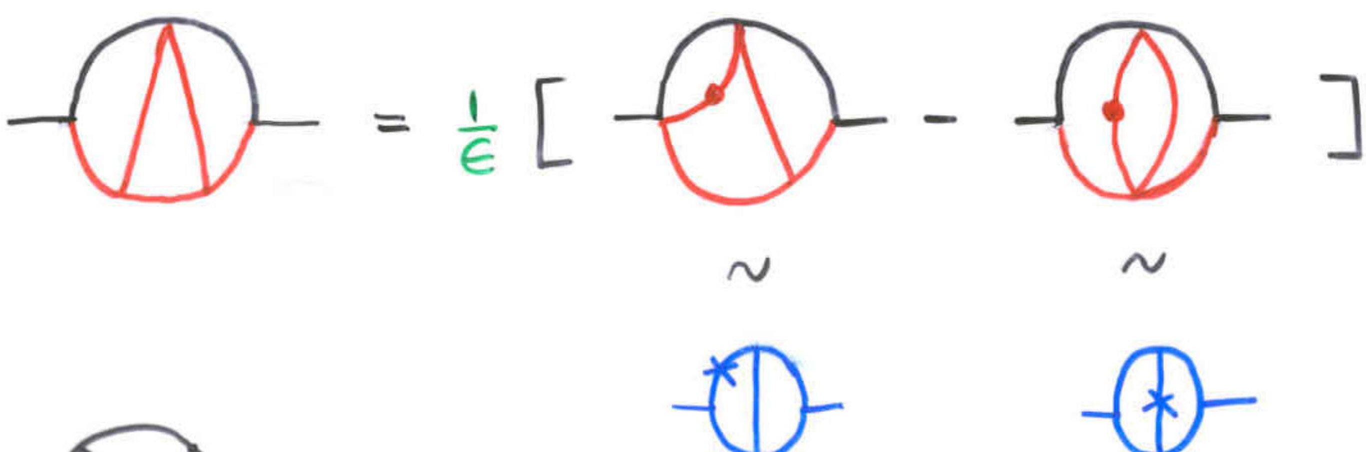
$\longrightarrow$ computer algebra

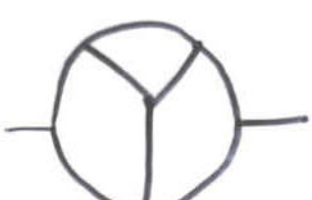
Triangle rule (4)

observation: this rec. relation applies to any triangular subloop !

e.g. 3-loop: $\left[D - 2n_5 - n_1 - n_4 = n_1 (5^- - 2^-) 1^+ + n_4 (5^- - 3^-) 4^+ \right]$





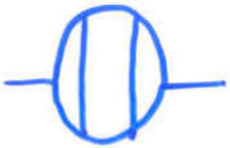


→ exercise !

3-loop self-energies (massless)

- just explained:

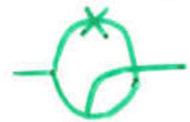
—×— $\hat{=}$ non-integer power




triangle rule $\Longrightarrow$


topology LA (ladder) BE (Mercedes-Benz)

Δ-rule ↓

 , 

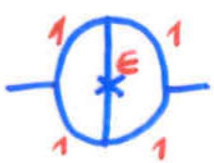
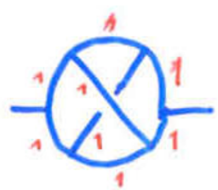
can not be reduced by Δ-rule!

-  → no triangular subloops!

NO (non-planar)

in fact: $NO(1,1,1,1,1,1,1,1) = 20\,\zeta(5) + \mathcal{O}(\epsilon)$

can not be reduced to simpler topology

 and  are <u>MASTER INTEGRALS</u>

The crucial idea

Any integral

$$I(n_1, \ldots, n_N) = \int d^Dk_1 \cdots d^Dk_\ell \, \frac{1}{(p_1^2 - m_1^2)^{n_1} \cdots\cdots (p_N^2 - m_N^2)^{n_N}}$$

can be written as

$$\boxed{I(n_1, \ldots, n_N) = c_1(D)\, I_1 + \cdots + c_M(D)\, I_M}$$

$I_1, \ldots, I_M$: finite set of master integrals (usually $n_i = 0$ or 1)

$c_i(D)$: rational functions of $D = 4 - 2\epsilon$

- universal problem: identify and evaluate $I_1, \ldots, I_M$
- specific problem: evaluate $c_i(D)$

Approaches

- **MINCER approach**
  * write down IBP identities
  * combine them to achieve reduction

→ successful for massless 3-loop self-energies

[Chetyrkin, Tkachov] [Larin, Tkachov, Vermaseren] → MINCER

massive 3-loop tadpoles

[Broadhurst] [Chetyrkin, Kühn, Steinhauser]

↳ MATAD

3-loop HQET

↳ GRINDER

[Grozin]

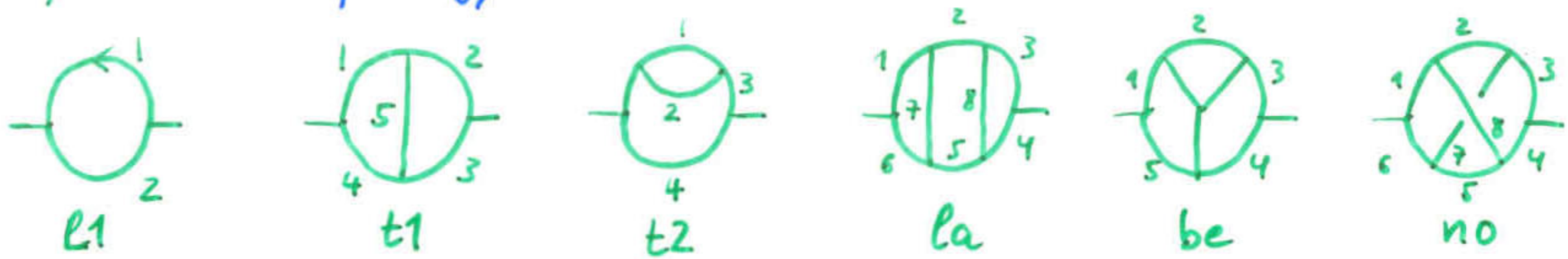
3-loop on-shell diagrams

 . . .

[Melnikov, v. Ritbergen]

About MINCER

- FORM package
- Input: * topology name:



* values of indices:

numerator allowed!

$$L \quad int = \frac{1}{(p_1\cdot p_1)^2 (p_2\cdot p_2)^1 (p_3\cdot p_3)^5 (p_4\cdot p_4)^1 (p_5\cdot p_5)^1 (p_6\cdot p_6)^2 (p_7\cdot p_7)^2 (p_8\cdot p_8)^1}$$

- MINCER then reduces this to master integrals and inserts their values
- The result: Laurent series in $\epsilon = \frac{4-D}{2}$

Applications of MINCER

- RG functions @ 4 loops [Chetyrkin, ...]
  [only pole needed → 3-loop suficient]
- moments of structure functions for DIS @NNLO
  [Larin, Vermaseren, v. Ritbergen, Rétey]
- more applications
  → tomorrow's talk on asymptotic expansions

Implementation of Laporta algorithm

- several private codes:
  [Gehrmann, Glover, ..., Czakon, Bern, ..., Chetyrkin, Sturm, ...]
- public code: AIR [Anastasiou, Lazopoulos]

  * MAPLE package
  * Input: • IBP identities
    • most complex integral to be evaluated
  * Generates reduction to most simple integrals for each integral [↳ master integrals]

- **Laporta approach**

- idea: for specific integral, e.g. $T_1(2,5,1,3,3)$,
  generate system of linear equations

e.g. $T_1(1,1,1,2,1)$: 

generated from simpler case through IBP, e.g.

IBP(1) on $T_1(1,1,1,1,1)$:

$$[D-4]\,T_1(1,1,1,1,1) + q^2\,T_1(1,1,1,2,1) - T_1(0,1,1,2,1) - T_1(0,1,1,1,2) - T_1(1,0,1,1,2) = 0$$

$$\Rightarrow \quad T_1(1,1,1,2,1) = \frac{1}{q^2}\left[-(D-4)\,T_1(1,1,1,1,1) + \ldots\right]$$

then, look for IBPs that generate $T_1(1,1,1,1,1)$
from simpler integrals
→ not the case!

but: IBP(3) on $T_1(1,1,1,1,1)$:

$$-T_1(2,0,1,1,1) + T_1(2,1,1,1,0) + q^2 T_1(1,1,1,2,1)$$
$$- T_1(0,1,1,2,1) - T_1(1,1,0,2,1) + T_1(1,0,1,1,2)$$
$$- T_1(0,1,1,1,2) = 0$$

$\Rightarrow$ $T_1(1,1,1,2,1)$ expressed through simpler integrals!

and: insertion into IBP(1) on $T_1(1,1,1,1,1)$

$\Rightarrow$ also $T_1(1,1,1,1,1)$ reduced!

Key issues of Laporta approach

- need complete system of identities

  does it exist?

  It should: $I(n_1, \ldots, n_N) = C_1 I_1 + \cdots + C_M I_M$

- in addition to IBP: Lorentz-Invariance

  $$I(q_i, \ldots) = I(q_i + \partial q_i, \ldots) \qquad q_i: \text{external momenta}$$

  $\rightarrow$ relations among $q_i \cdot \frac{\partial}{\partial q_j} I(q_i, \ldots)$

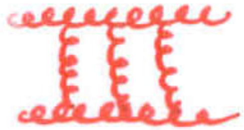
- need an ordering of integrals: complex ..... simple

  $I(10,8,4,9,2) \cdots\cdots I(1,0,1,1,1)$

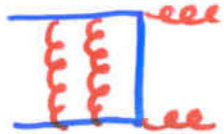
Applications of Laporta approach

- "double-box problems":

  * $gg \to gg$:

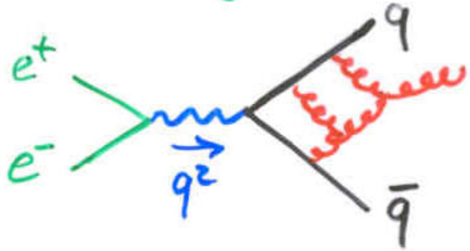
  

  $q\bar{q} \to gg$

  $\to$ part of $pp \to 2\,\text{jets}$ @ NNLO (2 scales: s, t)

  * $e^+e^- \to 3$ jets @ NNLO

  (3 scales: s, t, q^2)

  * ....

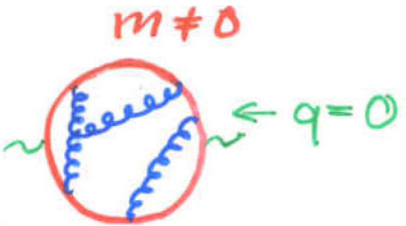
Huge activity!

[Remiddi, Gehrmann, Glover, Anastasiou, Oleari, Bern, Dixon, Kosower, Czakon, Gluza, ...]

master integrals: [Smirnov, Tausk, ...]

More applications

- "4-loop bubble problems"

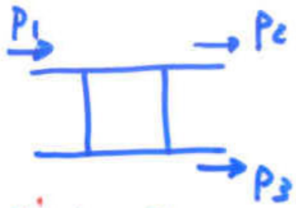


  * QCD β-function to 4-loop
    [v. Ritbergen, Larin, Vermaseren '97] ← "MINCER" approach!
    [Czakon '05]

  * moments of $\Pi(q^2)$
    ↳ derive bottom quark mass
    [Chetyrkin, Kühn, Sturm]
    [Czakon], [Schröder, Steinhauser]

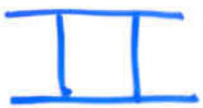
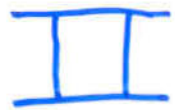
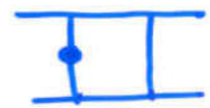
Differential Equations for Master Integrals

[Kotikov] [Remiddi et al] [Gehrmann, Remiddi]

example : 

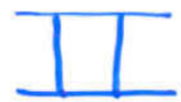
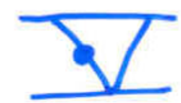
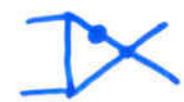
is a master integral

Lorentz-invariance:

$p_i^\mu \frac{\partial}{\partial p_i}$  $\longrightarrow$  , 

$=: I$

$\downarrow$ IBP

$\downarrow$ IBP

$=: C$

$$\Rightarrow \quad p_i \cdot \frac{\partial}{\partial p_i} I = I + C$$

Inhom. diff. eq.
for I !

Baikov approach

Starting point:

$$I(n_1,\dots,n_N) = c_1(D,n_1,\dots,n_N)\,I_1 + \dots + c_M(D,n_1,\dots,n_N)\,I_M$$

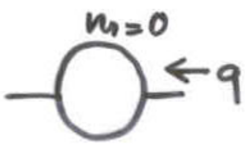
$I_1,\dots,I_M$: master integrals

$$c_i(D,n_1,\dots,n_N) = \int dx_1 \cdots \int dx_N \,\frac{[P(x)]^{(D-\ell-1)}}{x_1^{n_1}\cdots x^{n_N}}$$

[Baikov '96]

ℓ : # loops

$P(x)$: well-defined polynomial

e.g.: 

$\rightarrow\ P(x) = (q^2)^2 - 2q^2(x_1+x_2) + (x_1-x_2)^2$

$\int dx_i$: either contour
or over real domains

Applications of Baikov approach

- $e^+e^- \to$ hadrons @ $\mathcal{O}(\alpha_s^4)$

  requires evaluation of

[Baikov, Chetyrkin, Kühn]

- towards "algorithmization":

  [Smirnov, Steinhauser]

Gröbner basis approach

[Tarasov]

- idea:
  # of IBP identities large, but
  proper combination difficult
  → find minimal set of relations

- mapping of IBP identities onto diff. eq.'s
  → coupled set of DEs

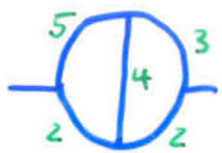
  → MAPLE, MATHEMATICA
  can find Gröbner basis.

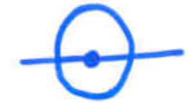
in progress (but great progress!)

Summary of Integration by Parts

- guiding idea:

$$I(n_1, \dots, n_N) = C_1(D, n_i)\, I_1 + \cdots + C(D, n_i)\, I_M$$

e.g. 

$=$ $\cdots$  $+$ $\cdots$ 

$I_1 = I(1,1,1,1,0)$ $\qquad$ $I_2 = I(0,1,0,1,2)$

- main problem: evaluate C_i

- various approaches:
  - * recursively (IBP + LI identities)
    - MINCER approach
    - Laporta approach
  - * directly: Baikov approach