Higher Order Massive Operator Matrix Elements and Heavy Quark Corrections to Deep-Inelastic Scattering

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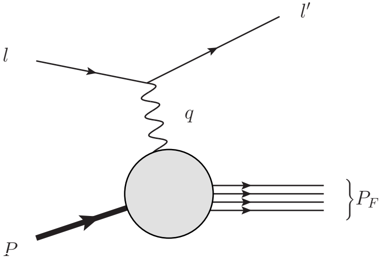
Deep-Inelastic Scattering



Define Virtuality Q^2 and variables x, y

$$Q^2 = -q^2\,, \quad x = \frac{Q^2}{2P\cdot q}\,, \quad y = \frac{P\cdot q}{P\cdot l}\,.$$

The inclusive differential cross section can be written in terms of a leptonic tensor and a hadronic tensor

$$l_0' \frac{d\sigma}{d^3 l'} = \frac{1}{4P\cdot l}\frac{\alpha^2}{Q^4} L_{\mu\nu} W^{\mu\nu}$$

The hadronic tensor reads

$$W_{\mu\nu} = \frac{1}{2x}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) F_L(x,Q^2) + \frac{2x}{Q^2}\left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu}\right) F_2(x,Q^2)\,.$$

The differential cross section in terms of structure functions reads

$$\frac{d\sigma}{dxdy} = \frac{2\pi\alpha^2}{xyQ^2}\left\{\left[1+(1-y)^2\right] F_2(x,Q^2) - y^2 F_L(x,Q^2)\right\}\,.$$

According to the factorization theorem the structure functions decompose into perturbative Wilson-coefficients and non-perturbative parton distribution functions (PDFs)

$$F_i(x, Q^2) = \sum_j C_i^j(x, Q^2) \otimes f_j(x, \mu^2), \quad i = 2, L.$$

The Mellin convolution $\otimes$ between two functions is defined as

$$[f \otimes g](x) = \int_0^1 dy \int_0^1 dz f(y) g(z) \delta(x - yz).$$

Introducing the Mellin transform of a function

$$\hat{f}(N) = \int_0^1 dx x^{N-1} f(x),$$

the Mellin convolution decomposes into in an ordinary product

$$\mathbf{M}[f \otimes g](N) = \mathbf{M}[f](N)\mathbf{M}[g](N).$$

In Mellin space the structure functions read

$$F_i(N, Q^2) = \sum_j C_i^j(N, Q^2) f_j(N, \mu^2) \,.$$

The Wilson coefficients can be written as sum of light and heavy flavour parts

$$C_i^j(N, Q^2, m^2) = \hat{C}_i^j(N, Q^2) + H_i^j(N, Q^2, m^2) \,.$$

The heavy flavour part decomposes into light flavour Wilson coefficients and massive operator matrix elements (OMEs)

$$H_i^j(N, Q^2, m^2) = \sum_i \hat{C}_i^j(N, Q^2) A_{ij}(N, m^2) \,.$$

Regularization and Renormalization

Loop integrals lead to divergencies in $D = 4$ dimensions $\rightarrow$ One needs to regularize the integrals
The prototype of the occuring momentum integrals is

$$\int \frac{d^D k}{(2\pi)^D} \frac{(k^2)^r}{(k^2+R^2)^m} = \frac{\Gamma(r+D/2)\Gamma(m-r-D/2)}{(4\pi)^{D/2}\Gamma(D/2)\Gamma(m)(R^2)^{m-r-D/2}} \,.$$

The Γ-function is an analytic function with a certain pole structure. We set $D = 4+\epsilon$ and expand in a Laurent-series. The analytic continuation of the dimension leads to the introduction of a scale via the substitution

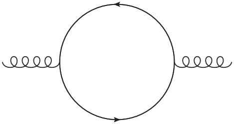
$$g \rightarrow g(\mu^2)^{-\epsilon/2} \,.$$

For convenience one introduces after the substitution $a_s = \frac{g^2}{(4\pi)^2}$. Poles in ε always occur in a combination with one factor per loop

$$S_\varepsilon = \exp\left\{\frac{1}{2}\varepsilon(\gamma_E - \ln(4\pi))\right\} \,.$$

In the $\overline{MS}$-regularization scheme the poles are subtracted in the form S_ε, and setting S_ε to unity after the calculation $\rightarrow$ Simplified result without any γ_E.

Vacuum Polarization



The analytic expression of the amputated vaccum polarization diagram is

$$\Pi^{\mu\nu}_{ab}(p) = -\,\mathrm{Tr}\int \frac{d^D k}{(2\pi)^D}\delta_{mj}\mathrm{i}\frac{(\not{k}+\not{p})+m}{(k+p)^2-m^2}\mathrm{i}g\gamma^\mu t^a_{jl}\delta_{ln}\mathrm{i}\frac{\not{k}+m}{k^2-m^2}\mathrm{i}g\gamma^\nu t^b_{nm}$$

$$= -\,4T_f g^2\delta_{ab}\int\frac{d^D k}{(2\pi)^D}\frac{(k^\mu+p^\mu)k^\nu+(k^\nu+p^\nu)k^\mu-(k+p)\cdot k g^{\mu\nu}+m^2 g^{\mu\nu}}{((k+p)^2-m^2)(k^2-m^2)}$$

Steps to calculate the integral:
1. Perfom Wick-rotation

$$k^0 = \mathrm{i}k^0_E\,,\quad k^2 = -k^2_E\,,\quad \int d^D k = \mathrm{i}\int d^D k_E\,,$$

2. Feynman-parametrization to combine denominators

$$\frac{1}{A_1^{j_1}\cdots A_n^{j_n}} = \frac{\Gamma(\sum_{i=1}^n j_i)}{\Gamma(j_1)\cdots\Gamma(j_n)} \int_0^1 \mathrm{d}x_1 \cdots \mathrm{d}x_n \frac{x_1^{j_1-1}\cdots x_n^{j_n-1}}{(x_1 A_1 + \cdots + x_n A_n)^{\sum_{i=1}^n j_i}} \delta\left(\sum_{i=1}^n x_i - 1\right),$$

$$\frac{1}{A^\alpha B^\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \mathrm{d}x \frac{x^{\alpha-1}(1-x)^{\beta-1}}{(xA+(1-x)B)^{\alpha+\beta}},$$

3. Use symmetric integration

$$\int \frac{d^D k}{(2\pi)^D} k^\mu f(k^2) = 0,$$

$$\int \frac{d^D k}{(2\pi)^D} k^\mu k^\nu f(k^2) = \frac{g^{\mu\nu}}{D} \int \frac{d^D k}{(2\pi)^D} k^2 f(k^2),$$

4. D-dimensional momentum integration .

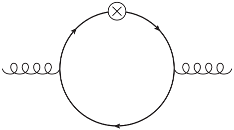
Result of this procedure

$$\Pi^{\mu\nu}_{ab}(p) = 8T_f g^2 \delta_{ab} i \frac{\Gamma(2-D/2)}{(4\pi)^{D/2}} (g^{\mu\nu}p^2 - p^\mu p^\nu) \int_0^1 dx x(1-x)[m^2 - x(1-x)p^2]^{D/2-2} \,.$$

We put p on shell in the integrand and perform the ε-expansion

$$\begin{aligned}\Pi^{\mu\nu}_{ab}(p) =& \frac{8}{3} \mathrm{i} T_f \frac{g^2}{(4\pi)^2} \delta_{ab}(g^{\mu\nu}p^2 - p^\mu p^\nu) S_\varepsilon \left(\frac{m^2}{\mu^2}\right)^{\varepsilon/2} \frac{1}{\varepsilon} \exp\left(\sum_{n=2}^{\infty} \frac{1}{n}\left(\frac{\varepsilon}{2}\right)^n \zeta_n\right) \\ =& \frac{8}{3} \mathrm{i} a_s T_f \delta_{ab}(g^{\mu\nu}p^2 - p^\mu p^\nu) S_\varepsilon \left(\frac{1}{\varepsilon} + \frac{1}{2}\ln\left(\frac{m^2}{\mu^2}\right) + \frac{\varepsilon}{8}\zeta_2\right) \,.\end{aligned}$$

Line-Insertion Diagram



The analytic expression of the amputated line insertion diagram reads

$$G^{(1),\mu\nu}_{Q,ab} = -\mathrm{Tr}\int \frac{d^D k}{(2\pi)^D} \mathrm{i}\frac{(\not{k}-\not{p})+m}{(k-p)^2-m^2}\delta_{ln}\mathrm{i}g\gamma^\mu t^a_{jl}$$
$$\times \mathrm{i}\frac{\not{k}+m}{k^2-m^2}\not{\Delta}(\Delta\cdot k)^{N-1}\mathrm{i}\frac{\not{k}+m}{k^2-m^2}\delta_{mj}\mathrm{i}g\gamma^\nu t^b_{nm}\,,$$

with Δ being a light-like 4-vector. We project this Greens function on the massive OME

$$A_{Qg} = \frac{1}{N_c^2-1}\frac{1}{D-2}(-g_{\mu\nu})\delta^{ab}(\Delta\cdot p)^{-N}G^{\mu\nu}_{Q,ab}\,.$$

We use the same steps as in the evaluation of the vacuum polarization diagram.
Only difference: effective vertex leads to more complicated numerator structure

$$(\Delta \cdot k)^N \quad \rightarrow \quad (\Delta \cdot k' + (1-x)\Delta \cdot p)^N = \sum_{n=0}^{N-1} \binom{N-1}{n} (1-x)^n (\Delta \cdot k')^{N-1-n} (\Delta \cdot p)^n \,.$$

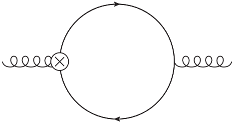
Using the fact that Δ is light-like vector and the symmetric integration the numerator simplifies. The remaining Feynman-parameter integrations lead to Beta-functions

$$B(x,y) = \int_0^1 dt t^{x-1}(1-t)^{y-1} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\,, \quad \mathrm{Re}(x), \mathrm{Re}(y) \geq 0\,.$$

The result for line insertion diagram is given by

$$A_{Qg}^{(1)} = -\,8 a_s T_f S_\varepsilon \left(\frac{m^2}{\mu^2}\right)^{\varepsilon/2} \frac{1}{(2+\varepsilon)\varepsilon} \exp\left(\sum_{n=2}^{\infty} \frac{1}{n} \left(\frac{\varepsilon}{2}\right)^n \zeta_n\right) \times \frac{2(N+1)(N+2) + \varepsilon(N^2+N+2)}{N(N+1)(N+2)}\,.$$

Vertex-Insertion Diagram



The analytical expression of the amputated vertex insertion diagram is given by

$$G_{Q,ab}^{(2),\mu\nu} = -\mathrm{Tr} \int \frac{d^D k}{(2\pi)^D} \mathrm{i} \frac{(\not{k} - \not{p}) + m}{(k-p)^2 - m^2} \delta_{ln} g t^a_{jl} \Delta^\mu \not{\Delta} \sum_{j=0}^{N-2} (\Delta \cdot (k-p))^j (\Delta \cdot k)^{N-2-j}$$
$$\times \mathrm{i} \frac{\not{k} + m}{k^2 - m^2} \delta_{mj} \mathrm{i} g \gamma^\nu t^b_{nm}$$

Again we project to the OME and performing the same steps as above lead to result

$$A_{Qg}^{(2)} = 32 a_s T_f S_\varepsilon \left(\frac{m^2}{\mu^2}\right)^{\varepsilon/2} \frac{1}{(2+\varepsilon)\varepsilon} \exp\left(\sum_{n=2}^{N} \frac{1}{n} \left(\frac{\varepsilon}{2}\right)^n \zeta_n\right) \frac{1}{(N+1)(N+2)} .$$

Only both diagrams together can be renormalized. Adding the unrenormalized results and ε-expanding yields

$$\begin{aligned}
\hat{A}_{Qg} =& \frac{1}{a_s}\left(A_{Qg}^{(1)} + A_{Qg}^{(2)}\right) \\
=& - T_f S_\varepsilon \left(\frac{m^2}{\mu^2}\right)^{\varepsilon/2} \frac{1}{\varepsilon} \exp\left(\sum_{n=2}^{N} \frac{1}{n}\left(\frac{\varepsilon}{2}\right)^n \zeta_n\right) \frac{8(N^2+N+2)}{N(N+1)(N+2)} \\
=& - T_f S_\varepsilon \left(\frac{m^2}{\mu^2}\right)^{\varepsilon/2} \left(\frac{1}{\varepsilon} + \frac{\zeta_2}{8}\varepsilon\right) \frac{8(N^2+N+2)}{N(N+1)(N+2)} .
\end{aligned}$$

The unrenormalized result can be written as

$$\hat{A}_{Qg} = S_\varepsilon \left(\frac{m^2}{\mu^2}\right)^{\varepsilon/2} \left[-\frac{1}{\varepsilon}\hat{P}_{qg} + a_{Qg} + \varepsilon \bar{a}_{Qg}\right]$$

with the splitting function

$$\hat{P}_{qg} = T_f \frac{8(N^2+N+2)}{N(N+1)(N+2)} , \quad a_{Qg} = 0 , \quad \bar{a}_{Qg} = -\frac{\zeta_2}{8}\hat{P}_{qg}$$

Expansion to the constant term in ε leads to

$$\hat{A}_{Qg} = S_\epsilon \left[-\frac{1}{\varepsilon} \hat{P}_{qg} - \frac{1}{2} \ln\left(\frac{m^2}{\mu^2}\right) \hat{P}_{qg} \right] .$$

In this case the renormalization procedure reduces to add the inverse renormalization factor

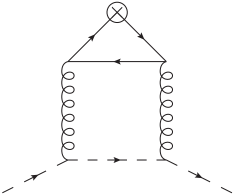
$$A^r_{Qg} = \hat{A}_{Qg} + Z^{-1}_{qg} ,$$

which is in this case

$$Z^{-1}_{qg} = \frac{S_\varepsilon}{\varepsilon} \hat{P}_{qg} .$$

The final renormalized one-loop OME reads

$$A^r_{Qg} = -\frac{1}{2} \ln\left(\frac{m^2}{\mu^2}\right) \hat{P}_{qg} .$$



For the computation of the two-loop graphs the Dirac-structure in the numerator has been neglected.
The scalar part of the loop integrals reads

$$I_1 = -g^4 (m^2)^2 \int \frac{d^D k}{(2\pi)^D} \frac{d^D l}{(2\pi)^D} \frac{1}{(p-l)^2} \frac{1}{(l^2)^2} \frac{1}{(k-l)^2 - m^2} \frac{1}{k^2 - m^2} (\Delta \cdot k)^N \frac{1}{k^2 - m^2} \, .$$

We perform the momentum integrals loop by loop doing same steps as in one-loop case yields Feynman-parameter integrals

$$\begin{aligned} I_1 = & \frac{\Gamma(6-D)}{(4\pi)^D} g^4 (m^2)^{-4+D} (\Delta \cdot p)^N \\ & \times \int_0^1 dx x^{N+3-D/2} (1-x)^{4-D/2} \int_0^1 dy y^{N+2} (1-y)^{D/2-4} \int_0^1 dz z^N (1-z) . \end{aligned}$$

$\Rightarrow$ Integrals decompose into Beta-functions

The ε-expansion has been performed with MAPLE

$$\begin{aligned} I_1 =& a_s^2 S_\varepsilon^2 \left(\frac{m^2}{\mu^2}\right)^\varepsilon (\Delta \cdot p)^N \Gamma(N+1) \\ & \times \exp(-\varepsilon\gamma_E) \frac{\Gamma(2-\varepsilon)\Gamma(N+2-\varepsilon/2)\Gamma(3-\varepsilon/2)\Gamma(-1+\varepsilon/2)}{\Gamma(N+5-\varepsilon)\Gamma(N+2+\varepsilon/2)} \\ =& a_s^2 S_\varepsilon^2 \left(\frac{m^2}{\mu^2}\right)^\varepsilon (\Delta \cdot p)^N \frac{1}{(N+1)(N+2)(N+3)(N+4)} \\ & \times \left[\frac{4}{\varepsilon} + (-5 + 4S_1(N+4) - 4S_1(N+1)) + \mathcal{O}(\varepsilon)\right] , \end{aligned} \tag{1}$$

with the single harmonic sums

$$S_1(N) = \sum_{k=1}^{N} \frac{1}{k} .$$

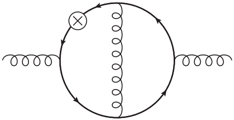
An expression for general multiple finite harmonic sums is given by

$$S_{a_1,\cdots,a_n}(N) = \sum_{k_1=1}^{N} \sum_{k_2=1}^{k_1} \cdots \sum_{k_n=1}^{k_{n-1}} \frac{\mathrm{sign}(a_1)^{k_1}}{k_1^{|a_1|}} \cdots \frac{\mathrm{sign}(a_n)^{k_n}}{k_n^{|a_n|}} \,.$$

Using the expression for the harmonic sums the result of the two-loop graph can be written as a purely rational function in the Mellin-moment N

$$I_1 = -\, a_s^2 S_\varepsilon^2 \left(\frac{m^2}{\mu^2}\right)^{\varepsilon} (\Delta \cdot p)^N \frac{1}{(N+1)(N+2)(N+3)(N+4)} \times \left[\frac{4}{\varepsilon} - \frac{5N^3+33N^2+58N+16}{(N+2)(N+3)(N+4)} + \mathcal{O}(\varepsilon)\right] .$$

2-Loop Graph with external Gluons



The scalar part of the two-loop diagram with external gluons reads

$$I_2 = -g^4(m^2)^2 \int \frac{d^D k}{(2\pi)^D} \frac{d^D l}{(2\pi)^D} \frac{1}{(p+k)^2 - m^2} \frac{1}{(p+l)^2 - m^2} \frac{1}{l^2 - m^2} \times \frac{1}{k^2 - m^2} (\Delta \cdot k)^N \frac{1}{k^2 - m^2} \frac{1}{(l-k)^2} .$$

The integration of the Feynman-parameter integrals is more involved than in the previous examples. An ntermediate result is given by

$$I_2 = \frac{\Gamma(6-D)}{(4\pi)^D} g^4 (m^2)^{-4+D} \int_0^1 dx dy x^{-3+D/2}(1-x)^{-2+D/2} y^{2-D/2}(1-y)^2 \left(1-y+\frac{y}{x}\right)^{-6+D} \times \int_0^1 dz_1 dz_2 (1-z_2)[z_2(1-y)+z_1 y]^N .$$

The z-integrals can be calculated in terms of elementary functions

$$I_2 = \frac{\Gamma(6-D)}{(4\pi)^D} g^4 (m^2)^{-4+D} \int_0^1 dx dy x^{-3+D/2}(1-x)^{-2+D/2} y^{1-D/2} \left(1-y+\frac{y}{x}\right)^{-6+D} \times \frac{1+y^N\left\{\left(\frac{1-y}{y}\right)^N (y-1)^3 + y^2[N(y-1)+2y-3]\right\}}{(N+1)(N+2)(N+3)} .$$

The remaining x-integration is of form of the hypergeometric function

$${}_2F_1(\alpha,\beta;\gamma;z) = \frac{1}{B(\beta,\gamma-\beta)} \int_0^1 dt t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha} , \quad \mathrm{Re}\gamma > \mathrm{Re}\beta > 0$$

The substitution $y = 1 - z$ yields

$$I_2 = \frac{\Gamma(6-D)}{(4\pi)^D} g^4 (m^2)^{-4+D} \int_0^1 dz B(4 - D/2, -1 + D/2) {}_2F_1\left(6 - D, 4 - D/2; 3, \frac{z}{z-1}\right)$$
$$\times (1-z)^{-5+D/2} \frac{1 - (1-z)^N \left\{ \left(\frac{z}{1-z}\right)^N z^3 + (1-z)^2 [Nz - 2(1-z) + 3] \right\}}{(N+1)(N+2)(N+3)} .$$

In order to solve the integral, one has to transform the argument via

$$ {}_2F_1\left(\alpha, \beta; \gamma, \frac{z}{z-1}\right) = (1-z)^\alpha {}_2F_1(\alpha, \gamma - \beta; \gamma; z) .$$

We use the iterative relation

$$\int_0^1 dz z^{\mu-1} (1-z)^{\nu-1} {}_2F_1(\alpha, \beta; \gamma; z) = B(\mu, \nu) {}_3F_2(\alpha, \beta, \mu; \gamma, \mu + \nu; 1) ,$$

with the generalized hypergeometric function ${}_3F_2$

The functions ${}_2F_1$ and ${}_3F_2$ are part of the function class ${}_pF_q$. They have a series representation for $|z| < 1$

$$ {}_pF_q(\alpha_1, \cdots, \alpha_p; \beta_1, \cdots, \beta_q; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_q)_k} \frac{z^k}{k!} $$

with the Pochhamer symbol

$$ (\alpha)_k = \frac{\Gamma(\alpha + k)}{\Gamma(\alpha)} . $$

Using the iterative relation and setting $D = 4 + \varepsilon$ we obtain

$$ \begin{aligned} I_2 =& a_s^2 S_\varepsilon \left(\frac{m^2}{\mu^2}\right)^{\varepsilon} \frac{\Gamma(2-\varepsilon) B(2-\varepsilon/2, 1+\varepsilon/2)}{(N+1)(N+2)(N+3)} \\ & \times \big[B(1, -\varepsilon/2) {}_3F_2(2-\varepsilon, 1+\varepsilon/2, 1; 3, 1-\varepsilon/2; 1) \\ & \quad - B(N+4, -\varepsilon/2) {}_3F_2(2-\varepsilon, 1+\varepsilon/2, N+4; 3, N+4-\varepsilon/2; 1) \\ & \quad - N B(2, N+2-\varepsilon/2) {}_3F_2(2-\varepsilon, 1+\varepsilon/2, 2; 3, N+4-\varepsilon/2; 1) \\ & \quad + 2 B(1, N+3-\varepsilon/2) {}_3F_2(2-\varepsilon, 1+\varepsilon/2, 1; 3, N+4-\varepsilon/2; 1) \\ & \quad - 3 B(1, N+2-\varepsilon/2) {}_3F_2(2-\varepsilon, 1+\varepsilon/2, 1; 3, N+3-\varepsilon/2; 1) \big] . \end{aligned} $$

The ε-expansion and infinite summation was done with the computer algebra package SIGMA. Our final result for the second loop graph is given by

$$I_2 = a_s^2 S_\varepsilon^2 \left(\frac{m^2}{\mu^2}\right)^\varepsilon \left\{ \frac{3 + 2N + 2N^2 + N^3}{(N+1)^3(N+2)^2(N+3)} \right.$$
$$\left. + \frac{1}{(N+1)^2(N+2)(N+3)} \left[\frac{3}{2}(N+1)S_2(N) + \frac{1}{2}(N+1)S_1^2(N) - NS_1(N)\right] \right\} .$$

Summary

- The substructure of the nucleus can be accessed through the deep-inelastic scattering process.
- The differential scattering cross section is related to structure functions, describing composition of nucleons.
- The structure functions decompose into the perturbative Wilson coefficients and non-perturbative parton distribution functions.
- The Wilson coefficients contain massive and massless contributions.
- The renormalized massive OME A_{Qg} has been calculated to one-loop order.
- Several scalar integrals at two-loop order have been computed.
- It is interesting to note that the evaluation of the integrals of OMEs generate naturally new mathematics.