Introduction to Lattice QCD

Fabio Bernardoni
NIC, DESY (Zeuthen)

2012 Summer Students

Outline

- QCD is our theory of the strong (nuclear) interactions; it is a Quantum Field Theory (QFT)

what is a QFT?

- Why do we need Lattice QCD

what can we compute in Lattice QCD?

- some technical details

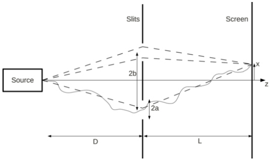
how is this done in practice?

From Superposition principle to Path integral

If a system is observed in state A and then in state B [with no intermediate observations], then the system must take all possible intermediate states between the two.

Double slit experiment:

- each path associated with a phase (amplitude)
- sum of two possible (classically) paths produces interference



$$\langle x_b | e^{-iHT/\hbar} | x_a \rangle = \sum_{\text{paths}} e^{i \cdot (phase)}$$

Physical paths satisfy (least action principle):

$$\frac{\delta}{\delta x(t)} \left(S[x(t)] \right) |_{x_{cl}} = 0$$

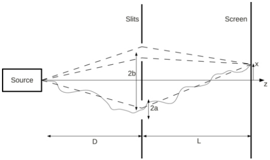
From Superposition principle to Path integral

If a system is observed in state A and then in state B [with no intermediate observations], then the system must take all possible intermediate states between the two.

Double slit experiment:

- each path associated with a phase (amplitude)
- sum of two possible (classically) paths produces interference
- adding a barrier with infinite number of slits should not change result

$\Rightarrow$ sum over all (infinite) paths, classical or not



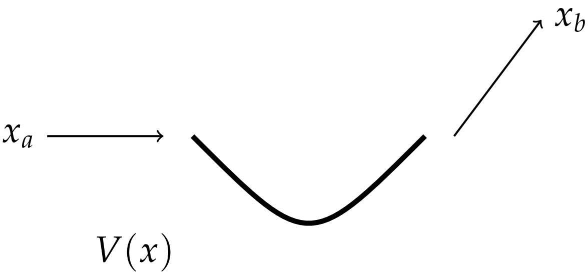
PATH INTEGRAL

$$\langle x_b | e^{-iHT/\hbar} | x_a \rangle = \int \mathcal{D}x(t) e^{iS[x(t)]/\hbar}$$

Destructive interference for $x_{odd}(t)$ if:

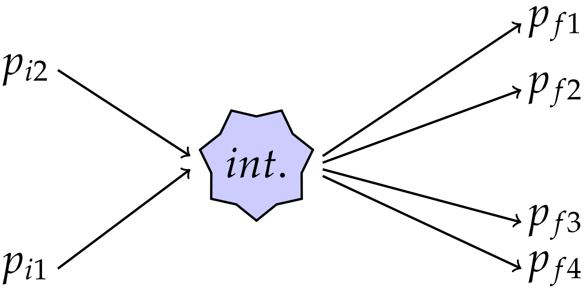
$$(S[x_{cl}(t)] - S[x_{odd}(t)]) \gg \hbar$$

Scattering in Quantum Mechanics



- number of particles conserved
- finite (small!) number of degrees of freedom (dof)
- macroscopic $V(x)$ given a priori

Relativistic Scattering



- local interaction $\Rightarrow$ initial and final states are (nearly) free
- $E = mc^2 \Rightarrow$ particle creation-annihilation
- intermediate states can have ∞ dof NEED a QFT

Theorist path to need of QFT

- non-relativistic energy momentum

$$E = p^2/(2m)$$

- Schrodinger equation

$$-\frac{\nabla^2}{2m}\Psi = i\partial_t\Psi$$

- solutions

$$\Psi(\vec{x},t) = e^{i(\vec{p}\cdot\vec{x}/\sqrt{2m}-\omega t)}$$

$$\omega = \vec{p}^2/(2m)$$

- relativistic energy momentum

$$E^2 = p^2 + m^2$$

- Klein-Gordon equation

$$(-\nabla^2 + \partial_t^2)\Psi = m^2\Psi$$

- solutions

$$\Psi(\vec{x},t) = e^{i(\vec{p}\cdot\vec{x}-\omega t)}$$

$$\omega = \pm\sqrt{\vec{p}^2 + m^2}$$

negative energy solutions

- also in interacting theory
- also if we try to obtain a first order equation using $E = \sqrt{p^2 + m^2}$, Dirac equation:

$$(\partial\!\!\!/ - m)\Psi = 0 \qquad \gamma\!\!\!/ = \gamma_\mu\partial_\mu$$

QFT

- QM for 1-particle (non-relativistic, $h = 1$):

$$\text{variables: } \vec{x},\, \vec{p} \qquad \text{quantization: } [\vec{x},\, \vec{p}] = i$$

- QFT (scalar particle):

$$\text{variables: } \phi(\vec{x}),\, \pi(\vec{x}) \qquad \text{quantization: } [\phi(\vec{x}),\, \pi(\vec{y})] = i\delta(\vec{x} - \vec{y})$$

$$[\phi(\vec{x}),\, \phi(\vec{y})] = 0 \qquad [\pi(\vec{x}),\, \pi(\vec{y})] = 0$$

  - $\phi(x)$ operator that creates particle in x
  - $\langle 0|T\phi(x)\phi(0)|0\rangle \neq 0$ even if x is spacelike (eg $x_0 = 0$, $\vec{x} \neq 0$)
  - $\phi(x)$ and $\phi(0)$ are different variables, they can be related only if x is timelike:

$$[\phi(x), \phi(0)] = 0 \quad \text{if } x^2 < 0 \quad \text{spacelike}$$

Path Integral, LSZ reduction formula and all that...

In QM we solve Sch. eq. for a given $V(x)$, we get what we want: $\langle x_b | x_a \rangle$.
QFT: what do we want, how we get it?

1. ${}_{out}\langle \vec{p}_1, ..., \vec{p}_n | \vec{q}_1, \ldots, \vec{q}_k \rangle_{in} =$ (LSZ formula)

$$\frac{\prod_{i,j} \int d^4x_i d^4y_j e^{ip_i \cdot x_i} e^{-iq_j \cdot y_j} \langle 0 | T\left(\phi(x_1) \ldots \phi(x_n) \phi(y_1) \ldots \phi(y_k)\right) | 0 \rangle}{\prod_{i=1}^{n} \left(\frac{i\sqrt{Z}}{p_i^2 - m^2 + i\epsilon} \right) \prod_{j=1}^{k} \left(\frac{i\sqrt{Z}}{q_j^2 - m^2 + i\epsilon} \right)}$$

2. denominator obtained from 2-pt functions (Källen-Lehmann)

$$\langle 0 | T\phi(x)\phi(0) | 0 \rangle = i \sum_{\alpha} \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \frac{|\langle 0 | \phi(0) | \alpha(\vec{0}) \rangle|^2}{p^2 - m(\alpha)^2 + i\epsilon} \, .$$

$$E_{\vec{p}}^2(\alpha) = m(\alpha)^2 + \vec{p}^2, \qquad Z_\alpha \equiv |\langle 0 | \phi(0) | \alpha(\vec{0}) \rangle|^2$$

3. correlation functions can be evaluated through a Path Integral:

$$\langle 0 | T\{\phi(x_1) \ldots \phi(x_N)\} | 0 \rangle = \frac{\int \mathcal{D}\phi \; \phi(x_1) \ldots \phi(x_N) e^{iS[\phi]}}{\int \mathcal{D}\phi e^{iS[\phi]}}$$

Summary

- to treat quantum relativistic systems we NEED QFT
- using a path integral we can compute the correlation functions $\langle 0|T\{\phi(x_1)\ldots\phi(x_N)\}|0\rangle$
- correlation functions contain all infos we need, eg scattering amplitudes

Summary

- to treat quantum relativistic systems we NEED QFT
- using a path integral we can compute the correlation functions $\langle 0|T\{\phi(x_1)\ldots\phi(x_N)\}|0\rangle$
- correlation functions contain all infos we need, eg scattering amplitudes

now let's see what kind of QFT QCD is...

Electromagnetism as a QFT (QED)

- it is Lorentz invariant
- it is gauge invariant
- it can be formulated in a Lagrangian formalism to have Lorentz symetries manifest:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$A_\mu \to A_\mu - \frac{1}{e} \partial_\mu \alpha(x)$$

Electromagnetism as a QFT (QED)

- it is Lorentz invariant
- it is gauge invariant
- it can be formulated in a Lagrangian formalism to have Lorentz symetries manifest:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$A_\mu \to A_\mu - \frac{1}{e} \partial_\mu \alpha(x)$$

- the other way around: $F^2_{\mu\nu}$ is the lowest dimensional operator Lorentz and gauge invariant

Electromagnetism as a QFT (QED)

- it is Lorentz invariant
- it is gauge invariant
- it can be formulated in a Lagrangian formalism to have Lorentz symetries manifest:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$A_\mu \to A_\mu - \frac{1}{e} \partial_\mu \alpha(x)$$

- the other way around: $F^2_{\mu\nu}$ is the lowest dimensional operator Lorentz and gauge invariant
- add fermion fields (spin 1/2) with gauge transformations:

$$\psi(x) \to U(x)\psi(x) = e^{ie\alpha(x)}\psi(x)$$

Electromagnetism as a QFT (QED)

- it is Lorentz invariant
- it is gauge invariant
- it can be formulated in a Lagrangian formalism to have Lorentz symetries manifest:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$A_\mu \to A_\mu - \frac{1}{e} \partial_\mu \alpha(x)$$

- the other way around: $F^2_{\mu\nu}$ is the lowest dimensional operator Lorentz and gauge invariant
- add fermion fields (spin 1/2) with gauge transformations:

$$\psi(x) \to U(x)\psi(x) = e^{ie\alpha(x)}\psi(x)$$

- Lorentz and gauge invariance fix the interaction lagrangian:

$$\mathcal{L}_{QED} = \overline{\psi}(\not{D} - m)\psi - \frac{1}{4} F^2_{\mu\nu} \quad D_\mu = \partial_\mu + ieA_\mu(x)$$

Electromagnetism as a QFT (QED)

- it is Lorentz invariant
- it is gauge invariant
- it can be formulated in a Lagrangian formalism to have Lorentz symetries manifest:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$A_\mu \to A_\mu - \frac{1}{e}\partial_\mu \alpha(x)$$

- the other way around: $F^2_{\mu\nu}$ is the lowest dimensional operator Lorentz and gauge invariant
- add fermion fields (spin 1/2) with gauge transformations:

$$\psi(x) \to U(x)\psi(x) = e^{ie\alpha(x)}\psi(x)$$

- Lorentz and gauge invariance fix the interaction lagrangian:

$$\mathcal{L}_{QED} = \overline{\psi}(\not{D} - m)\psi - \frac{1}{4}F^2_{\mu\nu} \quad D_\mu = \partial_\mu + ieA_\mu(x)$$

$$\not{D} = D_\mu\gamma^\mu \qquad \{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu} \qquad \gamma_\mu \text{ are } 4\times 4$$

Electromagnetism as a QFT (QED)

- it is Lorentz invariant
- it is gauge invariant
- it can be formulated in a Lagrangian formalism to have Lorentz symetries manifest:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$A_\mu \to A_\mu - \frac{1}{e} \partial_\mu \alpha(x)$$

- the other way around: $F^2_{\mu\nu}$ is the lowest dimensional operator Lorentz and gauge invariant
- add fermion fields (spin 1/2) with gauge transformations:

$$\psi(x) \to U(x)\psi(x) = e^{ie\alpha(x)}\psi(x)$$

- Lorentz and gauge invariance fix the interaction lagrangian:

$$\mathcal{L}_{QED} = \overline{\psi}(\not{D} - m)\psi - \frac{1}{4} F^2_{\mu\nu} \quad D_\mu = \partial_\mu + ieA_\mu(x)$$

Electromagnetism can be formulated as a gauge theory (QED)

QCD

Is QED the only gauge theory in the Universe?

QCD

- QED is a U(1) gauge theory $\Rightarrow$ ψ_α has only Lorentz index

$$(\gamma^\mu \psi)_\alpha = (\gamma^\mu)_{\alpha\beta}\psi_\beta \qquad (U(x)\psi(x))_\alpha = e^{ie\alpha(x)}\psi(x)_\alpha \qquad U(x) \in \mathrm{U(1)}$$

QCD

- QED is a U(1) gauge theory $\Rightarrow \psi_\alpha$ has only Lorentz index

$$(\gamma^\mu \psi)_\alpha = (\gamma^\mu)_{\alpha\beta}\psi_\beta \qquad (U(x)\psi(x))_\alpha = e^{ie\alpha(x)}\psi(x)_\alpha \qquad U(x) \in \text{U(1)}$$

- can we build a SU(N) gauge theory? ($\psi_{\alpha a}$ has Lorentz and gauge index)

$$(\gamma^\mu \psi)_{\alpha a} = (\gamma^\mu)_{\alpha\beta}\psi_{\beta a} \qquad (U(x)\psi(x))_{\alpha a} = U(x)_{ab}\psi(x)_{\alpha b} \qquad U(x) \in \text{SU(N)}$$

QCD

- QED is a U(1) gauge theory $\Rightarrow \psi_\alpha$ has only Lorentz index

$$(\gamma^\mu \psi)_\alpha = (\gamma^\mu)_{\alpha\beta}\psi_\beta \qquad (U(x)\psi(x))_\alpha = e^{ie\alpha(x)}\psi(x)_\alpha \qquad U(x) \in \mathrm{U(1)}$$

- can we build a SU(N) gauge theory? ($\psi_{\alpha a}$ has Lorentz and gauge index)

$$(\gamma^\mu \psi)_{\alpha a} = (\gamma^\mu)_{\alpha\beta}\psi_{\beta a} \qquad (U(x)\psi(x))_{\alpha a} = U(x)_{ab}\psi(x)_{\alpha b} \qquad U(x) \in \mathrm{SU(N)}$$

- YES! (we assume also CP invariance)

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu} + \overline{\psi}(i\not{D} - m)\psi$$

$$D_\mu = \partial_\mu + igA^a_\mu \tau^a \qquad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf^{abc}A^b_\mu A^c_\nu$$

  - $\tau_a \in$ SU(N); $[\tau_a, \tau_b] = if_{abc}\tau_c$
  - for $N > 1$ force carriers autointeract, unlike photons!

QCD

- QED is a U(1) gauge theory $\Rightarrow$ ψ_α has only Lorentz index

$$(\gamma^\mu\psi)_\alpha = (\gamma^\mu)_{\alpha\beta}\psi_\beta \qquad (U(x)\psi(x))_\alpha = e^{ie\alpha(x)}\psi(x)_\alpha \qquad U(x) \in \text{U(1)}$$

- can we build a SU(N) gauge theory? ($\psi_{\alpha a}$ has Lorentz and gauge index)

$$(\gamma^\mu\psi)_{\alpha a} = (\gamma^\mu)_{\alpha\beta}\psi_{\beta a} \qquad (U(x)\psi(x))_{\alpha a} = U(x)_{ab}\psi(x)_{\alpha b} \qquad U(x) \in \text{SU(N)}$$

- YES! (we assume also CP invariance)

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu} + \overline{\psi}(i\not{D} - m)\psi$$

$$D_\mu = \partial_\mu + igA^a_\mu\tau^a \qquad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf^{abc}A^b_\mu A^c_\nu$$

  - $\tau_a \in$ SU(N); $[\tau_a, \tau_b] = if_{abc}\tau_c$
  - for $N > 1$ force carriers autointeract, unlike photons!

theory of the strong interactions (QCD) is a SU(3) gauge theory

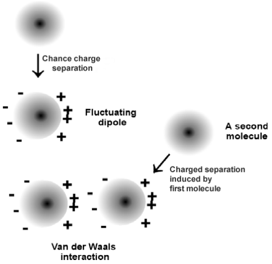
The strong interaction

Discovered as force that binds protons (p) and neutrons (n) into nuclei: Nuclear force

- stronger than electromagnetic force at distances $\sim$ fm stable nuclei exist
- strength rapidly (exponentially) decays with distance large nuclei are unstable

Today (QCD):

- written in terms of quarks (ψ) and gluons (A_μ)
- gluons are carriers of strong force (like photons)
- charge is called color
- strong force binds quarks and gluons in (p) and (n)
- asymptotic states are color singlets called hadrons (confinement)
- Nuclear force is the residual strong force analogous to Van der Walls forces between neutral atoms and molecules



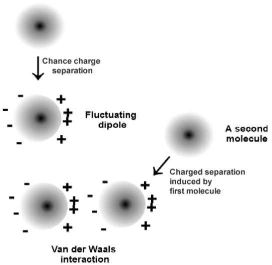
The strong interaction

Discovered as force that binds protons (p) and neutrons (n) into nuclei: Nuclear force

- stronger than electromagnetic force at distances $\sim$ fm stable nuclei exist
- strength rapidly (exponentially) decays with distance large nuclei are unstable

Today (QCD):

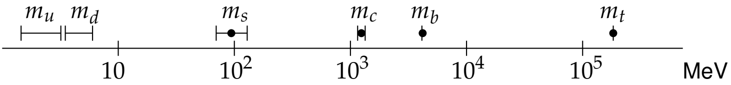
- written in terms of quarks (ψ) and gluons (A_μ)
- gluons are carriers of strong force (like photons)
- charge is called color
- strong force binds quarks and gluons in (p) and (n)
- asymptotic states are color singlets called hadrons (confinement)
- Nuclear force is the residual strong force analogous to Van der Walls forces between neutral atoms and molecules



How do we derive all these properties from QCD???

Quark content

In Nature there are 6 quarks:



ψ has now dimension: $N_f \times N_c \times 4$ ($N_f = 6$; $N_c = 3$)

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu} + \overline{\psi}(i\not{D} - \mathcal{M})\psi$$

Asymptotic states are color singlets (called hadrons). Some allowed interpolating operators are

- $\overline{\psi}_a \Gamma \psi_a$ (bosons, called mesons, eg $\pi^+ = \overline{\psi}^u_a \gamma_5 \psi^d_a$)
- $\epsilon_{cde}(C\gamma_5)_{\beta\gamma}\psi_{\alpha c}(\psi_{\beta d}\psi_{\gamma e} - \psi_{\beta d}\psi_{\gamma e})$ (fermions, called baryons, e.g. $p = \epsilon_{cde}(C\gamma_5)_{\beta\gamma}\psi^u_{\alpha c}(\psi^u_{\beta d}\psi^d_{\gamma e} - \psi^d_{\beta d}\psi^u_{\gamma e})$)

QCD path integral

Example: quark propagator.

$$\langle \psi(x)\overline{\psi}(0)\rangle = \frac{\int \mathcal{D}\psi\mathcal{D}\overline{\psi}\mathcal{D}A\ \psi(x)\overline{\psi}(0)e^{\int dx^4 \mathcal{L}_{QCD}}}{\int \mathcal{D}\psi\mathcal{D}\overline{\psi}\mathcal{D}Ae^{i\int dx^4 \mathcal{L}_{QCD}}}$$

Remember:

$$\mathcal{L}_{QCD} = -\frac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu} + \overline{\psi}(i\not{D} - \mathcal{M})\psi$$

$$D_\mu = \partial_\mu + igA^a_\mu\tau^a \qquad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf^{abc}A^b_\mu A^c_\nu$$

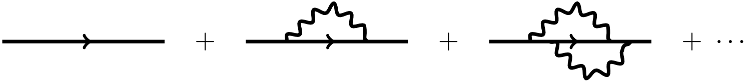
If $g \ll 1$ we can expand $\mathcal{L}_{QCD}$ inside the integral:

$$\langle \psi(x)\overline{\psi}(0)\rangle \simeq \frac{\int \mathcal{D}\psi\mathcal{D}\overline{\psi}\mathcal{D}A\ \psi(x)\overline{\psi}(0)(1 + ig\int dx^4\mathcal{L}_{int} - \frac{g^2}{2}(\int dx^4\mathcal{L}_{int})^2)e^{\int dx^4\mathcal{L}_{free}}}{\int \mathcal{D}\psi\mathcal{D}\overline{\psi}\mathcal{D}Ae^{i\int dx^4 \mathcal{L}_{QCD}}}$$

$\mathcal{L}_{free}$ is bilinear in ψ, $\overline{\psi}$ $\Rightarrow$ All gaussian integrals

Feynman diagrams

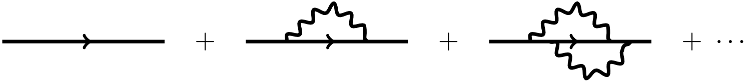
$$\langle\psi(x)\overline{\psi}(0)\rangle \simeq \frac{\int \mathcal{D}\psi\mathcal{D}\overline{\psi}\mathcal{D}A\ \psi(x)\overline{\psi}(0)(1+ig\int dx^4\mathcal{L}_{int} - \frac{g^2}{2}(\int dx^4\mathcal{L}_{int})^2)e^{\int dx^4\mathcal{L}_{free}}}{\int \mathcal{D}\psi\mathcal{D}\overline{\psi}\mathcal{D}Ae^{i\int dx^4\mathcal{L}_{QCD}}}$$



- mnemonic device to keep into account all terms coming from expanding the exponential
- Feynman rules allows how to write each diagram in terms of a product of polarization vectors, propagators, integrals over allowed internal momenta (in loops)

Feynman diagrams

$$\langle \psi(x)\overline{\psi}(0)\rangle \simeq \frac{\int \mathcal{D}\psi\mathcal{D}\overline{\psi}\mathcal{D}A\ \psi(x)\overline{\psi}(0)(1 + ig\int dx^4\mathcal{L}_{int} - \frac{g^2}{2}(\int dx^4\mathcal{L}_{int})^2)e^{\int dx^4\mathcal{L}_{free}}}{\int \mathcal{D}\psi\mathcal{D}\overline{\psi}\mathcal{D}Ae^{i\int dx^4\mathcal{L}_{QCD}}}$$



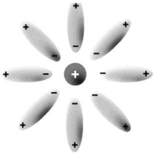
- mnemonic device to keep into account all terms coming from expanding the exponential
- Feynman rules allows how to write each diagram in terms of a product of polarization vectors, propagators, integrals over allowed internal momenta (in loops)

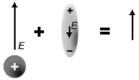
1. is g small in Nature?
2. how do we relate quark scattering amplitudes to hadrons scattering amplitudes?

The running coupling

In QED:

- electron-positron pairs can pop out of the vacuum
- they screen the electron charge
- at high momentum μ (or short distance) expect larger electric charge





$\Rightarrow$ $$\beta(e) = \mu \frac{\partial e(\mu)}{\partial \mu} > 0$$

In QCD:

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu$$

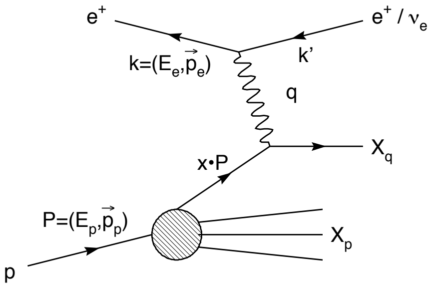
- gluons have nonzero color charge: autointeraction

$\Rightarrow$ charge bigger at large distances (confinement?)

$\Rightarrow$ charge smaller at short distances (asymptotic freedom)

$$g^2(\mu) = \frac{g^2(M^2)}{1-\beta(g(M^2))\log(\mu^2/M^2)/g(M^2)} \qquad \beta(g) \simeq -\frac{g^3}{(4\pi)^2}\left(\frac{11}{3}N_c - \frac{2}{3}N_f\right)$$

Asymptotic freedom, QCD tests at high energies



- if $Q^2 = -q^2 \gg 1$ GeV strong interactions not able to keep quark in proton ($\alpha(Q^2) \ll 1$)

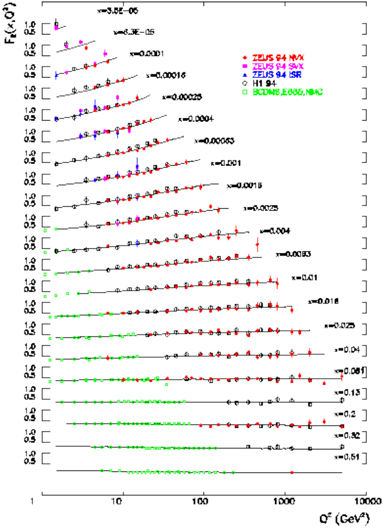
- collinear gluon production very favored $\Rightarrow$ Hadronization
- initial and final states are hadrons (jets) but we can compute amplitude as process happened for free quarks (Parton Model)

$$\frac{d\sigma}{dQ^2} = \int_0^1 d\xi \sum_f f_f(\xi) Q_f^2 \frac{2\pi\alpha^2}{Q^4}\left[1 + (1 - \frac{Q^2}{\xi s})^2\right]\theta(\xi s - Q^2)$$

$$p = \xi P$$

- one can improve precision and compute NLO perturbative corrections to the above (Altarelli-Parisi)

QCD as the theory of strong (and nuclear) interactions



- QCD in the perturbative regime tested in wide range of high energy experiments: success!
- Can we use the above theory to describe nuclear interactions (our starting point)? Hadron spectrum? Decays and scattering processes involving hadrons, at low energy?

WE NEED NON PERTURBATIVE METHODS!

QFT on a lattice

It is impossible to compute an infinite dimensional integral:

$$\langle 0|T\{\phi(x_1)\dots\phi(x_N)\}|0\rangle = \frac{\int \mathcal{D}\phi\ \phi(x_1)\dots\phi(x_N)e^{iS[\phi]}}{\int \mathcal{D}\phi e^{iS[\phi]}}$$

However:

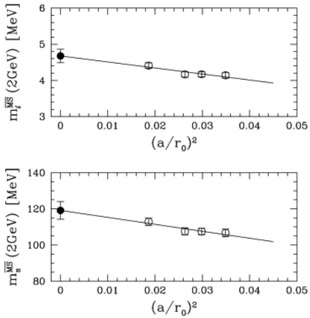
- we can discretize spacetime

$$(t,x,y,z) = (n_t a,\ n_x a,\ n_y a,\ n_z a) \quad n_i \in \mathcal{Z}$$

- consider a box of finite length

$$0 \le t,\ x,\ y,\ z \le Na = L$$

- take the limits $a \to 0$, $L \to \infty$ numerically



Monte Carlo Integration

Huge number of variables $O(V/a^4) \times d_{\mathcal{F}}(\phi)$: only feasible method is Monte Carlo.

1. perform a Wick-rotation

   evaluate the PI at imaginary times by analytic continuation

$$t \to it \qquad \int d^4x \to i \int d^4x \qquad \mathcal{L}_{Lorentz} \to \mathcal{L}_{Euc} \qquad e^{iS[\phi]} \to e^{-S[\phi]}$$

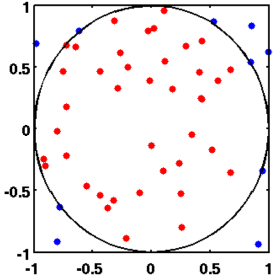
$\Rightarrow$ 3dim quantum system $\leftrightarrow$ 4dim statistical system

2. generate sample of ϕ configurations with probability distribution $P[\phi] = e^{-S[\phi]}$
3. evaluate correlation function on sample, compute average and statistical error

Typically $\sim O(1000)$ independent cfgs: $P[\phi]$ must satisfy

- is positive defined
- is bounded from above
- no strong oscillations

reasonable precision for a small set of observables



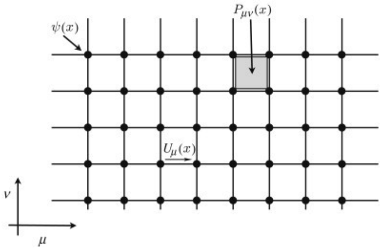
QCD on a lattice

Typical requirements when building $\mathcal{L}_{QCD}^{lat}$ on the lattice:

- $\mathcal{L}_{QCD}^{lat} \to \mathcal{L}_{QCD}$ when $a \to 0$
- hermiticity (to maintain transfer matrix)
- gauge invariance
- locality
- some symmetries of $\mathcal{L}_{QCD}$ have to be broken at $a \neq 0$ (eg: Lorentz): recovered in continuum limit

$\Rightarrow$ there is a lot of freedom in the choice!

$$\partial_\mu \psi(x) \to \frac{1}{2}\left[\psi(x+\hat{\mu}) - \psi(x-\hat{\mu})\right]$$



Hadron Spectrum

Källen-Lehmann parametrization of 2-point correlator:

$$\langle 0|T\phi(x)\phi(0)|0\rangle = i\sum_{\alpha}\int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \frac{|\langle 0|\phi(0)|\alpha(\vec{0})\rangle|^2}{p^2 - m(\alpha)^2 + i\epsilon}$$

Hadron Spectrum

...after Wick-rotation and projection over momentum $\vec{p}$:

$$c_2(x_0) = \int d^3\vec{x} e^{-i\vec{p}\cdot\vec{x}} \langle 0|T\phi(x)\phi(0)|0\rangle = \sum_{\alpha} \frac{Z_\alpha}{2E(\alpha)} e^{-E(\alpha)x_0} \xrightarrow[x_0\to\infty]{} \frac{Z_1}{2E(1)} e^{-E(1)x_0}$$

Hadron Spectrum

...after Wick-rotation and projection over momentum $\vec{p}$:

$$c_2(x_0) = \int d^3\vec{x} e^{-i\vec{p}\cdot\vec{x}} \langle 0|T\phi(x)\phi(0)|0\rangle = \sum_{\alpha} \frac{Z_\alpha}{2E(\alpha)} e^{-E(\alpha)x_0} \xrightarrow[x_0\to\infty]{} \frac{Z_1}{2E(1)} e^{-E(1)x_0}$$

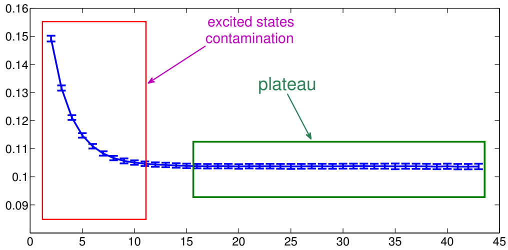
also valid for composite operators, eg $\phi(x)\gamma\phi(x)$

Suppose we want to compute the π^+ mass ($E(\vec{p}=0)$):

- build a composite operator made out of ψ, $\overline{\psi}$ and A_μ with the right quantum numbers (Lorentz, isospin, color)

  ψ, $\overline{\psi}$ and A_μ are colored, π are color singlets
- compute the 2-point function and project at $\vec{p} = 0$
- fit the mass from exponential decay at large time separations
- example $\pi(x) = \overline{\psi}\gamma_5\psi(x)$

Example: pion mass



- at $T = \infty$, $-\partial_0 \log c_2(x_0) \xrightarrow[x_0 \to \infty]{} m_\pi a$
- at finite T, $c_2(x_0) \xrightarrow[x_0 \text{large}]{} e^{-m_\pi x_0} + e^{-m_\pi (T - x_0)}$

$\Rightarrow$ I plot acosh $((c_2(x_0 + a) + c_2(x_0 - a))/(2c_2(x_0))) \xrightarrow[x_0 \to \infty]{} m_\pi a$

- weighted average in the plateau range gives $m_\pi a$
- $T \sim 6.2$ fm, $a \sim 0.065$ fm

Example: pion mass

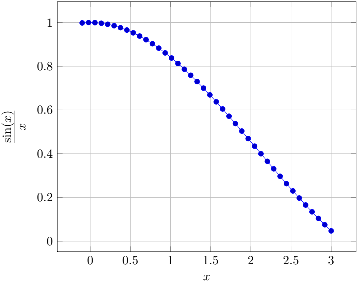
Looks easy...

Example: pion mass

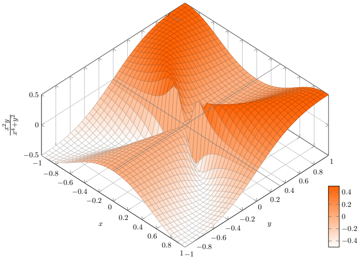
Looks easy...

1. how do we know a?
2. other hadrons? protons and neutrons in particular...

Continuum limit



- in 1d if $\lim_{x \to 0} \exists$ is unique



- $\lim$ depends on the path

$$\lim_{x,y \to 0} \frac{x^2 y}{x^4+y^2}$$

- $= 0$ if $y = kx$
- $= 0.5$ if $y = x^2$

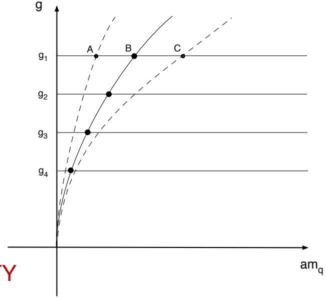
if we take the $a \to 0$ limit of the PI naively we obtain ∞

Continuum limit

Consider QCD, with u, d quarks, $m_u = m_d \Rightarrow g, m$ free parameters

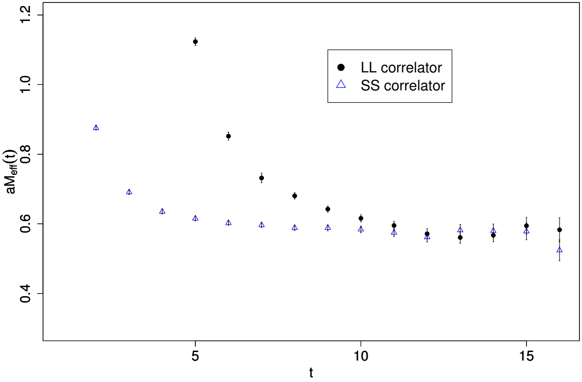
1. choose one $g(a)$
2. take two input variables, e.g. m_π, m_p
3. tune m until $\frac{m_\pi}{m_p}_{lat} = \left(\frac{m_\pi}{m_p}\right)_{phys}$
4. find a in physical units comparing $(m_\pi a)_{lat}$ with $(m_\pi)_{phys}$
5. close enough to continuum, any other observable $\mathcal{O}$ will satisfy:

   $\mathcal{O}_{lat} = \mathcal{O}_{phys} + O(a)$ RENORMALIZABILITY
6. repeat with smaller g (closer to $a = 0$)



We let $m(a)$ and $g(a)$ in such a way to maintain physics constant (up to $O(a)$).

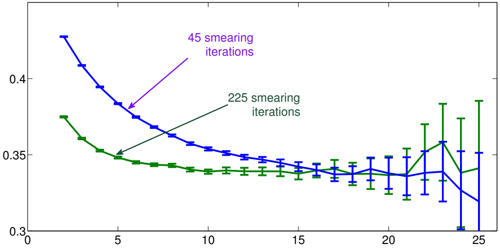
Example: proton mass



thanks to V. Drach

- interpolating field: $L(x) = \epsilon_{abc}((\psi_a^u)^T C\gamma_5 \psi_b^d)\psi_c^u(x)$
- noise to signal ratio increases at large separations
- invent another (non local) interpolating field $S(x)$ with less overlap with excited states

Example: B meson mass

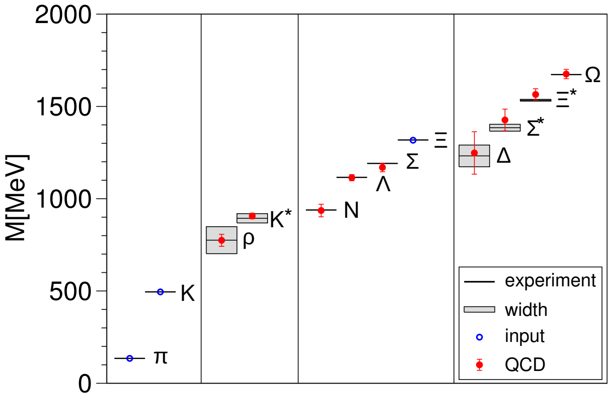


- B^+ is like π^+ but quark content: $u\bar{b}$ instead of $u\bar{d}$
- noise to signal ratio increases at large separations
- use smeared interpolating fields P_{hl}^k, $k = 1, \ldots, n$

$$P_{hl}^k = \overline{\psi}_l^{(k)} \gamma_0 \gamma_5 \psi_h \qquad \psi_l^{(k)}(x) = \left(1 + \kappa_G \, a^2 \, \Delta\right)^{R_k} \psi_l(x)$$

- the larger the radius the smaller the overlap with excited states

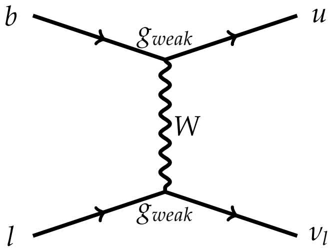
Spectrum



BMW collaboration, 2008

methods exist to extract masses of excited states (resonances): GEVP

Check unitarity of CKM-matrix

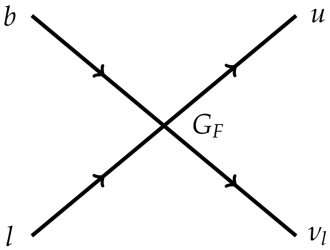


- flavor changing processes through exchange of W boson (weak interaction)
- happen at the quark level, but involve hadrons

At LO in g_{weak} and e amplitude factorizes into a strong and an EW part. Examples:

- $K \to \pi + e + \nu_e$ and $K \to \mu\nu_\mu$ (for $|V_{us}|$)
- $B \to \pi + l + \nu_l$ and $B \to \tau\nu_\tau$ (for $|V_{ub}|$)
- $\ldots$

Check unitarity of CKM-matrix



- flavor changing processes through exchange of W boson (weak interaction)
- happen at the quark level, but involve hadrons
- at small energy (wrt $M_W \sim 90$ GeV) and leading order in g_{weak} fermi effective theory

At LO in g_{weak} and e amplitude factorizes into a strong and an EW part. Examples:

- $K \to \pi + e + \nu_e$ and $K \to \mu\nu_\mu$ (for $|V_{us}|$)
- $B \to \pi + l + \nu_l$ and $B \to \tau\nu_\tau$ (for $|V_{ub}|$)
- ...

$B \to \tau\nu$ in the Standard Model

$$\mathcal{B}_{\rm SM}(B \to \tau\nu) = f_{\rm B}^2 |V_{ub}|^2 \frac{G_F^2 m_{\rm B} \tau_{\rm B}}{8\pi} m_\tau^2 \left[1 - \frac{m_\tau^2}{m_{\rm B}^2}\right]^2$$

- from lattice we need f_B: $\langle 0|\overline{\psi}_u \gamma_\mu \gamma_5 \psi_b |B(p)\rangle = -i f_B p_\mu$
- in perturbation theory (EW theory) we can compute $\langle \tau \overline{\nu}_\tau | \overline{\psi}_\tau \gamma_\mu \gamma_5 \psi_\nu | 0\rangle$
- using Källen-Lehmann again:

$$c(t) = \sum_{\vec{x}} \langle P_{hl}(x) P_{hl}(0)\rangle \xrightarrow[T\to\infty, t\to\infty]{} f_B^2 m_B e^{-M_B t}(1 + O(e^{-\Delta_{1,0} t})) \qquad (t = x_0)$$

$B \to \pi l \nu$ in the SM

At LO in α_{EM} and $m_l = m_\nu = 0$

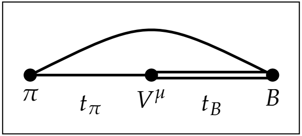
$$\frac{\mathrm{d}\Gamma}{\mathrm{d}q^2} = \frac{G_\mathrm{F}^2 |V_{ub}|^2}{192\pi^3 m_B^3} \lambda^{3/2}(q^2) \left| f_+(q^2) \right|^2 \qquad q^\mu = p_B^\mu - p_\pi^\mu$$

$$\langle \pi(p_\pi) | V^\mu | B(p_B) \rangle = f_+(q^2) \left[p_B^\mu + p_\pi^\mu - \frac{m_B^2 - M_\pi^2}{q^2} q^\mu \right] + f_0(q^2) \frac{m_B^2 - M_\pi^2}{q^2} q^\mu$$

Typically on the lattice one computes (B at rest):

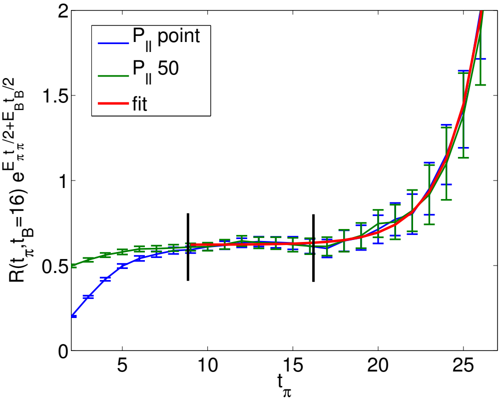
$$\langle \pi(p_\pi) | V^\mu | B(p_B) \rangle = \lim_{T\to\infty,\, t_{B,\pi}\to\infty} R(t_\pi, t_B) e^{E_\pi t_\pi/2} e^{m_B t_B/2}$$

$$R(t_\pi, t_B) \equiv \frac{\sum_{\vec{x}_\pi, \vec{x}_B} e^{-i\vec{p}\cdot\vec{x}_\pi} \langle P_{ll}(x_\pi + x_B) V^\mu(x_B) P_{hl}(0) \rangle}{\sqrt{\sum_{\vec{x}_\pi} e^{-i\vec{p}\cdot(\vec{x}_\pi)} \langle P_{ll}(x_\pi) P_{ll}(0) \rangle \sum_{\vec{x}_B} \langle P_{hl}(x_B) P_{hl}(0) \rangle}} .$$



- P_{ll} and P_{hl} are interpolating operators
- other ratios are possible

$R(t_\pi, t_B)$



- always smeared interpolating operators
- large finite T effects at large x_0 separations
- computation reliable for momenta $p_\pi \ll 1/a$

Conclusions 1

- Lattice QCD already helped testing QCD, and measuring parameters of the SM
- Need to compute some quantities with better precision to understand some discrepancies e.g. in determination CKM
  - for computational reasons one usually uses $m > m_{u,d}$ and extrapolates
    use ChPT, based on spontaneous chiral symmetry breaking
  - with present lattice spacings $a \sim 0.05$ fm cannot simulate b directly
    use effective theories (HQET, NRQCD), extrapolate from $m < m_b$
  - critical slowing down, topology freezing make it difficult to reduce a
    need better algorithms
- Simulations are run on machines (on parallel), with limited computational power a lot of work is done, trying to optimize programs, communication

Conclusions 2

Lattice is a regularization of a QFT, that can be used to approach other issues of particle physics:

- study beyond the SM models, like technicolor, quantum gravity...
- attack some issues as whether the SM can be the ultimate fundamental theory
  triviality of scalar theories
- study QCD in extreme situations of temperature and pressure
  understand neutron stars, heavy ion collisions