LHC Theory
Lecture 2: Energy Scales and LHC Observables

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Zeuthen, July 27, 2012

Outline

The Issue of Quantum Corrections

- In Lecture 1, we have learned that quantum corrections of scattering processes involve divergent integrals over a momentum k going around internal loops.
- We distinguish between UV divergences for $k \longrightarrow \infty$ and IR divergences for $k \longrightarrow 0$.
- In order to proceed, the loop integrals have to be regularized. There are various ways of doing this. The most obvious but not necessarily best choice is to introduce momentum cut-offs.

The Issue of Quantum Corrections

- In Lecture 1, we have learned that quantum corrections of scattering processes involve divergent integrals over a momentum k going around internal loops.
- We distinguish between UV divergences for $k \longrightarrow \infty$ and IR divergences for $k \longrightarrow 0$.
- In order to proceed, the loop integrals have to be regularized. There are various ways of doing this. The most obvious but not necessarily best choice is to introduce momentum cut-offs.
- Note also that in order to perform the loop integrals, we have to shift to a Euclidean signature. Therefore one uses

$\int_{-\infty}^{\infty} dk^0 f(k^0) = -\int_{i\infty}^{-i\infty} dk^0 f(k^0)$ (Wick rotation), and defines:

$$k^0 = ik_E^0, \quad k^i = k_E^i \quad \Rightarrow \quad -k^2 = k_E^2 = (k_E^0)^2 + (k_E^1)^2 + (k_E^2)^2 + (k_E^3)^2$$

Example: Consider the integral

$$\int \frac{d^4k}{(2\pi)^4} \left(\frac{1}{k^2 + i\delta} \right)^2 ,$$

which is UV and IR divergent!

This can easily be seen by shifting to Euclidean coordinates and redefining the integration variable. Then the above integral is proportional to $\int_0^\infty \frac{dx}{x}$.

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Use **dimensional regularization**:

Idea: Evaluate the loop integral in d dimensions instead of 4.

- If $d < 4$, the UV divergence vanishes.
- If $d > 4$, the IR divergence vanishes.
- We actually do not have to specify wether $d < 4$ ord $d > 4$. Just replace $4 \to d + 2\epsilon$, then the UV and IR divergences are regularized at the same time. The div. are transformed into poles $1/\epsilon$ and double poles $1/\epsilon^2$.
- Dimensional regularization preserves all symmetries of the theory, opposed to *cut-off regularization*, which breaks Lorentz invariance at the cut-off!

Calculating d-dimensional integrals:

- Sounds complicated, but it is not (if one knows the trick)!
- Using the Gaussian integral $\int dx\, e^{-x^2} = \sqrt{\pi}$, one can show that the area of the d-dimensional unit sphere is given by

$$\boxed{\int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}}$$

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- After Wick rotation, a typical momentum integral can be written in d-dimensonal spherical coordinates:

$$\int \frac{d^d k_E}{(2\pi)^d} \frac{1}{(k_E^2 + m^2)^2} = \int \frac{d\Omega_d}{(2\pi)^d} \int_0^\infty dk_E \frac{k_E^{d-1}}{(k_E^2 + m^2)^2}$$

$$= \int \frac{d\Omega_d}{(2\pi)^d} \frac{1}{2} \left(\frac{1}{m^2}\right)^{2-d/2} \int_0^1 dx\, x^{1-d/2} (1-x)^{d/2-1}$$

Here, we have substituted $x = m^2/(k_E^2 + m^2)$.

- Using $\int_0^1 x^{a-1}(1-x)^{b-1} = \Gamma(a)\Gamma(b)/\Gamma(a+b)$, we obtain

$$\int \frac{d^d k_E}{(2\pi)^d} \frac{1}{(k_E^2 + m^2)^2} = \frac{1}{(4\pi)^d/2} \frac{\Gamma(2-d/2)}{\Gamma(2)} \left(\frac{1}{m^2}\right)^{2-d/2}$$

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- In order to compensate for the change of the mass dimension in the integration measure, we multiply the d-dimensional integral with a prefactor μ^{4-d} with (mass) dimension $[\mu] = 1$. The scale μ is known as renormalization scale. We will talk about its meaning in the next section.

  Expanding the above result for small ϵ around $d = 4$ one obtains

$$\mu^{4-d} \int \frac{d^4 k_E}{(2\pi)^d} \frac{1}{(k_E^2 + m^2)^2}$$
$$\xrightarrow{d\to 4} \frac{1}{(4\pi)^2} \left(\frac{1}{\epsilon} - \gamma_E + \log(4\pi) + \log\left(\frac{\mu^2}{m^2}\right) + \mathcal{O}(\epsilon) \right)$$

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- You can find various master integrals in text books of QFT, for instance:

$$\int \frac{d^d k}{(2\pi)^d} \frac{\{1,\, k^2,\, k^\mu k^\nu,\, (k^2)^2,\, k^\mu k^\nu k^\sigma k^\tau\}}{(k^2-\Delta)^n}$$

Renormalization

In order to absorb UV divergences induced by quantum corrections, we have to redefine the parameters of the theory. Let us consider QED as an example. Remember from Lecture 1 that

$$\boxed{\mathcal{L}_{\mathrm{QED}} = -\frac{1}{4}\, F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(i\gamma^\mu\partial_\mu - m_0)\psi - e_0\,\overline{\psi}\gamma^\mu\psi A_\mu}$$

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Let us now define renormalization constants Z_i, renormalized fields Ψ_r and A_r^μ, as well as a renormalized coupling e and mass m:

$$\psi = \sqrt{Z_2}\,\psi_r, \quad A^\mu = \sqrt{Z_3}\,A_r^\mu, \quad \text{and} \quad e_0\,Z_2\sqrt{Z_3} = e\,Z_1.$$

Further define $\delta_i = Z_i - 1\ (i = 1,2,3)$ and $\delta_m = Z_2 m_0 - m$, then

$$\begin{aligned}\mathcal{L}_{\rm QED} = &-\frac{1}{4}\,(F_r^{\mu\nu})^2 + \overline{\psi}_r(i\gamma^\mu\partial_\mu - m)\psi_r - e\,\overline{\psi}_r\gamma^\mu\psi_r A_{r\mu}\\ &-\frac{1}{4}\,\delta_3(F_r^{\mu\nu})^2 + \overline{\psi}_r(i\delta_2\gamma^\mu\partial_\mu - \delta_m)\psi_r - e\delta_1\,\overline{\psi}_r\gamma^\mu\psi_r A_{r\mu}\end{aligned}$$

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- so-called counter terms

The idea of renormalization is to absorb divergent parts into the counter terms. Only the renormalized part contributes to an observable quantity!

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However, the definitions of the renormalization constants Z_i are not unique!
For instance, you can always shift constant pieces between e and e_0. One talks about different renormalization schemes!
$\rightarrow$ When theory is matched to experiment in order to determine the renormalized parameters, one always has to tell which renormalization scheme is used! One can shift from one scheme to another afterwards.

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→ When theory is matched to experiment in order to determine the renormalized parameters, one always has to tell which renormalization scheme is used! One can shift from one scheme to another afterwards.

2 examples for **dimensional regularization**:

- minimal subtraction (MS): Only poles $1/\epsilon$ are stored into counter terms.
- modified minimal subtraction ($\overline{\rm MS}$): The combination $1/\epsilon - \gamma_E + \log(4\pi)$ (which always appears for any process) is stored into the counter terms.

Field Strength Renormalization

In Lecture 1, we found that the two-point correlation function or *Feynman propagator* of a free (scalar) theory is given by:

$$\langle 0|T\Phi(x)\Phi(0)|0\rangle = \int \frac{d^4p}{(2\pi)^4}\,\frac{i\,e^{-ipx}}{p^2-m^2+i\delta},$$

Equivalently, we can write:

$$\int d^4x\, e^{ipx}\langle 0|T\Phi(x)\Phi(0)|0\rangle = \frac{i}{p^2-m^2+i\delta}.$$

Here, one has to use $\int \frac{d^4x}{(2\pi)^4}\, e^{i(p-p')x} = \delta^{(4)}(p-p')$

Within an interacting theory, we expect the two-point function to look like:

$$\int d^4x\, e^{ipx}\langle \Omega|T\Phi(x)\Phi(0)|\Omega\rangle = \frac{iZ}{p^2-m^2+i\delta} + \text{contrib. from multiple-particle radiation}$$

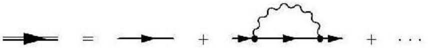
Example: The Electron Self-Energy

For fermions, one has the two-point function

$$\int d^4x\, e^{ipx} \langle \Omega | T \Psi(x) \overline{\Psi}(0) | \Omega \rangle = \frac{iZ_2(\not{p} + m)}{p^2 - m^2 + i\delta} + \text{particle radiation}$$

The factor Z_2 is called field strength renormalization.

QED diagramatically:



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QED diagramatically:

Applying Feynmanrules of QED, the second diagram can be written as

$$\frac{i(\not{p} + m_0)}{p^2 - m_0^2} \left(\underbrace{\int \frac{d^4k}{(2\pi)^4} (-ie\gamma^\mu) \frac{i(\not{p} \pm \not{k} - m_0)}{(p \pm k)^2 - m_0^2 + i\delta} (-ie\gamma_\mu) \frac{-i}{k^2 + i\delta}}_{\equiv -i\Sigma_2(p)} \right) \frac{i(\not{p} + m_0)}{p^2 - m_0^2}$$

!!! Note that we distinguish between m (phys. electron mass) in the full propagator and the unphysical parameter m_0 in the pert. expansion !!!

The integral $\Sigma_2(p)$ is UV and IR divergent. It could be made convergent by replacing

$$\frac{1}{k^2 + i\delta} \longrightarrow \frac{1}{k^2 - \mu^2 + i\delta} - \frac{1}{k^2 - \Lambda^2 + i\delta}$$

This is known as *Pauli-Villars regularization*. The parameter μ regulizes the IR divergence by introducing a small photon mass. The second propagator compensates the UV divergence of the first one. Λ is called the *Pauli Villars regulator*. The original photon propagator is re-obtained in the limit $\Lambda \to \infty$ and $\mu \to 0$. Using this regularization, we obtain ($\alpha = e^2/(4\pi)$)

$$\Sigma_2(p) = \frac{\alpha}{2\pi} \int_0^1 dx\,(2m_0 - x\not{p})\,\log\left(\frac{x\Lambda^2}{(1-x)m_0^2 + x\mu^2 - x(1-x)p^2}\right).$$

Obviously, this expression is infinite for $\Lambda \to \infty$.

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Obviously, this expression is infinite for $\Lambda \to \infty$.

Using *dimensional regularization* with $2\epsilon = 4 - d$, we obtain

$$\Sigma_2(p) = \mu^{2\epsilon}\,\frac{\alpha}{(4\pi)^{d/2-1}} \int_0^1 dx\,\frac{2\big((2-\epsilon)m_0 - (1-\epsilon)x\not{p}\big)\Gamma(2-d/2)}{\big((1-x)m_0^2 + x\mu^2 - x(1-x)p^2\big)^{2-d/2}}.$$

(Note that also the Dirac algebra is modified in d dimensions!)

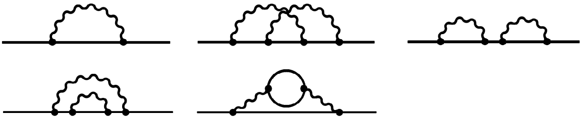
Next, we want to find a systematic way to sum the whole perturbation series. You can convince yourself, that all possible Feynman diagrams are covered by

$$\int d^4x\, e^{ipx} \langle\Omega| T\Phi(x)\Phi(0)|\Omega\rangle =$$

$$P + P(-i\Sigma(p))P + P(-i\Sigma(p))P(-i\Sigma(p))P + \cdots, \text{ where } P = \frac{i(\not{p} + m_0)}{p^2 - m_0^2}$$

(let us drop $+i\epsilon$ by now), and $\Sigma(p)$ is the sum over all so-called one-particle irreducible (1PI) diagrams

One-particle irreducible (1P1) diagrams:



1PI no **1PI**

Let us further write $\frac{i(\not{p}+m_0)}{p^2-m_0^2} \overset{\not{p}^2=p^2}{=} \frac{i}{\not{p}-m_0}$ and $\Sigma(p)=\Sigma(\not{p})$, then

$$\int d^4x\, e^{ipx}\langle\Omega|T\Phi(x)\Phi(0)|\Omega\rangle =$$

$$\frac{i}{\not{p}-m_0}+\frac{i}{\not{p}-m_0}\frac{\Sigma(\not{p})}{\not{p}-m_0}+\frac{i}{\not{p}-m_0}\left(\frac{\Sigma(\not{p})}{\not{p}-m_0}\right)^2+\cdots = \frac{i}{\not{p}-m_0-\Sigma(\not{p})}$$

(Geometric series)

The physical mass m is defined as the pole of the full propagator, which is the solution of:

$$\boxed{\left(\not{p}-m_0-\Sigma(\not{p})\right)\Big|_{\not{p}=m}=0}$$

- For unstable particles like the muon or the τ-lepton, $\Sigma(\not{p})$ is a complex quantity. The imaginary part gives rise to a width.
- $\Sigma(\not{p})$ has to be determined order by order in powers of α. At LO we have $\Sigma(\not{p})=\Sigma_2(p)$.

- Close to the pole, the denominator of the full propagator can be expanded as:

$$(\not{p} - m)\left(1 - \frac{d\Sigma}{d\not{p}}\bigg|_{\not{p}=m}\right) + \mathcal{O}\left((\not{p} - m)^2\right) = \frac{p^2 - m^2}{\not{p} + m}\left(1 - \frac{d\Sigma}{d\not{p}}\bigg|_{\not{p}=m}\right) + \cdots$$

Combining this result with the definition of the field strength renomalization Z_2 (see above), we find:

$$\boxed{Z_2^{-1} = 1 - \frac{d\Sigma}{d\not{p}}\bigg|_{\not{p}=m}}$$

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- At LO in α, the difference between the bare mass m_0 and the pole mass m is given by: (use pole mass definition in first step and $\Sigma_2 = \mathcal{O}(\alpha)$ in the second)

$$\delta m \equiv m - m_0 = \Sigma_2(\not{p} = m) = \Sigma_2(\not{p} = m_0)\big(1 + \mathcal{O}(\alpha)\big)$$

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$$\overset{P.V.}{=} \frac{\alpha}{2\pi} m_0 \int_0^1 dx\,(2 - x)\log\left(\frac{x\Lambda^2}{(1-x)^2 m_0^2 + x\mu^2}\right) \xrightarrow{\Lambda\to\infty} \frac{3\alpha}{4\pi} m_0 \log\left(\frac{\Lambda^2}{m_0^2}\right)$$

$$\overset{d.r.}{=} \frac{2\alpha}{(4\pi)^{1-\epsilon}} m_0\, \mu^{2\epsilon} \int_0^1 dx\, \frac{\big(2 - x - \epsilon(1-x)\big)\Gamma(\epsilon)}{\left((1-x)^2 m_0^2 + x\mu^2\right)^{\epsilon}} \xrightarrow{\epsilon\to 0} \frac{3\alpha}{4\pi} m_0 \frac{1}{\epsilon}$$

Remarks:

- δm is divergent for $\Lambda \to \infty / \epsilon \to 0$. As m is the observed particle mass, m_0 is an infinite quantity. As it is just a parameter in the theory, this is not forbidden in first place. However, if we compare to experiment, we have to replace $m_0 = m + \mathcal{O}(\alpha)$, where the „small"$\mathcal{O}(\alpha)$ correction is actually infinite. Thus, the validity of pertubation theory is no longer clear. Netherless, the perturbation series can be rearranged such, that m_0 is eliminated for m everywhere! This is done by renormalization!

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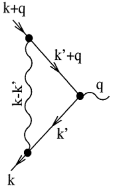
- The perturbative expansion of Z_2 also involves an infinite term, when we take $\Lambda \to \infty/\epsilon \to 0$. However, this term compensates a divergent piece of the so-called vertex correction at the same order in α!



$\Gamma^\mu(q=0) = Z_1^{-1}\gamma^\mu$, Γ^μ includes the vertex corrections

- One can prove in general that to all orders in perturbation theory $Z_1 = Z_2$!

The Photon Self-Energy

We can repeat the whole discussion and compute the self-energy of a photon. This gives rise to a charge renormalization Z_3.

LO:

$$= \frac{-ig_{\mu\sigma}}{p^2} \underbrace{(-1) \int \frac{d^4k}{(2\pi)^4} \operatorname{tr}\left[(-ie\gamma^\sigma)\frac{i(\not{k}+m)}{k^2-m^2}(-ie\gamma^\tau)\frac{i(\not{k}+\not{p}+m)}{(k+p)^2-m^2}\right]}_{\equiv\, i\Pi_2^{\sigma\tau}(p^2)} \frac{-ig_{\tau\nu}}{p^2}$$

p : photon momentum, k : loop momentum; The trace has to be taken w.r.t. the product of γ-matrices, and we have to include a factor (-1) for a closed fermion loop (this is also a Feynman rule).

Sum over 1PI diagrams:

The sum over all 1PI diagrams is denoted as $\Pi^{\mu\nu}(p)$ and has to satisfy $p_\mu \Pi^{\mu\nu}(p) = 0$ (Ward identity, follows from the *classical continuity equation* $\partial_\mu j^\mu = 0$, which says that charge is conserved!). Thus we can write:

$$\Pi^{\mu\nu}(p) = (p^2 g^{\mu\nu} - p^\mu p^\nu)\Pi(p^2)$$

$$= \frac{-ig_{\mu\nu}}{p^2} + \frac{-ig_{\mu\sigma}}{p^2}\left[i(p^2 g^{\sigma\tau} - p^\sigma p^\tau)\Pi(p^2)\right]\frac{-ig_{\tau\nu}}{p^2} + \cdots = \ldots = \frac{-ig_{\mu\nu}}{p^2(1-\Pi(p^2))}$$

[More details are given in Peskin, Schroeder, chap. 7.5]

charge renormalization: $\quad \dfrac{1}{1-\Pi(0)} \equiv Z_3 \quad$ (residue of the $p^2 = 0$ pole)

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The sum over all 1PI diagrams is denoted as $\Pi^{\mu\nu}(p)$ and has to satisfy $p_\mu \Pi^{\mu\nu}(p) = 0$ (Ward identity, follows from the *classical continuity equation* $\partial_\mu j^\mu = 0$, which says that charge is conserved!). Thus we can write:

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$$= \frac{-ig_{\mu\nu}}{p^2} + \frac{-ig_{\mu\sigma}}{p^2}\left[i(p^2 g^{\sigma\tau} - p^\sigma p^\tau)\Pi(p^2)\right]\frac{-ig_{\tau\nu}}{p^2} + \cdots = \ldots = \frac{-ig_{\mu\nu}}{p^2(1-\Pi(p^2))}$$

[More details are given in Peskin, Schroeder, chap. 7.5]

charge renormalization: $\quad \dfrac{1}{1-\Pi(0)} \equiv Z_3 \quad$ (residue of the $p^2 = 0$ pole)

LO: $\quad \Pi_2(p^2) = -\mu^{4-d}\dfrac{8\alpha}{(4\pi)^{d/2-1}}\displaystyle\int_0^1 dx\,(1-x)\,\frac{\Gamma(2-d/2)}{\left(m_0^2 - x(1-x)p^2\right)^{2-d/2}}$

$$\xrightarrow{d\to 4} -\frac{2\alpha}{\pi}\int_0^1 dx\,(1-x)\left(\frac{1}{\epsilon} - \gamma_E + \log(4\pi) + \log(\mu^2) - \log(m^2 - x(1-x)p^2)\right)$$

Renormalizable Theories

Within a renormalizable QFT, one has to fix a finite number of counter terms ($\rightarrow$ experimental input needed!) in order to make predictions for further experiments.

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Does this work for any Lagrangian?

No!

For instance it does not work for a four-fermion operator $(\overline{\Psi}\Psi)(\overline{\Psi}\Psi)$ or an operator like $\overline{\Psi}\Psi A^{\mu}\partial_{\mu}\partial_{\nu}A^{\nu}$.

For these operators, every order in the perturbative expansion creates a new operator, which was not there before. As a consequence, there are infinitely many counter terms required and the theory can not be fixed!

What is the criterion?

Answer: The mass dimension of the operator under consideration.

Note that also fields have a mass dimension. This is clear if we look at the action S, which, in natural units, is dimesionsless!

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Let us again take QED as example (in unitary gauge):

$$\underbrace{S}_{0} = \int \underbrace{d^4x}_{-4} \left\{ \frac{1}{2} A_\mu \Big(\underbrace{\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu}_{2} \Big) A_\nu - e\overline{\psi}\gamma^\mu A_\mu \psi + \overline{\psi} \big(\underbrace{i\gamma^\mu \partial_\mu - m}_{1} \big) \psi \right\}$$

We conclude that $[A_\mu] = 1$ and $[\Psi] = 3/2$.
The coupling e on the other hand is dimensionless!

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The coupling e on the other hand is dimensionless!

Renormalizable operators have mass dimension ≤ 4.

$\Leftrightarrow$ The coefficient of a ren. operator has non-negative mass dimension.

Remarks:

- Operators with mass dim. > 4 are called higher dimensional operators. They only show up in so-called effective field theories with a hard momentum cut-off Λ ($\to$ no UV divergences).
- In order to obtain the correct physical dimension, the coefficients of higher dim. operators involve inverse powers of the cut-off scale Λ. Therefore, these ops. vanish in the limit $\Lambda \to \infty$ (renormalizable theory).
- The criterium of renormalizability allows only for a finite number of terms in the Lagrangian. In fact, the QED und QCD Lagrangians are fixed by: Lorentz invariance, gauge invariance, parity ($\to$ no terms like $\epsilon_{\mu\nu\sigma\tau} F^{\mu\nu} F^{\tau\sigma}$), and renormalizabilty!
- The four-fermion operator has mass dimension 6 and thus should be multiplied with $1/\Lambda^2$. There is a nice historical example where this operator is used: Fermi's theory of β decay which is nowadays replaced by the theory of weak interactions

The Running Coupling

Remember that we have introduced the energy scale μ, in order to keep the physiscal (mass) dimension of the d-dimensional loop integral equal to 4:

$$\boxed{\int d^4k \longrightarrow \mu^{4-d} \int d^dk}$$

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- As evident from $\Pi_2(p^2)$ in the previous section, Z_g is a function of μ.
- The bare coupling g_0 obviously is independent of μ.

$\Rightarrow$ It follows that (even though dimensionless), g_R is also a function of μ, such that all μ-dependent terms cancel in the r.h.s. of the above equation.

- Define $\beta(g_R) \equiv \mu \frac{d}{d\mu} g_R$ and calculate

$$\mu \frac{d}{d\mu} g_0 = 0 = \mu \frac{d}{d\mu}(Z_g)\mu^\epsilon g_R + \mu \frac{d}{d\mu}(\mu^\epsilon) Z_g g_R + \beta(g_R)\mu^\epsilon Z_g$$
$$= \left(g_R Z_g^{-1} \mu \frac{d}{d\mu} Z_g + \epsilon g_R + \beta(g_R) \right) Z_g \mu^\epsilon$$

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- Z_g is a function of $\alpha = g_R^2/(4\pi)$! For the leading quantum correction β_0 we can write

$$Z_g^{-1} \mu \frac{d}{d\mu} Z_g = \frac{g_R^2}{(4\pi)^2} \beta_0 = \frac{\alpha}{4\pi} \beta_0$$

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- For $d = 4$ ($\epsilon = 0$), we obtain (using $\mu \frac{d}{d\mu}\alpha = \frac{g_R}{2\pi}\mu \frac{d}{d\mu} g_R$)

$$\boxed{\mu \frac{d}{d\mu} \frac{\alpha}{4\pi} = \frac{d}{d\log(\mu)} \frac{\alpha}{4\pi} = -2\beta_0 \left(\frac{\alpha}{4\pi}\right)^2}$$

The solution of the latter equation is given by

$$\boxed{\frac{\alpha(\mu)}{4\pi} = \frac{\alpha(\mu_0)}{4\pi} \frac{1}{1 + \frac{\alpha(\mu_0)}{4\pi}\beta_0 \ln\left(\frac{\mu^2}{\mu_0^2}\right)}}$$

or, choosing μ_0 such that $1 = \frac{\alpha(\mu_0)}{4\pi}\beta_0 \ln\left(\frac{\mu_0^2}{\Lambda^2}\right)$,

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Interpretation?

$$\frac{\alpha(\mu)}{4\pi} = \frac{\alpha(\mu_0)}{4\pi} \frac{1}{1 + \frac{\alpha(\mu_0)}{4\pi}\beta_0 \ln\left(\frac{\mu^2}{\mu_0^2}\right)}$$

- $\rightarrow$ If we know α from a measurement with the center of mass (cms) energy μ_0, we can calculate its value at the energy μ.
- The sign of the beta-fuction $\beta(g_R) = -\beta_0 g_R^3/(4\pi)^2$ is crucial!
  If $\beta_0 > 0$ ($\beta(g_R) < 0$), α decreases with increasing energy and vice verca.

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$$\frac{\alpha(\mu)}{4\pi} = \frac{1}{\beta_0 \ln\left(\frac{\mu^2}{\Lambda^2}\right)}$$

- For $\mu \rightarrow \Lambda$, the r.h.s. is divergent! However, this divergence is artificial: Starting from some $\alpha(\mu) < 1$, the coupling becomes $\mathcal{O}(1)$ before reaching the divergence! $\rightarrow$ Perturbation theory can no longer be applied and the above formula becomes invalid!
- The pole at $\mu = \Lambda$ is called the Landau pole, and is understood as the scale where the theory becomes strongly coupled.

We say that the coupling runs with energy, but what if we used a different regularization sheme without the issue of a dimensionfull coupling g, which has to be replaced by g_R?

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$\longrightarrow$ We would have needed some different regulator (with mass dimension 1) in the propagator or simply a cut-off scale in the momentum integral. The independence of the full cross section w.r.t. to these scales would lead to the same conclusion!

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The running of the SM couplings:

- For weak and electromagnetic interactions, the beta function is positive. The Landau pole of QED is at about 10^{277} GeV.
- The beta function of QCD is negative! The Landau pole of QCD is at about 200 MeV!

Remark: The sign of the QCD-beta function (calculated in the late 70ths by *Gross*, *Politzer*, and *Wilczek*) was worth a Nobel price in 2004! Actually, *t'Hooft* calculated it before but did not publish. However, he got a Nobel price in 1999 for his proof of the renormalizability of the electroweak theory.

Confinement vs. Asymptotic Freedom

QCD predicts that the strong coupling $\alpha_s = g_s^2/(4\pi)$ **increases with decreasing energy and vice verca! This explains why we do observe quarks as fundamental particles in collider experiments, but not at low energies:**

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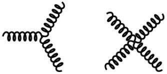
QCD predicts that the strong coupling $\alpha_s = g_s^2/(4\pi)$ increases with decreasing energy and vice verca! This explains why we do observe quarks as fundamental particles in collider experiments, but not at low energies:

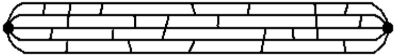
- At energies of the Z-mass ($= 91\,\text{GeV}/c^2$), we measure $\alpha_s(m_Z) \approx 1.2$ and are well in the perturbative regime. Quarks are weakly coupled und thus observable as individual particals. $\rightarrow$ Assymptotic freedom
- At 1 GeV, we measure $\alpha_s(\mu) \approx 0.4$. Going below 1 GeV the perturbativity of QCD breaks down. In the nucleus at rest, we have α_s of $\mathcal{O}(1)$.
- Further remember that gluons interact with each other.



In combination with the strong coupling, we are now able to understand the so-called quark confinement, e.g. the fact that only color-neutral bound states (=hadrons) are observed at low energies!

Confinement

Consider the interaction of two color-charged (anti)quarks (lines are gluons):



- When the quarks are separeted, the gluons form a strongly coupled web.
- The energy which is stored in the web increases linearily with the separation distance of the quarks (like a rubber band).
- At some distance, the web provides enough energy to create a new quark-antiquark pair:

Conclusion: Whenever one tries to seperate color-charged particles at low energies, the linear potential creates a new color-neutral bound state.

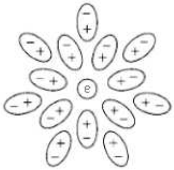
QCD bound states

For the color-charge, there is a nice analogon to the spectral colors of light (therefore the name!)

- In the vector representation of $SU(3)$ (quarks), there are 3 colors: Lets call them red, blue & green.
- There is also a conjugate representation for the anti-quarks with the colors anti-red, anti-blue & anti-green.
- Obviously, the combination color and anticolor is color-neutral.
- Also (known from the mixing of light), the combination red-blue-green is color neutral (same for anti colors).
- Therefore, the most simple bound states of QCD are **mesons** $q_i \bar{q}_{\bar{i}}$, **baryons** $\epsilon^{ijk} q_i q_j q_k$, and **anti baryons** $\epsilon^{\bar{i}\bar{j}\bar{k}} \bar{q}_{\bar{i}} \bar{q}_{\bar{j}} \bar{q}_{\bar{k}}$.

Charge Sreening

For QED, we find a positive beta function.
$\Rightarrow$ The electromagnetic coupling increases with energy. This can be understood by an intuitive picture:



• The electron charge is sreened by virtual e^+/e^- pairs, which are created and annihilated in the quantum vacuum.

• If we increase the energy of our scattering experiment, we penetrate positron-electron cloud and thus observe a stronger charge.

The running mass

The self-energy corrections to fermion propagators will cause a *running mass*!

For the case of the electron, we computed the field strength normalization Z_2 to leading order, where $\delta_m = Z_2 m_0 - m$. Let us now define

$$m_0 = Z_m m \,, \qquad \gamma = \frac{\mu}{Z_m}\frac{dZ_m}{d\mu} \equiv \gamma_0 \frac{\alpha}{4\pi} + \gamma_1 \left(\frac{\alpha}{4\pi}\right)^2 = \cdots$$

The function γ is called anomalous dimension. Typically one considers $\alpha = \alpha_s$.

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The function γ is called anomalous dimension. Typically one considers $\alpha = \alpha_s$.

Again, the bare parameter m_0 does not depend on μ. Therefore

$$\mu \frac{d}{d\mu} m_0 = 0 = Z_m \left(\gamma + \mu \frac{d}{d\mu} m\right) \qquad \stackrel{\text{LO}}{\Longrightarrow} \qquad \boxed{\mu \frac{d}{d\mu} m = -\gamma_0 \frac{\alpha}{4\pi} m}$$

On the other hand we already know that $\mu \frac{d}{d\mu} = -2\beta_0 \left(\frac{\alpha(\mu)}{4\pi}\right)^2 \frac{d}{d\alpha(\mu)/(4\pi)}$.

$$\alpha(\mu)\frac{d}{d\alpha(\mu)}m = \frac{\gamma_0}{2\beta_0}m \qquad \Longrightarrow \qquad m(\mu) = \left(\frac{\alpha(\mu)}{\alpha(\mu_0)}\right)^{\frac{\gamma_0}{2\beta_0}} m(\mu_0)$$

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Conclusion:
The scale dependence of α causes a scale dependence of the mass. We observe that fermion masses become smaller, when extracted at higher energies!

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Final remarks on the running of $\alpha_{(s)}$ and m:

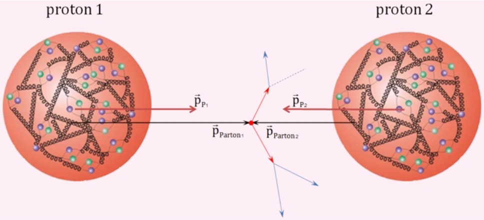
- For quarks, the leading contributions to the running mass come from the running of the strong coupling α_s. Contributions from the running of the electromagnetic coupling are small, as α itself is very small.
- Both, the theoretical predictions for $\alpha_s(\mu)$ and $m(\mu)$ depend on the renormalization sheme. This dependence would vanish if we could evaluate the full perturbation series. In practice, one distinguishes between $\overline{\text{MS}}$-mass, pole mass, etc.
- The particle masses given by the **particle data group (PDG)** correspond to the masses at the production threshold of the respective particle.

Theory Errors

The renormalization scale μ can be used to get an estimate on the unknown contributions of higher order corrections, and thus a theory error to a fixed order calculation!

- Obviously, the exact theoretical prediction for an arbitrary scattering cross section (full perturbation series) must not depend on μ. On the other hand, this is not true for a truncated series!
- There are good choices for μ (the series converges quickly), and bad choices (the series converges slowly).
- It turns out that a good choice is $\mu \approx \hat{s}$, where $\hat{s}$ is the partonic cms energy.
- We expect the scale dependence to decrease, the more orders are taken into account.
- If we vary μ against its choosen value, for instance $\hat{s}/2 < \mu < 2\hat{s}$, we get an estimate on the contribution of higher order corrections.
- Of course, this variation and the choice of scales is arbitrary and there is no statistical interpretation. Nethertheless, it proved to produce serious estimates, which were confirmed by higher order calculations.

Consider again the picture of proton-proton interactions:



- So far, we discussed how partonic cross sections can be calculated from scattering amplitudes of the interacting partons.
- Now we need a handle on which interactions we expect in proton-proton collisions. In other words, we need to know the propabilities for all possible interactions in a *pp* collision.
- Maybe you are not surprised to hear that these propabilities depend on the cms energy of the collision.

Parton Distribution Functions (PDFs)

Def.: **parton distribution function** $f_p(x, \mu)$; $f_p(x, \mu)dx$ gives the propability to find a parton p with longitudinal momentum fraction x of the proton's total momentum at the energy scale μ.

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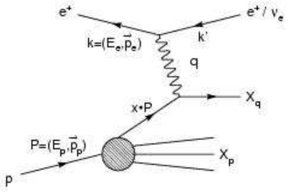
- Without going into detail we note that PDFs are obtained from **deep inelastic scattering (DIS)** experiments with electron-proton collisions. (Experiments have been performed at SLAC, CERN, Fermilab and DESY).
- As the electron has no inner structure and no strong interaction, it is the perfect candidate to probe the distribution of quarks inside the proton!



[picture taken from desy.de]

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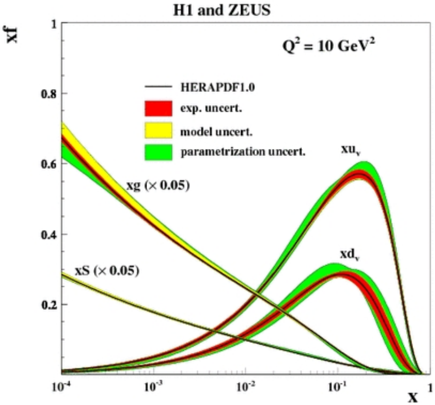
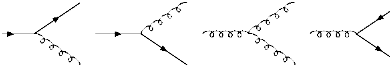


[picture taken from desy.de, published in JHEP **1001** (2010) 109, also available on arXiv:0911.0884 [hep-ex]]

Parton Evolution

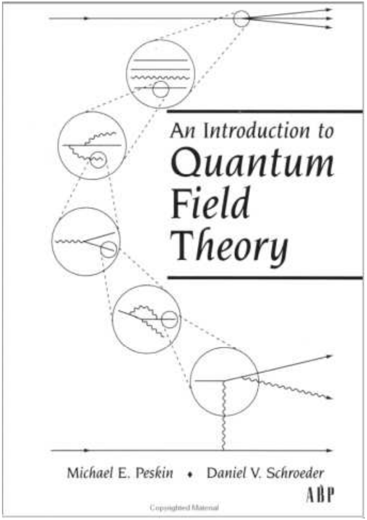
Why do the PDFs depend on the cms energy of the scattering experiment?

- A parton (quark, anti-quark or gluon) moving inside the proton can split into other partons.



- The splitting propabilty can be calculated from QCD. Here, we define a splitting function $P_{ij}(z)$ which gives the propability that a parton of type i with mom. p converts into a parton of type j, carrying the fraction z of p.
- Obviously, the calculations of the splitting functions involve the strong coupling α_s, which is a function of the energy μ.
- The so-called parton evolution along the energy μ is described by a set of differential equations, the *Altarelli-Parisi equations*, where the PDFs $f_i(x/z, \mu)$ are convoluted with the splitting functions $P_{ij}(z)$. If you know the PDFs at a scale μ from experiment, you can compute them for a different scale μ' by using these equations.

Parton Evolution



The parton evolution can be understood as follows:

In a scattering experiment you probe the inner structure of the proton. With increasing scattering energy, you resolve more and more substructures. New virtual particles show up! Thus, **a collider serves as a microscope for subatomaric distances!** However, the measurement itself e.g. the scattering process, is so to say part of what you measure. As a consequence, your measurement depends on the scattering energy.

Hadronic Cross Section

Consider as ex. **top-quark pair production** plus some remnant X:

$$pp \to t\bar{t} + X$$

At the partonic level you have:

$$q\bar{q} \to t\bar{t} + X \qquad \text{and} \qquad gg \to t\bar{t} + X$$

We define

- the hadronic cms energy s
- the partonic cms energy $\hat{s} = x_1 x_2 s$
  (x_1, x_2 denote the parton's momentum fractions)
- dimensionless variables $\rho_h = \frac{4m_t^2}{s}$ and $\rho = \frac{4m_t^2}{\hat{s}}$

Factorization Approach

We can factorize the **hadronic cross section** into the PDFs and the partonic cross section $\hat{\sigma}_{ij}$, which describes the hard (=high energetic) interaction:

$$\sigma_{pp\to t\bar{t}X}(\rho_h, m_t^2) = \sum_{ij=q,\bar{q},g} \int_0^1 dx_1 \int_0^1 dx_2 \int_{\rho_h}^1 \delta\left(\frac{\rho_h}{x_1 x_2} - \rho\right) f_i(x_1, \mu_f^2)\, f_j(x_2, \mu_f^2)\, \hat{\sigma}_{ij}(\rho, m_t^2, \mu_f^2, \mu_r^2)$$

- We have to integrate over all possible parton mom. fractions x_1 and x_2.
- The delta function relates the partonic to the hadronic cms energy.
- The factorization approach neglets small interferences between the hard interaction and the parton evolution.

We distinguish between

- factorization scale μ_f: scale at which we evaluate the PDFs as well as real radiation corrections in higher orders of perturbation theory.
- renormalization scale μ_r: used in renormalization of UV-divergences caused by virtual (loop) corrections, see discussion above.

Jets

What do we expect after the collision has taken place?

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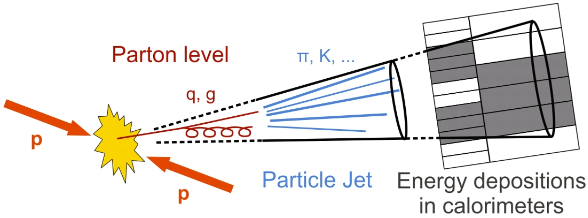
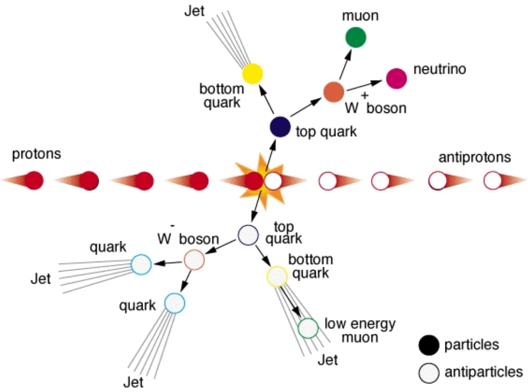


[picture from the CMS homepage: http://cms.web.cern.ch]

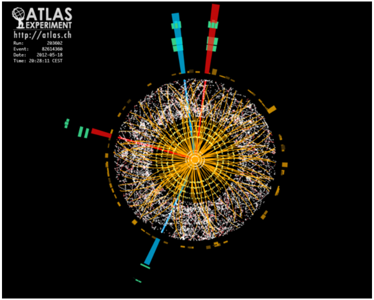
Heavy particles, which are primary produced, will decay inside the detector. Color-charged particles will hadronize. This means that the decay products form colour-neutral bound states, as the energy goes down due to the various splittings. One observes a whole bunch of decay products which is called a jet.

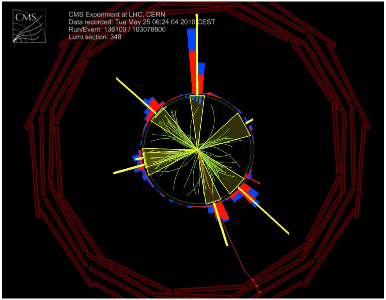
A typical top- anti top event at the Tevatron ($p\bar{p}$ collision):



For the LHC, just replace the antiprotons by protons.

- Particle tracks (curved, due to the magnetic field) are displayed by in the inner detector (→ momentum reconstruction).
- The stable or „long-living“particles are stopped via emmision of bremsstrahlung in the calorimeter (→ energy measurement).
  There are hadronic and leptonic calorimeters.
- Muons (heavy electrons) manage to travel through the inner detector layers and produce signals in the so-called muon chambers.

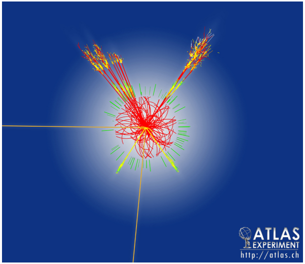




Event Simulation

In order to compare theory to experiment, one has to **simulate events**. This is done with so-called **event generators**.

- Event generators produce parton showers, using Monte Carlo methods.
- At high energies, perturbative QCD can be applied.
- At low energies where hadronization begins, different approaches are required!



Simulated Higgs event from the ATLAS homepage

Why New Physics (NP)?

The **Standard Model of Particle Physics** explains all forces (besides gravity) and elementary particles, which have been probed by collider experiments so far. It has been tested to energies up to several hundered GeV.

Do we expect it to hold for all energies?

Why New Physics (NP)?

The **Standard Model of Particle Physics** explains all forces (besides gravity) and elementary particles, which have been probed by collider experiments so far. It has been tested to energies up to several hundered GeV.

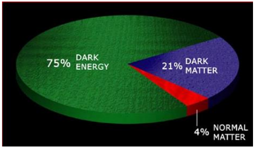
Do we expect it to hold for all energies?

Rather not.

There are various reasons...

... from experimental Astrophysics:

- From the rotation of galaxies and certain gravity-lensing effects, one concludes that there is more matter we actually see → **Dark matter**. This could be explained by weakly interacting massive particle (WIMPs). However, there is no candidate in the SM!
- The observed accelarated expansion of the universe further asks for an unknown form of energy. As we don't know anything about it, we call it **Dark energy**.



- The observed universe consits of matter and radiation. There is no anti-matter. After the big-bang, nearly all matter annihilated with anti-matter. For some reason, there must have been more matter at the beginning. Indeed, the SM predicts a matter/anti-matter asymmetry in weak decays! However, this is not sufficient. The LHCb experiment is designed to adress this question.

... from the experimental observation that neutrinos have masses:

The SM predicts neutrino to be massless. In fact, no neutrino masses have been determined so far. On the other hand one has observed the phenomenon of so-called neutrino oscillations. This requires a non-vanishing mass.

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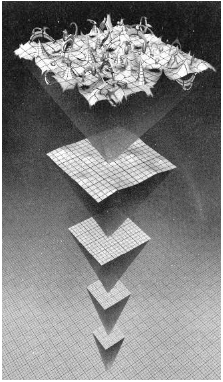
There are various ways to obtain masses:

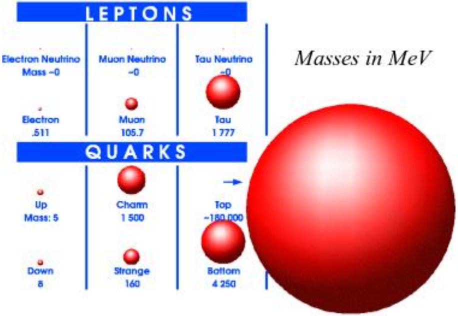
- by adding right-handed neutrinos, which are neutral w.r.t. to all SM forces: This is a trivial extension of the SM. I personally would still call it standard model.
- by allowing for lepton-number violating processes: This allows for so-called Majorana masses of neutral particles. The lepton-number is a conserved quantity in the SM. However, this is not a necesarrily needed. In order to answer this question, people are searching for neutrinoless double-beta decays, which would violate lepton number.
- by simply assuming that the SM is an effective theory with cut-off Λ: Then you can write a dimension 5 operator $\bar{\nu}_L H^* H \nu_L/\Lambda$ consisting of (left-handed) neutrinos and the Higgs field, which produces a mass via the Higgs mechanism.

... from the expectation, that quantum gravity effects should become important at the Planck length.

The SM does not describe gravity, which is completely negligible for elem. particle interactions. It assumes a *smooth spacetime*. On the other hand, general relativity predicts that masses and energy bend the space. If the cms energy of an interaction is high enough (about the Planck scale $M_{\mathrm{Pl}} = \sqrt{\hbar c/G} \approx 10^{19}\mathrm{GeV}$), it would create a black hole (which immediately decays again). In that energy region, effects of gravity have to be taken into account, as the assumption of a smooth spacetime is no longer true.



Finally, there is no theoretical prediction for the values of the SM input parameters (18 of them are independent). They have to be taken from experiment. Especially, the strong hierarchy of the observed fermion masses is non-understood:



[taken from www-d0.fnal.gov]

On the other hand, the ratio m_W/m_Z is predicted as a function of the so-called weak-mixing or Weinberg angle θ_W, which also relates the strength of the e.m. and weak interactions. Here, theory and experiment match within errors!

If we now assume, that the SM has a cut-off Λ, it does not have any UV divergences by construction. Nethertheless, we still have to distinguish between bare parameters in the Lagrangian and physical parameters, which take into account the quantum effects of the theory.

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If we compute quantum corrections to the bare Higgs mass m_{h0} due to a virtual particle of mass m, we find

$$\delta m_h^2 = -a\,\Lambda^2 + b\,m \ln\left(\frac{\Lambda}{m}\right) + \cdots$$

On the other hand we want to have

$$m_h^2 = {m_{h0}}^2 + \delta m_h^2 \approx (125.5\,\text{GeV})^2$$

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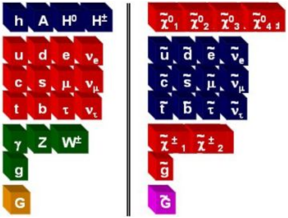
  This is considered to be *unnatural* and one speaks of the *hierarchy problem* of the SM.
- If however $m \leq \Lambda \approx 1\,\mathrm{TeV}$, there is no tuning! That is why we expect NP within the reach of the LHC!

Supersymmetry

Symmetry between fermions and bosons:

- For each SM spin 1 boson, there is an addtional (spin 1/2) fermion with the same quantum numbers (colour, electroweak isospin, charge).
- For each SM fermion, there is an additional spin 0 boson.
- There are two complex Higgs doublets giving rise to 5 different Higgs fields (2 of them charged).



[taken from etp.physik.uni-muenchen.de]

Spins: 0 $\frac{1}{2}$ 1 $\frac{3}{2}$ 2

Supersymmetry

The SUSY partners need to be heavy, as they have not been observed yet.
$\Rightarrow$ SUSY must not be an exact symmetry of nature. Otherwise, the additional particles would have the same masses as their respective SM partners.

SUSY is an interesting extension of the SM because

- it cancels the Λ^2 term in δm_h^2, because the quantum corrections due to the superpartners involve Λ^2 with the opposite sign!
- It provides a candidate for dark matter, the lightest so-called Neutralino (spin 0), which is stable.

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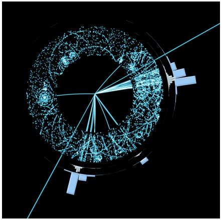
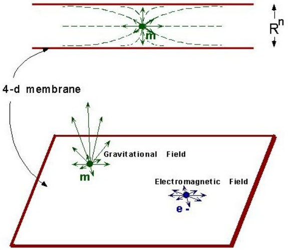
Collider signals:

The Neutralino would escape detection. A typical SUSY signal is therefore missing transverse Energy, which can be reconstructed from momentum conservation. On the right, you see a simulated event of the ATLAS collaboration.



Extra dimensions

Idea: Gravity propagates in n compact extra dimensions and thus gets deluted. [Arkani-Hamed, Dimopoulos, Dvali (ADD), 1998]



[picture from www.stanford.edu]

The „true“ Planckscale $M_{\mathrm{Pl}(4+n)}$ of the higher dimensional theory is close to the electroweak scale m_W of the SM. $\rightarrow$ no hierarchy problem (HP). The extra dim. have to be smaller than the current bound $R \leq 44\mu m$, to which the $1/r$-potential of gravity has been testet.

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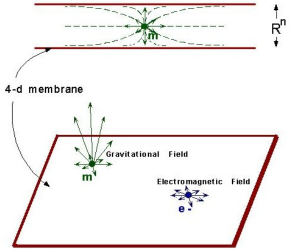

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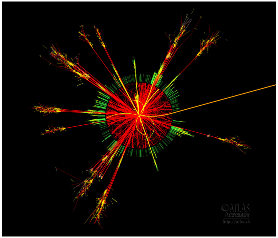
Alternative Ansatz [Randall, Sundrum, 1999]: One curved or „warped“ extra dimension of the Planck size. This also solves the HP and can be used to explain the fermion mass hierarchies, if all SM fields live in the full „bulk“.

Collider Signals

Large extra dimensions (ADD):

Black hole production!

As the *n*-dim. Planckscale is within the reach of the LHC, microscopic black holes are created which immediately decay „democratically“to SM particles.



Warped Extra dimensions:

Besides black holes, one may observe a spectrum of so-called **Kaluza-Klein (KK) excitations** with masses in the TeV regime.

KK excitations always arise, when SM particles and gravitons are locked into a compact dimension. In analogy to QM particles in a box, there will be ground states (associated with the SM particles) and excitations! These are also there in the ADD modell, but the spectrum is very dense such that one observes a continous enhancement of the cross section but no peaks!

Summary & Outlook

In order to calculate cross sections for LHC processes, the problem is factorized into various pieces.

- The partonic cross section: describes the hard scattering interaction; It can be calculated perturbatively with the help of Feynman rules. The theoretical uncertainty can be estimated by variation of the renormalization scale.
- The initial parton distribution: taken from DIS experiments; It can be applied to LHC physics with the concept of parton evolution (Altarelli Parisi equations).
- The parton showering: Achieved by event generators based on Monte Carlo simmulations, which use LO perturbative QCD as input.

The factorization works because the individual processes appearing at different energy scales have tiny interference with each other.

Summary & Outlook

For high precision predictions, one also has to consider bound-state effects at lower energies. These can be obtained by non-perturbative approximations like non-relativistic QCD or lattice QCD.

The physics of baryons (3-quark bound states) and mesons (2 quarks) can also be desribed with the help of chiral perturbation theory, which assumes the quark-bound states as the fundamental degrees of freedom.

Literature:

- M. E. Peskin & D. V. Schroeder, *An Introduction to Quantum Field Theory*, Westview Press
- S. Weinzierl, Mathematical aspects of particle physics, Lecture script 2007