LHC Theory
Lecture 1: Calculation of Scattering Cross Sections

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Outline

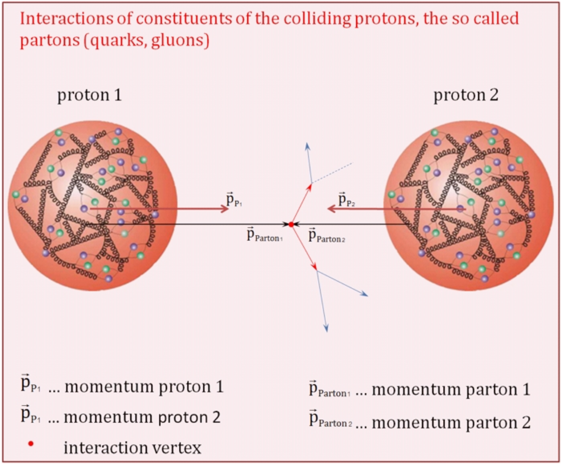
1. Proton-Proton Collisions at the LHC
2. The Basic Playground: Quantum Field Theory (QFT)
3. Perturbation Theory

The LHC

basic facts:

- Proton-proton collider: bunches of protons are brought to collision at a center of mass (cms) energy of several TeV/c^2 (Tera $\equiv T = 10^{12}$).
- About 115 billion particles per bunch of $\sim$ 7cm length and the diameter of a hair!
- As protons are very small objects (ϕ of about 1fm$=10^{-15}$m), nearly all particles miss each other, except for a few events (one in the most ideal case). see http://www.lhc-facts.ch for more details! (homepage in German)
- Protons are no elementary particles. The LHC provides enough energy to resolve the inner structure of the proton.
- The interaction of protons has to be understood as interactions among its constituents, which are called partons.
- The partons consist of quarks and gluons. The former are charged fermions ($+2/3$ for up-type and $-1/3$ for down-type quarks) which strongly interact which each other through the exchange of gluons.

Parton interactions



[picture taken from kjende.web.cern.ch]

When the partons collide, they are annihilated and their energy is coverted to „*something else*".

Remember: $$E = \gamma m c^2, \quad \gamma = \sqrt{1 - v^2/c^2} \equiv \sqrt{1 - \beta^2}$$

The job of theorists is to predict „*something else*" from theory!

Of course, the theory which we call the **Standard Modell of elementary particle physics (SM)** was motivated by older experiments at lower energies.

We had e^+/e^- collisions (LEP/CERN), electron-proton collisions (SLAC,DESY), proton-antiproton collisions (Tevatron), and many more.

Thus, the task is to extrapolate the SM to LHC energies for *pp* collisions, calculate its prediction, and compare with the new measurements.

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Thus, the task is to extrapolate the SM to LHC energies for *pp* collisions, calculate its prediction, and compare with the new measurements.

We are looking for:

- Evidences for the production and subsequent decay of the Higgs particle, the last missing ingredient of the SM. (It seems that we were lucky!)
- Deviations from the SM predictions at highest energies, which give hints for physics beyond the SM. (No evidences yet!)

The Cross Section

The observable quantity, which allows the comparison between theory and experiment, is the scattering cross section σ.

From the experimental side it is defined by:

$$\text{Number of scattering events} = \sigma \, l_1 l_2 \int d^2x \, \rho_1(x)\rho_2(x)$$

Here, l_i and $\rho_i(x)$ denote the length and particle densities of the bunch i. The densities are integrated over the bunch area.

For constant densities (which is however not the case in real beams), the formula simplifies to:

$$\text{Number of scattering events} = \frac{\sigma \, N_1 N_2}{A}$$

$N_{1/2}$: numbers of particles per bunch; A: common area of the bunches.

The differential cross section:

Usually, we wish to measure not only what the final-state particles are, but also the momenta of those. Therefore, one defines the *differential cross cection* $d\sigma/(d^3 p_1 ... p_n)$.

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Theoretical expression:

$$d\sigma = \frac{1}{2E_a 2E_b |v_a - v_b|} \left(\Pi_f \frac{d^3 p_f}{(2\pi)^3 2E_f} \right) |\mathcal{M}(p_a, p_b \rightarrow \{p_f\})|^2 (2\pi)^4 \delta \left(p_a + p_b - \sum_f p_f \right)$$

$|v_a - v_b|$: relative velocity (=longitudinal momentum/energy) of the beams viewed from the lab frame

p_a, p_b, E_a, E_b: initial state momenta and energies; p_f: final state momenta

$\mathcal{M}(p_a, p_b \rightarrow \{p_f\})$: scattering amplitude, $\{p_f\} = p_1, p_2, ..., p_n$

The Scattering amplitude

The square of the scattering amplitude $|\mathcal{M}|^2$ is, as usual, a measure for the scattering propability. However, ordinary (that is non-relativistic) quantum mechanics can not be used, as the particles have high velocity!

What do we expect?

The Scattering amplitude

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What do we expect?

The amplitude should be given by the expaction value of some interaction operator $\mathcal{O}$ (sandwiched between an initial and a final multi-particle state):

$$\boxed{\langle\{p_f\}|\mathcal{O}|p_a, p_b\rangle = (2\pi)^4\delta\left(p_a + p_b - \sum_f p_f\right)\mathcal{M}(p_a, p_b \to \{p_f\})}$$

Here, the momenta $p_a,\ p_b$ should be undersood as central values of wave packets. The next section deals with the question, how the operator $\mathcal{O}$ looks like and how we define a state $|...\rangle$.

Partonic vs. Hadronic Cross Section

For LHC physics, an additional complication arises, as we do not know which parts of the protons interact with each other. One distinguishes between a

- *partonic cross section* $\hat{\sigma}$: has to be calculated seperately for individual annihilation and scattering processes.
- *hadronic cross section* σ: sums over all possible initial states weighted with the propability to find them inside the proton.

In this lecture, we will always refer to partonic processes. The connection to hadronic cross sections and thus *real* LHC observables is discussed in the second lecture.

Quantum Field Theory (QFT)

- Relativistic (quantum) theory for the movement and interactions of elementary particles
- Particles are represented by fields. What is meant by this?
  $\longrightarrow$ Discuss below!
- QFT's are commonly formulated in so-called natural units featuring $\hbar = c = 1$.
  $\longrightarrow$ Mass, Energy, momentum, inverse time and length scales have the same physical dimension, known as mass dimension:
  $[m] = [E] = [p] = [t^{-1}] = [l^{-1}] = 1$

To get started, consider a classical field theory, the *Maxwell Theory* of electro-magnetism.

Maxwell Theory: covariant notation

Maxwell's equations are invariant under Lorentz transformations.
This can be seen, when we introduce a so-called covariant notation:

$$A(x) = (\Phi(t,\vec{x}), A^1(t,\vec{x}), A^2(t,\vec{x}), A^3(t,\vec{x}))^T \equiv A^\mu$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\partial_t, \vec{\nabla}\right), \quad \partial^\mu = \frac{\partial}{\partial x_\mu} = \left(\partial_t, -\vec{\nabla}\right)$$

- A^μ: four-potential; Φ, $\vec{A}$: scalar and vector potential
- Define field strength tensor: $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

Maxwell's equations in the vacuum (and natural units):

$$\boxed{\partial_\mu F^{\mu\nu} = j^\nu\,, \qquad \epsilon_{\mu\nu\sigma\tau}\partial^\nu F^{\sigma\tau} = 0}$$

Maxwell Theory: Lagrange density

Maxwell's equations can be derived from a Lagrange density $\mathcal{L}$ through variation of the action.

$$S = \int d^4x\, \mathcal{L}(A_\mu(x), \partial_\nu A_\mu(x), x) \quad \xrightarrow{\delta S[A_\mu]} \quad \text{Euler-Lagrange eq.:}$$

$$\left(\frac{\partial}{\partial A_\mu} - \partial_\nu \frac{\partial}{\partial(\partial_\nu A_\mu)} \right) \mathcal{L}(A_\mu(x), \partial_\nu A_\mu(x), x) = 0$$

$$\boxed{\mathcal{L}_{\text{Maxwell}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + j_\mu A^\mu}$$

Symmetry properties:

- *Lorentz invariance*: All Lorentz indices are contracted.
- *Gauge invariance*: The replacement $A^\mu(x) \to A^\mu - \partial^\mu \chi$ does not change $\mathcal{L}$. $\chi(x)$ is an arbitrary scalar potential.

$$\boxed{\mathcal{L}_{\mathrm{Maxwell}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + j_\mu A^\mu}$$

Interpretation of A^μ and j^μ within a quantum theory?

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Interpretation of A^μ and j^μ within a quantum theory?

We know that electro-magnetic radiation is nothing else but light!
$\Rightarrow$ A^μ should be related to the relativistic description of the photon.

$\Rightarrow$ Maxwell's eq. are understood as equations of motion (EOMs) for massless „spin"-1 particles! (Question: Why do I use quotation marks for spin?)

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We know that the current j^μ describes the interaction of electro-magnetic radiation with an external charge. At the fundamental level, this should be interpreted as the interaction of a photon with charged particles, such as electrons!

$\Rightarrow$ We need a relativistic field theory for fermions!

Dirac Theory: (Very) short summary

- Need a representation of the Lorentzgroup for fields with 2 degrees of freedom (dof) corresponding to spin up/down.

- One finds two independent so-called spinor representations which describe particle & anti-particle!

- Define the Lorentz vector $\sigma_\mu = \hat{\sigma}^\mu \equiv (\sigma_0, \sigma_1, \sigma_2, \sigma_3)$, where $\sigma_{i=1,2,3}$ denote the Pauli matrices, and $\sigma_0 = \mathbf{1}_{2\times 2}$.

- The two spinor repr. are combined by introducing (4×4) Dirac matrices

$$\gamma^\mu = \begin{pmatrix} \mathbf{0} & \sigma^\mu \\ \hat{\sigma}^\mu & \mathbf{0} \end{pmatrix}$$

and defining a complex 4-component-spinor $\psi(x)$, which describes both fermions and anti-fermions.

- The relativistic EOM for fermions, the Dirac equation, reads:

$$\boxed{(i\gamma^\mu \partial_\mu - m\,\mathbf{1}_{4\times 4})\psi(x) = 0}$$

Dirac Theory: Lagrange density

Introduce the Dirac conjugate: $\overline{\psi} \equiv \psi^\dagger \gamma^0$

Using $\gamma^0 (\gamma^\mu)^\dagger \gamma^0 = \gamma^\mu$, there is also a conjugate Dirac equation:

$$\overline{\psi}(x)(i\gamma^\mu \overleftarrow{\partial_\mu} + m\,\mathbf{1}_{4\times 4}) = 0$$

The Dirac Lagrangian reads (using $\overleftrightarrow{\partial_\mu} \equiv \partial_\mu - \overleftarrow{\partial_\mu}$):

$$\mathcal{L}_{\rm Dirac} = \overline{\psi}(x)\left(\frac{i}{2}\gamma^\mu \overleftrightarrow{\partial_\mu} - m\,\mathbf{1}_{4\times 4}\right)\psi(x)$$

or, applying integration by parts w.r.t. $\int d^4x$:

$$\mathcal{L}_{\rm Dirac} = \overline{\psi}(x)\left(i\gamma^\mu \partial_\mu - m\,\mathbf{1}_{4\times 4}\right)\psi(x)$$

The Gauge Principle

We are still looking for an interpretation of the current j^μ on the quantum level:

$$\text{Naive guess:} \quad j^\mu \propto \overline{\psi}\gamma^\mu\psi$$

Indeed, the guess is right, but can we derive this from theory?

Idea: Try to apply a gauge transformation to $\psi(x)$.

Obviously, $\mathcal{L}_{\mathrm{Dirac}}$ is invariant under a global phase rotation:

$$\psi(x) \to e^{i\alpha}\psi(x)$$

However, it is not invariant under a local one:

$$\psi(x) \to e^{i\alpha(x)}\psi(x)$$

The invariance under local transformations fails because of the derivative ∂_μ! In order to obtain a covariant theory, one has to replace the derivative by a so-called *covariant derivative*. A general ansatz is given by

$$D_\mu = \partial_\mu - igV_\mu(x),$$

with g some constant and $V_\mu(x)$ some vector field. If we require the transformation property

$$V_\mu(x) \to V_\mu(x) + \frac{1}{g}\,\partial_\mu \alpha(x),$$

the (modified) Dirac theory is covariant w.r.t. local phace rotations of $\psi(x)$.

However, the latter transf. equals the gauge transf. of the Maxwell field $A_\mu(x)$!

$$\boxed{\text{Identify} \quad V_\mu(x) = A_\mu(x) \quad \text{and} \quad g = e\,Q}$$

Here, e is the elementary charge of the electron and Q the charge quantum number ($= -1$ for the electron).

Quantum Electro Dynamics (QED)

Combining Maxwell's Theory with the (covariant) Dirac Theory, we obtain a local relativistic theory of electro-magnetism:

$$\boxed{\mathcal{L}_{\mathrm{QED}} = -\frac{1}{4}\, F_{\mu\nu}(x)F^{\mu\nu}(x) + i\,\overline{\psi}(x)\gamma^{\mu}D_{\mu}\psi(x) - m\,\overline{\psi}(x)\psi(x)}$$

The covariant derivative D_μ gives rise to the interaction term

$$\mathcal{L}_{\mathrm{int}} = -e\,\overline{\psi}(x)\gamma^{\mu}A_{\mu}(x)\psi(x),$$

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!!! Note that up to know, the discussion also holds for classical field theories. In order to obtain a quantum theory, the next step is the so-called field quantization. However, some remarks should be made before. **!!!**

Masses in the theory

Question: We have a mass term $m\overline{\psi}\psi$ for fermions. Why don't we write a mass term $m_A A_\mu A^\mu$ for photons???

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Gauge invariance!!!

A mass term $m_A A_\mu A^\mu$ would violate local gauge inv. and thus is forbidden.

$\Rightarrow$ QED **predicts** that the photon is massless!

Indeed, gauge symmetry is more than an accidental property of $\mathcal{L}$, it is rather a **construction principle for QFTs**.

HOLY COW

GAUGE PRINCIPLE

Generalization of QED

The symmetry group of the QED gauge transformation is the unitary group $U(1)$. The *infinitesimal* $U(1)$ transformation acting on $\psi(x)$ is given by

$$U(1): \quad e^{i\alpha(x)} = 1 + i\alpha(x) + \ ...$$

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$\Rightarrow$ The infinitesimal transformations are given by

$$SU(3): \quad e^{i\theta^a(x)T^a} = 1 + i\theta^a(x)T^a + \; ...$$

where in general T^a are $N \times N$ matrices, which generate the SU(N) transformation. For $SU(N)$ one has $N^2 - 1$ generators.

$SU(3): \quad a = 1, ..., 8$ (index of adjoint representation)

The generators do not commute: $\quad [T^a, T^b] = if^{abc}T^c$

Quantum Chromo Dynamics (QCD)

Generalize covariant derivative: $D_\mu = \partial_\mu - ig_s\, T^a G_\mu^a$

Generalize field strength tensor: $G_{\mu\nu}^a = \partial_\mu G_\nu^a - \partial_\nu G_\mu^a + g_s f^{abc} G_\mu^b G_\nu^c$

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Write down a gauge invariant Lagrange density:

$$\mathcal{L}_{\mathrm{QCD}} = -\frac{1}{4}\, G^a_{\mu\nu}(x) G^{a\,\mu\nu}(x) + i\,\overline{\psi}_i(x)\gamma^\mu D_\mu \psi_i(x) - m\,\overline{\psi}_i(x)\psi_i(x)$$

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Write down a gauge invariant Lagrange density:

$$\mathcal{L}_{\rm QCD} = -\frac{1}{4}\, G^a_{\mu\nu}(x) G^{a\,\mu\nu}(x) + i\,\overline{\psi}_i(x)\gamma^\mu D_\mu \psi_i(x) - m\,\overline{\psi}_i(x)\psi_i(x)$$

(A summation over a and i is understood.)

Remarks:

- The three copies (ψ_r, ψ_g, ψ_b) of ψ are known as colors.
- The gauge fields G^a_μ $(a = 1, ..., 8)$ are known as gluons.
- The theory is known as Quantum chromo dynamics (QCD) and describes the strong interaction within the nucleus.

QCD vs. QED

$\mathcal{L}_{\rm QCD}$ and $\mathcal{L}_{\rm QED}$ differ

- by the couplings $g_s\, T^a \;\Leftrightarrow\; e\, Q$
- by self interactions of gluons, which are absent for photons:

$$g_s f^{abc} \partial_\mu G^a_\nu G^{b\,\mu} G^{c\,\nu} \quad \text{and} \quad g_s^2 f^{abc} f^{ade} G^b_\mu G^c_\nu G^d_\mu G^e_\nu$$

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Important: After quantization, the so-called triple and quartic gluon couplings give rise to a different phenomenology of strong and electromagnetic interactions. The e.m. force creates a *Coulomb potential* $\sim 1/r$, where the strong force creates a *linear potential* $\sim r$! $\quad \Rightarrow$ Discuss later

Weak Interactions

The theory of weak interactions is based on the sym. group $SU(2)$. However, there are problems! From experiment we know that

- Weak gauge bosons $W_\mu^\pm$ and Z_μ are massive! $\longleftarrow$ forbidden by gauge invariance!
- Weak interactions violate parity. This means they distinguish between fermions with the spin axis pointing into the direction of movement (right-handed), and the opposite case (left-handed). Thus, one has to assign different (weak) charge-quantum numbers. A fermion-mass term however combines both types. Therefore, also fermion masses are forbidden by gauge invariance!

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Solution: Generate mass terms via gauge invariant interactions with a scalar background field, which transforms as a doublet under $SU(2)$ and develops a vacuum expectation value (VEV) different from zero. $\Rightarrow$ Higgs mechanism

The so-called Higgs field can be excited and gives rise to a massive spin 0 particle, the Higgs particle.

Simplified Higgs model (coupled to $U(1)$ instead of $SU(2)$):

- Complex scalar field $\phi(x) = \sigma(x) + i\varphi(x) \equiv \phi_0 + h(x) + i\varphi(x)$

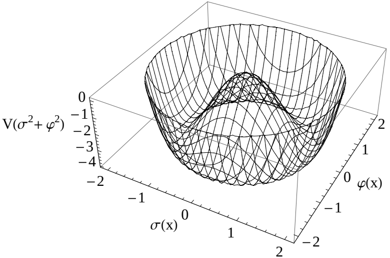
$$\mathcal{L} = \frac{1}{2}\,\partial_\mu \phi^*(x)\partial^\mu \phi(x) + \mu^2|\phi(x)|^2 - \lambda|\phi(x)|^4$$

- Higgs potential

$$V(\phi) = -\mu^2|\phi(x)|^2 + \lambda|\phi(x)|^4$$

- For $\mu^2 > 0$ and $\lambda > 0$ there is a minimum different from zero:

$$\phi_0 = \left(\frac{\mu^2}{2\lambda}\right)^{1/2}$$



- The field $\varphi(x)$ is the dof along the minimum $\phi(x) = \phi_0$, the Higgs field $h(x)$ is the dof in the perpendicular direction.

- The coupling of a $U(1)$ gauge field to ϕ_0 forms a mass term.

The Standard Model of Elementary Particle Physics (SM)

The SM of elementary particle physics is a QFT based on the symmetry group

$$\boxed{SU(3)_c \times SU(2)_L \times U(1)_Y}$$

$SU(3)_c$: gauge group of QCD (c=color).

$SU(2) \times U(1)_Y$: gauge group for so-called electroweak interactions, which contain both weak and electro-magnetic interactions.

- Y is called hypercharge and is related to the electric charge Q and the weak isospin T^3 with respect to $SU(2)_L$: $Q = T^3 + Y$

  (You may also find the definition $Q = T^3 + Y/2$ in the literature.)
- The subscript L indicates, that the coupling is to left-handed fermions.

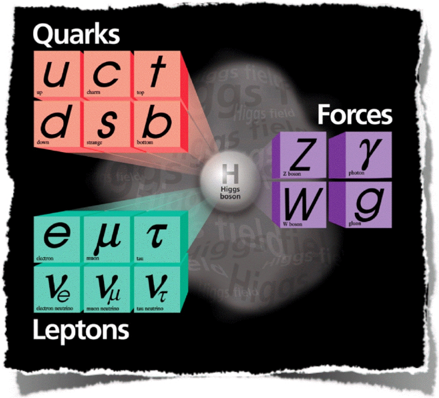
After electroweak symmetry breaking (EWSB) through couplings to the Higgs VEV, the symmetry of the theory's ground state is reduced to

$$\boxed{SU(3)_c \times SU(2)_L \times U(1)_Y \stackrel{EWSB}{\longrightarrow} SU(3)_c \times U(1)_{e.m.}}$$

$\Longrightarrow$ 8 massless gluons, 1 massless photon, and 3 massive bosons $(W^{\pm}, Z)$

Fermions: The SM fermions (spin 1/2) are classified into leptons (no color-charge ⇔ neutral w.r.t. $SU(3)_c$), and quarks (color-charged).

Bosons: Apart from the gauge bosons g^a, γ (helicity 1) and $W^{\pm}, Z$ (spin 1), there is the Higgs boson (spin 0) as remnant of the EWSB mechanism.



[picture taken from kip.uni-heidelberg.de]

Interim Conclusion

In a relativistic field theory, we have to distinguish between fields, which live in different representations of the Lorentz group:

- Vector fields (A^μ, etc.) live in the vector representation. They transform as $A^\mu \to A'^\mu = \Lambda^{\mu\nu} A_\nu$ ($\Lambda =$ Lorentz transf.)
- Spinor fields live in the spinor representation. Without going into detail here, we note that γ^μ transforms as a vector.
- Scalar fields are invariant under Lorentz transformations (trivial representation).

In a QFT, the above fields are identified with spin 1, spin 1/2 and spin 0 particles, respectively!

Field Quantization

Until now, we did not specify the fields $A_\mu(x)$, $\psi(x)$, etc. from the mathematical side. We just conjectured that they should somehow describe the particles of a local quantum field theory.

(The classical interpretation $A^\mu = (\phi, \vec{A})^T$ does not apply to a local QFT.)

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There are two distinct ways of field quantization, which however lead to the same physical results:

- *Canonical quantization*: based on an algebra of so-called creation and annihilation operators within the Hamilton operator.
- *Path integral quantization*: Functional approach; uses Lagrange density or, more precisely, the classical action and interpretes the fields as integration variables of a path integral.

Canonical Quantization

Let us cosider the most simple example, the quantization of a neutral spin-0 particle without interaction: Introduce the (real) *Klein-Gordon field* $\phi(x)$, where

$$\boxed{\mathcal{L}_{\mathrm{KG}} = \frac{1}{2}\Big(\partial_\mu\phi(x)\partial^\mu\phi(x) - m^2\phi^2(x)\Big)}$$

with the **classical EOM**: $(\partial_\mu\partial^\mu + m^2)\phi(x) = 0$

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Let us cosider the most simple example, the quantization of a neutral spin-0 particle without interaction: Introduce the (real) *Klein-Gordon field* $\phi(x)$, where

$$\mathcal{L}_{\rm KG} = \frac{1}{2}\Big(\partial_\mu\phi(x)\partial^\mu\phi(x) - m^2\phi^2(x)\Big)$$

with the **classical EOM**: $(\partial_\mu\partial^\mu + m^2)\phi(x) = 0$

Momentum density conjugate: $\pi(x) = \dfrac{\partial\mathcal{L}}{\partial(\partial_0\phi)} = \partial^0\phi(x)$

Hamilton density $\left(\text{Hamilton operator: } H = \int d^3x\,\mathcal{H}\right)$

$$\mathcal{H} = (\pi\partial_0\phi - \mathcal{L})\Big|_{\partial^0\phi=\partial_0\phi=\pi(x)} = \frac{1}{2}\left(\pi^2(x) + (\vec{\nabla}\phi)^2 + m^2\phi^2(x)\right)$$

Quantization:

Replace functions $\phi(x)$ and $\pi(x)$ by operators $\Phi(x)$ and $\Pi(x)$ satisfying commutation relations:

$$[\Phi(x), \Pi(y)]_{x^0=y^0} = i\delta(\vec{x} - \vec{y})\,, \quad [\Phi(x), \Phi(y)]_{x^0=y^0} = [\Pi(x), \Pi(y)]_{x^0=y^0} = 0$$

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When inserted into the classical EOM, the **field operator** should satisfy the on-shell condition $p^2 = m^2$ (particle on the mass shell). Make the **Ansatz**:

$$\boxed{\Phi_{\text{free}}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{2E_p} \left(a(p)\, e^{-ipx} + a^\dagger(p)\, e^{ipx} \right)}$$

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Remark: Though not obvious, the integration $\int \frac{d^3p}{2E_p}$ is Lorentz invariant. One can show in general that $\int d^4p\, \delta(p^2 - m^2)\Theta(p^0) \cdots = \int \frac{d^3p}{2E_p} \cdots$

From the above Ansatz it follows that

$$[a(p), a^\dagger(p')] = 2E_p\delta(\vec{p}-\vec{p}')\,, \qquad [a(p), a(p')] = [a^\dagger(p), a^\dagger(p')] = 0\,.$$

In order to obtain the Hamilton operator, we calculate

$$\Pi(x) = \partial^0 \Phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{2E_p} \left(-iE_p\right) \left(a(p)\, e^{-ipx} - a^\dagger(p)\, e^{ipx}\right)$$

and, using $\frac{1}{(2\pi)^3} \int d^3x\, e^{i(\vec{p}-\vec{p}\,')\vec{x}} = \delta(\vec{p}-\vec{p}\,')$ and $a(p) \equiv a_p$, we find

$$\boxed{H = \frac{1}{2} \int \frac{d^3p}{2E_p} E_p \left(a_p^\dagger a_p + a_p a_p^\dagger\right) = \int \frac{d^3p}{2E_p} E_p \left(a_p^\dagger a_p + \frac{1}{2}[a_p, a_p^\dagger]\right)}$$

The last term is proportional to $\delta(0)$ and thus infinite. It corresponds to the sum over all modes of the zero-point energy $E_p/2$.
As only energy differences can be measured, this term is skipped!

Note that a state $|p\rangle$ is defined via

$$|p\rangle = a^\dagger(p)|0\rangle, \qquad a(p)|p\rangle = |0\rangle, \qquad \langle 0|0\rangle = 1$$

You can easily show that $\langle 0|\Phi(x)|p\rangle = e^{-ipx}$.

Therefore, $a_p^\dagger$ and a_p are interpreted as creation and annihilation operators for particels with momentum p.

Comparison to non-relativistic QM

The analytic form of the Hamiltonian

$$H = \int \frac{d^3p}{2E_p} \, E_p \Big(a_p^\dagger a_p + \frac{1}{2}[a_p, a_p^\dagger] \Big)$$

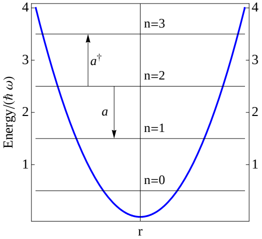
and the algebra of the creation and annihilation operators have a mathematical analogon in non-relativistic QM:

The quantum-mechanical harmonic oscillator

Define ladder operators a and $a^\dagger$

$$H = \hbar\omega \left(a^\dagger a + \frac{1}{2} \right) , \quad [a, a^\dagger] = 1 , \quad a\,|0\rangle = 0 ,$$

$$a^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle , \quad a\,|n\rangle = \sqrt{n-1}\,|n-1\rangle$$



Quantization of particles with spin $\neq 0$:

Fermions (spin $\frac{1}{2}$):

- need seperate creation and annihilation operatos for particles and anti-particles (also needed for a complex Klein-Gordon field!)
- creation/annih. ops. depend on spin $\Rightarrow$ Sum over spins required
- creation/annih. ops. accompanied by (4-component) Dirac spinors, which describe ingoing particles and outgoing anti-particles or vice verca
- All commutators are replaced by anti-commutators $\{a, b\} \equiv ab + ba$. This is crucial in order to obtain the correct statistical behavior of fermions!

Gauge Bosons („spin“1):

- creation/annih. ops. depend on polarization $\Rightarrow$ Sum over pol. required
- creation/annih. ops. accompanied by polarization vectors
- The „naive“ quantization produces a contradiction to Maxwell's equation. Reason: The photon (also the gluon) has 2 physical helicity states ± 1. The field A^μ however has 4 dofs. There is some redundance in the description of the photon. This can be removed by adding the term $-\frac{1}{2\xi}(\partial_\mu A^\mu(x))^2$ to $\mathcal{L}(x)$, where ξ is an arbitrary real parameter.

Correlation Functions

Our goal is to calculate so-called correlation functions, which relate quantum fields at different points of the space-time, and give the expectation value w.r.t. to the ground state $|\Omega\rangle$ of the theory. The latter is referred to as the quantum vacuum. For instance, the *two-point correlation function* is defined by

$$\langle\Omega|T\Phi(x)\Phi(y)|\Omega\rangle\,,$$

where T denotes time ordering. $T\Phi(x)\Phi(y) = \Theta(x^0 - y^0)\,\Phi(x)\Phi(y) + \Theta(y^0 - x^0)\,\Phi(y)\Phi(x)$

If we have interactions, $|\Omega\rangle$ differs from the ground state of the free theory $|0\rangle$!

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Without interactions, we can easily compute the *two-point function*, which goes under the name *Feynman propagator*. It describes the free movement of a particle from a point x in space-time to y, and is given by:

$$\langle 0|T\Phi_{\text{free}}(x)\Phi_{\text{free}}(y)|0\rangle = D_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\,\frac{i\,e^{-ip(x-y)}}{p^2 - m^2 + i\epsilon}$$

One has to use the integral representation $\Theta(u) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} d\lambda\,\frac{e^{i\lambda u}}{\lambda - i\epsilon}$ and $\Theta(-u) = -\frac{1}{2\pi i}\int_{-\infty}^{+\infty} d\lambda\,\frac{e^{i\lambda u}}{\lambda + i\epsilon}$.

Now, we want to study a quantum theory including interactions.
!!! This will change the quantum fields as well as the vacuum !!!

We keep the above example and add an interaction term:

$$H = H_0 + H_{\text{int}} \,, \quad H_0 = \int d^3x \, \frac{1}{2} \left(\Pi^2 + (\vec{\nabla}\Phi)^2 + m^2\Phi^2 \right) \,, \quad H_{\text{int}} = \int d^3x \, \frac{\lambda}{4!} \Phi^4$$

Our goal is to express $\Phi(t, \vec{x})$ and $|\Omega\rangle$ in terms of $\Phi_{\text{free}}(x)$ and $|0\rangle$, from which we can directly compute correlation funcions.

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We start from a fixed time t_0, switch to the *Heisenberg picture* and evaluate the *time evolution* for $t \neq t_0$:

$$\boxed{\Phi(t,\vec{x}) = e^{iH(t-t_0)}\Phi(t_0,\vec{x})e^{-iH(t-t_0)} \overset{\lambda=0}{=} \Phi_{\text{free}}(x) \equiv \Phi_I(t,\vec{x})}$$

$$\left(\Phi(t_0,\vec{x}) = \frac{1}{(2\pi)^{3/2}}\int\frac{d^3p}{2E_p}\left(a_p\,e^{i\vec{p}\cdot\vec{x}} + a_p^\dagger\,e^{-i\vec{p}\cdot\vec{x}}\right)\,, \quad I \text{ stands for } \textit{interaction picure}\right)$$

Thus, the interacting field $\Phi(t,\vec{x})$ can be written as

$$\boxed{\Phi(t,\vec{x}) = U^\dagger(t,t_0)\Phi_I(t,\vec{x})U(t,t_0), \qquad U(t,t_0) = e^{iH_0(t-t_0)}e^{-iH(t-t_0)}}$$

With $H_{\text{int}} = H - H_0$, it is now easy to show that

$$i\frac{\partial}{\partial t}U(t,t_0) = H_I(t)U(t,t_0)\,, \quad H_I(t) = e^{iH_0(t-t_0)}H_{\text{int}}\,e^{-iH_0(t-t_0)} = \int d^3x\,\frac{\lambda}{4!}\Phi_I^4$$

This equation also holds, if we replace t_0 by some other time t'. With $U(t_1,t_2)U(t_2,t_3) = U(t_1,t_3)$, we have

$$U(t,t') = U(t,t_0)U(t_0,t') = e^{iH_0(t-t_0)}e^{-iH(t-t')}e^{-iH_0(t'-t_0)}$$

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The general time evolution operator $U(t,t')$ satisfies

$$\boxed{i\frac{\partial}{\partial t}U(t,t') = H_I(t)U(t,t')\,, \qquad U(t,t) = 1}$$

The solution is given by

$$\boxed{U(t,t') = T\left\{\exp\left[-i\int_{t'}^{t} dt''\,H_I(t'')\right]\right\}}$$

The quantum vacuum $|\Omega\rangle$:

Starting point: Time evolution of $|0\rangle$

$$e^{-iHT}|0\rangle = \sum_{n=0}^{\infty} e^{-iE_nT}|n\rangle\langle n|0\rangle = e^{-iE_0T}|\Omega\rangle\langle\Omega|0\rangle + \sum_{n=1}^{\infty} e^{-iE_nT}|n\rangle\langle n|0\rangle$$

Here we have used $\sum_n |n\rangle\langle n| = 1$ (full set of states) and $H|n\rangle = E_n|n\rangle$.

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$$\Rightarrow \quad |\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \left(e^{-iE_0T}\langle\Omega|0\rangle\right)^{-1} e^{-iHT}|0\rangle$$

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One can now shift T by a small constant t_0 and thus, after a short calculation, involve the time evolution operator into the above expression:

$$|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \left(e^{-iE_0(t_0-(-T))}\langle\Omega|0\rangle\right)^{-1} U(t_0,-T)|0\rangle$$

$$\langle\Omega| = \lim_{T\to\infty(1-i\epsilon)} \langle 0|U(T,t_0)\left(e^{-iE_0(T-t_0)}\langle 0|\Omega\rangle\right)^{-1}$$

Master Formulas

Using $\langle\Omega|\Omega\rangle = 1$, we finally obtain the two-point function of the interacting theory in terms of free fields Φ_I and the vacuum of the free theory $|0\rangle$:

$$\langle\Omega| T\Phi(x)\Phi(y)|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0| T\{\Phi_I(x)\Phi_I(y)U(T,-T)\}|0\rangle}{\langle 0| T\{U(T,-T)\}|0\rangle}$$

$$= \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0| T\left\{\Phi_I(x)\Phi_I(y)\exp\left[-i\int_{-T}^{T} dt\, H_I(t)\right]\right\}|0\rangle}{\langle 0| T\left\{\exp\left[-i\int_{-T}^{T} dt\, H_I(t)\right]\right\}|0\rangle}$$

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$$\begin{aligned}\langle\Omega|T\Phi(x)\Phi(y)|\Omega\rangle &= \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|T\{\Phi_I(x)\Phi_I(y)U(T,-T)\}|0\rangle}{\langle 0|T\{U(T,-T)\}|0\rangle} \\ &= \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|T\left\{\Phi_I(x)\Phi_I(y)\exp\left[-i\int_{-T}^{T} dt\, H_I(t)\right]\right\}|0\rangle}{\langle 0|T\left\{\exp\left[-i\int_{-T}^{T} dt\, H_I(t)\right]\right\}|0\rangle}\end{aligned}$$

Two-point function in the path integral formalism:

$$\langle\Omega|T\Phi(x)\Phi(y)|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\int\mathcal{D}\phi\,\phi(x)\phi(y)\exp\left[i\int_{-T}^{T} d^4x\,\mathcal{L}\right]}{\int\mathcal{D}\phi\,\exp\left[i\int_{-T}^{T} d^4x\,\mathcal{L}\right]}$$

(Functional integral over the classical field ϕ)

The generalization of a two-point function to an *n*-point function is obvious: Just include *n* field operators in the nominator.

The key ingredient of these master formulas is the exponent. Its expansion involves infitely many terms with different numbers of field operators!
Unfortunately, nobody knows how to calculate this quantity analytically in a closed form. If you find a way, the Nobel price is for sure!

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At least not how to calculate exactly...

Main idea of perturbation theory:
Expand the exact correlation functions of the full theory in terms of a small parameter.

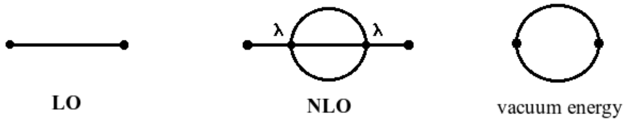
Example: $e^{-ix} = 1 - ix + (-i)^2 x^2 + \mathcal{O}(x^3)$

If $|x| \ll 1$ you can approximate the exponent by skipping higher orders in x. Here, you can do a similar thing, if you have a small coupling constant $\lambda \ll 1$!

Interpretation of the perturbative expansion:

- Leading order (LO): The exponent reduces to 1. The correlation function is that of the free theory (no quantum corrections through interaction with the quantum vacuum).
- Next-to-leading order (NL0): Interaction with the quantum vacuum. For the example of the ϕ^4 theory, 4-point interactions are involved.

Remark: The nominator of the master formula produces also terms, where the vacuum just interacts with itself. Those are cancelled by the denominator.

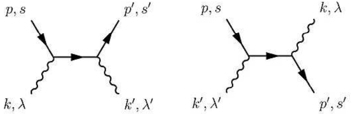


Feynman diagrams

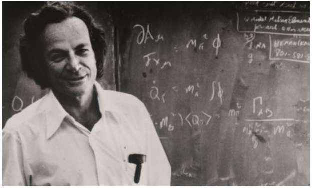
- In order to calculate correlation functions, we expand the exact formula in the coupling constant and restrict ourselves to the lowest orders.
Due to *Feynman*, this approximation can be understood diagramatically. We already saw diagrams for a scalar theory on the previous page.

- Now consider **QED**. Take **Compton scattering** as an example:

At LO we have the *Feynman diagrams*:



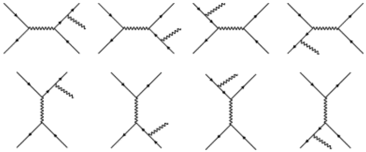
electron with momentum p and spin s,
photon with momentum k and polarization λ



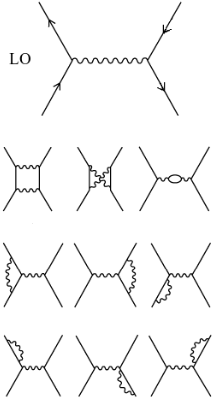
Feynman diagrams at NLO

Example: e^+/e^--**annihilation**

The NLO consists of radiation diagrams (real corrections) and loop diagrams (virtual corrections). The latter can not be understood in a classical theory and are therefore known as quantum corrections!



real corrections



+ *„transposed diagrams"*

virtual corrections

Feynman rules

- Correlation functions of QED can now be evaluated by expansion in the fine-structure constant $\alpha = \frac{e^2}{4\pi} \approx \frac{1}{137}$. ($\alpha = \frac{e^2}{4\pi\hbar c\epsilon_0}$ in SI-units)

Feynman rules

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- At LO, the **scattering amplitude** for e^+/e^--annihilation is proportional to

$$\mathcal{M}_{e^+/e^-} \propto \langle 0|\overline{\Psi}(x)A_\mu(x)\Psi(x)\overline{\Psi}(y)A^\mu(y)\Psi(y)|0\rangle$$

  One can evaluate this expression by using the algebra of creation and annihilation operators. However, this method becomes very awkward starting from NLO.

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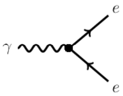
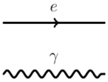
- Scattering amplitudes are nowadays computed by making use of Feynman diagrams. Therefore one has to derive rules, how these diagrams are to be translated into mathematical expressions. These rules are called Feynman rules.

How to obtain Feynman rules:

By inspection of the above diagrams, one realizes that they consist of the building blocks

vertices and *lines*





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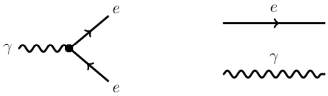

These are expected from the action $S = \int d^4x\,\mathcal{L}$, which we can write as

$$S = \int d^4x \left\{ \frac{1}{2} A_\mu \left(\partial^2 g^{\mu\nu} - \left(1 - \frac{1}{\xi}\right) \partial^\mu \partial^\nu \right) A_\nu - e\overline{\psi}\gamma^\mu A_\mu \psi + \overline{\psi}(i\gamma^\mu \partial_\mu - m)\psi \right\}$$

Here, we have used integration by parts to rewrite the term $-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2$

- The first term gives rise to the propagation of a (massless) photon.
- The second term describes a coupling of strength e.
- The third term gives rise to the prop. of an electron/positron of mass m.

Propagators (lines between vertices):
Change to momentum space:

$$S_{\text{Gauge}} = \int \frac{d^4k}{(2\pi)^4}\,\frac{1}{2}\,\widetilde{A}_\mu \underbrace{\left(-k^2 g^{\mu\nu} + \left(1-\frac{1}{\xi}\right)k^\mu k^\nu\right)}_{\equiv K^{\mu\nu}} \widetilde{A}_\nu$$

The *Feynman propagator* can now directly be obtained as the inverse of $K_{\mu\nu}$:

$$K_{\mu\nu}\widetilde{D}_F^{\nu\sigma}(k) \stackrel{!}{=} i\delta_\mu^\sigma \quad \Rightarrow \quad \widetilde{D}_F^{\mu\nu}(k) = \frac{-i}{k^2+i\epsilon}\left(g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right)$$

Remarks:

- The term $+i\epsilon$ is obtained by the more accurate derivation according to the previous section.
- Note that without the $1/\xi$-term, $K_{\mu\nu}$ would be singular and thus would not have an inverse!
- For QCD, we have to multiply the above expression with δ^{ab}, which means that the gluon does not change its color configuration during free propagation.

The same short-cut can be taken fo fermions where we obtain

$$\widetilde{D}_F(p) = \frac{i}{\not{p} - m + i\epsilon} = \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}\,, \qquad \not{p} \equiv \gamma^\mu p_\mu\,, \quad \not{p}^2 = p^2$$

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Outer states:

- polarization vectors $\epsilon_\lambda^\mu(p)$ for photons or gluons
- seperate momentum-space spinors $u^s(p)$ and $v^s(p)$ for fermions and anti-fermions (no propagators for lines ending as external state!)
- For unpolarized cross sections one has to sum over spins

$$\sum_s u^s(p)\bar{u}^s(p) = \not{p} + m, \quad \sum_s v^s(p)\bar{v}^s(p) = \not{p} - m.$$

There is also a polarization sum for gauge bosons.

Remark: The quantization of QCD involves a further (unphysical) dof, which does not exist as external state, but can be pair-produced and annihilated within virtual corrections. There are also rules (vertex and propagator).

> **Feynman rules serve as a recipe and can be derived for any QFT!**

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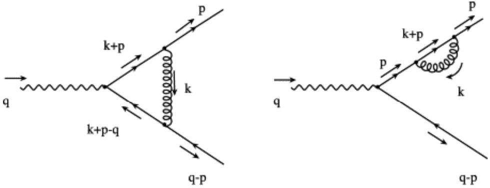
Feynman rules serve as a recipe and can be derived for any QFT!

Further proceeding:

- Write down the amplitude using Feynman rules and square.
- Avarage over initial spins and sum over outgoing spins (and polarizations). Use known results for the sum of squared spinors and polarization vectors.
- The squaring of the amplitude produces traces of gamma matrices. There are algebraic relations to boil them down to elements of the Minkowski metric $g^{\mu\nu}$ and/or antisymmetric tensors $\epsilon^{\mu\nu\sigma\tau}$. These relations are known as Dirac algebra.
- Express the momentums in terms of the center of mass energy, the masses of the scattered particles and the scattering angle.
- All these steps can be teached to a computer.
- One crucial step is still missing!

Loop integrals



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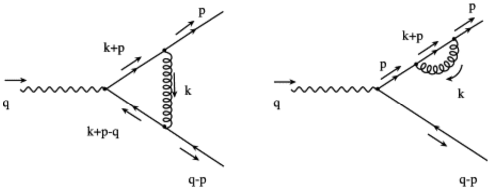

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But: Loop integrals may diverge for $k \rightarrow 0$ (infrared divergence) and for $k \rightarrow \infty$ (ultraviolet divergence)!

After regularization of the integral, the infrared divergences cancel with divergent parts of real emission amplitudes, when the momentum of the radiated photon or gluon goes to zero. The UV divergences stay and ask for a redefinition of the input parameters to get a meaningful theory.

$\Longrightarrow$ QFT has to be renormalized!

Renormalization has an important impact on the phenomenology: Masses and couplings are no longer constants, but depend on the energy scale, where they are measured!!!

We will talk about so-called *running masses and couplings* in Lecture 2.

When the theory is renormalized, there is still the task of calculating the loop integral. At NLO this is „rather easy“(not really, there are some tricks required). At NNLO it really gets complicated (but interesting!).

Summary

• In Section 1, we have learned that the object of interest for the comparism of theory and experiment is the (differential) cross section for the scattering of partons. From this, we have to calculate the hadonic cross section, which describes the interaction of protons → Lecture 2.

• In Section 2, we have summarized the main ideas of QFT, starting from Lagrange densities of classical field theories. We have seen how the fields can be quantized and how they have to be interpreted.

• In Section 3, we scetched how the scattering amplitude is calculated. The calculation is based on perturbation theory, which allows for an approximate treatment of the interacting theory under the assumption of a small coupling constant. We ended in noting that higher order quantum corrections require the calculation of loop integrals. These have to be regularized and the theory asks for a renormalization.

Final Remark:

When theorists (in elementary particle physics) talk of „**a theory**“, they always refer to the **Lagrange density** $\mathcal{L}$, which stores all physical information (in combination with a given renormalization prescription). The framework of **QFT** provides the rules how to calculate observable quantities starting from $\mathcal{L}$.

Literature:

- M. E. Peskin & D. V. Schroeder, *An Introduction to Quantum Field Theory*, Westview Press
- F. Scheck, *Theoretische Physik 4: Quantisierte Felder*, Springer