Solution of Evolution Equations in Leading and Higher Twist

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Project Report

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Abstract

We study the evolution equations for operators of leading and higher twist in scalar ϕ^3-theory. The evolution equations for operators of leading twist $\tau = 4$ are solved. At next to leading twist, $\tau = 6$, so-called "spin-towers" occur due to the mixing of operators. There are linear dependencies between tree-level operator matrix elements, which are evaluated for general spin. In the limit of large spin, the coefficients depend only on the second quantum number. The operator mixing matrices are then calculated up to the first order in the coupling constant and are diagonalized. The eigenvalue spectrum is real and shows saturation as the spin grows.

1 Introduction

Deeply inelastic scattering experiments have been conducted since the 1960's in order to examine the nucleon sub-structure. In these experiments, typically charged lepton or neutrino beams are scattered off the target. From the outgoing energy and momenta, the structure of the scattering object can be determined.

The SLAC-MIT experiments revealed the sub-structure of the nucleon. The scattering centers were identified with the quarks of the static quark model proposed in 1964 by M. Gell-Mann and G. Zweig [1].

The cross-section for unpolarized single-photon lepton-nucleon scattering can be described in terms of the two structure functions $F_2(\nu, Q^2)$ and $F_L(\nu, Q^2)$, that depend only on the transferred energy ν and the squared momentum Q^2 between the lepton and the nucleon. At low values of Q^2 the whole proton recoils. The cross-section decreases until at very high virtualities scattering off the constituents becomes most probable.

The experiments [2] found that in the Bjorken-limit $Q^2 \to \infty, \quad \nu \to \infty, \quad Q^2/\nu = \text{const.}$ the structure functions do not depend on the absolute values of Q^2 and ν, but only on their ratio, which is described by the Bjorken-variable $x = Q^2/2M\nu$. This behavior is called scaling. It had been predicted by Bjorken in 1969 [3]. A phenomenological explanation was given by R. Feynman, who introduced the quark parton model [4]. Here, the proton is described as a set of point-like partons, which are quasi-free during the very short interaction time. Hence, the total cross section can be considered as the coherent sum of all the elastic electron-parton cross-sections.

Quantum Chromodynamics (QCD) was proposed a candidate theory of the strong interaction [5] and was proved to be asymptotically free [6], [7]. This property enables to conduct perturbative calculations at high enough energy scales, where the coupling constant is small enough.

The parton model is only valid for $x \not\ll 1$, $x \not\approx 1$ and high Q^2, which is equivalent to the domain of the lowest twist. For large virtualities Q^2, the light-cone expansion allows a factorization into Wilson-coefficients and operator matrix elements [8]. Here, the input-densities have to be calculated non-perturbatively whereas the evolution kernel can be calculated in perturbative QCD, cf. e.g. [9].

If $x \approx 1$, the quark contributions dominate the parton distribution. In the small x region the gluons rise steeply which leads to a strong growth of the structure function. If this rise continued further, the unitarity of the inclusive scattering amplitude would be violated. Contributions beyond those of leading twist are therefore required to preserve unitarity. Higher twist corrections also occur in polarized lepton-proton scattering, cf. e.g. [10]. Dynamical effects of higher twist has been considered e.g in the References [11]. The study of higher twist contributions to deeply inelastic scattering provides detailed test of predictions in QCD [12]. The eigenvalues of QCD mixing matrices have been considered in [13].

In this paper, scalar ϕ^3-theory in $D = 6$ dimensions is studied. This theory shares two important features with QCD. First it is asymptotically free and second it has a 3-boson vertex, which in the case of QCD dominates the gluo-dynamics [14].

We start by considering the unpolarized deeply inelastic scattering process in section 2. In section 3 we introduce the light-cone expansion. Leading twist $\tau = 4$ in scalar ϕ^3-theory is considered in section 4.We use the light-cone expansion to solve the evolution equation of the parton density distributions up to first order in the coupling constant. This corresponds to the operators of leading twist. In section 5 operators of next to leading twist ($\tau = 6$) are considered, closely following [15]. There are linear dependencies between the tree-level operator matrix elements, that limit the number of independent elements at a given spin. We obtain numerically, that the coefficients depend only on the second quantum number in the limit of high spin. In order to get an evolution equation, the anomalous dimension matrix is calculated. This leads to spin-towers, i.e. the mixing of more and more operators as the total spin rises. Then we diagonalize the anomalous dimension matrix, and examine the eigenvalue spectrum. The spectrum turns out to be real and saturates as the spin increases. Section 6 contains the summery. Some technical details such as Feynman rules, operator relations and anomalous dimension matrices are given in appendices.

2 Deeply inelastic scattering

2.1 Deeply inelastic scattering at sub-lowest order

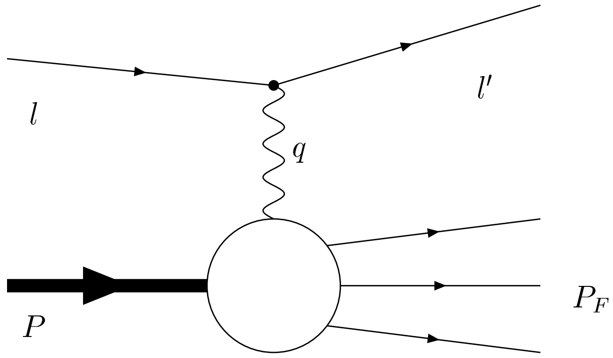

Figure 1: Schematic scetch of deeply inelastic scattering

Deeply inelastic lepton-nucleon scattering (DIS) provides a clean way to investigate the structure of the nucleon [16]. The process is depicted schematically in Figure 1. The kinematic variables that describe the process are

$$Q^2 = -q^2 = -(l - l')^2, \qquad \nu = \frac{P.q}{M^2} \tag{1}$$

$$x = \frac{Q^2}{2P.q}, \qquad y = \frac{P.q}{P.l} \tag{2}$$

$$W^2 = (P + Q)^2 = M_p^2 + 2P.q - Q^2 \ . \tag{3}$$

Here l and l' denote the incoming and outgoing lepton momenta and P is the nucleon momentum. The space-like virtuality $Q^2 = -q^2$ describes the squared momentum transfer q from the lepton to the nucleon, whereas the invariant ν is the transferred energy. M denotes the nucleon mass. It is convenient to express the scattering process in terms of the dimensionless Bjorken-variables $x, y \in [0; 1]$. In regions, where the hadronic mass W^2 is high enough, the process hadronic final state clearly indicates an inelastic process

In the following, we consider the region of single photon exchange only. Effects due to the weak bosons and two-photon exchange lead to a more involved picture, where QED- and electroweak corrections have to be taken into account. For these contributions more hadronic structure functions have to be considered.

At tree-level, the matrix element of the deep-inelastic process is given by

$$M_{fi} = \frac{e^2}{q^2} \bar{u}(l, \lambda) \gamma^\mu u(l', \lambda') \langle P_F | J_\mu | P, \sigma \rangle \ . \tag{4}$$

Here e denotes the electric charge of the lepton. The differential inclusive cross-section reads

$$l_0' \frac{d\sigma}{d^3 l'} = \frac{1}{32(2\pi)^3 (l.P)} \sum_{\lambda, \lambda', \sigma, F} (2\pi)^4 \delta^4 (P_F + l' - P - l) |M_{fi}|^2 = \frac{1}{4P.l} \frac{\alpha^2}{Q^4} L^{\mu\nu} W_{\mu\nu} \ . \tag{5}$$

$L_{\mu\nu}$ and $W_{\mu\nu}$ denote the leptonic and the hadronic tensor, respectively. The leptonic tensor can be calculated analytically using the trace formalism,

$$L^{\mu\nu}(l, l') = 4(l_\mu l'_\nu + l'_\mu l_\nu - \frac{Q^2}{2} g_{\mu\nu}) \tag{6}$$

As unpolarized scattering is considered here, we average the initial helicities and sum over the lepton helicities. To construct the hadronic tensor, we consider the electromagnetic quark current

$$J_\mu^{em}(\xi) = \sum_f \bar{\Psi}_f(\xi) \gamma_\mu \lambda^{em} \Psi_f(\xi) \ . \tag{7}$$

Here $\Psi_f(\xi)$ are the quark fields and λ^{em} is the corresponding matrix of the $SU(3)$-group. In the unpolarized case the hadronic tensor is given by

$$W^{\mu\nu}(P,q) = \frac{1}{4\pi}\int d^4\xi \exp(iq\xi)\langle P|J_\mu^{em}(\xi)J_\nu^{em}(0)|P\rangle \quad , \tag{8}$$

which is the Fourier-transformation of a composite operator expectation value. Interchanging of the currents leads to $P_F^2 < M^2$ which is unphysical, so this contribution must vanish. Hence, the operator product $J_\mu^{em}(\xi)J_\nu^{em}(0)$ may be replaced by the commutator $[J_\mu^{em}(\xi), J_\nu^{em}(0)]$.

2.2 Structure functions

Since in the unpolarized case all antisymmetric parts of the hadronic tensor $W_{\mu\nu}$ will be cancelled by the symmetric leptonic tensor, the hadronic tensor for pure photon exchange only contains contributions proportional to the tensors

$$g_{\mu\nu}, \quad q_\mu q_\nu, \quad P_\mu P_\nu, \text{ and } q_\nu P_\mu + q_\mu P_\nu \ . \tag{9}$$

Two (scalar) structure functions F_2 and F_L are sufficient to describe the hadronic tensor. The requirements of electromagnetic gauge invariance,

$$q_\mu W^{\mu\nu} = q_\nu W^{\mu\nu} = 0 \quad , \tag{10}$$

lead to the structure

$$W^{\mu\nu}(P,q,x) = \frac{1}{2x}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)F_L(x,Q^2) + \frac{2x}{Q^2}\left(P_\mu P_\nu + \frac{P_\nu + P_\mu + q_\mu P_\nu}{2x} - \frac{Q^2}{4x^2}g_{\mu\nu}\right)F_2(x,Q^2) \ . \tag{11}$$

The contraction of $W^{\mu\nu}$ with the metric tensor and $P_\mu P_\nu$ yields

$$P_\mu P_\nu W^{\mu\nu} = -\frac{1}{2x}\frac{(P.q)^2}{q^2}F_L \qquad \text{and} \qquad g_{\mu\nu}W^{\mu\nu} = \frac{D-1}{2x}F_L - \frac{D-2}{2x}F_2 \ . \tag{12}$$

From these relations we obtain the structure functions F_L and F_2. Here D denotes the dimensionality of the space-time used for dimensional regularization[1]. Often, another structure function, that is not independent of the latter two, is introduced:

$$F_1(x,Q^2) = \frac{1}{2x}\left[F_2(x,Q^2) - F_L(x,Q^2)\right] \ . \tag{13}$$

2.3 Parton model and scaling

In the Bjorken-limit the hadronic structure functions at lowest order of the coupling constant of QCD obey Bjorken-scaling:

$$Q^2 \to \infty, \qquad \nu \to \infty, \qquad x = \text{const.} \Rightarrow F_i(x,Q^2) \to F_i(x) \quad , \tag{14}$$

i.e. the two structure functions depend only on the ratio ν/Q^2, but not on the absolute values [3], [2]. Furthermore the scattering of leptons off the nucleon bears clear evidence of scattering from point-like scattering-centers inside the nucleon. Feynman [4] introduced the parton-model, in which the partons are described by individual momenta p_i in terms of the momentum fractions z_i. Individual distribution functions f_i and partonic scaling variables ξ_i, analogous to the Bjorken-variable x of the nucleon, are introduced as well,

$$p_i = z_i P, \qquad \xi_i = \frac{Q^2}{2p_i.q_i} \ . \tag{15}$$

In the collinear parton model, $p_i \parallel P$ is assumed and $\xi_i z_i = x$ holds.

[1] $D = 4 + \varepsilon$ is considered in the case of QCD. Later, in scalar ϕ^3-theory $D = 6 + \varepsilon$ will be used.

An estimate of the ratio between the interaction-time and the lifetime can be made, cf. [17]. In the infinite momentum frame $P \gg M, P \gg \sqrt{k_\perp^2}$ the ratio is given by

$$\frac{\tau_{int}}{\tau_{life}} \approx \frac{Q^2 xP}{(1-x)^2 2k_\perp^2 xP} = \frac{Q^2}{(1-x)^2 2k_\perp^2} \; . \tag{16}$$

Here $k_\perp$ is the transverse parton momentum. If x is neither too close to 0 nor 1, and $k_\perp^2$ is much smaller than the virtuality Q^2, DIS will give information on the individual partons. Since the numerator and the denominator in Eq. (16) are proportional to xP, one has to demand that $x \not\ll 1$. Small values of x yield $\bar{P} = xP$ and the approximations which are possible in the infinite momentum frame can no longer be made. Likewise the single parton picture fails in the region $x \approx 1$ and/or $k_\perp^2 \approx Q^2$.

Conversely, approaching these domains, one expects the excitation of other degrees of freedom, which describe the nucleon sub-structure as partonic correlation functions due to the higher twist operators discussed below. If the condition

$$\frac{\tau_{int}}{\tau_{life}} \ll 1 \tag{17}$$

holds, the single-parton model can be applied, because in this case the time-scales separate and the individual partons can be considered as quasi-stable during the interaction. Now the hadronic tensor $W_{\mu\nu}$ is expressed in terms of the parton distributions in the region, where the single parton picture is applicable. Introducing the non-perturbative partonic density functions $f_i, \bar{f}_i$ and the partonic tensor $w^i_{\mu\nu}$, the hadronic tensor has the form of a Mellin-convolution:

$$W_{\mu\nu}(Q^2) = \frac{1}{4}\sum_i \int_0^1 dz \int_0^1 d\tau z \left[f_i(z) + \bar{f}_i(z)\right] w^i_{\mu\nu}(\tau, Q^2)\delta(x - z\tau) \tag{18}$$

$$w^i_{\mu\nu} = \frac{2\pi e_i^2}{q.p^i}\delta(1-\tau)\left(2p^i_\mu p^i\nu = p^i_\mu q^i_\nu - g_{\mu\nu} q.p^i\right) \; . \tag{19}$$

The structure functions at lowest order in the coupling constant are given by

$$F_L(x, Q^2) = 0 \qquad \text{and} \qquad F_2(x, Q^2) = \sum_i e_i^2 x \left(f_i(x) + \bar{f}_i(x)\right) \; . \tag{20}$$

Here the structure functions do not depend on the scale Q^2 and the Callan-Gross-relation [19] holds:

$$F_L(x, Q^2) = F_2(x, Q^2) - 2xF_1(x, Q^2) = 0 \; . \tag{21}$$

3 The light-cone expansion

Operator products of the type $\phi(x)\phi(y)$ will generally diverge as $x \to y$. According to Wilson [8] products of this type can be expanded into series, in which the remaining operators will be local.

We had shown before that the hadronic tensor can be written as the Fourier-integral of the current-current commutator (8). According to the Rieman-Lebesgue theorem, for large values of Q^2 this integral is dominated by the contributions, in which $q.\xi \approx 0$, because only here the oscillation is slow. In the Bjorken-limit (14) the product obeys

$$q.\xi \approx 0 \Rightarrow \xi^2 = \xi_\mu \xi^\mu \approx 0 \; . \tag{22}$$

Hence the relevant contributions to the integral stem from light-like vectors ξ. This property is called light-cone-dominance.

The commutator $[J^{em}_\mu(\xi), J^{em}_\nu(0)]$ is therefore expanded for values of ξ near the light-cone,

$$[J^{em}_\mu(\xi), J^{em}_\nu(0)] = \sum_{i,N,\tau} c^N_{i,\tau}(\xi^2, \mu^2)\xi_{\mu_1}...\xi_{\mu_N} O^{\mu_1...\mu_N}_{i,\tau}(0, \mu^2) \; . \tag{23}$$

The canonical dimension of the Wilson-coefficients d_c is given by

$$d_c = N - d_O + 2d_J = -\tau + 2d_J \ . \tag{24}$$

Here N denotes the spin of the operator, d_O is the canonical dimension of the operator and $2d_J$ is the dimension of the current product. Since d_J is constant, the quantity that describes the order of the expansion is the twist [20]

$$\tau = d_O - N \ . \tag{25}$$

In the Bjorken-limit, the partonic distribution function is identified with the operator matrix element (OME) of leading twist:

$$\langle P|[J(\xi), J(0)]|P\rangle_i(N) = f_i(x, Q^2) \ . \tag{26}$$

Now the partonic distribution function is expressed in terms of its Mellin transform F_i^N, which can be expressed as the Mellin-convolution of the Wilson-coefficients and the OMEs:

$$\begin{aligned} F_i(x,Q^2) &= [c_{i,j} \otimes f_j](x) && (27)\\ F_i^N(Q^2) &= \int_0^1 dx x^{N-1} F_i(x,Q^2) && (28)\\ &= c_{i,\tau}^N \otimes \langle P|O_{i,\tau}^N|P\rangle \ . && (29) \end{aligned}$$

4 Leading twist operators in ϕ^3-theory

4.1 Evolution of operator matrix elements

Let us derive the evolution equation for operator matrix elements. We follow Ref. [22]. The light cone expansion of the current commutator reads

$$G = [J(\xi), J(0)] = \sum_{i,N,\tau} c_{i,\tau}^N(\xi^2, \mu^2)\xi_{\mu_1}...\xi_{\mu_N} O_{i,\tau}^{\mu_1...\mu_N}(0,\mu^2) \ . \tag{30}$$

To obtain a compact notation, we introduce the operator

$$\mathcal{D} = \left(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g_R} - \gamma_m m_r \frac{\mu\partial}{\partial m_r}\right)\Bigg|_{g,m} \ . \tag{31}$$

If the currents are not conserved, their anomalous dimension does not vanish. The electromagnetic current is conserved, whereas the scalar current is not. Therefore, in scalar ϕ^3-theory the current anomalous dimension has to be kept in the renormalization group equation (RGE)

$$(\mathcal{D} + 2\gamma_J - n\gamma)\, G = 0 \ . \tag{32}$$

The RGE of the operator $O_{i,\tau}$ is given by

$$\left(\mathcal{D} + \gamma_{O_{i,\tau}} - n\gamma\right) O_{i,\tau} = 0 \ , \tag{33}$$

where $\gamma_{O_{i,\tau}}$ denotes the operator's anomalous dimension and $Z_{O_{i,\tau}}$ is the respective Z-factor,

$$\begin{aligned} \gamma_{O_{i,\tau}} &= \mu\frac{\partial}{\partial\mu} Z_{O_{i,\tau}}^{-1}\Bigg|_{g,m} , \qquad Z_O = \frac{\langle P|O_{i,\tau}|P\rangle_r}{\langle P|O_{i,\tau}|P\rangle} && (34)\\ \Rightarrow (\gamma_{O_{i,\tau}})_{ij} &= -(Z_{O_{i,\tau}}^{-1})_{ik}\, \frac{\partial}{\partial \ln\mu} (Z_{O_{i,\tau}})_{kj}\Bigg|_{g,m} \ . && (35) \end{aligned}$$

The RGE of the Wilson-coefficients is obtained by inserting (30) into (32) and by applying the product rule. It reads

$$\left(\mathcal{D} + 2\gamma_J - \gamma_{O_{i,\tau}}\right) c_{i,\tau}^N = 0 \ . \tag{36}$$

The bare parton density $f(N)$ does not depend on the renormalization scale μ. This property and the other two RGE yield

$$\mu\frac{\partial}{\partial\mu}f(\mu^2,N) \;=\; -\gamma_{O_{i,\tau}}(N)f(\mu^2,N)\ . \tag{37}$$

We change the variable from μ to $a(\mu)$ by inserting the running coupling

$$a(\mu^2) = \frac{a_0}{1+\beta_0 a_0\ln(\mu^2/\mu_0^2)}\ ,\qquad a_0 = a(\mu_0) \tag{38}$$

in order to refer to an experimentally more convenient quantity. The anomalous dimension is expanded into the Taylor series in the coupling constant

$$\gamma_{O_{i,\tau}}(N) = \gamma^0_{O_{i,\tau}}(N)a + \mathcal{O}(a^2)\ . \tag{39}$$

The evolution equation of the scalar parton densities and its solution read now

$$\frac{df(\mu^2,N)}{da} \;=\; \frac{1}{a}\frac{\gamma^0_{O_{i,\tau}}(N)}{2\beta_0}f(\mu^2,N) \tag{40}$$

$$\Rightarrow f(\mu^2,N) \;=\; f_0(\mu^2,N)\left(\frac{a(\mu^2)}{a(\mu_0^2)}\right)^{\gamma^0_{O_{i,\tau}}(N)/2\beta_0}\ . \tag{41}$$

This result enables us to calculate the parton density function from the operator anomalous dimension at all scales, once it is determined at a reference scale μ_0^2.

4.2 Leading twist anomalous dimension in scalar ϕ^3-theory

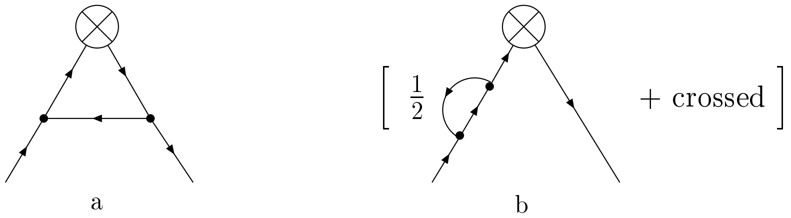

Figure 2: Leading twist contributions of scalar ϕ^3-theory.

Let us now calculate the leading twist anomalous dimension in scalar ϕ^3-theory in $D = 6$ dimensions. In this case, only the two types of diagrams, shown in Figure 2, contribute. The scalar fields are massless $m = 0$, so we have to assume off-shell external particles $-p^2 > 0$, to set the scale for the process. Δ is an arbitrary light-like vector, such that $\Delta^2 = 0$. The evaluation of the diagrams yields

$$S_a \;=\; \frac{ig^2}{(4\pi)^3}(\Delta.p)^N\left[-\frac{2}{(N+1)(N+2)}\right]\frac{1}{\varepsilon}+\mathcal{O}(1) \tag{42}$$

$$S_b \;=\; \frac{ig^2}{(4\pi)^3}(\Delta.p)^N\left[\frac{1}{6}\right]\frac{1}{\varepsilon}+\mathcal{O}(1)\ . \tag{43}$$

The total amplitude of the process is given by

$$S = S_a + 2\cdot\frac{1}{2}S_b\ .$$

The anomalous dimension is obtaind from the pole terms. It can also be expressed in terms of a splitting function [14].

$$\gamma^{(0)}_{\tau=4} = \frac{g^2}{(4\pi)^3}\left[\frac{1}{6}-\frac{2}{(N+1)(N+2)}\right] = \frac{g^2}{(4\pi)^3}\mathbf{M}\left[\frac{1}{6}\delta(1-x)-2x(1-x)\right](N)\ . \tag{44}$$

5 Higher twist in ϕ^3-theory

5.1 Operators

To this point, only contributions of the lowest twist $\tau = 4$ have been considered. We will now examine the structur of operators of higher twist. Generally, there are two ways to construct operators of given twist and spin.

1. The first type of operators contain only fields and differential operators. The twist is twice the number of fields, due to the canonical field dimension of two in ϕ^3-theory. The spin is equivalent to the number of differential operators. Further quantum numbers describe the number of differential operators between the fields.

2. Another class of operators is created by contracting two derivatives. This yields d'Alembertian operators $\Box = \partial_\mu \partial^\mu$, each of which adds an additional field's dimension to the twist.

In the case of the next-to-leading twist $\tau = 6$, spin n and second quantum number m, type 1 operators have the structure

$$O_1 =^{\mu_1 \cdots \mu_n} (y) = \phi(y) \partial^{\mu_1} ... \partial^{\mu_m} \phi(y) \partial^{\mu_{m+1}} ... \partial^{\mu_n} \phi(y) + \text{trace-terms} + \text{symmetrization} \ . \tag{45}$$

Type 2 operators with one Lorentz-contraction do not have a second quantum number. Their structure is the following

$$O_2^{\mu_1 \cdots \mu_n}(y) = -\frac{1}{2} \phi(y) \partial^{\mu_1} ... \partial^{\mu_n} \partial^\lambda \partial_\lambda \phi(y) + \ \text{trace-terms} + \text{symmetrization} \ . \tag{46}$$

Thus the operators $O_2^{\mu_1 \cdots \mu_n}$ and $O_1^{\mu_1 \cdots \mu_n}$ have the same twist and spin. For operators of the same quantum numbers, mixing occurs generally.

The equation of motion follows from the Lagrangian

$$\mathcal{L} = \frac{1}{2} \left[\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2 \right] - \frac{g}{3!} \phi^3 \qquad \Longrightarrow \qquad \Box \phi + \frac{g}{2} \phi^2 = 0 \ \ , \tag{47}$$

and gives a direct relation between the operators of type 1 and 2. For instance, for the trivial case $n = m = 0$ we find

$$O_1^{(0,0)} = \phi^3, \qquad O_2^{(0)} = \phi \Box \phi = -\frac{g}{2} O_1^{(0,0)} \ . \tag{48}$$

5.2 Feynman-rules

For easier Feynman rules, the operators are contracted with source-terms $\Delta_{\mu_1} ... \Delta_{\mu_n}$ where $\Delta^2 = 0$. This yields operators that now depend only on the number and position of the fields w.r.t the derivatives:

$$O_I^{(n,m)}(y) = \Delta_{\mu_1} ... \Delta_{\mu_n} O_I^{\mu_1 \cdots \mu_n}(y), \qquad O_J^{(n)}(y) = \Delta_{\mu_1} ... \Delta_{\mu_n} O_J^{\mu_1 \cdots \mu_n}(y) \tag{49}$$

Now, operator matrix elements (OMEs) are defined by sandwiching the operators between the in-states $\langle k_1 |$ and $\langle k_1, k_2 - k_1 |$ and the out-state $| k_2 \rangle$:

$$\begin{aligned} A_{II}^{(n,m)} &= \langle k_1, k_2 - k_1 | O_I^{(n,m)} | k_2 \rangle, \qquad & A_{IJ}^{(n)} &= \langle k_1 | O_I^{(n,m)} | k_2 \rangle \\ A_{JJ}^{(n)} &= \langle k_1, k_2 - k_1 | O_J^{(n)} | k_2 \rangle, \qquad & A_{JI}^{(n,m)} &= \langle k_1 | O_J^{(n)} | k_2 \rangle \end{aligned} \tag{50}$$

The Feynman-rules at lowest order are given by

$$\begin{aligned} O_I \to A_{II}^{0,(n,m)}(\Delta.k_1, \Delta.k_2) \ &= \ \frac{g}{6} \left[(\Delta.k_1)^m (\Delta.(k_2 - k_1))^{n-m} + (\Delta.(k_2 - k_1))^m (-\Delta.k_2)^{n-m} + \right. \\ & \quad + (-\Delta.k_2)^m (\Delta.k_1)^{n-m} + (\Delta.(k_2 - k_1))^m (\Delta.k_1))^{n-m} + \\ & \quad \left. + \ (-\Delta.k_2)^m (\Delta.(k_2 - k_1))^{n-m} + (\Delta.k_1)^m (-\Delta.k_2)^{n-m} \right] \qquad (51) \\ O_J \to A_{JJ}^{0,(n)}(\Delta.k_1, \Delta.k_2) \ &= \ k_1^2 (\Delta.k_1)^n \ . \qquad (52) \end{aligned}$$

5.3 Properties of $A_{II}^{0,(n,m)}$

For the OME $A_{II}^{0,(n,m)}$ the following identities hold:

$$
\begin{aligned}
A_{II}^{0,(n,m)}(\Delta.k_1,\Delta.k_2) &= A_{II}^{0,(n,n-m)}(\Delta.k_1,\Delta.k_2) && (53)\\
A_{II}^{0,(n,0)}(\Delta.k_1,\Delta.k_2) &= -2A_{II}^{0,(n,1)}(\Delta.k_1,\Delta.k_2) && (54)\\
A_{II}^{0,(n,m)}(\Delta.k_1,\Delta.k_2) &= A_{II}^{0,(n,m)}(\Delta.k_1,\Delta.(k_1-k_2)) \ . && (55)
\end{aligned}
$$

These identities limit the number of independent operators and enable us to find relations between different OMEs of the same spin and twist. Another symmetry in the arguments occurs, because in this case all physically relevant quantities have even total spin.

$$
A_{II}^{0,(2n,m)}(\Delta.k_1,\Delta.k_2) = A_{II}^{0,(2n,2n-m)}(\Delta.k_2,\Delta.k_1) \ . \qquad (56)
$$

Since the arguments are scalar, the objects $A_{II}^{0,(n,m)}(\Delta.k_1,\Delta.k_2)$ are polynoms. Polynoms make up a vector space, whose elements can be expressed as linear combinations of few basis elements. The latter can be chosen in different ways. For example a basis of symmetric functions could be used, due to the symmetry expressed in Eq. (56). We will not do so at this point and use a basis of OMEs instead:

$$
A_{II}^{0,(n,m)}(\Delta.k_1,\Delta.k_2) = \sum_{m'} c^{(n,m')} A_{II}^{0,(n,m')}(\Delta.k_2,\Delta.k_1) \qquad (57)
$$

Here m' runs over the linearly independent OMEs. Up to $n=4$ there is one independent OME only, which can be conveniently taken as $A_{II}^{0,(n,0)}$. At higher spin, polynomials occur that cannot be expressed as a linear combination of the ones that have been obtained before. They are therefore added to the basis. From $n=6$ to $n=10$, all OMEs can be expressed by $A_{II}^{0,(n,0)}$ and $A_{II}^{0,(n,2)}$, at spin $n=12$ the OME $A_{II}^{0,(n,4)}$ appears.

An algorithm has been implemented to calculate the linear dependencies for all spins. The relations up to spin $n=16$ are collected in Appendix B. We found numerically that there are reappearing patterns in the OMEs' coefficients. The number of OMEs for which this holds, grows with the spin. In the limit of large spin the linear dependencies between the OMEs are therefore general and do not to depend on the value of the spin, but only on the second quantum number. For the lowest values of the second quantum number these general coefficients are given by

$$
\begin{aligned}
A_{II}^{0,(\infty,1)} &= -\frac{1}{2}A_{II}^{0,(\infty,0)}\\
A_{II}^{0,(\infty,3)} &= \frac{1}{4}A_{II}^{0,(\infty,0)} - \frac{3}{2}A_{II}^{0,(\infty,2)}\\
A_{II}^{0,(\infty,5)} &= -\frac{1}{2}A_{II}^{0,(\infty,0)} + \frac{5}{2}A_{II}^{0,(\infty,2)} - \frac{5}{2}A_{II}^{0,(\infty,4)}
\end{aligned} \qquad (58)
$$

5.4 Anomalous dimensions of the operator matrix elements

The 1-loop-corrections contributing to $A_{II}^{0,(n,m)}$ are shown in Figure 3. The conservation of momentum requires that $p_1+p_2+p_3=0$. In the calculation it is necessary to assume that all external particles have space-like momenta off the mass-shell ($-p_i^2>0$). The total amplitude is given by

$$
\begin{aligned}
&A_{II}^{(1),(n,m)}(\Delta.p_1,\Delta.p_2) = -a\left[\frac{2}{\varepsilon}+L\right]\times \qquad (59)\\
&\times\left[\left\{\sum_{l=0}^{n-m}\binom{n-m}{l}\frac{1}{(l+m+1)(l+m+2)}\times(-1)^{l+n}\sum_{j=0}^{m+l}(-1)^j A_{II}^{0,(n,j)}(\Delta.p_1,\Delta.p_2)\right\}+\right.\\
&+\left\{\frac{1}{(m+1)(m+2)}\times(-1)^m\sum_{l=0}^{m}(-1)^l A_{II}^{0,(n,l)}(\Delta.p_1,\Delta.p_2)\right\}+\\
&\left.+\left\{\frac{1}{(n-m+1)(n-m+2)}\times(-1)^{n-m}\sum_{l=0}^{n-m}(-1)^l A_{II}^{0,(n,l)}(\Delta.p_1,\Delta.p_2)\right\}\right] \ .
\end{aligned}
$$

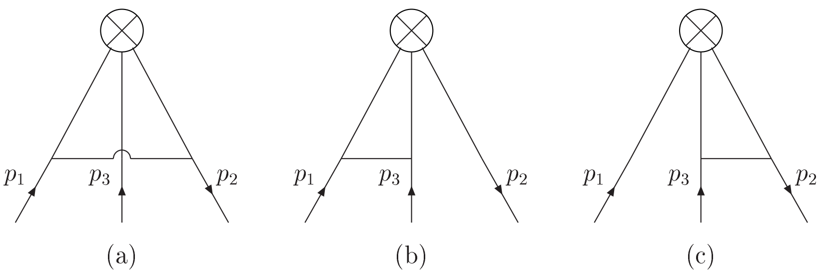

Figure 3: 1-loop corrections to $A_{II}^{0,(n,m)}$.

Here, all constant terms are collected in

$$L = \gamma_E - \ln(4\pi) + \ln\left(\frac{-p_3^2}{\mu^2}\right) + \mathcal{O}(1) \ . \tag{60}$$

Operator mixing occurs if there is more than one independent operator at a given spin. In this case, the anomalous dimension is given by the matrix $\gamma_{II}^{(m,m')}$. The matrix can be obtained by solving the linear system given by

$$\begin{aligned} A_{II}^{(1),(n,m)}(\Delta.k_1, \Delta.k_2) \quad + \quad & (Z_1 - 1) A_{II}^{0,(n,m)}(\Delta.k_1, \Delta.k_2) = \\ & -\frac{a}{2}\left[\frac{2}{\varepsilon} + L\right] \sum_{m'} \left(\gamma_{II}^{(n)}\right)^{mm'} A_{II}^{0,(n,m')}(\Delta.k_1, \Delta.k_2) \ . \end{aligned} \tag{61}$$

Here $Z_1 = 1 + \frac{a}{\varepsilon}$. It has been shown before that more and more linearly independent operators occur as the spin grows. Since more operators of the same twist yet different spin take part in the mixing, the anomalous dimension matrix will spread out. This phenomenon is known as a spin tower.

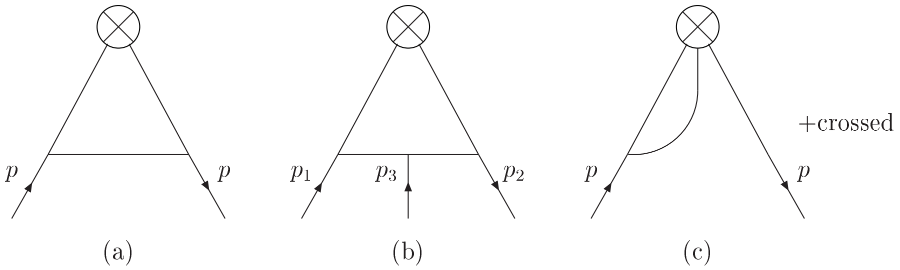

Figure 4: 1-loop corrections to (a) $A_{JJ}^{0,(n)}$, (b) $A_{JI}^{0,(n,m)}$, and (c) $A_{IJ}^{0,(n,m)}$

The 1-loop correction of the operator matrix element $A_{JJ}^{(n,m)}$ is shown in Figure 4a. The result is

$$A_{JJ}^{(n)}(\Delta.p) = a\left[\frac{2}{\varepsilon} + L\right] \frac{1}{(n+2)(n+3)} A^{0,(n)} \ . \tag{62}$$

There is no mixing in this case, so the anomalous dimension matrix has just one element. It can be obtained from

$$A_{JJ}^{(n)}(\Delta.k_1) + (Z_3 - 1) A_{JJ}^{0,(n)}(\Delta.k_1) \quad = \quad -\frac{a_s}{2}\left[\frac{2}{\varepsilon} + L\right] \gamma_{JJ}^{(n)} A_{JJ}^{0,(n)}(\Delta.k_1) \tag{63}$$

$$\Rightarrow \gamma_{JJ}^{(n)} = -\frac{2}{(n+2)(n+3)} - \frac{1}{6} \ . \tag{64}$$

Figure 4b shows the only contribution to the 1-loop correction of the operator matrix element $A^{(n,m)}_{JJ}$. The result can be expressed in terms of the OME $A^{0,(n,m)}_{II}$.

$$A^{(n)}_{JI}(\Delta.k_1,\Delta.k_2) \;=\; -a\Big[\frac{2}{\varepsilon}+L\Big]\frac{3}{(n+1)(n+2)}\sum_{m=0}^{n}(-1)^{n-m}A^{(0),(n,m)}_{II}(\Delta.k_1,\Delta.k_2) \quad (65)$$

$$= -\frac{a_s}{2}\Big[\frac{2}{\varepsilon}+L_\mu\Big]\sum_{m'}\left(\gamma^{(n)}_{JI}\right)^{m'}(-1)^{n-m'}A^{(0),(n,m')}_{II}(\Delta.k_1,\Delta.k_2)\,. \quad (66)$$

If we reduce this to the linearly independent operators only, we obtain the elements of the anomalous dimension matrix. In this case there are several entries, due to the mixing of A_{II}.
The 1-loop correction of the operator matrix element $A^{(n,m)}_{JJ}$ is shown in Figure 4b. Again, the result can be expressed in terms of $A^{0,(n,m)}_{II}$ and the elements of the anomalous dimension matrix can be calculated in a closed form:

$$A^{(n,m)}_{IJ}(\Delta.k_1) \;=\; -\frac{a}{2}\Big[\frac{2}{\varepsilon}+L\Big]\left(\gamma^{(n)}_{IJ}\right)^{m}A^{0,(n)}_{JJ}(\Delta.k_1) \quad (67)$$

$$\left(\gamma^{(n)}_{IJ}\right)^{m} \;=\; -\frac{2}{3}\Big[\frac{(-1)^m}{(n-m+2)(n-m+3)}+\frac{(-1)^{n-m}}{(m+2)(m+3)}+ \quad (68)$$

$$+\sum_{i=0}^{n-m+1}\binom{n-m+1}{i}\frac{(-1)^i}{i+m+2}\Big]\,. \quad (69)$$

5.5 Complete anomalous dimension matrix

Now all parts of the anomalous dimension matrix have been calculated. This matrix is the connection between the linearly independent OMEs at tree-level and the 1-loop corrections. For spins $n<4$ all partial matrices consist of one element only. The lowest spin anomalous dimension matrices are

$$\gamma^{(0)}=\begin{bmatrix}2 & 3\\ -\frac{1}{3} & -\frac{1}{2}\end{bmatrix}\rightarrow\begin{bmatrix}0 & 0\\ 0 & \frac{3}{2}\end{bmatrix}\ \text{and}\ \gamma^{(2)}=\begin{bmatrix}\frac{5}{6} & \frac{5}{4}\\ -\frac{8}{45} & -\frac{4}{15}\end{bmatrix}\rightarrow\begin{bmatrix}0 & 0\\ 0 & \frac{17}{30}\end{bmatrix}\,. \quad (70)$$

The relations similar to Eq. (48) that stem from the equation of motion are mirrored in the remaining linear dependencies. They are removed by diagonalization, which means the change to a basis of linearly independent operators. At spin $n=6$, there is more than one operator $A^{0,(n,m)}_{II}$ in the basis, hence a spin tower appears. The anomalous dimension matrix is now

$$\gamma^{(6)}=\begin{bmatrix}\frac{11}{56} & \frac{39}{112} & \frac{33}{112}\\ \frac{1}{4} & -\frac{23}{40} & \frac{3}{8}\\ -\frac{7}{54} & -\frac{191}{3780} & -\frac{7}{36}\end{bmatrix}\rightarrow\begin{bmatrix}0 & 0 & 0\\ 0 & -\frac{361}{1260}+\frac{2\sqrt{3754}}{315} & 0\\ 0 & 0 & -\frac{361}{1260}-\frac{2\sqrt{3754}}{315}\end{bmatrix}\,. \quad (71)$$

In order to calculate the matrices and eigenvalues at all values of the spin, an algorithm has been implemented using computer algebra software. The matrices up to spin $n=10$ calculated that way are collected in Appendix C.

5.6 The eigenvalue spectrum

The eigenvalue spectrum of the anomalous dimension matrix is plotted in Figure 5 as a function of the spin. Notably, the eigenvalues are real numbers. They can be grouped into sets according to their values. All of these sets decrease and show a saturation as the spin grows. We can conclude from this, that in the limit of large spins, the eigenvalues are a general property of the operator and depend only on the second quantum number.

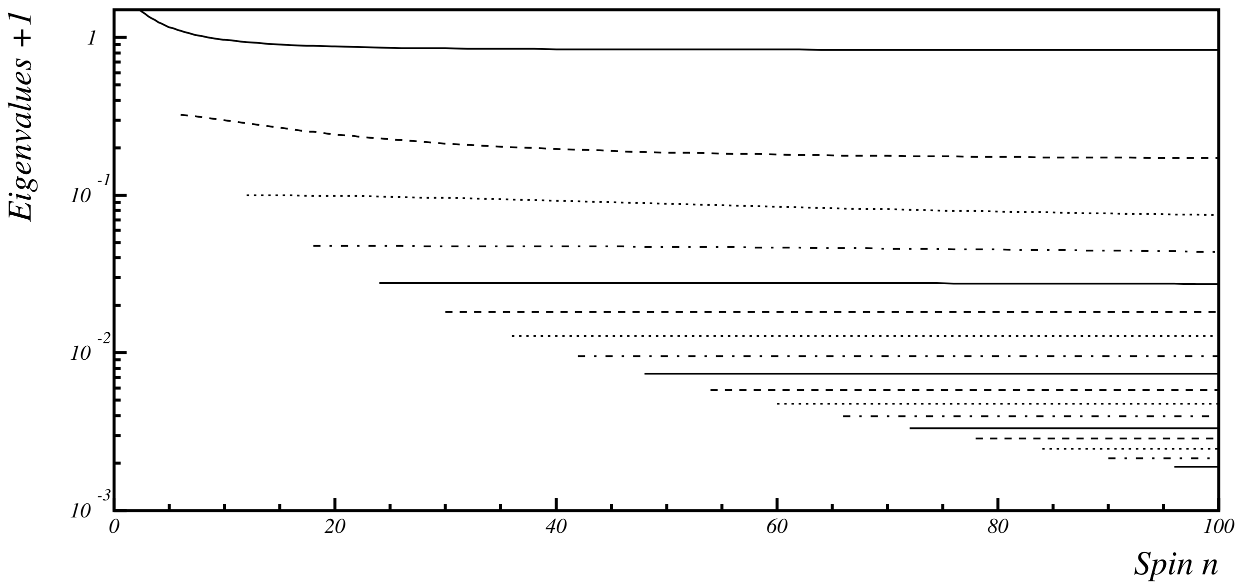

Figure 5: Eigenvalues of the anomalous dimension matrix.

5.7 The evolution equation for twist $\tau = 6$

The running coupling constant and the leading order evolution equation for the partonic distribution function are given by

$$\frac{da}{d\ln\mu^2} = \beta = -\sum_{l=0}^{\infty} a^{l+2}\beta_l, \qquad \frac{df_i(x,Q^2)}{d\ln\mu^2} = \left(\frac{a}{a_0}\right)^{-\gamma_0(N)} f_i(N)\ . \tag{72}$$

It has a form similar to Eq. (40), but here the anomalous dimension is matrix-valued. The evolution equation is obtained to all orders as an expansion around the leading order equation, cf. [9]. An eigenvalue-decomposition has to be applied to deal with the matrix in the exponent.

6 Summary and Conclusions

The parton model in deeply inelastic scattering is valid only if $\tau_{life} \ll /\tau_{int}$. In this case, it yields a sufficient description at leading twist. Higher twist has to be taken account in the regions, where $x \not\ll 1$, $x \not\approx 1$ or at low values of Q^2. We have studied the leading twist $\tau = 4$ and the first higher twist $\tau = 6$ in scalar ϕ^3-theory in $D = 6$ dimensions as a simplified model to gluo-dynamics (QCD) at first order in the coupling constant. The evolution equation for $\tau = 4$ operators been found and solved. The $\tau = 6$ operator mixing problem has been explored for local operators, which leads to so-called "spin-towers". I.e. more and more operators contribute to the mixing as the spin rises. The linear dependencies between tree-level operator matrix elements have been examined and the coefficients have been evaluated. An algorithm has been implemented to do these calculations. We then saw reappearing coefficients, which depend only on the second quantum number in the limit of large spin.

The mixing matrices have been calculated. Their diagonalization led to a spectrum of real eigenvalues, in which all sets saturate as the spin grows.

At this point, it would be desirable to resum the spin-towers into 2-variable evolution equations.

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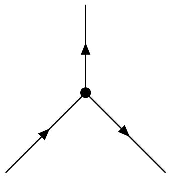
A Feynman-Rules of scalar ϕ^3-theory

All calculations assume $D = 6 + \varepsilon$. The Feynman-Rules used here follow [22].



g

$\frac{1}{m^2 - k^2 + i0}$

Figure 6: Feynman-rules of ϕ^3-theory

B Relation between operator matrix elements $A_{II}^{0,(n,m)}$

The operator matrix elements $A_{II}^{0,(n,m)}(\Delta.k_1, \Delta.k_2)$ at twist $\tau = 6$ in scalar ϕ^3 -theory in $D = 6$ dimensions can be constructed as linear combinations of a few basis-elements. Up to spin $n = 16$, the relations are given below.

Spin $n = 0$

$$A_{II}^{0,(0,0)} = 1 \tag{B.1}$$

Spin $n = 2$

$$\begin{aligned} A_{II}^{0,(2,0)} &= A_{II}^{0,(2,2)} \\ A_{II}^{0,(2,1)} &= -\frac{1}{2}A_{II}^{0,(2,0)} \end{aligned} \tag{B.2}$$

Spin $n = 4$

$$\begin{aligned} A_{II}^{0,(4,0)} &= A_{II}^{0,(4,4)} \\ A_{II}^{0,(4,1)} &= A_{II}^{0,(2,3)} = -\frac{1}{2}A_{II}^{0,(4,0)} \\ A_{II}^{0,(4,2)} &= \frac{1}{2}A_{II}^{0,(4,0)} \end{aligned} \tag{B.3}$$

Spin $n = 6$

$$\begin{aligned} A_{II}^{0,(6,0)} &= A_{II}^{0,(6,6)} \\ A_{II}^{0,(6,1)} &= A_{II}^{0,(6,5)} = -\frac{1}{2}A_{II}^{0,(6,0)} \\ A_{II}^{0,(6,2)} &= A_{II}^{0,(6,4)} \\ A_{II}^{0,(6,3)} &= \frac{1}{4}A_{II}^{0,(6,0)} - \frac{3}{2}A_{II}^{0,(6,2)} \end{aligned} \tag{B.4}$$

Spin $n = 8$

$$\begin{aligned} A_{II}^{0,(8,0)} &= A_{II}^{0,(8,8)} \\ A_{II}^{0,(8,1)} &= A_{II}^{0,(8,7)} = -\frac{1}{2}A_{II}^{0,(8,0)} \\ A_{II}^{0,(8,2)} &= A_{II}^{0,(8,6)} \\ A_{II}^{0,(8,3)} &= A_{II}^{0,(8,5)} = \frac{1}{4}A_{II}^{0,(8,0)} - \frac{3}{2}A_{II}^{0,(8,2)} \\ A_{II}^{0,(8,4)} &= -\frac{3}{10}A_{II}^{0,(8,0)} + \frac{8}{5}A_{II}^{0,(8,2)} \end{aligned} \tag{B.5}$$

Spin $n = 10$

$$\begin{aligned} A_{II}^{0,(10,0)} &= A_{II}^{0,(10,10)} \\ A_{II}^{0,(10,1)} &= A_{II}^{0,(10,9)} = -\frac{1}{2}A_{II}^{0,(10,0)} \\ A_{II}^{0,(10,2)} &= A_{II}^{0,(10,8)} \\ A_{II}^{0,(10,3)} &= A_{II}^{0,(10,7)} = \frac{1}{4}A_{II}^{0,(10,0)} - \frac{3}{2}A_{II}^{0,(10,2)} \\ A_{II}^{0,(10,4)} &= A_{II}^{0,(10,6)} = -\frac{5}{14}A_{II}^{0,(10,0)} - \frac{12}{7}A_{II}^{0,(10,2)} \\ A_{II}^{0,(10,5)} &= -\frac{11}{28}A_{II}^{0,(10,0)} + \frac{25}{14}A_{II}^{0,(10,2)} \end{aligned} \tag{B.6}$$

Spin $n = 12$

$$
\begin{aligned}
A_{II}^{0,(12,0)} &= A_{II}^{0,(12,12)} \\
A_{II}^{0,(12,1)} &= A_{II}^{0,(12,11)} &&= -\frac{1}{2}A_{II}^{0,(12,0)} \\
A_{II}^{0,(12,2)} &= A_{II}^{0,(12,10)} \\
A_{II}^{0,(12,3)} &= A_{II}^{0,(12,9)} &&= \frac{1}{4}A_{II}^{0,(12,0)} - \frac{3}{2}A_{II}^{0,(12,2)} \\
A_{II}^{0,(12,4)} &= A_{II}^{0,(12,8)} \\
A_{II}^{0,(12,5)} &= A_{II}^{0,(12,7)} &&= -\frac{1}{2}A_{II}^{0,(12,0)} + \frac{5}{2}A_{II}^{0,(12,2)} - \frac{5}{2}A_{II}^{0,(12,4)} \\
A_{II}^{0,(12,6)} & &&= \frac{3}{4}A_{II}^{0,(12,0)} - \frac{26}{7}A_{II}^{0,(12,2)} + \frac{45}{14}A_{II}^{0,(12,4)}
\end{aligned}
\tag{B.7}
$$

Spin $n = 14$

$$
\begin{aligned}
A_{II}^{0,(14,0)} &= A_{II}^{0,(14,14)} \\
A_{II}^{0,(14,1)} &= A_{II}^{0,(14,13)} &&= -\frac{1}{2}A_{II}^{0,(14,0)} \\
A_{II}^{0,(14,2)} &= A_{II}^{0,(14,12)} \\
A_{II}^{0,(14,3)} &= A_{II}^{0,(14,11)} &&= \frac{1}{4}A_{II}^{0,(14,0)} - \frac{3}{2}A_{II}^{0,(14,2)} \\
A_{II}^{0,(14,4)} &= A_{II}^{0,(14,10)} \\
A_{II}^{0,(14,5)} &= A_{II}^{0,(14,9)} &&= -\frac{1}{2}A_{II}^{0,(14,0)} + \frac{5}{2}A_{II}^{0,(14,2)} - \frac{5}{2}A_{II}^{0,(14,4)} \\
A_{II}^{0,(14,6)} &= A_{II}^{0,(14,8)} &&= \frac{10}{11}A_{II}^{0,(14,0)} - \frac{148}{33}A_{II}^{0,(14,2)} + \frac{11}{3}A_{II}^{0,(14,4)} \\
A_{II}^{0,(14,7)} & &&= -\frac{93}{88}A_{II}^{0,(14,0)} + \frac{343}{66}A_{II}^{0,(14,2)} - \frac{49}{12}A_{II}^{0,(14,4)}
\end{aligned}
\tag{B.8}
$$

Spin $n = 16$

$$
\begin{aligned}
A_{II}^{0,(16,0)} &= A_{II}^{0,(16,16)} \\
A_{II}^{0,(16,1)} &= A_{II}^{0,(16,15)} &&= -\frac{1}{2}A_{II}^{0,(16,0)} \\
A_{II}^{0,(16,2)} &= A_{II}^{0,(16,14)} \\
A_{II}^{0,(16,3)} &= A_{II}^{0,(16,13)} &&= \frac{1}{4}A_{II}^{0,(16,0)} - \frac{3}{2}A_{II}^{0,(16,2)} \\
A_{II}^{0,(16,4)} &= A_{II}^{0,(16,12)} \\
A_{II}^{0,(16,5)} &= A_{II}^{0,(16,11)} &&= -\frac{1}{2}A_{II}^{0,(16,0)} + \frac{5}{2}A_{II}^{0,(16,2)} - \frac{5}{2}A_{II}^{0,(16,4)} \\
A_{II}^{0,(16,6)} &= A_{II}^{0,(16,10)} &&= \frac{51}{52}A_{II}^{0,(16,0)} - \frac{690}{143}A_{II}^{0,(16,2)} + \frac{85}{22}A_{II}^{0,(16,4)} \\
A_{II}^{0,(16,7)} &= A_{II}^{0,(16,9)} &&= -\frac{17}{13}A_{II}^{0,(16,0)} + \frac{1827}{286}A_{II}^{0,(16,2)} - \frac{105}{22}A_{II}^{0,(16,4)} \\
A_{II}^{0,(16,8)} & &&= \frac{37}{26}A_{II}^{0,(16,0)} - \frac{992}{143}A_{II}^{0,(16,2)} + \frac{56}{11}A_{II}^{0,(16,4)}
\end{aligned}
\tag{B.9}
$$

C Anomalous dimension matrices of twist $\tau = 6$

The anomalous dimension matrices at twist $\tau = 6$ in scalar ϕ^3 -theory in $D = 6$ dimensions describe the mixing of operators of the types O_I and O_J. The matrices up to spin $n = 10$ are given below.

Spin $n = 0$

$$\begin{aligned} \gamma^{(0)} &= \begin{bmatrix} 2 & 3 \\ -\frac{1}{3} & -\frac{1}{2} \end{bmatrix} \\ \to \begin{bmatrix} 0 & 0 \\ 0 & \frac{3}{2} \end{bmatrix} &\approx \begin{bmatrix} 0 & 0 \\ 0 & 1.5 \end{bmatrix} \end{aligned} \tag{C.1}$$

Spin $n = 2$

$$\begin{aligned} \gamma^{(2)} &= \begin{bmatrix} \frac{5}{6} & \frac{5}{4} \\ -\frac{8}{45} & -\frac{4}{15} \end{bmatrix} \\ \to \begin{bmatrix} 0 & 0 \\ 0 & \frac{17}{30} \end{bmatrix} &\approx \begin{bmatrix} 0 & 0 \\ 0 & 0.567 \end{bmatrix} \end{aligned} \tag{C.2}$$

Spin $n = 4$

$$\begin{aligned} \gamma^{(4)} &= \begin{bmatrix} \frac{7}{15} & \frac{7}{10} \\ -\frac{1}{7} & -\frac{3}{14} \end{bmatrix} \\ \to \begin{bmatrix} 0 & 0 \\ 0 & \frac{53}{210} \end{bmatrix} &\approx \begin{bmatrix} 0 & 0 \\ 0 & 0.252 \end{bmatrix} \end{aligned} \tag{C.3}$$

Spin $n = 6$

$$\gamma^{(6)} = \begin{bmatrix} \frac{11}{56} & \frac{39}{112} & \frac{33}{112} \\ \frac{1}{4} & -\frac{23}{40} & \frac{3}{8} \\ -\frac{7}{54} & -\frac{191}{3780} & -\frac{7}{36} \end{bmatrix} \tag{C.4}$$

$$\to \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{361}{1260} + \frac{2\sqrt{3754}}{315} & 0 \\ 0 & 0 & -\frac{361}{1260} - \frac{2\sqrt{3754}}{315} \end{bmatrix} \approx \begin{bmatrix} 0 & 0 & 0 \\ 0 & -0.103 & 0 \\ 0 & 0 & -0.676 \end{bmatrix}$$

Spin $n = 8$

$$\gamma^{(8)} = \begin{bmatrix} \frac{22}{225} & \frac{127}{450} & \frac{11}{75} \\ \frac{22}{75} & -\frac{611}{1050} & \frac{11}{25} \\ -\frac{61}{495} & -\frac{64}{1485} & -\frac{61}{330} \end{bmatrix} \tag{C.5}$$

$$\to \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{1159}{3465} + \frac{\sqrt{6004909}}{6930} & 0 \\ 0 & 0 & -\frac{1159}{3465} - \frac{\sqrt{6004909}}{6930} \end{bmatrix} \approx \begin{bmatrix} 0 & 0 & 0 \\ 0 & -0.019 & 0 \\ 0 & 0 & -0.688 \end{bmatrix}$$

Spin $n = 10$

$$\gamma^{(10)} = \begin{bmatrix} \frac{13}{308} & \frac{2329}{9240} & \frac{39}{616} \\ \frac{13}{42} & -\frac{751}{1260} & \frac{13}{28} \\ -\frac{14}{117} & -\frac{17}{429} & -\frac{7}{39} \end{bmatrix} \tag{C.6}$$

$$\to \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{16516}{45045} + \frac{\sqrt{910902994}}{90090} & 0 \\ 0 & 0 & -\frac{16516}{45045} - \frac{\sqrt{910902994}}{90090} \end{bmatrix} \approx \begin{bmatrix} 0 & 0 & 0 \\ 0 & -0.032 & 0 \\ 0 & 0 & -0.702 \end{bmatrix}$$

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