HIGHER ORDER CALCULATIONS IN PERTURBATION THEORY

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Gonzalez Lopez, Jenifer

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ABSTRACT: In this report we present an effective method - the Integration by Parts Identities - which is very useful for performing higher order perturbative calculations. It is implemented in a program, AIR, for the automated reduction of large systems of integrals, whose algorithm was first proposed by Laporta and was implemented in MAPLE by Anastasiou and Lazopoulos. To illustrate the method we solve a particular case: the massless self-energy Feynman diagram, by considering perturbation theory until second-order. First of all, we treat the lowest-order diagram for this process, the one-loop one. After that, we consider two-loop self-energy diagrams, which involve more complex integrals to be reduced.

Contents

1 Introduction

Our goal throughout this paper is to show the efficiency of an actual method for reducing large systems of integrals to master integrals. It consists of a program whose algorithm was first proposed by Laporta and was implemented in MAPLE by Anastasiou and Lazopoulos.

In most cases, the way to solve physical problems is based on perturbation theory. Specifically, in particle physics, it is necessary to work in perturbation theory in order to describe well the experimental data. It is often not enough to work in lowest-order of perturbation theory; to explain, as precisely as possible, the physical situation one has to take higher orders into account. The problem comes up when the order of the correction is higher and higher; in this case, large number of integrals have to be solved and a good method is needed to make the work feasible. It is at this point, where the program we use here does a great job; it is based on an algorithm which reduces the number of integrals to a few master integrals which are the only that need to be solved. In order to show it, we deal here with a particular problem, the massless self-energy diagram.

In Section 2 we explain the main characteristics of the program. In Section 3 we solve the scalar part of the massless self-energy diagram; the one-loop diagram is treated in 3.1 and two-loops corrections are calculated in 3.2. In Section 4 we show the conclusions of our work.

2 AIR description

Of main interest in solving a physical problem is the development of automated methods applicable to every process and order in perturbation theory. The first requirement is one of the advantages that this program provides; it is a MAPLE program, based on an algorithm which was first proposed by Laporta and was implemented in MAPLE by Anastasiou and Lazopoulos, for Automated Integral Reductions, AIR. It reduces the number of integrals to a few master integrals so it is only necessary to solve these last ones. It uses the method of Gauss elimination to solve large systems of equations, satisfied by the coupled algebraic identities. For the moment, reduction algorithms can not be extended to arbitrarily large calculations but many phenomenological problems can be treated using this program. We present here only a basic description of the algorithm and the program, AIR[1].

Integrals belonging to the same topology (with common propagators) satisfy linear algebraic identities and the IBP (integration by parts) method[2] is suitable to derive them; its idea is to multiply the integrand with a loop or

[1] A detailed description can be found in Ref. [4, 5, 6]

[2] cf. [4]

external momentum and differentiate it with the loop momenta. In virtue of Gauss theorem, these derivatives integrate to zero, being each one a generic IBP identity for the topology. Based on this method, Laporta introduced an automated algorithm for the reduction of generic loop amplitudes. By solving a large system of IBP equations, it reduces one-by-one the integrals. The algorithm requires successive generation of identities, with increasing complexity. The algorithm substitute the seeds one by one in the generic expression obtained from the IBP identities; this produces subsystems of coupled algebraic equations. The equations on each system are ordered according to their complexity, employing *priority criteria*, and then the algorithm uses Gauss elimination to make the reduction. By a recursive use of this procedure the algorithm expresses the complicated loop integrals in terms of master ones.

In multi-loop calculations, large number of IBP identities is produced and it is a difficult task to implement efficiently the Laporta´s algorithm in a program. The program we use here, AIR, implements the algorithm in the following way; as input, is only required to introduce the IBP equations for each topology. Also, can be added information about the vanishing integrals of the topology and the master integrals (when they are known), as well as information about parameters for treating large expressions. This program generates the seeds automatically. This is accomplished by the routine *SEEDGEN*, which takes the shape:

SEEDGEN("filename", maxtop, [min N_{prop}**,max** N_{prop}**], [min** N_{-}**,max** N_{-}**], [min** N_{+}**,max** N_{+}**])**.

It means that the user must also provide the range of *number of propagators* (N_{prop}), *sum of positive indices* (N_{+}), *sum of negative indices* (N_{-}) and *maxtop* (which is a list of integers indicating the propagators that can appear with positive powers in the seed with the highest priority). It is quite easy to choose these values, only by inspecting the integrals that are required in the process. The definition of this variables is the following:

$$N_{prop} = \sum_i \Theta(\nu_i), \tag{1}$$

$$N_{+} = \sum_i \Theta(\nu_i)(\nu_i - 1), \tag{2}$$

$$N_{-} = -\sum_i \Theta(-\nu_i)\nu_i, \tag{3}$$

where our amplitudes are expressed in the form $B(\nu_1, \nu_2, \cdots, \nu_r)$, being ν_i the exponents of the propagators and $\Theta(x) = 1$ for $x > 0$ and $\Theta(x) = 0$ for $x \leq 0$. In any case, this notation will be more understandable when we show a concrete example in the following section.

The program is, basically, a collection of MAPLE routines which generate seeds, substitute them into the IBP identities producing the IBP equations, which generate coupled subsystems of equations which are reduced using Gauss elimination. This is the basic part of the program and is the only we use here, since our problem does not require more complicated processing.

3 Solving the massless self-energy diagram

As we mentioned before, we study as example the massless self-energy diagram. The main reason to treat the self-energy diagram is that it is the simplest one and it is already well studied, so we can check our results and to compare our method with previous treatments, showing how it improves them and makes more easy the calculations; however, in spite of being the simplest one, the number of equations that one must solve, when higher orders in perturbation theory are considered, is large enough to make the problem not trivial. Another motivation is the fact that it is one of the processes which helps "relate" a *bare* particle to a *physical* one and that it is a part of others more complicated diagrams, which contain the self-energy diagram as part of them. Many important physical problems, like the calculation of decay rates or even differential cross-sections can be expressed through self-energy type diagrams.

We also should mention here that this kind of diagrams, loop diagrams, generate divergent integrals for the amplitudes describing the processes. It comes because it is necessary to integrate to all possible values for the internal momenta, which leads to divergent integrations. To overcome this difficulties two main steps are involved:

Regularization: modification of the theory so that it remains finite and well-defined in all orders of perturbation theory.

Renormalization: consists of relating the properties of the physical particles to those of the bare (non-interacting) ones and expressing the predictions of the theory in terms of the masses and charges of the physical particles. Going back to the original theory, the initial infinities are translated now to the relations (totally unobservable) between bare and physical particles and the observables of the theory are expressed in terms of the measured charges and masses, remaining finite when the original theory is recovered.

We do not show the second step here, because it goes out of the scope of this report; however, it is necessary to carry out the *regularization* to solve our integrals. There are many regularization processes and the criterion to choose a good one is that the regularized theory should preserve as many

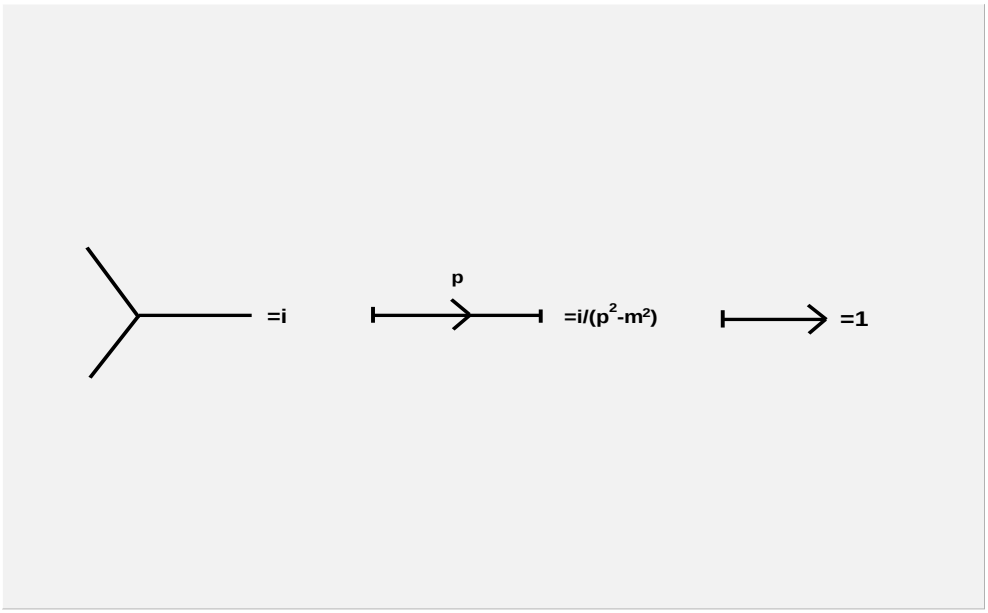


Fig. 1. Feynman rules in ϕ^3 theory.

symmetries as possible. We work here with *dimensional regularization*; this is the most suitable for gauge theories since nothing, except that the space-time is not 4-dimensional, is violated. It is based on the fact that a divergent multiple integral may be made convergent by changing the number of multiple integrals; because of it, throughout this paper we have all our integrals expressed in terms of arbitrary space-time dimension, d.

It is important to note that there are not references, along the paper, to specific Feynman factors[3]. It is due to we are working here in the ϕ^3 theory, which generates the scalar part of the problem; its Feynman rules are shown in *Fig.*[1].

It is not a physical treatment, but we are interested here only in the scalar part of the diagram to show how AIR works. The physical solution (QED, QCD) can be obtained only by considering the right metrics and Feynman factors for each model and concrete problem, which do not add relevant information to the item we are interested in throughout this paper.

The Feynman amplitude M for a given process takes the form

$$M = \sum_{n=1}^{\infty} M^{(n)} \tag{4}$$

where $M^{(n)}$ comes from the nth order perturbation term $S^{(n)}$ in the S-matrix

[3]We are not going to give here a deep description about Feynman diagrams and rules because it goes out of the scope of this paper. For a detailed study, Ref.[2].

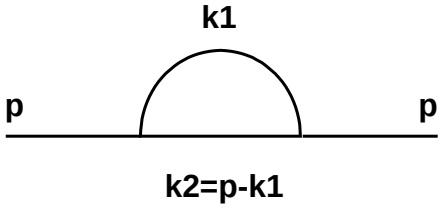


Fig.2. One-loop self-energy Feynman diagram.

expansion[4]. In order to obtain the Feynman amplitude $M^{(n)}$ is needed to consider all topologically different, connected Feynman diagrams, with n vertices, in momentum space, obtaining from the Feynman rules the contribution from each vertex to the amplitude $M^{(n)}$.

In our case, we calculate the self-energy diagram considering radiative corrections until second-order, constructed by modifying the lowest-order graph in all possible ways by adding one more loop to the first-order diagram. The Feynman graphs for this processes are shown in *Fig.*[2, 3]. *Fig.*[2] represents the one-loop Feynman diagram and *Fig.*[3] shows the second-order radiative corrections contributions. In all processes, we denote with p the external momentum and k_i all internal momenta.

It is necessary, as well, to say that we neglect all the masses here. In some cases, for a physical treatment of the problem they have to be taken into account, but they are irrelevant to our goal here.

3.1 One-loop self-energy diagram

In this section, we solve the massless self-energy Feynman diagram in the lowest non-vanishing order in perturbation theory, one-loop diagram. In order to do that, the first step is to regularize the theory. Using dimensional regularization as well as the Feynman rules for our particular theory, the amplitude of the process is as follows:

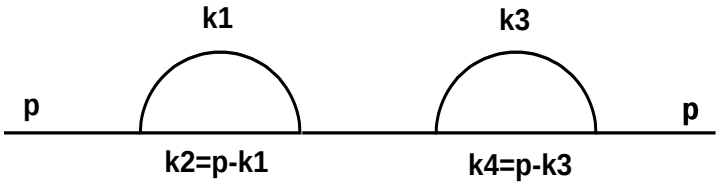
$$B(1,1) = \int d^d k \frac{1}{k^2 (k-p)^2}. \tag{5}$$

Looking at this integral, it is easy to check that it belongs to the following topology:

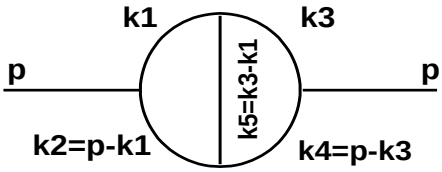
$$B(\nu_1, \nu_2) = \int d^d k \frac{1}{[k^2]^{\nu_1} \, [(k-p)^2]^{\nu_2}}. \tag{6}$$

Knowing that, the following step is to use AIR in order to express our

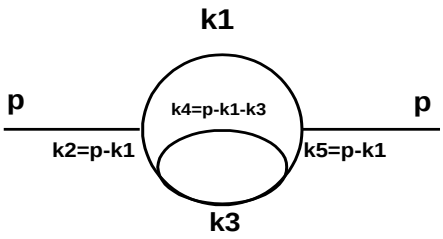
[4]cf. [2] for more information



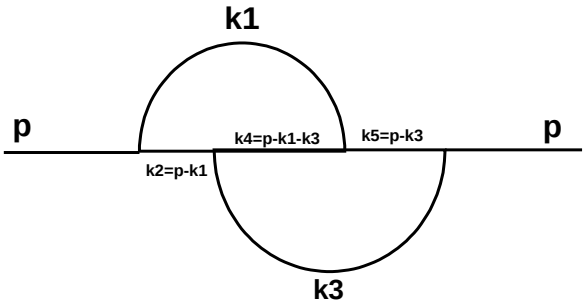
$(3.a).M_1$



$(3.b).M_2$



$(3.c).M_3$



$(3.d).M_4$

Fig. 3. Two-loops self-energy Feynman diagrams.

amplitude in terms of master integrals. As we explained before, it is only necessary to introduce the IBP identities as input, as well as the parameters needed for the routine *SEEDGEN*. Looking to our amplitude, $B(1,1)$, we decided to select the next values for these parameters:

$$maxtop = [1,2] \tag{7}$$

$$N_{prop} \subset [0,2] \tag{8}$$

$$N_{+} \subset [0,2] \tag{9}$$

$$N_{-} \subset [0,2] \tag{10}$$

Now, we have to derive the IBP identities for this topology. In order to do that, we use the IBP method:

$$0 = \int d^d k \frac{\partial}{\partial k^\mu} \frac{\eta^\mu}{[k^2]^{\nu_1} [(k-p)^2]^{\nu_2}}, \tag{11}$$

where we can choose η freely to obtain two linearly independent equations. We have chosen $\eta = k, k-p$ in this case. Doing the calculations with this election, two possible IBP identities for this topology can be:

$$R1 : 0 = s\nu_2 B(\nu_1, \nu_2+1) + (d-2\nu_1-\nu_2)B(\nu_1,\nu_2) - \nu_2 B(\nu_1-1,\nu_2+1) \tag{12}$$

$$R2 : 0 = s\nu_1 B(\nu_1+1, \nu_2) + (d-\nu_1-2\nu_2)B(\nu_1,\nu_2) - \nu_1 B(\nu_1+1,\nu_2-1), \tag{13}$$

where s is the Mandelstam variable, defined as $s = p^2$.

Implementing this input on AIR, the program shows that $B(1,1)$ is a *master integral*. In this case, the following step is to solve it. Using 'Feynman parametrization':

$$\frac{1}{AB} = \int_0^1 \frac{dx}{[xA + (1-x)B]^2}, \tag{14}$$

and calling $A = (k-p)^2$, $B = k^2$, the integral can be written:

$$B(1,1) = \int d^d k \int_0^1 dx \frac{1}{[x(k-p)^2 + (1-x)k^2]^2}. \tag{15}$$

To ensure the convergence, it is necessary to keep $d < 3$ and, in this case, it is possible to interchange k and x integrations:

$$B(1,1) = \int_0^1 dx \int d^d k \frac{1}{[x(k-p)^2 + (1-x)k^2]^2}, \tag{16}$$

and making a rearrangement in the denominator of this expression, it takes the shape:

$$B(1,1) = \int_0^1 dx \int d^d k \frac{1}{[(k-xp)^2 + x(1-x)p^2]^2}. \tag{17}$$

Since dimensional regularization preserves translational invariance, we can do a shift in k: $k^{`} = k - xp$, to make more easy the integration:

$$B(1,1) = \int_0^1 dx \int d^d k^{`} \frac{1}{[k^{`2} + x(1-x)p^2]^2}. \tag{18}$$

But $k^{`}$ integration should be performed in the Minkowski space, so we can make a $90^{\text{o}}$-rotation in the $k_0^{`}$-plane, *Wick rotation*, and to transform this integration into Euclidean. The change is:

$$\begin{cases} k_0^{`} = ir_0 \quad r_0 \text{ real} \\ \vec{k}^{`} = \vec{r} \end{cases} \tag{19}$$

so,

$$\begin{cases} d^d k^{`} = i d^d r \\ k^{`2} = -r^2 \quad r^2 = r_0^2 + \vec{r}^2 \end{cases} \tag{20}$$

and denoting $L \equiv -x(1-x)p^2$, the expression becomes:

$$B(1,1) = i \int_0^1 dx \int d^d r \frac{1}{[r^2 + L]^2}. \tag{21}$$

The integral is no singular only for $L > 0$ so, according to this requirement, it is necessary to keep $p^2 < 0$, space-like region. Considering that and using the d-dimensional polar coordinate system:

$$\begin{cases} r^0 = r\cos\theta_1 \\ r^1 = r\cos\theta_2 \sin\theta_1 \\ \cdots \\ r^{d-1} = r\sin\theta_{d-1}\cdots\sin\theta_1 \end{cases} \tag{22}$$

where

$$d^d r = r^{d-1} dr d\Omega_d \tag{23}$$

and

$$d\Omega_d = \prod_{l=1}^{d-1} (\sin\theta_l)^{d-1-l} d\theta_l \tag{24}$$

the integral can be expressed:

$$B(1,1) = i \int d\Omega_d \int_0^1 dx \int_0^\infty dr \frac{r^{d-1}}{[r^2+L]^2}. \tag{25}$$

Considering now Beta and Gamma functions definitions as well as the angular integration:

$$B(p,q) = \int_0^\infty dt \frac{t^{p-1}}{[1+t]^{p+q}} \tag{26}$$

$$B(p,q) = \int_0^1 dx x^{p-1}(1-x)^{q-1} \tag{27}$$

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \tag{28}$$

$$\int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} \tag{29}$$

we get the final expression for our amplitude:

$$B(1,1) = i(-p^2)^{d/2-2}\pi^{d/2}\Gamma(2-d/2)B(d/2-1,d/2-1). \tag{30}$$

3.2 Two-loops self-energy diagram

Now, we have reached the point we were interested in from the beginning. In this section we calculate second order corrections to the massless self-energy Feynman diagram. Four different diagrams contribute to the amplitude of the process at this order in perturbation theory [See Fig. 3], and they can be classified into different topologies. Nevertheless, the first diagram does not have to be solved since the result is already known; it is proportional to the one-loop self-energy diagram squared:

$$M_1 = \frac{1}{p^2}[B(1,1)]^2 = (-p^2)^{d-5}\pi^d[\Gamma(2-d/2)B(d/2-1,d/2-1)]^2, \tag{31}$$

Taken it into account, there are three diagrams to solve and they belong to only two different topologies, named:

$$TP_1 \equiv B(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5)$$
$$= \int d^dk_1 d^dk_3 \frac{1}{\left[k_1^2\right]^{\nu_1}\left[(k_1-p)^2\right]^{\nu_2}\left[k_3^2\right]^{\nu_3}\left[(k_3-p)^2\right]^{\nu_4}\left[(k_1-k_3)^2\right]^{\nu_5}}, \tag{32}$$

$$TP_2 \equiv \bar{B}(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5)$$
$$= \int d^dk_1 d^dk_3 \frac{1}{\left[k_1^2\right]^{\nu_1}\left[(k_1-p)^2\right]^{\nu_2}\left[k_3^2\right]^{\nu_3}\left[(k_3-p)^2\right]^{\nu_4}\left[(k_1+k_3-p)^2\right]^{\nu_5}}. \tag{33}$$

In order to obtain the amplitude, we need the following three integrals:

$$M_2 = B(1,1,1,1,1) \tag{34}$$
$$M_3 = \bar{B}(1,2,1,0,1) \tag{35}$$
$$M_5 = \bar{B}(1,1,1,1,1). \tag{36}$$

The first step is to find algebraic equations for the integrals of each topology. In order to do that, we use the IBP (Integration by Parts) method; in virtue of Gauss theorem, if we multiply the integrand with a vector (loop or external momentum) and differentiate it with the loop momentum, we know that these total derivatives integrate to zero:

$$0 = \int d^dk_1 d^dk_3 \frac{\partial}{\partial k_i^\mu} \frac{\eta^\mu}{\left[k_1^2\right]^{\nu_1}\left[(k_1-p)^2\right]^{\nu_2}\left[k_3^2\right]^{\nu_3}\left[(k_3-p)^2\right]^{\nu_4}\left[(k_1-k_3)^2\right]^{\nu_5}}, \tag{37}$$
$$0 = \int d^dk_1 d^dk_3 \frac{\partial}{\partial k_i^\mu} \frac{\eta^\mu}{\left[k_1^2\right]^{\nu_1}\left[(k_1-p)^2\right]^{\nu_2}\left[k_3^2\right]^{\nu_3}\left[(k_3-p)^2\right]^{\nu_4}\left[(k_1+k_3-p)^2\right]^{\nu_5}}. \tag{38}$$

Here $i = 1, 3$ and we can choose η freely to obtain six linearly independent equations, the IBP identities. We have chosen in our work $\eta = k_1, k_3, p$.

After the calculations with the derivatives we obtain the IBP identities for both topologies:

Topology 1:

$$\begin{aligned} T_1 : 0 = {} & (d-2\nu_1)B(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5) \\ & - \nu_2[B(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5) + B(\nu_1-1,\nu_2+1,\nu_3,\nu_4,\nu_5) \\ & - sB(\nu_1,\nu_2+1,\nu_3,\nu_4,\nu_5)] \\ & - \nu_5[B(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5) + B(\nu_1-1,\nu_2,\nu_3,\nu_4,\nu_5+1) \\ & - B(\nu_1,\nu_2,\nu_3-1,\nu_4,\nu_5+1)] \end{aligned} \tag{39}$$

$$
\begin{aligned}
T_2 : 0 = & -\nu_1[B(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5) + sB(\nu_1+1,\nu_2,\nu_3,\nu_4,\nu_5) \\
& - B(\nu_1+1,\nu_2-1,\nu_3,\nu_4,\nu_5)] \\
& - \nu_2[-B(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5) - sB(\nu_1,\nu_2+1,\nu_3,\nu_4,\nu_5) \\
& + B(\nu_1-1,\nu_2+1,\nu_3,\nu_4,\nu_5)] \\
& - \nu_5[B(\nu_1-1,\nu_2,\nu_3,\nu_4,\nu_5+1) - B(\nu_1,\nu_2-1,\nu_3,\nu_4,\nu_5+1) \\
& - B(\nu_1,\nu_2,\nu_3-1,\nu_4,\nu_5+1) \\
& + B(\nu_1,\nu_2,\nu_3,\nu_4-1,\nu_5+1)] \qquad (40)
\end{aligned}
$$

$$
\begin{aligned}
T_3 : 0 = & -\nu_1[B(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5) - B(\nu_1+1,\nu_2,\nu_3,\nu_4,\nu_5-1) \\
& + B(\nu_1+1,\nu_2,\nu_3-1,\nu_4,\nu_5)] \\
& - \nu_2[-B(\nu_1,\nu_2+1,\nu_3,\nu_4,\nu_5-1) + B(\nu_1,\nu_2+1,\nu_3,\nu_4-1,\nu_5) \\
& + B(\nu_1-1,\nu_2+1,\nu_3,\nu_4,\nu_5) - sB(\nu_1,\nu_2+1,\nu_3,\nu_4,\nu_5)] \\
& - \nu_5[-B(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5) + B(\nu_1-1,\nu_2,\nu_3,\nu_4,\nu_5+1) \\
& - B(\nu_1,\nu_2,\nu_3-1,\nu_4,\nu_5+1)] \qquad (41)
\end{aligned}
$$

$$
\begin{aligned}
T_4 : 0 = & -\nu_3[B(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5) - B(\nu_1,\nu_2,\nu_3+1,\nu_4,\nu_5-1) \\
& + B(\nu_1-1,\nu_2,\nu_3+1,\nu_4,\nu_5)] \\
& - \nu_4[B(\nu_1,\nu_2,\nu_3-1,\nu_4+1,\nu_5) + B(\nu_1,\nu_2-1,\nu_3,\nu_4+1,\nu_5) \\
& - B(\nu_1,\nu_2,\nu_3,\nu_4+1,\nu_5-1) - sB(\nu_1,\nu_2,\nu_3,\nu_4+1,\nu_5)] \\
& - \nu_5[-B(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5) - B(\nu_1-1,\nu_2,\nu_3,\nu_4,\nu_5+1) \\
& + B(\nu_1,\nu_2,\nu_3-1,\nu_4,\nu_5+1)] \qquad (42)
\end{aligned}
$$

$$
\begin{aligned}
T_5 : 0 = & -\nu_3[B(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5) - B(\nu_1,\nu_2,\nu_3+1,\nu_4-1,\nu_5) \\
& + sB(\nu_1,\nu_2,\nu_3+1,\nu_4,\nu_5)] \\
& - \nu_4[-B(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5) + B(\nu_1,\nu_2,\nu_3-1,\nu_4+1,\nu_5) \\
& - sB(\nu_1,\nu_2,\nu_3,\nu_4+1,\nu_5) \\
& - \nu_5[-B(\nu_1-1,\nu_2,\nu_3,\nu_4,\nu_5+1) + B(\nu_1,\nu_2-1,\nu_3,\nu_4,\nu_5+1) \\
& + B(\nu_1,\nu_2,\nu_3-1,\nu_4,\nu_5+1) \\
& - B(\nu_1,\nu_2,\nu_3,\nu_4-1,\nu_5+1)] \qquad (43)
\end{aligned}
$$

$$
\begin{aligned}
T_6 : 0 = & (d-2\nu_3)B(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5) \\
& - \nu_4[B(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5) + B(\nu_1,\nu_2,\nu_3-1,\nu_4+1,\nu_5) \\
& - sB(\nu_1,\nu_2,\nu_3,\nu_4+1,\nu_5)] \\
& - \nu_5[B(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5) - B(\nu_1-1,\nu_2,\nu_3,\nu_4,\nu_5+1) \\
& + B(\nu_1,\nu_2,\nu_3-1,\nu_4,\nu_5+1)] \qquad (44)
\end{aligned}
$$

Topology 2:

$$
\begin{aligned}
\bar{T}_1 : 0 = & (d-2\nu_1)\bar{B}(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5) \\
& - \nu_2[\bar{B}(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5) + \bar{B}(\nu_1-1,\nu_2+1,\nu_3,\nu_4,\nu_5) \\
& - s\bar{B}(\nu_1,\nu_2+1,\nu_3,\nu_4,\nu_5)] \\
& - \nu_5[\bar{B}(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5) + \bar{B}(\nu_1-1,\nu_2,\nu_3,\nu_4,\nu_5+1) \\
& - \bar{B}(\nu_1,\nu_2,\nu_3,\nu_4-1,\nu_5+1)] \qquad (45)
\end{aligned}
$$

$$
\begin{aligned}
\bar{T}_2 : 0 = & -\nu_1[\bar{B}(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5) + s\bar{B}(\nu_1+1,\nu_2,\nu_3,\nu_4,\nu_5) \\
& - \bar{B}(\nu_1+1,\nu_2-1,\nu_3,\nu_4,\nu_5)] \\
& - \nu_2[-\bar{B}(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5) - s\bar{B}(\nu_1,\nu_2+1,\nu_3,\nu_4,\nu_5) \\
& + \bar{B}(\nu_1-1,\nu_2+1,\nu_3,\nu_4,\nu_5)] \\
& - \nu_5[\bar{B}(\nu_1-1,\nu_2,\nu_3,\nu_4,\nu_5+1) - \bar{B}(\nu_1,\nu_2-1,\nu_3,\nu_4,\nu_5+1) \\
& + \bar{B}(\nu_1,\nu_2,\nu_3-1,\nu_4,\nu_5+1) \\
& - \bar{B}(\nu_1,\nu_2,\nu_3,\nu_4-1,\nu_5+1)] \qquad (46)
\end{aligned}
$$

$$
\begin{aligned}
\bar{T}_3 : 0 = & -\nu_1[\bar{B}(\nu_1+1,\nu_2,\nu_3,\nu_4,\nu_5-1) - \bar{B}(\nu_1+1,\nu_2-1,\nu_3,\nu_4,\nu_5) \\
& - \bar{B}(\nu_1+1,\nu_2,\nu_3,\nu_4-1,\nu_5) + s\bar{B}(\nu_1+1,\nu_2,\nu_3,\nu_4,\nu_5)] \\
& - \nu_2[-\bar{B}(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5) + \bar{B}(\nu_1,\nu_2+1,\nu_3,\nu_4,\nu_5-1) \\
& - \bar{B}(\nu_1,\nu_2+1,\nu_3-1,\nu_4,\nu_5)] \\
& - \nu_5[\bar{B}(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5) + \bar{B}(\nu_1,\nu_2,\nu_3-1,\nu_4,\nu_5+1) \\
& - \bar{B}(\nu_1,\nu_2-1,\nu_3,\nu_4,\nu_5+1)] \qquad (47)
\end{aligned}
$$

$$\begin{aligned}\bar{T}_4 : 0 = & -\nu_3[-\bar{B}(\nu_1,\nu_2-1,\nu_3+1,\nu_4,\nu_5)+s\bar{B}(\nu_1,\nu_2,\nu_3+1,\nu_4,\nu_5)\\ & +\bar{B}(\nu_1,\nu_2,\nu_3+1,\nu_4,\nu_5-1)-\bar{B}(\nu_1,\nu_2,\nu_3+1,\nu_4-1,\nu_5)]\\ & -\nu_4[-\bar{B}(\nu_1-1,\nu_2,\nu_3,\nu_4+1,\nu_5)+\bar{B}(\nu_1,\nu_2,\nu_3,\nu_4+1,\nu_5-1)\\ & -\bar{B}(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5)]\\ & -\nu_5[\bar{B}(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5)-\bar{B}(\nu_1,\nu_2,\nu_3,\nu_4-1,\nu_5+1)\\ & +\bar{B}(\nu_1-1,\nu_2,\nu_3,\nu_4,\nu_5+1)]\end{aligned} \tag{48}$$

$$\begin{aligned}\bar{T}_5 : 0 = & -\nu_3[\bar{B}(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5)-\bar{B}(\nu_1,\nu_2,\nu_3+1,\nu_4-1,\nu_5)\\ & +s\bar{B}(\nu_1,\nu_2,\nu_3+1,\nu_4,\nu_5)]\\ & -\nu_4[-\bar{B}(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5)+\bar{B}(\nu_1,\nu_2,\nu_3-1,\nu_4+1,\nu_5)\\ & -s\bar{B}(\nu_1,\nu_2,\nu_3,\nu_4+1,\nu_5)]\\ & -\nu_5[\bar{B}(\nu_1-1,\nu_2,\nu_3,\nu_4,\nu_5+1)-\bar{B}(\nu_1,\nu_2-1,\nu_3,\nu_4,\nu_5+1)\\ & +\bar{B}(\nu_1,\nu_2,\nu_3-1,\nu_4,\nu_5+1)-\bar{B}(\nu_1,\nu_2,\nu_3,\nu_4-1,\nu_5+1)]\end{aligned} \tag{49}$$

$$\begin{aligned}\bar{T}_6 : 0 = & (d-2\nu_3)\bar{B}(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5)\\ & -\nu_4[\bar{B}(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5)+\bar{B}(\nu_1,\nu_2,\nu_3-1,\nu_4+1,\nu_5)\\ & -s\bar{B}(\nu_1,\nu_2,\nu_3,\nu_4+1,\nu_5)]\\ & -\nu_5[\bar{B}(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5)+\bar{B}(\nu_1,\nu_2,\nu_3-1,\nu_4,\nu_5+1)\\ & -\bar{B}(\nu_1,\nu_2-1,\nu_3,\nu_4,\nu_5+1)]\end{aligned} \tag{50}$$

The next point is to implement the IBP identities on AIR and the parameters for the routine *SEEDGEN*. In this case, we select:

$$maxtop = [1,2,3,4,5] \tag{51}$$

$$N_{prop} \subset [0,5] \tag{52}$$

$$N_{+} \subset [0,2] \tag{53}$$

$$N_{-} \subset [0,2] \tag{54}$$

and, the program produces the solution:

$$M_2 = B(1,1,1,1,1) = -2B(1,1,1,1,0)\frac{(d-3)}{s(d-4)} + B(0,1,1,0,1)\frac{(3d-8)(3d-10)}{s^2(d-4)^2} + B(1,0,0,1,1)\frac{(3d-8)(3d-10)}{s^2(d-4)^2} \quad (55)$$

$$M_3 = \bar{B}(1,2,1,0,1) = \bar{B}(1,0,1,0,1)\frac{(3d-8)(3d-10)}{s^2(d-4)(d-6)} \quad (56)$$

$$M_5 = \bar{B}(1,1,1,1,1) = -2\bar{B}(1,1,1,1,0)\frac{(d-3)}{s(d-4)} + \bar{B}(0,1,0,1,1)\frac{(3d-8)(3d-10)}{s^2(d-4)^2} + \bar{B}(1,0,1,0,1)\frac{(3d-8)(3d-10)}{s^2(d-4)^2} \quad (57)$$

Now, it is necessary to solve the master integrals. In order to do that, we proceed like in the previous section, when we calculated $B(1,1)$. That is, employing *Feynman parametrization*, *Wick rotation*, d-dimensional polar coordinate system and *Gamma* and *Beta* functions relations. First of all, before to solve all these master integrals, we realize that not all of them are different. It can be shown by making variable changes, and the result is that:

$$\bar{B}(1,1,1,1,0) = B(1,1,1,1,0) \quad (58)$$

$$B(1,0,0,1,1) = B(0,1,1,0,1) \quad (59)$$

$$\bar{B}(0,1,0,1,1) = \bar{B}(1,0,1,0,1). \quad (60)$$

Also, after solve these three master integrals, we noticed that:

$$\bar{B}(1,0,1,0,1) = B(0,1,1,0,1), \quad (61)$$

so, we need only to calculate two master integrals, being:

$$B(1,1,1,1,0) = \int d^dk_1 d^dk_3 \frac{1}{k_1^2(k_1-p)^2k_3^2(k_3-p)^2} \quad (62)$$

$$B(0,1,1,0,1) = \int d^dk_1 d^dk_3 \frac{1}{(k_1-p)^2k_3^2(k_1-k_3)^2}. \quad (63)$$

The way to do that is:

• *Solving* $B(1,1,1,1,0)$:

This integral can be separated into two independent ones, one involving k_1 and the other k_3. Making it:

$$B(1,1,1,1,0) = \int d^d k_1 \frac{1}{k_1^2 (k_1 - p)^2} \int d^d k_3 \frac{1}{k_3^2 (k_3 - p)^2}. \tag{64}$$

If we look at this last expression, we can check that it is just $[B(1,1)]^2$ so, using the result obtained in the previous section for $B(1,1)$, the final value is:

$$B(1,1,1,1,0) = -(-p^2)^{d-4} \pi^d \left[\Gamma(2 - d/2) B(d/2 - 1, d/2 - 1)\right]^2. \tag{65}$$

• *Solving* $B(0,1,1,0,1)$:

The first step in order to solve it, is to separate again both integrals, solving firstly the integral in k_3. We realize again that this is one of type $B(1,1)$ where now $p \equiv k_1$. If we do that, our initial integral takes the shape:

$$B(0,1,1,0,1) = i\pi^{d/2} (-1)^{d/2-2} \Gamma(2 - d/2) B(d/2 - 1, d/2 - 1) I \tag{66}$$

where

$$I = \int d^d k_1 \frac{1}{(k_1^2)^{2-d/2} (k_1 - p)^2}. \tag{67}$$

The next step is to solve I. To this end, we use *Feynman parametrization*, now in the general case:

$$\prod_{i=1}^{n} \frac{1}{A_i^{\alpha_i}} = \frac{\Gamma(\alpha)}{\prod_{i=1}^{n} \Gamma(\alpha_i)} \int_0^1 \left(\prod_{i=1}^{n} dx_i x_i^{\alpha_i} - 1 \right) \frac{\delta(1 - x)}{(\sum_{i=1}^{n} x_i A_i)^{\alpha}} \tag{68}$$

being α_i arbitrary complex numbers and

$$\alpha = \sum_{i=1}^{n} \alpha_i \qquad x = \sum_{i=1}^{n} x_i. \tag{69}$$

If we do that, I can be written in the way:

$$I = \frac{\Gamma(3 - d/2)}{\Gamma(2 - d/2)} \int d^d k_1 \int_0^1 dx_2 \frac{(1 - x_2)^{1-d/2}}{\left[(1 - x_2)(k_1^2) + x_2 (k_1 - p)^2\right]^{3-d/2}}. \tag{70}$$

We can change one more time the integration order and rearrange the denominator on I, having now the expression for this integral:

$$I = \frac{\Gamma(3-d/2)}{\Gamma(2-d/2)} \int_0^1 dx_2 (1-x_2)^{1-d/2} \bar{I} \tag{71}$$

with,

$$\bar{I} = \int d^d k_1 \frac{1}{[(k_1 - x_2 p)^2 + x_2(1-x_2)p^2]^{3-d/2}}. \tag{72}$$

The next step is to solve $\bar{I}$. First of all we make a variable change, $k' \equiv k_1 - x_2 p$, and after that a *Wick rotation*, obtaining:

$$\bar{I} = i \int d^d r \frac{1}{(r^2 + L)^{3-d/2}} \tag{73}$$

where, like before, $L = -x_2(1-x_2)p^2$ and

$$d^d r = r^{d-1} dr d\Omega_d \tag{74}$$

$$d\Omega_d = \prod_{l=1}^{d-1} (\sin\theta_l)^{d-1-l} d\theta_l. \tag{75}$$

If we do now the change $t \equiv r^2/L$ and use the Beta function definition, the final expression for $\bar{I}$ is:

$$\bar{I} = i\pi^{d/2} L^{d-3} \frac{B(d/2, 3-d)}{\Gamma(d/2)}. \tag{76}$$

Now we have to substitute this expression on I and make the x_2 integral. We use again the Beta function definition obtaining, finally, an expression for I:

$$I = i\pi^{d/2} (-p^2)^{d-3} \frac{\Gamma(3-d/2)}{\Gamma(2-d/2)\Gamma(d/2)} \beta(d/2, 3-d)\beta(d-2, d/2-1). \tag{77}$$

Substituting I on $B(0,1,1,0,1)$, the solution for this integral becomes:

$$\begin{aligned} B(0,1,1,0,1) = (-)^{d/2-1} \pi^d (-p^2)^{d-3} \frac{\Gamma(3-d/2)}{\Gamma(d/2)} B(d/2-1, d/2-1) \\ \times B(d/2, 3-d) B(d-2, d/2-1). \end{aligned} \tag{78}$$

At this point, we are in a position of to write the final expression for our

three scalar amplitudes:

$$M_2 = (-p^2)^{d-4}\pi^d \frac{2(d-3)}{s(d-4)} \left[\Gamma(2-d/2)B(d/2-1,d/2-1)\right]^2 + (-)^{d/2-1}\pi^d(-p^2)^{d-3}\frac{2(3d-8)(3d-10)}{s^2(d-4)^2}\frac{\Gamma(3-d/2)}{\Gamma(d/2)} \times B(d/2-1,d/2-1)B(d/2,3-d)B(d-2,d/2-1) \quad (79)$$

$$M_3 = (-)^{d/2-1}\pi^d(-p^2)^{d-3}\frac{(3d-8)(3d-10)}{(d-4)(d-6)s^2}\frac{\Gamma(3-d/2)}{\Gamma(d/2)} \times B(d/2-1,d/2-1)B(d/2,3-d)B(d-2,d/2-1) \quad (80)$$

$$M_5 \equiv M_2 = (-p^2)^{d-4}\pi^d \frac{2(d-3)}{s(d-4)} \left[\Gamma(2-d/2)B(d/2-1,d/2-1)\right]^2 + (-)^{d/2-1}\pi^d(-p^2)^{d-3}\frac{2(3d-8)(3d-10)}{s^2(d-4)^2}\frac{\Gamma(3-d/2)}{\Gamma(d/2)} \times B(d/2-1,d/2-1)B(d/2,3-d)B(d-2,d/2-1). \quad (81)$$

4 Conclusions

When higher orders in perturbation theory are taken into account, the integrals defining the amplitudes become more difficult. It implies that the number of IBP equations needed to solve them increases very fast and the IBP identities acquire more and more complicated expressions so to find a solution for the problem is a very tedious task; typical multi-loop calculations require an enormous number of IBP equations, $10^5 - 10^6$. Even in the most simple case, the self-energy diagram, the number of equations is huge when we treat second-order corrections in perturbation theory and the expressions do not look simple, as we can check by inspecting the previous section.

The effectiveness of AIR is capable of solving enormous systems of equations spending a small time and reducing our initial complicated integrals to only a few master ones which are much more easy to deal with. The great advance is that the algorithm works in an automated way, permitting to use it in completely general problems; and, although by the moment it can not solve problems until arbitrarily high orders in perturbation theory, it can be used to solve the great majority of physical cases.

In the example we treat here, AIR gets to express our initial amplitudes in terms of only three master integrals, for each topology. The final result is even more easy, since the master integrals are not all different; comparing them, it is easy to prove that there are only two different master integrals which must be solved in this problem. These masters are not trivial but are also not too complicated, allowing us to find a compact expression for the solution. The reduction has been made in a short time, less than 500 seconds for each topology, and all this considering only the simplest use of the program, that is, without to consider the masking algorithms and the vanishing conditions; only introducing as input the IBP identities for each topology and the parameters for the routine *SEEDGEN*.

From this study is therefore clear that this program is very useful and efficient; it is interesting in actual particle physics calculations, since higher energies are being generated each time, which implies that higher orders in perturbation theory calculations have to be considered to reproduce the actual and future experimental data generated by the experimentalists.

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