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Heavy Flavor Coefficient Functions at LO QCD in the Limit $Q^2 \gg m^2$

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Abstract

Quark-antiquark production by virtual-photon-gluon fusion in deeply inelastic scattering is treated at LO. The corresponding Wilson coefficients are derived by the exact massive calculation and by Operator product expansion. The results are compared and discussed.

1 Introduction

Deeply inelastic scattering (DIS) events of leptons off protons have led to important information on the structure of the proton. They serve as experimental evidence for the quark parton model and the correctness of QCD [1–4]. The information which can be extracted by measuring cross sections of DIS events is contained in the structure functions $F_2(x, Q^2)$ and $F_L(x, Q^2)$ of the proton. For pure photon exchange in the naive parton model [5,6] they depend only on the scaling variable Bjorken-x, whereas at higher orders scale-breaking effects are contributing. These structure functions are related to the parton densities inside the proton and to the coefficient functions of the subprocess of DIS considered. Therefore the treatment of the DIS cross section is separated in two parts. The coefficient functions can be calculated by applying perturbative QCD for sufficiently large values of Q^2, whereas the parton densities can not be calculated perturbatively. They have to be determined by fits to experimental data. Especially interesting is the gluon density, since the gluons carry a large amount of the momentum of the proton and have large impact on scale-breaking effects at smaller values of x. The gluon density can be measured using heavy-quark production in DIS. This process is given by the photon-gluon fusion mechanism at leading order(LO). One is therefore led to the study of the heavy flavor coefficient functions and to heavy flavor production. Heavy flavor production is by itself an interesting subject, since the lightest heavy quark is the charm quark with $m_c \approx 1.5$GeV. The charm quark is a constituent of the D–Meson and the J/Ψ–mesons. The study of the D–meson could give us more knowledge about the Cabibbo-Kobayashi-Maskawa matrix, which is a fundamental quantity of nature.

In this paper, heavy flavor production for deeply inelastic electron-proton scattering is treated at LO. The heavy flavor Wilson coefficients in the spin averaged case are calculated following two approaches. In the 4-dimensional massive case, the heavy flavor coefficients can be calculated directly. If one considers the limit $Q^2 \gg m_c^2$ of the result derived, mass singularities are encountered. If the quark mass is set equal to zero at the beginning, one will encounter these singularities directly. They will be treated in the $\overline{\text{MS}}$-scheme [7] and the operator product expansion (OPE) [8] will be applied to factorize the effect of quark masses in the asymptotic range. One of the main objectives in this paper is to show and understand that at LO, there are a priori no pole terms for $m_c > 0$. Large mass logarithms only occur in the limit $Q^2 \gg m^2$. The deeper reason for this is that in the OPE the pole terms of the operator matrix element and of the massless case cancel exactly. This is a non trivial result, since usually pole terms emerge which have to be absorbed by factorization into the un-renormalized parton densities.

The paper is organized as follows. Section 2 will give explanations of the terms used and basic notions, such as the naive parton model, factorization and splitting functions. In Section 3 the calculation of the LO massive Wilson coefficient is performed and in section 4 the massless case is treated in the $\overline{\text{MS}}$-scheme. In Section 5 the 1-loop operator matrix element $A_{Qg}^{(1)}$ is derived and the technique of the operator product expansion will be used to derive the massive results in the limit $Q^2 \gg m^2$. Section 6 contains the conclusion.

2 LO Wilson coefficients for massive quark production in *ep* -scaling

2.1 Probing single partons

The interaction of a virtual photon with a proton gives rise to the question which conditions must be fulfilled in order for the photon to be able to interact with the partons inside the proton. It is clear that the time for such an interaction, τ_{int}, should be considerably shorter than the average life time of a parton, τ_{life}, i.e. $\tau_{\text{int}}/\tau_{\text{life}} \ll 1$.
It is convenient to define certain reference frames, in which partons can be considered as "free" particles during the interaction time. The kinematics of the process can then be treated independently of forces within the proton. One way is to introduce the notion of an infinite momentum frame [9, 10], Figure 1. This means that the modulus of the three momentum of the proton will be taken to be very large $|P| \gg M$.

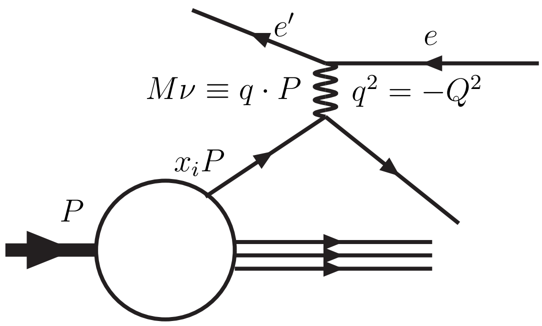


Figure 1: Deeply inelastic electron-proton scattering in the infinite momentum (P_∞)frame

In this limit, the expansion of the relativistic four momentum yields

$$P^\mu = \left(P + \frac{M^2}{2 \cdot P}; P, 0, 0 \right) \; . \tag{1}$$

This expansion gives a non-covariant representation at large P. The partons carry a certain fractional momentum x_i of the total proton momentum $x_i \cdot P$, $\sum_i x_i = 1$), in the P_∞-frame.
Due to the time-energy uncertainty relation, the life time of the individual partons is estimated to be inversely proportional to the energy fluctuations of the partons

$$\tau_{\text{life}} \simeq \frac{1}{\sum_i E_i - E} \; . \tag{2}$$

Here E_i and E denote the energy of the partons and the energy of the proton, respectively. By introducing the two momentum $\vec{k}_{\perp i}$ of the partons perpendicular to the direction of motion of the proton as given in (1), this can be rewritten via

$$\sum_i E_i - E \;\; = \;\; \sum_i \sqrt{{x_i}^2 P^2 + {M_i}^2 + \vec{k}^2_{\perp i}} - \sqrt{P^2 + M^2} \; , \tag{3}$$

where M_i denotes the mass of the i-th parton. Further calculation yields

$$\sum_i E_i - E \;\; = \;\; \sum_i x_i P \sqrt{1 + \frac{{M_i}^2 + k^2_{\perp i}}{{x_i}^2 P^2}} - P - \frac{M^2}{2P}$$

$$
\begin{aligned}
&\approx \sum_i x_i P + \sum_i \frac{{M_i}^2 + k_{\perp i}^2}{2x_i P} - P - \frac{M^2}{2P} \\
&= \frac{1}{2P} \cdot \left(\sum_i \frac{{M_i}^2 + k_{\perp i}^2}{x_i} - M^2 \right) \ . \qquad (4)
\end{aligned}
$$

The parton life time is then given by

$$
\tau_{\text{life}} = \frac{2P}{\sum_i \frac{{M_i}^2 + {k_{\perp i}}^2}{x_i} - M^2} \ . \qquad (5)
$$

In the cms of the collision, one can make a certain ansatz for the four momentum q of the photon,

$$
q = (q_0; q_3, \vec{q}_\perp) \ , \qquad (6)
$$

where $\vec{q}_\perp$ denotes the transversal momentum of the photon with respect to the proton. In this system $Q^2 = -q^2$ turns out to obey

$$
Q^2 \approx |\vec{q}_\perp| \ . \qquad (7)
$$

Keeping in mind that $P \gg Q^2$ and introducing Bjorken-x:

$$
x \ \equiv \ \frac{Q^2}{2q \cdot p} \ , \qquad (8)
$$

one sets

$$
\begin{aligned}
q_0 &= \frac{Q^2}{4P} \cdot \frac{1-x}{x} \qquad (9) \\
q_3 &= -\frac{Q^2}{4P} \cdot \frac{1+x}{x} \ , \qquad (10) \\
& \qquad (11)
\end{aligned}
$$

and thus obtains

$$
-q^2 \ = \ Q^2 = (q_3 - q_0)(q_3 + q_0) + {q_\perp}^2 = \frac{Q^4}{4xP^2} + {q_\perp}^2 \approx {q_\perp}^2 \ . \qquad (12)
$$

For the characteristic time scale using (9) and the time-energy uncertainty relation one obtains

$$
\tau_{\text{int}} \ \simeq \ \frac{1}{q_0} = \frac{4Px}{Q^2(1-x)} \ . \qquad (13)
$$

Together with (5) one finds

$$
\frac{\tau_{\text{int}}}{\tau_{\text{life}}} \ = \ \frac{Q^2(1-x)}{2x\left(\sum_i (M_i^2 + k_{\perp i}^2)/x_i - M^2\right)} \ . \qquad (14)
$$

This expression is independent of P. The above procedure allows therefore to estimate the probability of deeply inelastic scattering to occur independently of the momentum of the proton. Let $k_{\perp 1}^2 = k_{\perp 1}^2 = k_\perp^2$, $M_i^2 \approx 0$, $x_1 + x_2 = 1$,cf. [10]. It follows that

$$
\frac{\tau_{\text{int}}}{\tau_{\text{life}}} \approx \frac{Q^2}{(1-x)^2} \cdot \frac{1}{2{k_\perp}^2} \ . \qquad (15)
$$

If $k_\perp^2 = 1\ \mathrm{GeV}^2$ and $Q^2 = 20\ \mathrm{GeV}^2$ one obtains: $\tau_{\mathrm{life}}/\tau_{\mathrm{int}} \approx 5$. On the other hand, if $x \to 0, 1$ it follows that τ_{life} goes to zero as well, contrary to the basic assumptions.
The above example therefore leads to the conclusion, that deeply inelastic scattering probes single partons if the virtual mass of the virtual photon is much larger than the transverse momenta squared of the partons and Bjorken-x is neither close to zero or one.

2.2 Factorization

A useful method for the treatment of cross sections in high-energy scattering is based on the equivalent photon method by Fermi, Williams and Weizsäcker [11]. Consider the process $e+p \to e' + p'$

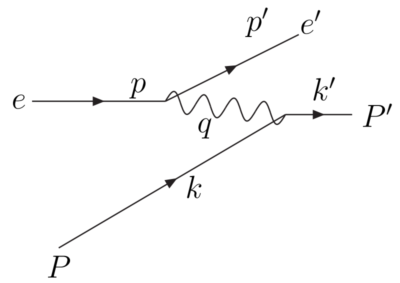


Figure 2: ep scattering in LO

As an example, let $q^2 \approx 0$,i.e. the virtual photon is quasi-real. The idea is to factorize the process into a leptonic and a hadronic part and to extract mass singularities emerging. The first process describes the electron emitting a photon, whereas the latter describes the reaction of the virtual photon with the proton. The cross section can now be written as

$$\sigma(ep) \approx \sigma(e \to \gamma^* e') \otimes \sigma(\gamma^* p \to p') \ , \tag{16}$$

where $\otimes$ denotes the Mellin-convolution, cf. Section 2.3. The differential cross-section reads

$$d\sigma(ep) = \frac{1}{8kp} \cdot \frac{e^2 W^{\mu\nu} L_{\mu\nu}}{q^4} \cdot \frac{d^2 p'}{(2\pi)^3 (2E')} \ . \tag{17}$$

$L_{\mu\nu}$ denotes the leptonic and $W^{\mu\nu}$ the hadronic tensor. This method has the advantage, that $W^{\mu\nu}$ is sufficient in order to calculate the structure functions of the partons inside the proton. The general form of the hadronic tensor in the spin averaged case is given by [10]

$$W_{\mu\nu} = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) W_1 + \frac{1}{M^2}\left(p_\mu - \frac{p\cdot q}{q^2} q_\mu\right)\left(p_\nu - \frac{p\cdot q}{q^2} q_\nu\right) W_2 \ , \tag{18}$$

where P_μ is the four momentum of the proton and W_1 and W_2 are the structure functions. They depend on Q^2 and of the energy loss of the electron in the nucleon's rest frame, denoted by $\nu = E' - E$. In deriving (18) one has made use of the fact that the leptonic tensor is symmetric and its contraction with the photon four momentum is equal to zero due to gauge invariance [10]. Therefore the hadronic tensor also has to obey these properties. In the case considered, the leptonic tensor is given by

$$L_{\mu\nu} = 4\left[\frac{q^2}{2}g_{\mu\nu} + p'_\mu p_\nu + p_\mu p'_\nu\right] \ . \tag{19}$$

Since the virtual photon is quasi–real, the structure functions of the hadronic tensor can be approximated by

$$W_1(0,Q^2) \approx W_2(0,Q^2) \ . \tag{20}$$

Therefore, the differential cross section (17) depends on one structure function only,

$$\begin{aligned} d\sigma_{ep} &= -W_1\frac{a}{sq^2}dq^2dy\frac{1+(1-y)^2}{y^2} && (21)\\ &= \sigma^{\gamma p}\cdot f_{\gamma/e}(y)dy \ , && (22)\end{aligned}$$

where the Bjorken-variable y has been used

$$y = \frac{q\cdot k}{p\cdot k} \ . \tag{23}$$

After performing the integration over q, the function $f_{\gamma/e}(y)$ is given by

$$f_{\gamma/e}(y) = 2a\cdot\frac{1+(1-y)^2}{y}\cdot\ln\left(\frac{Q^2_{max}}{Q^2_{min}}\right) \ , \tag{24}$$

with $a = \alpha/4\pi$ and $\alpha = e^2/4\pi$, the fine structure constant. Here one refers to the minimal and maximal values of Q^2, with $Q^2_{min} = m_e^2 y/(1-y)$. In the equivalent photon approximation the flux factor $\sigma(e \to \gamma^* e')$ and the cross section $\sigma(\gamma^* p \to p')$ factorize in cross section. This is due to a sufficiently long life-time of the quasi–real photon. In a very similar way this picture applies to the factorization of long–lived partons inside protons, cf. Section 5. The function $f_{\gamma/e}(z)$ is called the splitting function of an electron into a photon at LO QED.

2.3 LO splitting function $P_{qg}(z)$

2.3.1 Mellin space

A very useful transformation often used for massless scattering processes is the Mellin transformation [12]. It operates on the function space $C^\infty(]0,1[)$. Here Bjorken-x defines interval $]0,1[$. The N-th Mellin moment of the function $f \in C^\infty(]0,1[)$ for N integer is defined by

$$M[f](N) \equiv \int_0^1 dz z^{N-1} f(z) \ . \tag{25}$$

The Mellin-convolution is defined by

$$[f\otimes g](x) \equiv \int_0^1 dz\int_0^1 dy\delta(x-yz)f(z)g(y) \ , \tag{26}$$

which obeys the relation

$$M[f\otimes g](N) = M[f](N)\cdot M[g](N) \ . \tag{27}$$

2.3.2 Splitting functions

An example of a splitting, the collinear transition probability of massless particles into each other, has already been encountered in Section 2.2. Let $M_N(Q^2) \equiv M[q_i(x,Q^2)](N)$ be the N-th Mellin moment of the quark number density $q_i(x,Q^2)$. The Q^2 dependence of the Mellin moment is then found in [13]

$$M_N(Q^2) = M_N(0)\left[\frac{a_s(0)}{a_s(Q^2)}\right]^{-\gamma^N/2\beta_0} . \tag{28}$$

Here $a_s(t)$ is the running coupling constant of QCD, $\beta_0 = 11 - 2N_f/3$ for $SU(3)_C$ and N_f quark flavors and γ^N denotes the anomalous dimension. They relate the evolution of the Mellin moments of the quark density to the running coupling constant. One rewrites (28) as a differential equation. The inverse Mellin transform provides the x–space representation

$$\frac{dM_N(Q^2)}{dQ^2} = -\frac{1}{2}a_s(Q^2)\gamma^N M_N(Q^2) \tag{29}$$

$$\frac{dq(x,Q^2)}{dQ^2} = 2a_s(Q^2)P(x)\otimes q(x,Q^2) , \tag{30}$$

where $P(x)$ is the splitting function for $q \to q$, with

$$\gamma^N \equiv -4\int_0^1 dz z^{N-1}P(z) . \tag{31}$$

Eq. (30) can be rewritten in x-space as

$$q(x,Q^2) + dq(x,Q^2) = \int_0^1 dy \int_0^1 dz\delta(zy-x)q(y,Q^2)[\delta(z-1) + 2a_sP(z)dQ^2] . \tag{32}$$

The interpretation of this equation is that, given a quark with momentum fraction y, there is a chance that it radiates a gluon, thereby reducing its momentum from y to x. The z-dependent integrand is the probability density of finding a quark inside a quark with fraction z of the parent momentum [7]. Thus $2a_sP(z)$ is the variation of this density per unit Q^2 at order a_s. From these considerations stems the term splitting function, since it describes the probability of particles "splitting" up into other particles. In general one has to include all flavors and the gluon density as well [13].
In section 2.2 the splitting function $P_{\gamma/e}$ has been derived already. In the present study the splitting function P_{qg}, i.e. the splitting function using old-fashioned perturbation theory for a gluon producing a quark-antiquark pair, occurs. The derivation of this splitting function is given in [13] and reads

$$P_{qg}(z) = 8T_R(z^2 + (1-z)^z) . \tag{33}$$

Here T_R denotes the color factor for $SU(N)_C$, $T_R = 1/2$ for $SU(3)_C$. Its Mellin transform is given by

$$\int_0^1 dz z^{N-1}P_{qg}(z) = 8T_R\frac{N^2+N+2}{N(N+1)(N+2)} . \tag{34}$$

2.4 The projections of the hadronic tensor

We consider the scattering process $\gamma^* + G \to q + \overline{q}$, Figure 3.

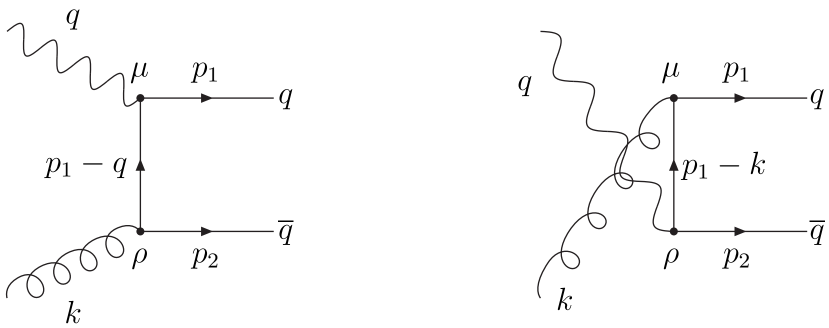


Fig.3.: Feynman diagrams contributing to the subprocess $\gamma^* + g \to q + \overline{q}$ at LO

This process is contributes to deeply inelastic scattering, Figure 1, at $O(a_s)$. The four-momenta of the virtual photon, the gluon and the quark and antiquark are given by q, k, p_1, p_2. The Mandelstam-variables are defined by

$$\begin{aligned} s &= (q+k)^2 = (p_1+p_2)^2 = -Q^2 + 2q \cdot k + k^2 = m_1^2 + m_2^2 + p_1 \cdot p_2 & (35) \\ u &= (q-p_2)^2 = (k-p_1)^2 = m_1^2 - 2k \cdot p_1 + k^2 = m_2^2 - 2q \cdot p_2 - Q^2 & (36) \\ t &= (q-p_1)^2 = (k-p_2)^2 = -Q^2 - 2q \cdot p_1 + m_1^2 = m_2^2 - 2k \cdot p_2 + k^2 \ , & (37) \end{aligned}$$

with

$$s + t + u = -Q^2 + m_1^2 + m_2^2 \ . \tag{38}$$

In the present case we specify these relations with $k^2 = 0$, $p_1^2 = m_1^2 = p_2^2 = m_2^2 = m^2$. The total momentum transferred by the virtual photon is given by $q^2 = -Q^2$. The invariant matrix element squared for deeply inelastic ep–scattering, Figure 1, is given by [14]

$$\overline{|M|}^2 = \frac{e^4}{Q^4} L^{\mu\nu} W_{\mu\nu} \ . \tag{39}$$

The general form of the leptonic contribution to the matrix element in this case is given by

$$\begin{aligned} L_{\mu\nu} &= \frac{1}{2} tr[(\not{l} + m_e)\gamma_\mu(\not{l}' + m_e)\gamma_\nu] \\ &= 2(l_\mu l'_\nu + l'_\mu l_\nu - l \cdot l' g_{\mu\nu}) \ , \end{aligned} \tag{40}$$

for $m_e \to 0$. As in (18), the hadronic tensor in the spin averaged case reads

$$W_{\mu\nu} = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) W_1 + \frac{1}{M^2}\left(P_\mu - \frac{P \cdot q}{q^2} q_\mu\right)\left(P_\nu - \frac{P \cdot q}{q^2} q_\nu\right) W_2 \ . \tag{41}$$

Here W_1 and W_2 denote the unpolarized structure functions, which are not accessible using perturbative methods only. We consider the Bjorken limit [14]

$$Q^2 \to \infty, \quad \nu = \frac{q \cdot p}{M} \to \infty \ , \ \frac{Q^2}{\nu} = const. \tag{42}$$

Experimentally it has been observed, that in this limit the structure functions depend only on the ratio of transferred momentum and energy, Bjorken-x, where

$$x = \frac{Q^2}{2M\nu} = \frac{Q^2}{sy} = \frac{Q^2}{2q \cdot P} \ . \tag{43}$$

Having performed this limit simplifies matters, since now the mass of the proton and the energy transfer are absorbed in two newly defined structure functions F_2 and F_1. In the free parton model, they obey exact scaling and are functions of x only. However, at higher orders scaling violation effects have to be taken into account. They are given by [14]

$$\begin{aligned} W_1 &\to F_1(x, Q^2) & (44)\\ \frac{\nu}{M} \cdot W_2 &\to F_2(x, Q^2) \ . & (45) \end{aligned}$$

We now derive relations between projections of the hadronic tensor and the structure functions. In the general case of D dimensions

$$g^{\mu\nu} g_{\mu\nu} = D \ . \tag{46}$$

holds.
One considers the projections

$$\begin{aligned} g^{\mu\nu} W_{\mu\nu} &= (-D+1)\, W_1 + \frac{1}{M^2}\left(P^\nu - \frac{P \cdot q}{q^2} q^\nu\right)\left(P_\nu - \frac{P \cdot q}{q^2} q_\nu\right) W_2 \\ &= (-D+1) 3W_1 + \frac{1}{M^2}\left(-\frac{(k \cdot q)^2}{q^2}\right) W_2 \\ &= (-D+1) 3W_1 + \frac{P \cdot q}{2xM^2} W_2 \\ &= (-D+1) 3W_1 + \frac{1}{2x} \cdot \frac{\nu \cdot W_2}{M} \\ &\to \frac{1}{2x}\left[F_2 - 2x(1-D)F_1\right] \ . \end{aligned} \tag{47}$$

Contracting the hadronic tensor with the D-dimensional momentum of the proton in a similar manner and defining the longitudinal structure function

$$F_L \equiv (F_2 - xF_1) \tag{48}$$

gives the following relations [14]

$$\begin{aligned} \frac{1}{x} F_2 &= \frac{2}{D-2}\left[4(D-1)\frac{x^2}{Q^2} P^\mu P^\nu W_{\mu\nu} - g^{\mu\nu} W_{\mu\nu}\right] & (49)\\ F_L &= \frac{8x^3}{Q^2} P^\mu P^\nu W_{\mu\nu} \ . & (50) \end{aligned}$$

Thus the hadronic tensor is expressed in terms of two structure functions depending only on x and Q^2.

3 LO Wilson coefficients for massive quark production in ep–scaling

3.1 The Wilson coefficients f_2 and f_L

The question now arises, whether it is possible to calculate the structure functions F_2 and F_L for the subprocess $\gamma^* + G \to q + \overline{q}$. It is clear that since one has introduced the proton four momentum P in the above calculation, it has to be replaced by the gluon momentum k. One arrives at the notion of Wilson coefficients [14–17], which relate the gluon–photon reaction to the overall scattering process via the gluon density $G(y)$. As has been mentioned in section 2.4, the hadronic tensor cannot be calculated with perturbative methods only. The gluon density is the non-perturbative quantity which has to be considered and to be fitted from experimental data. The variable y now denotes the fraction of the proton momentum carried by the gluon. One introduces the scaling variable z for the subprocess considered

$$z = \frac{Q^2}{2k \cdot q} = \frac{Q^2}{2(yP) \cdot q} = \frac{x}{y} \ . \tag{51}$$

The Wilson coefficients obey the same projection relations as the structure functions (49) and (50), except that they depend on the scaling variable z. The convolution relating the structure functions, the Wilson coefficients, and the parton density reads [18]

$$F_{2,L}(x, Q^2) = \int_{ax}^{1} \frac{dy}{y} G(y) f_{2,L}\left(\frac{x}{y}, Q^2\right) \ . \tag{52}$$

The cms velocity v of the produced quarks and the threshold a are given by [18]

$$a \equiv 1 + 4\frac{m^2}{Q^2} \ , \tag{53}$$

$$v = \left(1 - 4\frac{m^2}{Q^2}\frac{z}{1-z}\right)^{1/2} \ . \tag{54}$$

The definition of v implies that

$$1 - 4\frac{m^2}{Q^2}\frac{z}{1-z} \geq 0 \ . \tag{55}$$

It follows

$$z \leq \left(1 + 4\frac{m^2}{Q^2}\right)^{-1} = \frac{1}{a} \tag{56}$$

$$\Leftrightarrow \frac{x}{y} \leq \frac{1}{a} \Leftrightarrow ax \leq y \ . \tag{57}$$

The lower bound therefore ensures that the velocity is real. In order to calculate $f_{2,L}$ one has to consider the subprocess $\gamma^* + g \to q + \overline{q}$, which is shown in Figure 2. The functions $f_{2,L}$ have some similarity to a cross section. They obey similar projection relations as the hadronic tensor. Using the Feynman rules of [20], the matrix element for this process reads

$$M_{\rho\mu} = \overline{u}(p_2) i g_s \gamma_\rho i \frac{\not{p}_1 - \not{q} + m}{(p_1 - q)^2 - m^2} i\gamma_\mu v(p_1) + \overline{u}(p_2) i g_s \gamma_\rho i \frac{\not{p}_1 - \not{k} + m}{(p_1 - k)^2 - m^2} i\gamma_\mu v(p_1) \ . \tag{58}$$

Here u and v denote the bi-spinors for particles and antiparticles, respectively. $\overline{u}$ and $\overline{v}$ are their conjugate bi-spinors. They satisfy the Dirac-equation:

$$(\not{p} - m)u(p) = 0 \quad \overline{u}(p)(\not{p} - m) = 0 \tag{59}$$
$$(\not{p} + m)v(p) = 0 \quad \overline{v}(p)(\not{p} + m) = 0 \ . \tag{60}$$

The bi-spinors and polarization vectors are normalized as

$$\sum_\sigma u(p,\sigma)\overline{u}(p,\sigma) = \not{p} + m \tag{61}$$
$$\sum_\sigma v(p,\sigma)\overline{v}(p,\sigma) = \not{p} - m \tag{62}$$
$$\sum_\lambda \epsilon^\mu(k,\lambda)\epsilon^\nu(k,\lambda) = -g^{\mu\nu} \ . \tag{63}$$

Squaring the matrix element yields a quantity depending on four indices, $H^{\rho\mu\sigma\nu} := M^{\rho\mu}M^{*\sigma\nu}$. Since one considers the case of unpolarized gluons one has to sum over the polarization states of the gluons, yielding a contraction with $-g_{\rho\sigma}$. This averaging gives an overall factor $1/(D-2)$. It stems from the fact that a vector particle has $D-2$ spin degrees of freedom in D dimensions. The quark-antiquark momenta are integrated over because we calculate the inclusive cross section. Since all integrals converge, D is set equal to 4. Keeping in mind that the Wilson coefficients obey equivalent projection relations as the structure functions, one arrives at

$$f_2(x,Q^2) = \frac{e_q^2 \cdot T_R}{2} \int dR_2 \left(g_{\mu\nu} - 12\frac{x^2}{Q^2}k_\mu k_\nu \right) H^{\mu\nu\rho\sigma} g_{\rho\sigma} \tag{64}$$
$$f_L(x,Q^2) = \frac{e_q^2 \cdot T_R}{2} \int dR_2 \left(-8\frac{x^2}{Q^2}k_\mu k_\nu H^{\mu\nu\rho\sigma} g_{\rho\sigma} \right) \ . \tag{65}$$

The phase space element is given in Appendix A. The expression for the Wilson coefficients simplifies, cf. (A.24)

$$f_2(x,Q^2) = \frac{e_q^2 \cdot T_R}{2} \int_{t^-}^{t^+} \frac{dt}{64\pi s p_{cm}^2} \left(g_{\mu\nu} - 12\frac{x^2}{Q^2}k_\mu k_\nu \right) H^{\mu\nu\rho\sigma} g_{\rho\sigma} \tag{66}$$
$$f_L(x,Q^2) = \frac{e_q^2 \cdot T_R}{2} \int_{t^-}^{t^+} dR_2 \frac{dt}{64\pi s p_{cm}^2} \left(-8\frac{x^2}{Q^2}k_\mu k_\nu H^{\mu\nu\rho\sigma} g_{\rho\sigma} \right) \ . \tag{67}$$

The integration bounds $t^\pm$ are determined by the physical regions of the phase space and are given in (A.25). The calculation has been done using the computer algebra program `FORM` [19]. Having performed the t–integration one obtains the Wilson coefficients

$$\begin{aligned} f_2(z,Q^2) &= 8e_q^2 T_R a_s \cdot \left\{ v\left[-\frac{1}{2} + 4z - 4z^2 + \frac{m^2}{Q^2}(2z^2 - z) \right] \right. \\ &\quad \left. + \left[-\frac{1}{2} + z - z^2 + 2\frac{m^2}{Q^2}(3z^2 - z) - 4\frac{m^4}{Q^4}z^2 \right] \ln\left(\frac{1-v}{1+v} \right) \right\} \end{aligned} \tag{68}$$
$$f_L(z,Q^2) = 16e_q^2 T_R a_s \cdot \left[v \cdot z(1-z) + 2\frac{m^2}{Q^2}z^2 \ln\left(\frac{1-v}{1+v} \right) \right] \ . \tag{69}$$

Comparing the results to [18] yields the same results except for a factor of x/Q^2, which is due to different definitions of F_2 and F_L, see (52).

3.2 The Limit $Q^2 \gg m^2$

One now considers the limit $Q^2 \gg m^2$, i.e. one takes $m \to 0$ in (68) and (69). Since for small masses

$$v = 1 - \frac{2m^2}{Q^2}\frac{z}{1-z} + O\left(\frac{m^4}{Q^4}\right), \tag{70}$$

the logarithm becomes

$$\ln\left(\frac{1-v}{1+v}\right) = \ln\left(\frac{m^2}{Q^2}\right) + \ln\left(\frac{z}{1-z}\right) + O\left(\frac{m^2}{Q^2}\right). \tag{71}$$

Using the Taylor–expansion in m^2/q^2, one arrives at

$$\begin{aligned}
\hat{f}_2(z,Q^2) &= 4e_q^2 T_R a_s \cdot \left\{8z(1-z) - 1 + [z^2+(1-z)^2]\ln\left(\frac{Q^2}{m^2}\right)\right.\\
&\quad \left. +[z^2+(1-z)^2]\ln\left(\frac{1-z}{z}\right)\right\}\\
&\quad +4e_q^2 T_R a_s \frac{m^2}{Q^2}\left\{-10z^2 - z + 4(3z^2-z)\left[\ln\left(\frac{z}{1-z}\right) + \ln\left(\frac{m^2}{Q^2}\right)\right]\right\} \qquad (72)\\
\hat{f}_L(z,Q^2) &= 16e_q^2 T_R a_s \cdot z(1-z)\\
&\quad +16e_q^2 T_R a_s z^2 \frac{m^2}{Q^2}\left\{-1 + \left[\ln\left(\frac{z}{1-z}\right) + \ln\left(\frac{m^2}{Q^2}\right)\right]\right\} + 32e_q^2 T_R a_s \frac{m^4}{Q^4}\frac{z^3}{1-z}\,. \qquad (73)
\end{aligned}$$

However, the terms of higher order in m^2/Q^2 will not be considered throughout the rest of this study and are only given to better understand section 3.3. In the massless limit one encounters a collinear singularity through $\ln(Q^2/m^2)$. The name collinear can be understood if one looks at Figure 3. For $m \to 0$ the angel between p_2 and $p_1 - q$ approaches zero, i.e. p_2 becomes parallel to k and hence collinear.

3.3 Numerical Results

Numerical results for the relations derived in Sections 3.1 and 3.2 will now be given. It is useful to study the ratios of (68) to (72) and (69) to (73). One defines

$$\begin{aligned}
R_L(z) &\equiv \frac{f_L(z)}{\hat{f}_L(z)} \qquad (74)\\
R_2(z) &\equiv \frac{f_2(z)}{\hat{f}_2(z)} \qquad (75)\\
R_L\left(\xi \equiv \frac{Q^2}{m^2}\right) &\equiv \frac{f_L(\xi)}{\hat{f}_L(\xi)} \qquad (76)\\
R_2\left(\xi \equiv \frac{Q^2}{m^2}\right) &\equiv \frac{f_L(\xi)}{\hat{f}_L(\xi)}\,, \qquad (77)
\end{aligned}$$

where we refer to the variable $\xi := Q^2/m^2$. These functions, as well as $f_2(z)$ and $f_L(z)$, were calculated numerically using a `FORTRAN` program and are given in Figures 4 to 9.

Figure 4 and 5 give the un-expanded Wilson coefficients. The running coupling constant has been set to $a_s = 0.2$ and $T_R = 0.5$. The quark charge has not been assigned a value. One sees that both Wilson coefficients give higher contributions for higher energies, f_2 being almost by a factor 20 larger than f_L. For f_2 these contributions lie in the range of small z and for f_L in the range of large z.

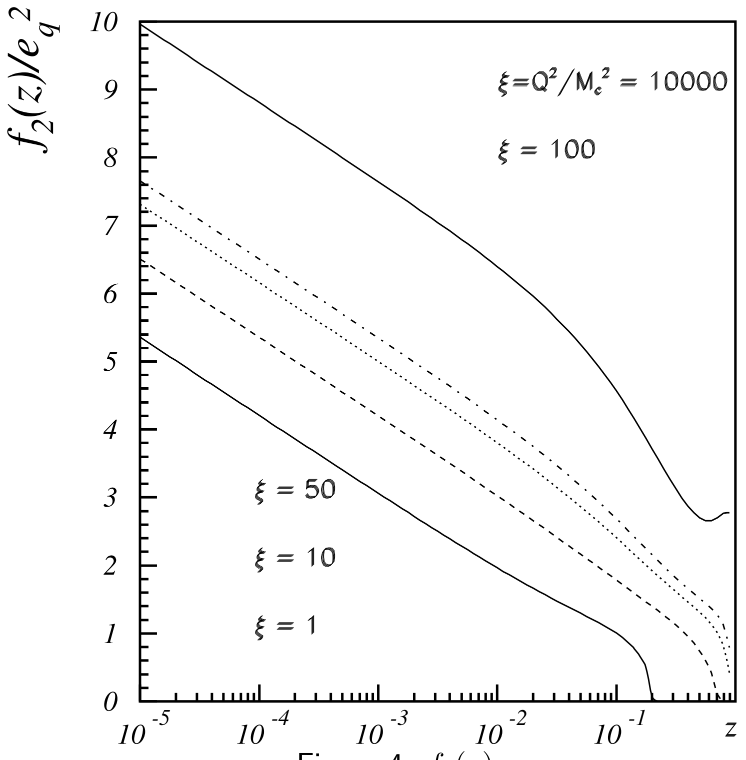


Figure 4: $f_2(z)$

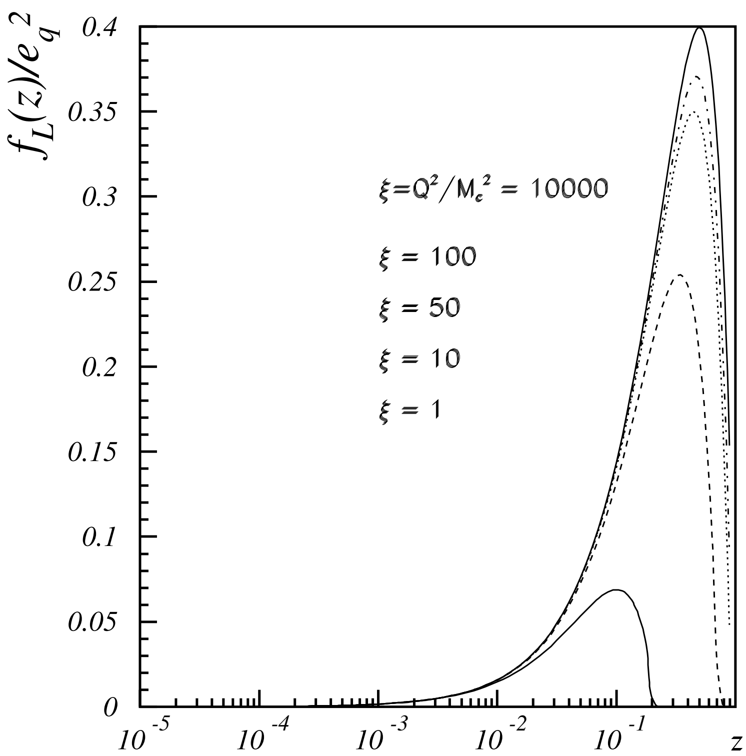


Figure 5: $f_L(z)$

In Figures 6 and 7 one can see that approximately for $z < 0.2$, $R_2(z)$ is close to 1, whereas for $R_L(z)$ this holds holds only for $z < 0.05$. From these plots one can infer that for small z the asymptotic formulas (72) and (73) coincide with the exact ones for any value of ξ. However, the difference is much larger for R_L, since the ξ–terms present in (73) are all of higher order, contrary to (72).

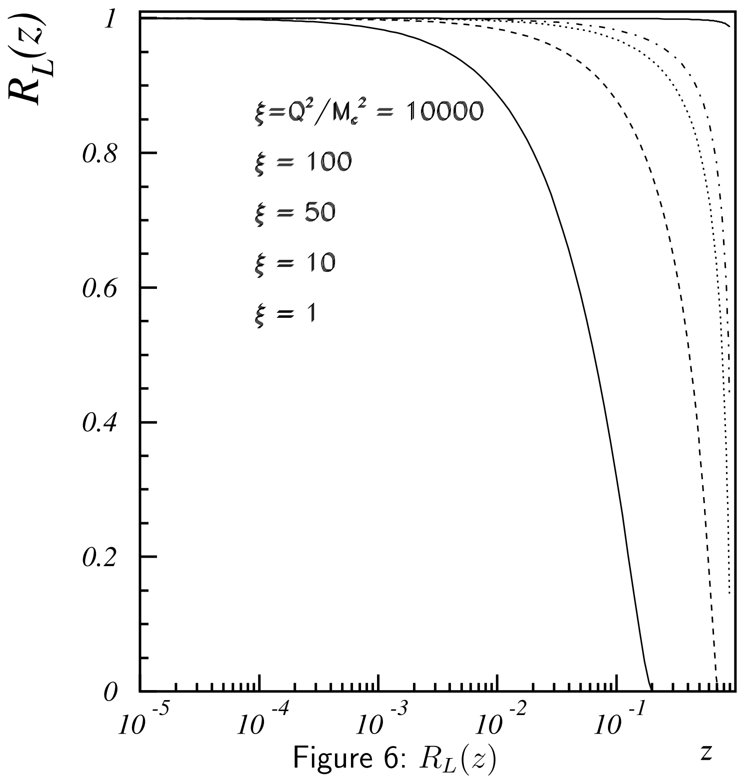


Figure 6: $R_L(z)$

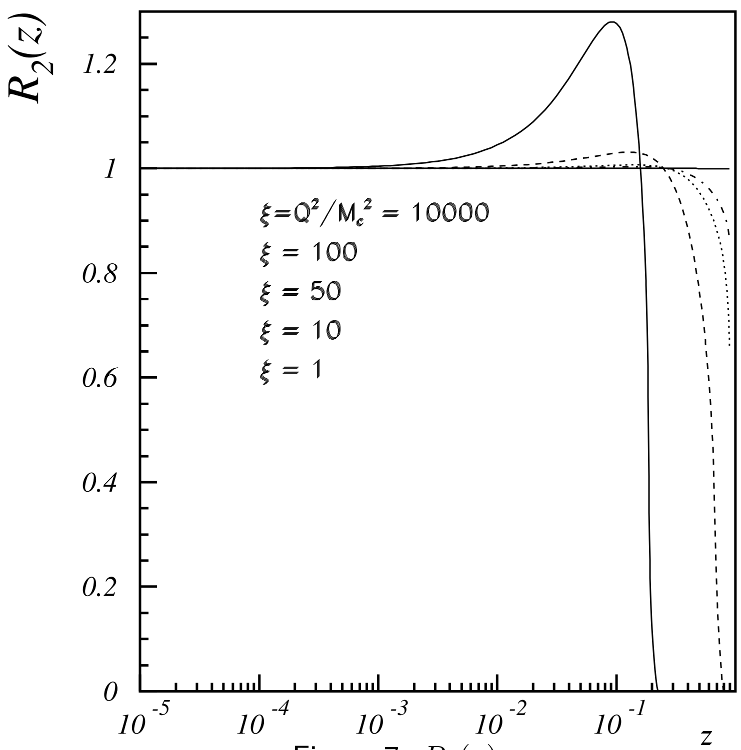


Figure 7: $R_2(z)$

Figure 8 and 9 show that for $\xi > 50$, $R_2(\xi)$ is close to one, whereas the same holds for $R_L(\xi)$ only if $\xi > 500$. The difference is explained as for Figures 6 and 7. Whereas the expansion of f_2 still contains logarithmic ξ–terms, this does not hold for f_L. Therefore the difference is larger, since no corrections due to the lowest order in ξ are made in $\hat{f}_L$, see 73.

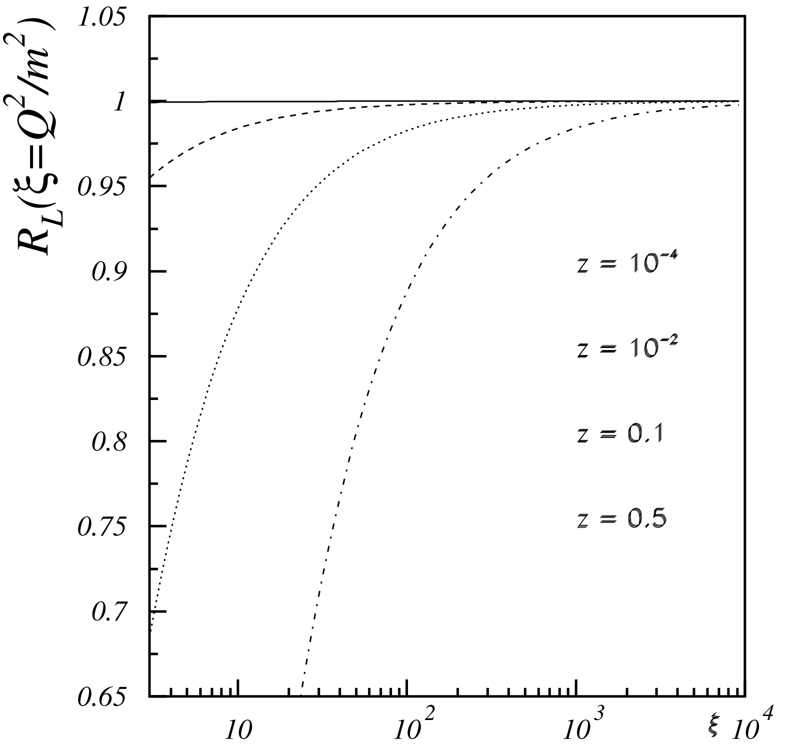


Figure 8: $R_L(\xi)$

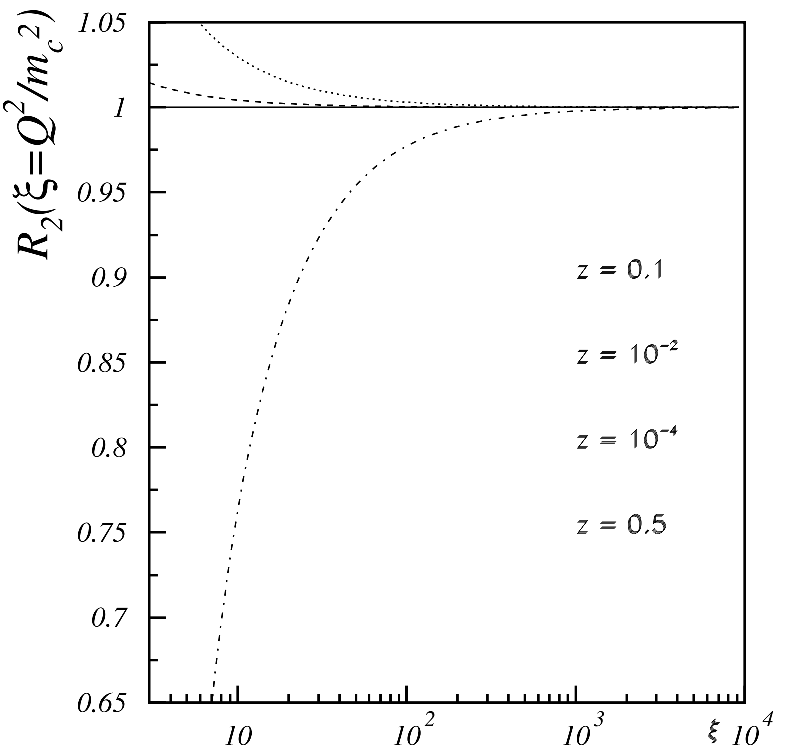


Figure 9: $R_2(\xi)$

4 LO Wilson Coefficients for massless quark ep-pair production in the $\overline{\text{MS}}$ scheme

4.1 $\overline{\text{MS}}$ scheme

When evaluating the momentum integrals of Feynman-diagrams, one is bound to encounter singularities. In order to be able to evaluate these divergent integrals and to calculate finite physical quantities, one introduces the concept of renormalization and factorization [7, 15–17]. The idea is to extract the diverging parts in a certain way to absorb them into a priori non-renormalized quantities, e.g. the coupling constant or the gluon density, which are as well divergent and not observable. The final result is free of singularities. There are several ways to do this, usually based on dimensional regularization. One leaves the four-dimensionality of space-time and works in the general case of D dimensions. Evaluating the divergent integrals of a re-normalizable field theory yields that these integrals are divergent in 4 dimensions, but convergent for $D \neq 4$. Hence one may perform all calculations in D dimensions. In this study the convention $D = 4 + \epsilon$ is used. The limit $\epsilon \to 0$ will then yield the desired result for $D = 4$. Since one has already encountered singularities when trying to calculate the integral in 4 dimensions, it is clear that the result should diverge when ϵ is set to zero. The advantage is that the divergent parts can be identified as ϵ^{-k}–poles with k integer. The general concept is to add suitable counter terms, which compensate these divergencies. The $\overline{\text{MS}}$ scheme [7] is based on the observation, that the poles usually emerge as terms

$$-\frac{1}{\epsilon} + \frac{1}{2}[\gamma_E - \ln(4\pi)] \; . \tag{78}$$

Since the finite terms in (78) are universal, they may be subtracted along with the pole terms. One can rewrite the expression

$$\begin{aligned} -\frac{1}{\epsilon} + \frac{1}{2}[\gamma_E - \ln(4\pi)] &= \left(\frac{1}{\epsilon}\right)\left[-1 + \left(\gamma_E - \ln(4\pi)\right)\frac{\epsilon}{2}\right] \\ &= S_\epsilon \frac{1}{\epsilon} \; , \end{aligned} \tag{79}$$

where one has to keep in mind that orders of ϵ with positive exponents are negligible after all singularities are accounted for. Here S_ϵ is defined by

$$S_\epsilon \equiv \exp\left[\frac{\epsilon}{2}(\gamma_E - \ln 4\pi)\right] \; , \tag{80}$$

the spherical factor which is subtracted in the $\overline{\text{MS}}$ scheme contrary to the MS scheme, where only the pole terms are removed.

4.2 LO Wilson coefficients in the massless case

We consider again the process described in Figure 3 but set $m \equiv 0$. Therefore singularities will appear, which are dealt with in the $\overline{\text{MS}}$ scheme. The matrix element is again given by (58) and the projections of the hadronic tensor by (49) and (50). The relations for the D–dimensional γ–matrices can be found in [20]. The phase space integral is given by (A.19), where y has to be integrated from -1 to 1. The sub-system cross sections $\hat{C}_i$ are given by

$$\begin{aligned}\hat{C}_{2,g}(x,Q^2) &= \frac{2e_q^2\cdot T_R}{(D-2)^2}\int_{-1}^{1}\frac{dy(p'_{cm})^{D-3}}{32\pi s p_{cm}}\left[g_{\mu\nu}-4\left(D-1\right)\frac{x^2}{Q^2}k_\mu k_\nu\right]H^{\mu\nu\rho\sigma}g_{\rho\sigma}\\ &\quad\cdot\frac{(1-y^2)^{D/2-2}}{2^{D-4}\pi^{D/2-2}\Gamma(D/2-1)} \qquad (81)\\ \hat{C}_{L,g}(x,Q^2) &= \frac{e_q^2\cdot T_R}{D-2}\int_{-1}^{1}\frac{dy(p'_{cm})^{D-3}}{32\pi s p_{cm}}\left(-8\frac{x^2}{Q^2}k_\mu k_\nu H^{\mu\nu\rho\sigma}g_{\rho\sigma}\right)\\ &\quad\cdot\frac{(1-y^2)^{D/2-2}}{2^{D-4}\pi^{D/2-2}\Gamma(D/2-1)}\ . \qquad (82)\end{aligned}$$

The calculation has again been done using `FORM` [19]. Another aspect has to be kept in mind. In 4 dimensions, the strong coupling constant g_s is dimensionless. However, in D dimensions it becomes dimensionful [26],

$$g_s^2 = \frac{16\pi^2}{\mu^\epsilon}\cdot a_s\ , \qquad (83)$$

with a_s being dimensionless. After performing the ϵ–expansion, the result of the calculation can be written as

$$\begin{aligned}\hat{C}_{2,g} &= e_q^2 T_R a_s\Bigg\{P_{qg}(z)\cdot\left[\frac{1}{\epsilon}+\frac{1}{2}\left(\gamma_E-\ln(4\pi)\right)\right]\\ &\quad+\frac{1}{2}P_{qg}(z)\left[\ln\left(\frac{1-z}{z}\right)+\ln\left(\frac{Q^2}{\mu^2}\right)-4\right]+12\Bigg\} \qquad (84)\\ \hat{C}_{L,g} &= 16e_q^2T_Ra_sz(1-z)\ , \qquad (85)\end{aligned}$$

where the splitting function for gluon-quark splitting as described in section 2.3 has been introduced

$$P_{qg}(z) = 8T_R(z^2+(1-z)^2)\ . \qquad (86)$$

After re-expanding this quantity by extracting the factor S_ϵ, see (80), one arrives at

$$\begin{aligned}\hat{C}_{2,g} &= e_q^2T_Ra_sS_\epsilon\left\{\frac{P_{qg}(z)}{\epsilon}+\frac{P_{qg}(z)}{2}\left[\ln\left(\frac{1-z}{z}\right)+\ln\left(\frac{Q^2}{\mu^2}\right)-4\right]+12\right\} \qquad (87)\\ \hat{C}_{L,g} &= 16e_q^2T_Ra_sz(1-z)\ , \qquad (88)\end{aligned}$$

and the Wilson coefficients are given by the finite terms without ϵ-poles. These results are the same as have been derived by [15–17]. The further use of these coefficients functions will be described in section 5.

5 1-loop massive operator matrix elements

5.1 Heavy flavor Wilson coefficients

In this section another way to achieve the results derived in section 3 will be followed. The various relations are taken from [15–17] and [8] mainly. The hadronic tensor reads

$$W_{\mu\nu} = \frac{1}{4\pi}\int d^4z e^{iqz}\langle P\mid[J_\mu(z),J_\nu(0)]\mid P\rangle\ . \qquad (89)$$

One applies the OPE for the commutator of the two electromagnetic currents as z goes to zero, i.e. near the light cone, which yield the dominant contribution as $Q^2 \to \infty$,

$$\lim_{z\to 0}[J(z), J(0)] = \sum_k \sum_m \hat{C}_k^{(m)}(z^2) z_{\mu_1}....z_{\mu_m} O_k^{\mu_1...\mu_m}(0) , \tag{90}$$

where the distributions are the Fourier-transforms of the light parton coefficient functions $\hat{C}_k^{(m)}(Q^2/\mu^2)$. In the case considered one adds indices $i = 2, L$ for the expansion coefficients. The local operators $O_k^{\mu_1...\mu_m}(0)$ are given by

$$O_q^{\mu_1....\mu_m} = \frac{1}{2} i^{m-1} S[\overline{\Psi}(x)\gamma^{\mu_1} D^{\mu_2}....D^{\mu_m}\Psi(x)] + \mathsf{TraceTerms} , \tag{91}$$

where $D_\mu = \partial_\mu + i g_s T_a A_\mu^a$ denotes the covariant derivative and S is the symmetrization operator for the Lorentz indices μ_i. We now calculate the Wilson coefficients. The coefficient functions $C_{(2,L),g}$ were already derived in the massless case, whereas the local operators form the matrix elements (OME)

$$\tilde{A}_{kl}\left(\frac{m^2}{\mu^2}, \epsilon\right) \equiv \langle l \mid O_k \mid l \rangle . \tag{92}$$

Here the proton state $\mid P\rangle$ emerging in (89) has been replaced by the gluon state $\mid l\rangle$ since only this subprocess is considered. The heavy flavor coefficient functions are denoted by

$$H_{(2,L),g}\left(z, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) , \tag{93}$$

and can be expanded into a perturbative series

$$H_{(2,L),g}\left(z, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) = \sum_{k=1}^{\infty} a_s^k H_{(2,L),g}^{(k)}\left(z, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) . \tag{94}$$

In the limit $Q^2 \gg m^2$ one finds the following relation for the k–th order coefficients in a_s

$$H_{(2,L),g}^{(k)}\left(z, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) = \sum_{l=0}^{k} a_{(2,L),g}^{(k,l)}\left(\frac{m^2}{\mu^2}, z\right) \ln^l\left(\frac{Q^2}{m^2}\right) . \tag{95}$$

The heavy quark coefficient functions factorize as

$$H_{(2,L),g}\left(\frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) = A_{Qg}\left(\frac{m^2}{\mu^2}\right) \otimes C_{(2,L),g}\left(\frac{Q^2}{\mu^2}\right) , \tag{96}$$

due to mass factorization in the limit $Q^2 \gg m^2$ [15–17]. All contributions but the power corrections $\propto (m^2/Q^2)^k$, $k > 0$, can be described in this way. Here $C_{(2,L),g}(Q^2/\mu^2)$ denotes the massless Wilson–coefficients in the $\overline{\mathrm{MS}}$–scheme and $A_{Qg}(m^2/\mu^2)$ the operator matrix element. The structure functions are obtained by

$$F_{(2,L),g}(x, Q^2, m^2) = x \int_x^{z_{max}} \frac{dz}{z} G\left(\frac{x}{z}\right) H_{(2,L),g}\left(z, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) , \tag{97}$$

where $z_{max} = Q^2/(4m^2 + Q^2)$ factorization scale, which one can choose to be equal to the renormalization scale of the strong coupling constant $a_s(\mu^2)$. In LO one obtains the Born–approximation or the purely photon–gluon fusion process, which has already been described

$$\gamma^* + g \to q + \overline{q} . \tag{98}$$

Now perturbation theory can be performed for the quantities in (96), leading to the contributions to the coefficient functions $H_{(2,L)}$

$$A_{Q,g}\left(z,\frac{\mu^2}{m^2}\right) = \sum_{k=1}^{\infty} a_s^k A_{Q,g}^{(k)}\left(z,\frac{\mu^2}{m^2}\right) \tag{99}$$

$$C_{(2,L),g}\left(z,\frac{Q^2}{\mu^2}\right) = \sum_{k=1}^{\infty} a_s^k C_{(2,L),g}^{(k)}\left(z,\frac{Q^2}{\mu^2}\right) . \tag{100}$$

Up to $O(a_s)$ the coefficient functions are given by

$$C_{L,g} = a_s C_L^{(1),g} \tag{101}$$

$$C_{2,g} = 1 + a_s C_2^{(1),g} \tag{102}$$

in Mellin space which have been calculated in (87) and (88). One obtains

$$H_{L,g}^{(1)}\left(\frac{Q^2}{m^2},\frac{m^2}{\mu^2}\right) = \hat{C}_{L,g}^{(1)}\left(\frac{Q^2}{\mu^2}\right) \tag{103}$$

$$H_{2,g}^{(1)}\left(\frac{Q^2}{m^2},\frac{m^2}{\mu^2}\right) = A_{Qg}^{(1)} + \hat{C}_{2,g}^{(1)}\left(\frac{Q^2}{\mu^2}\right) . \tag{104}$$

The OME $A_{Qg}^{(1)}$ is already renormalized and can only be collinear divergent. One applies D–dimensional regularization to isolate the ϵ–poles. The renormalized OME A_{Qg} can be evaluated by a perturbation series. The general structure of the OME, [15–17], will now be reviewed briefly. One has to perform renormalization to deal with ultraviolet (UV) and collinear (C) divergencies. Having performed mass and charge renormalization, one can remove the UV divergencies by the operator renormalization constants $Z_{ik}(\epsilon_{UV}, a_s)$, whereas the collinear divergencies are removed by the transition functions $\Gamma(\epsilon_C, a_s)$. For convenience and without loss of generality, one sets $\epsilon_C = \epsilon_{UV} = \epsilon$. Up to LO, this reads

$$A_{ij} = \delta_{ij} + a_s\left[A_{ij}^{(1)} + Z_{ij}^{-1,(1)} + \Gamma_{ij}^{-1,(1)}\right] \tag{105}$$

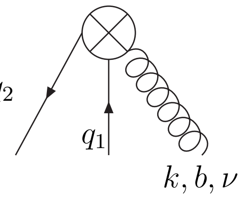
$$Z_{ij} = \delta_{ij} + a_s S_\epsilon\left[-\frac{1}{\epsilon}P_{ij}^{(0)}\right] \tag{106}$$

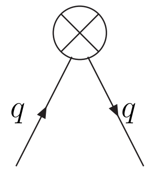
$$\Gamma_{ij} = \delta_{ij} + a_s S_\epsilon\left[\frac{1}{\epsilon}P_{ij}^{(0)}\right] . \tag{107}$$

Here i, j denote the parton types. as turns out later for the case considered, one will not have to deal with these renormalization group coefficients explicitly, since all divergencies will cancel exactly.

5.2 The calculation of the operator matrix element

In order to calculate the un-renormalized operator matrix elements $\tilde{A}_{Qg}^{(1)}$ to $O(a_s)$, the Feynman-rules for composite operators [20] as in Figure 10 and the Feynman diagrams as in Figure 11 have to be considered





$g t^a_{kj} \Delta^\mu \not{\Delta} \sum_{j=0}^{N-2} (\Delta \cdot q_1)^j (\Delta \cdot q_2)^{N-j-2}$ $\qquad$ $\not{\Delta}(\Delta \cdot q)^{N-1}$

Δ: arbitrary 4-vector, $\Delta^2 = 0$

Figure 10: Feynman rules for composite operators

Figure 11: Feynman Diagramms contributing at LO to the operator matrix elements

Additional to diagrams (α_1) and (β_1), two more diagrams α_2 and β_2 contribute at LO, which differ by the direction of the internal momenta. The Mellin-moment is given by N, μ denotes the Lorentz-index for the gluon, a its color-index and k, j are the color indices of the quarks. The respective Green's functions are [15–17]

$$\langle 0 \mid T(A^a_\mu(x) O_Q^{\mu_1 \dots \mu_N}(0) A^b_\nu(y)) \mid 0 \rangle \ . \tag{108}$$

Here A^b_ν denotes the external gluon field. The composite operators are traceless and symmetric but since one is not interested in trace terms emerging when computing the Green's function, they are projected out by multiplying with an external source $J_{\mu_1 .. \mu_N} = \Delta_{\mu_1} .. \Delta_{\mu_n}$, where $\Delta^2 = 0$. Since one usually works in momentum space, the Fourier transform is performed and extracting the effect of the external Gluon field, the following quantity will be calculated [15–17]:

$$G^{ab}_{Q\mu\nu} \ . \tag{109}$$

Averaging over color and spin has to be performed, giving factors $1/(N_C^2 - 1)$ and $1/(D-2)$. Including these factors, the contribution to the operator matrix element is given by

$$\tilde{A}^{(m)}_{Qg} = \frac{\delta^{ab}}{N_C^2 - 1} \frac{-g^{\mu\nu}}{D-2} (\Delta \cdot k)^{-N} G^{ab}_{Q\mu\nu} \ . \tag{110}$$

As an example, the explicit calculation diagram β_1 will be given. One obtains

$$\begin{aligned} & (-g_{\mu\nu}) G^{ab}_{Q\mu\nu,\alpha_1} \\ = & \int \frac{d^D q}{(2\pi)^D} (g_{\mu\nu}) tr \left[i g_s \gamma_\mu t^a_{mj} \delta_{jm} \frac{i}{\not{q} - m} \not{\Delta} (\Delta \cdot q)^{N-1} \delta_{lh} \frac{i}{\not{q} - m} i g_s \gamma_\nu t^b_{hf} \frac{i}{\not{q} - \not{k} - m} \delta_{fm} \right] \end{aligned}$$

$$
\begin{aligned}
&= ig_s^2 t^a_{kj} t^b_{kj} \int \frac{d^D q}{(2\pi)^D} \frac{(\Delta\cdot q)^{N-1}}{(q^2-m^2)^2((q-k)^2-m)} \cdot \\
&\quad tr\left[\gamma^\nu(\not{q}+m)\Delta(\not{q}+m)\gamma_\nu(\not{q}-\not{k}+m)\right] \\
&= ig_s^2 t^a_{kj} t^b_{kj} \int \frac{d^D q}{(2\pi)^D} \frac{(\Delta\cdot q)^{N-1}}{(q^2-m^2)^2((q-k)^2-m)} \cdot \\
&\quad \Big[(2-D)(q^2)(\Delta\cdot q+\Delta\cdot k)+\Delta\cdot q(m^2-2k\cdot q)-m^2\Delta\cdot k)+2m^2\Delta\cdot qD\Big] \ . \qquad (111)
\end{aligned}
$$

The factor of -1 emerges since β_1 has one closed fermion line [20]. In order to evaluate the integral, one uses Feynman parameterization and integral relations given in [20]. One arrives at the result

$$
(-g_{\mu\nu})G^{ab}_{Q\mu\nu,\beta_1} = -4g_s^2 t^a_{kl} t^b_{kl} \frac{\Gamma(2-D/2)}{(16\pi^2)^{D/4}}(\Delta\cdot k)^N m^{D-4}\frac{2(N+2-N)-D(N^2+2+N)}{N(N+1)(N+2)} \ , \qquad (112)
$$

As we study unpolarized scattering, N is even [8]. The calculation of $G^{ab}_{Q\mu\nu,\alpha_{1,2}}$ yields

$$
(-g_{\mu\nu})G^{ab}_{Q\mu\nu\alpha_{1,2}} = -16g_s^2 t^a_{kj} t^b_{kj} \frac{\Gamma(2-D/2)}{(16\pi^2)^{D/4}}(\Delta\cdot k)^N m^{D-4}\frac{1}{(N+1)(N+2)} \ . \qquad (113)
$$

To obtain $\tilde{A}^{(1)Qg}$ one has to add all 4 diagrams and include a symmetry factor of 1/2 since diagrams α_1, α_2 and β_1, $beta_2$ are a priori equal, since N is even.

$$
\begin{aligned}
\tilde{A}^{(1)}_{Qg} &= \frac{1}{2}g_s^2 T_R \frac{\Gamma(2-D/2)}{(16\pi^2)^{D/2}} m^{D-4}\frac{8(N^2+N+2)}{N(N+1)(N+2)} \\
&= \frac{1}{2}\frac{16(\pi)^2 a_s}{\mu^\epsilon}\frac{\Gamma(-\epsilon/2)}{(16\pi^2)^{1+\epsilon/4}} m^\epsilon P_{qg}(z) \\
&= \frac{1}{2}a_s\left(\frac{m^2}{\mu^2}\right)^{\epsilon/2}\frac{\Gamma(-\epsilon/2)}{(4\pi)^{\epsilon/2}}P_{qg}(z) \ . \qquad (114) \\
& \qquad (115)
\end{aligned}
$$

Here we transformed back from Mellin space to z space via (25). We factor out S_ϵ to apply the $\overline{\text{MS}}$ later,

$$
\begin{aligned}
\tilde{A}^{(1)}_{Qg} &= T_R a_s S_\epsilon \left(\frac{m^2}{\mu^2}\right)^{\epsilon/2}\left[-\frac{1}{\epsilon}P^{(0)}_{qg} + a^{(1)}_{qg} + \epsilon\overline{a}^{(1)}_{qg}\right] \qquad (116) \\
&= T_R a_s S_\epsilon \left(\frac{m^2}{\mu^2}\right)^{\epsilon/2}\left[-\frac{1}{\epsilon}P^{(0)}_{qg} - \frac{\zeta(2)}{8}\epsilon\right] \ . \qquad (117)
\end{aligned}
$$

The coefficient $a^{(1)}_{qg}$ of order ϵ^0 vanishes. The renormalized OME is then given by dropping S_ϵ and the pole terms in ϵ after having expanded $(m^2/\mu^2)^{\epsilon/2}$. However, we keep those two terms and only perform the ϵ-expansion and dropping terms in ϵ

$$
\tilde{A}^{(1)}_{Qg} = T_R a_s S_\epsilon\left[-\frac{P_{qg}(z)}{\epsilon} - \ln\left(\frac{m^2}{\mu^2}\right)\frac{P_{qg}(z)}{2}\right] \ . \qquad (118)
$$

5.3 The heavy flavor Wilson coefficients for $\mathbf{Q^2 \gg m^2}$

Using the massless results of section 4 and the OME computed in the previous section, the heavy quark coefficient functions $H_{2,L}$ will now be computed up to order a_s. Inserting (87) , (88) and (118) into (104) one finally arrives at

$$H_{L,g}^{(1)}\left(\frac{Q^2}{m^2},\frac{m^2}{\mu^2}\right) = 16 e_q^2 T_R a_s S_\epsilon z(1-z) \tag{119}$$

$$\begin{aligned} H_{2,g}^{(1)}\left(\frac{Q^2}{m^2},\frac{m^2}{\mu^2}\right) &= e_q^2 T_R a_s S_\epsilon \Bigg\{ \frac{P_{qg}(z)}{2}\left[\frac{2}{\epsilon} + \ln\left(\frac{1-z}{z}\right) + \ln\left(\frac{Q^2}{\mu^2}\right) - 4\right] + 12 \\ &\quad - \frac{P_{qg}(z)}{\epsilon} - \left(\frac{m^2}{\mu^2}\right)\frac{P_{qg}(z)}{2}\Bigg\} \\ &= e_q^2 T_R a_s S_\epsilon \Bigg\{ \frac{P_{qg}(z)}{2}\left[\ln\left(\frac{1-z}{z}\right) + \ln\left(\frac{Q^2}{m^2}\right) - 4\right] + 12\Bigg\} . \end{aligned} \tag{120}$$

These two final results are equal to (73) and (72) if $S_\epsilon = 1$, which is the case for $\epsilon \to 0$. These results are the same as in [15–17]. This is a non trivial result, since in the above formulae the pole terms of the OPM and the Wilson coefficients both cancel for this process and there is no correction proportional to ϵ^0. This gives the explanation for the observation of section 3 that the massive result is not divergent and agrees with the massless result in the $\overline{\text{MS}}$. The reason for this is that there are no collinear divergencies due to the finite mass of the heavy quarks produced and the ϵ^0–term in $\tilde{A}_{Qg}^{(1)}$ vanishes since the D– dependence there is non–polynomial but exponential and due to the Γ–function.

6 Conclusion

The heavy quark coefficient functions for deeply inelastic lepton-proton scattering have been calculated up to LO in a_s in the asymptotic limit $Q^2 \gg m^2$. The massive coefficient functions were calculated at LO for $D = 4$ exactly and no poles emerged, only a logarithmic mass singularity when performing the asymptotic limit. The factorization theorem was then applied to calculate the coefficient functions in the asymptotic limit as a sum of the operator matrix elements and the massless coefficient functions. The ϵ–poles of these two quantities canceled exactly, which gives a deeper understanding of the first result, since a priori divergencies would have to be expected. Numerical analysis was then performed to study the range in which the asymptotic limit is valid. It turned out that for small z it is accurate, whereas it gives too large results for $z \to 1$. One can estimate that the asymptotic result for f_2 is close to the massive result if $z < 0.2$, whereas the result for f_L is not that good, giving only accurate results for $z < 0.05$. The same holds for the dependence of the Wilson coefficients on the ratio of mass and momentum transfer, $\xi = Q^2/m^2$. The asymptotic limit gives good results for f_2 if $\xi > 50$, for f_L if $\xi > 500$. The reason was found to be that in the asymptotic limit there are no mass terms left in f_L.

Appendix

A Kinematics of 2-to-2 particle scattering

Let q, k and p_1, p_2 denote the 4-momenta of the incoming and the outgoing particles. Their masses are given by m_q, m_k and m_1, m_2, respectively. In the cms-frame one can write without loss of generality

$$\begin{aligned} q &= (E_q; 0, 0, p_{cm}) \quad k = (E_k; 0, 0, -p_{cm}) &(A.1)\\ p_1 &= (E_1; \vec{p}^{\,\prime}_{cm}) \quad p_1 = (E_1; -\vec{p}^{\,\prime}_{cm}) \; . &(A.2)\end{aligned}$$

The above quantities can be expressed by [21]

$$\begin{aligned} p_{cm} &= \frac{\lambda^{1/2}(s, m_q^2, m_k^2)}{2\sqrt{s}} &(A.3)\\ |\vec{p}^{\,\prime}_{cm}| &= \frac{\lambda^{1/2}(s, m_1^2, m_2^2)}{2\sqrt{s}} &(A.4)\\ E_q &= \frac{s + m_q^2 - m_k^2}{2\sqrt{s}} &(A.5)\\ E_k &= \frac{s + m_k^2 - m_q^2}{2\sqrt{s}} &(A.6)\\ E_1 &= \frac{s + m_1^2 - m_2^2}{2\sqrt{s}} &(A.7)\\ E_2 &= \frac{s + m_2^2 - m_1^2}{2\sqrt{s}} \; , &(A.8)\end{aligned}$$

where the kinematical function λ is given by

$$\lambda(x, y, z) = (x - y - z)^2 - 4yz \; . \tag{A.9}$$

Here s is the cms energy squared, see 35. The two–particle phase space integral in D dimensions is given by

$$\int dR_2 = \int \frac{d^{D-1}p_1}{(2\pi)^{D-1}(2E_1)} \frac{d^{D-1}p_2}{(2\pi)^{D-1}(2E_2)} \frac{1}{4q\cdot k} (2\pi)^D \delta^D(p_1 + p_2 - q - k) \; . \tag{A.10}$$

Using the D–dimensional δ–function and integrating over p_4 yields

$$\int dR_2 = \int d^{D-1}p_1 \frac{(2\pi)^{2-D}}{4E_1E_2} \frac{1}{4q\cdot k} \delta^D(E_1 + E_2 - E_q - E_k) \; . \tag{A.11}$$

The integration over p_3 can be performed by introducing D–dimensional spherical coordinates. Jacobi's determinant for this transformation is given by [22]

$$J = \frac{\partial \vec{p}_1}{\partial(|\vec{p}_1|, \vec{\phi})} = p_1^{D-2} \sin^{D-3}(\phi_1) \sin^{D-4}(\phi_2) ... \sin(\phi_{D-3}) \; . \tag{A.12}$$

Defining

$$\begin{aligned} y &:= \cos(\phi_{D-3}) &(A.13)\\ \phi_{D-3} &\equiv \theta_{qp_1} \; , &(A.14)\end{aligned}$$

the scattering angle between q and p_1 and integrating over all other angles gives

$$\int dR_2 = \int \frac{dp_1 dy 2\pi^{2-D}}{4E_1 E_2} \frac{1}{4q\cdot k} \frac{(2\pi)^{D/2-1}}{\Gamma(D/2-1)} p_1^{D-2}(1-y^2)^{D/2-2}\delta^D(E_1+E_2-E_q-E_k) \ . \quad (A.15)$$

Finally one integrates over the modulus of $\vec{p}_1$

$$\int dR_2 = \int \frac{dy 2^{3-D}\pi^{1-D/2}}{4q\cdot k} \frac{(1-y^2)^{D/2-2}}{\Gamma(D/2-1)} \frac{(p'_{cm})^{D-3}}{4\sqrt{s}} \ . \quad (A.16)$$

Now one may specialize to the process considered. The mass of the gluon is equal to zero and the mass of the virtual photon is given by $q^2 = -Q^2$. Since the same flavor of quarks is produced, their masses are equal. From (A.8) it follows that

$$p_{cm} = \frac{s+Q^2}{2\sqrt{s}} = \frac{q\cdot k}{\sqrt{s}} \quad (A.17)$$

$$p'_{cm} = \frac{1}{2\sqrt{s}} \ . \quad (A.18)$$

The final expression for the D–dimensional case now reads

$$\int dR_2 = \int \frac{(p'_{cm})^{D-3}}{32\pi s p_{cm}} \frac{(1-y^2)^{D/2-2}}{2^{D-4}\pi^{D/2-2}\Gamma(D/2-1)} dy \ . \quad (A.19)$$

In the remainder we use y as reference variable, this is done for the massless case in section 3. Since $y = \cos\theta_{qp_1}$, it has to be integrated from -1 to 1. It is useful to recall the relations of y and the Mandelstam–variable t

$$t = -\frac{Q^2}{2x}(1-y) \ . \quad (A.20)$$

Due to this it is possible to evaluate the integral by integrating over y, obtaining an expression, which contains Bjorken–x. In the massive case, section 2, the phase space integral evaluates as follows. For $D=4$ one obtains

$$\int dR_2 = \int \frac{1}{32s} \frac{p'_{cm}}{p_{cm}} dy \ . \quad (A.21)$$

We recall the definition of t

$$t = (q-p_1)^2 = -Q^2 + m_1^2 - 2E_q E_1 + 2p_{cm}p'_{cm}\cos(\theta_{qp_1}) \quad (A.22)$$

$$\Longrightarrow dt = 2p_{cm}p'_{cm} d\cos(\theta_{qp_1}) = 2p_{cm}p'_{cm} dy \quad (A.23)$$

and change variables from y to t and use the physical region of t as in [21] gives

$$\int dR_2 = \int_{t^-}^{t^+} \frac{dt}{64\pi s p_{cm}^2}, \quad (A.24)$$

$$t^\pm = \frac{s+Q^2}{2} \pm \frac{s+Q^2}{2}\cdot v \ , \quad (A.25)$$

where v is the cms velocity (54).

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