Schwinger-Model

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Abstract

In this report we want to give an impression of the importance of path integrals, their connection with statistical mechanics and finally turn to the Schwinger model in 2-dimensions. The goal is also to present some numerical results in the discretized the Schwinger model.

Chapter 1

Path integrals and Quantum Mechanics

1.1 Path integrals

Assume we want to determine the probability of the path that a particle may take in quantum mechanics. The initial point is

$$x_a = x(t = t_a) \tag{1.1}$$

and the final point is x_b at the chosen time t_b. For simplicity we restrict ourselves to the particle moving in one dimension. The position of any time can be specified by a coordinate x as a function of t. The path we mean by the function $x(t)$. The goal is to find the probability of each possible path for the particle. In the classical regime we usually use the *principle of the least action* ($\delta S = 0$) to determine the classical path, which is given of course. It can be derived as follows:

$$S = \int_{t_a}^{t_b} L(\dot{x}, x, t)dt \tag{1.2}$$

L is the lagrangian of the system. For a particle moving in a potential $V(x, t)$ the lagrangian is:

$$L = \frac{m}{2}\dot{x}^2 - V(x, t) \tag{1.3}$$

The most interesting case for us will be the transition from the classical into the quantum regime. The probability $P(b, a)$ to go from $x_a$to x_b is the

absolute square of an amplitude $K(b,a)$ to go from x_a to x_b. The amplitude (which we can call kernel) is the sum of the contribution of each path. Following the super-position principle:

$$K(b,a) = \sum_{over\ all\ paths} \varphi[x(t)] \tag{1.4}$$

Each contribution of the path has a phase factor proportional to the action S:

$$\varphi[x(t)] = conste^{i/\hbar S[x(t)]} \tag{1.5}$$

We can also write the kernel in the integral notation.

$$K(b,a) \sim \iint \cdots \int_{a=x_a}^{b=x_b} \varphi[x_i(t)] dx_1 dx_2 \cdots dx_{N-1} \tag{1.6}$$

Which was created by dividing the independent variable t into steps of infinitesimal width. We obtain a set of values t_i. At each time we select a point x_i and integrate through all possibilities. The path is constructed by taking the multiple integral over all values x_i. Having in mind (1.4) we can write the kernel as follows:

$$K(b,a) = \int_a^b e^{i/\hbar S[x(t)]} Dx(t) \tag{1.7}$$

The sum over all paths written as in (1.6) we shall call the path integral.

1.2 Connection between classical statistical mechanics and quantum theory

Our aim is to derive the following formula:

$$K(b,a) =< x_b|e^{-HT/\hbar}|x_a > \sim \int_a^b e^{-S[x(t)]/\hbar} Dx(t) \tag{1.8}$$

which is the Euclidean (imaginary time) version of Feynman's path integral. The vector $|x_a>$ and $|x_b>$ are position eigenstates. H is the Hamiltonian operator. We can expand (1.8) in a complete set of eigenstates.

$$H|n >= E_n|n > \tag{1.9}$$

and we obtain:

$$K(b,a) = \sum_n e^{-E_n T/\hbar} < x_b|n >< n|x_a > \tag{1.10}$$

We consider a system with one degree of freedom. The most interesting case in quantum theory is a matrix element $< x_b, t_b|x_a, t_a >$ - an amplitude- to go from position x_a at the time t_a to position x_b at the time t_b. The eigenvectors in Heisenberg picture are defined as:

$$|x, t >= e^{iHt}|x > \tag{1.11}$$

Following the definition (1.9) we can write the transition matrix as (one should consider that we are no longer in Euclidean time!):

$$< x_b, t_b|x_a, t_a >=< x_b|e^{-iH(t_b-t_a)}|x_a > \tag{1.12}$$

We divide $t_b - t_a$ into small time intervals $(t_b - t_a)/N$:

$$e^{iH(t_b-t_a)} = e^{iH(t_b-t_{n-1})}e^{iH(t_{n-1}-t_{n-2})} \cdots e^{iH(t_1-t_a)} \tag{1.13}$$

After inserting the complete sets of eigenstates between the factors:

$$< x_b|e^{iH(t_b-t_a)}|x_a >= \int dx_1 \cdots \int dx_{n-1} < x_b|e^{iH(t_b-t_a)/N}|x_{n-1} > \\ < x_{n-1}|e^{iH(t_b-t_a)/N}|x_{n-2} > \cdots < x_1|e^{iH(t_1-t_a)}|x_a > \tag{1.14}$$

To obtain the formula (1.8) we should write the Hamiltonian as a function of momentum squared- kinematic part, and space variable- x- just as $H = \frac{1}{2}p^2 + V(x)$. The next step is to write (1.14) within a momentum- space transition factor- e^{ipx} and to integrate over momentum (dp_i). Fortunatelly we can give the analytical result of momentum integrals- cause they are all squared. The left part in the exponential factor is the integral of the lagrangian, which lead us to the formula (1.8).

We usually use the follow equation to measure the correlation functions:

$$Z = \int dx < x|e^{HT/\hbar}|x >= Tre^{-HT/\hbar} \tag{1.15}$$

The 2-point correlation function:

$$\Gamma^{(2)} = Tre^{-HT/\hbar}x(t_1)x(t_2)/Z \tag{1.16}$$

Using this formula we can calculate the expectation value of an operator in the time limit $T \to \infty$

$$\lim_{T\to\infty} < \Gamma^{(2)} >= \frac{\sum_n e^{-E_n T/\hbar < n|x(t_1)x(t_2)|n>}}{\sum_n e^{-E_n T/\hbar}} \sim e^{-E_0 T/\hbar} \tag{1.17}$$

In the next section we will discuss the Schwinger model using the formalism given above.

1.3 The Schwinger model

For the Schwinger model we consider the action given below:

$$S = \int d^2x [\sum_{f=1}^{2} \bar{\psi}_f (i\gamma^\mu D_\mu - m)\psi_f - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}] \tag{1.18}$$

where $\bar{\psi}$ and ψ stands for fermion field. $\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ is for electromagnetic field strength tensor, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The next step is to turn from the continuum into the lattice with the boundary conditions. The constructed lattice is 2 dimensional- space time lattice. 'a' stands for a lattice spacing. We can come to the continuum via $a \to 0$. For the numerical computing we define the parameter $\beta \sim \frac{1}{a^2}$ which $\beta \to \infty$ in the continuum.

As we remember the amplitude is proportional to the integral of e^{-S} depending on gauge and fermionic field. Our goal is to calculate the mass of the pion. To do this we consider the operator $\hat{O}_x = \bar{\psi}_x \gamma_5 \psi_x$ and examine the following correlation function $< \hat{O}_t \hat{O}_0 >$ which we can develop into the complete set of eigenstates $\sum_n < 0|\hat{O}_t|n >< n|\hat{O}_0|0 >= \sum_n | < 0|\hat{O}_0|n > |^2 e^{-E_n t}$ which in the limit $\lim_{t\to\infty}$ and projecting to zero momentum gives $\sim e^{-m_\pi t}$.

When we keep in mind (1.15) and (1.16) we come to the point that to calculate the m_π we need to diagonalize the matrix and sum over all eigenvalues and eigenfuctions to reach our goal.

$$\lim_{t\to\infty} < \hat{O}_t \hat{O}_0 > \sim \lim_{t\to\infty} < \sum_{\lambda_i} \frac{\psi_i^\alpha \bar{\psi}_i^\beta}{\lambda_i} \sum_{\lambda_j} \frac{\psi_j^\alpha \bar{\psi}_j^\beta}{\lambda_j} > \sim e^{-m_\pi t} \tag{1.19}$$

The interpretation of amplitude is the creation of particle at time 0, propagation for time t and annihilation at time t.

This formalism lead us to the mass we were looking for.

1.4 The topological charge- the index theorem on the lattice

It is very interesting to add an information about chiral anomaly and how it implements our investigation. In the previous section we considered a set of eigenvalues and eigenvectors. Imagine that some of the eigenvectors can have vanishing eigenvalues ($\lambda_n = 0$) on condition that: $\gamma_5\varphi_n = \pm\varphi_n$. As a result we have n_+ and n_- eigenvectors- zero modes. The subtraction of number of 'right' and 'left' zero modes gives us the Pontryagin index ν which is the direct information of chiral anomaly. The index is so called topological current, it is always an integer number depending on number of zero modes.

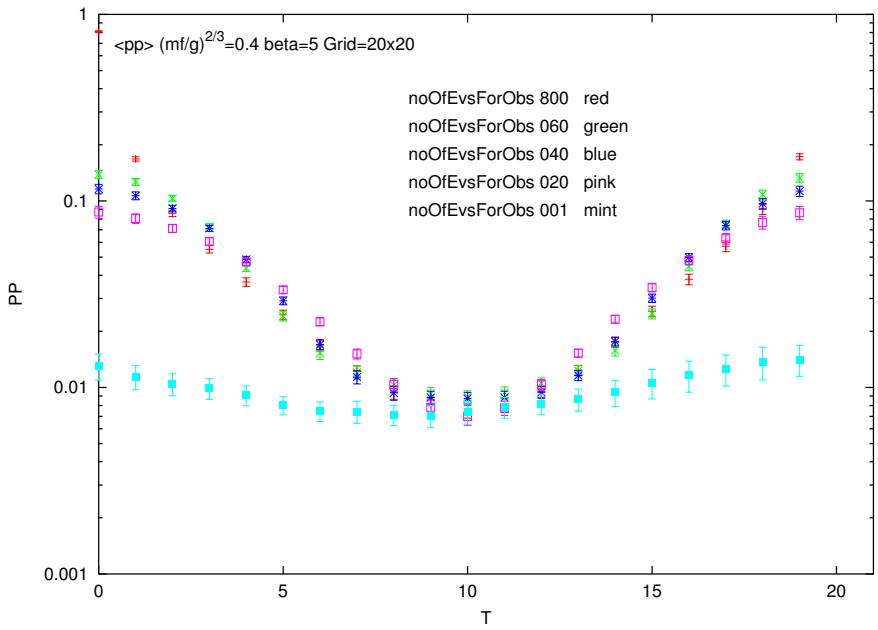
We want to know, how important the zero modes are in the calculation of the pion mass. To obtain this one should keep in mind that the eigenvalues we get from calculating the matrix D consist of values of zero mass and a quark mass itself: $\lambda_i = \lambda_i^{m=0} + m_{quark}$. As we remember (1.19) the pion mass depends on the sum of $\frac{1}{\lambda_i\lambda_j}$ which for zero modes is equal to $\frac{1}{m_{quark}^2}$. As one can see the smaller the mass of the quark the more important zero modes are.

Chapter 2

Simulation

2.1 The correlation function

As we saw in the previous chapter, we can calculate the correlator out of the eigenvalues and -vectors of the matrix D. There already exists a program, which does exactly this and fits a cosh-function to the data of the correlator. To reduce the amount of computer time for this calculation, we now want to check, if we really need all eigenvalues and -vectors to calculate the correlator. Perhaps it will be enough, to take only the lowest n ($n \in N$) out of them. So we add to the existing program the possibility to sort the eigenvalues and -vectors according to the norm of the eigenvalues and then use only the lowest n eigenvalues and -vectors to calculate the correlator. For several numbers n of used eigenvalues and -vectors we calculate the correlator on a 20x20 Grid. The full number of eigenvalues in a 20x20 Spinor-field is of course 20x20x2=800.



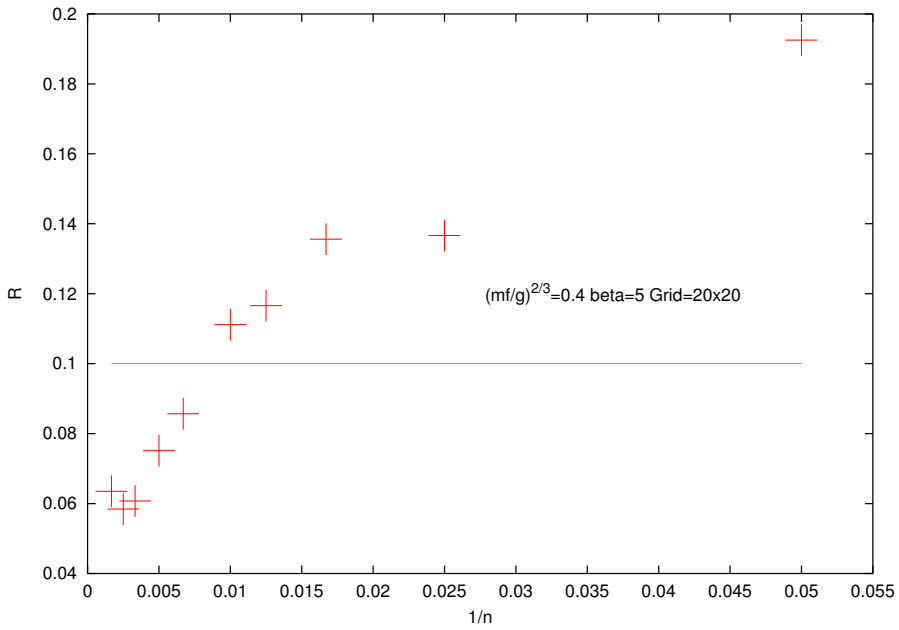
2.2 The dependency of the correlation function on the amount of used eigenvalues

To determine the amount of needed eigenvalues and -vectors for several lattice spacings a, we generated the data (eigenvalues and -vectors) with fixed $Z = (\frac{m_f}{a})^{\frac{2}{3}} = 0.4$ for five different values of $\beta = \{1, 2, 3, 4, 5\}$, which is proportional to $\frac{1}{a^2}$. So going to the continuum $a \to 0$ means to go to higher β. To determine the dependency of the number of needed eigenvalues and -vectors on the lattice, we need a criteria, in which case we want to accept the calculated correlator. A good parameter for this will be the relative difference to the exact result:

$$R = \frac{1}{T} \sum_t \frac{< pp(t) >^{exact} - < pp(t) >^{calculated}}{< pp(t) >^{exact}} \tag{2.1}$$

In respect to the typical error of the used data, we set the ratio R to 10%, at which point we would accept the accuracy of computing the correlator.

Now let us have a view on such a function $R(\frac{1}{n})$ for $\beta = 5$:

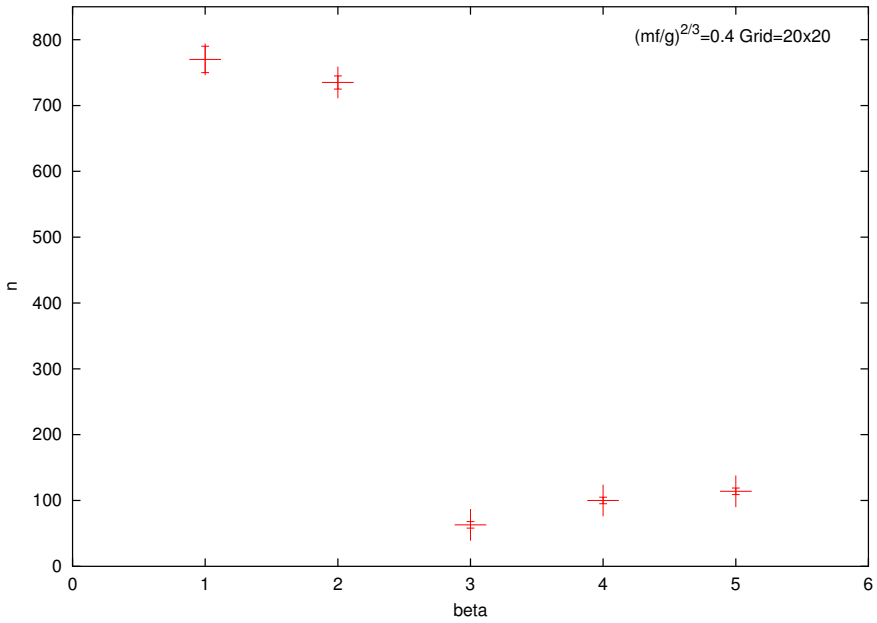


By assuming a smooth and monotonous function to the data points with $\frac{1}{n}$, we can estimate the needed number of eigenvalues and -vectors and the correlated errors by graphical methods.

2.3 The dependency on the lattice spacing

As the result we obtain the function $n(\beta)$:

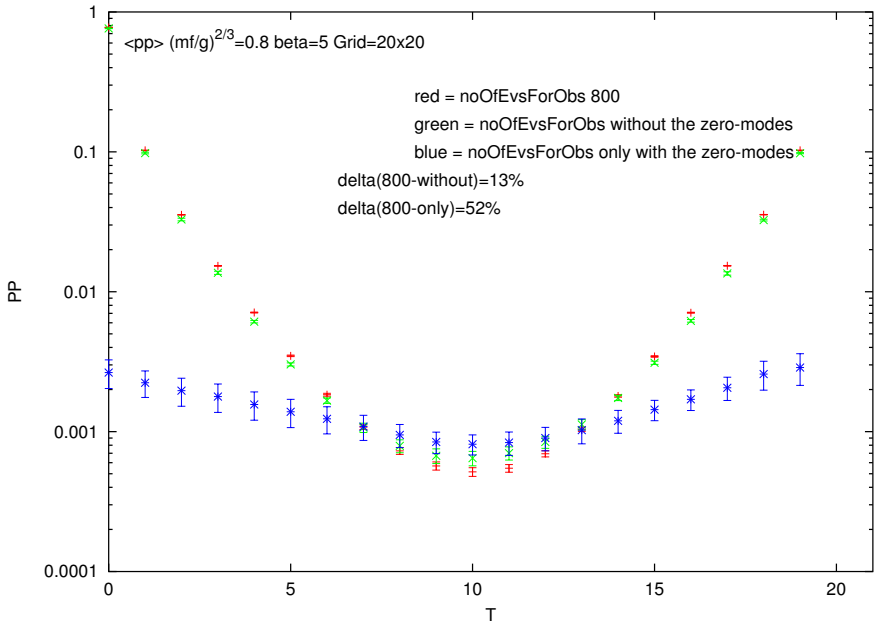
β	n
1	770±20
2	735±10
3	63± 5
4	100± 5
5	114± 5



This result don't show the estimated smooth dependency but a sharp transition of the lattice spacing. So: what have we done? By increasing the β, we decrease the lattice space a and we decrease the physical volume $L = Na$ (N=Number of Gridpoints). Perhaps it would be a better idea to always use the same constant physical volume by varying β such that $L = Na = N\frac{1}{\sqrt{\beta}} = const.$ and then calculate the relative amount of needed eigenvalues and -vectors in dependency on β.

2.4 The zero-modes experiments

For a better understanding of the physical content of the zero-modes (s. section 1.4) we also want to calculate their influence on the correlation function. For this we used the generated data (eigenvalue and -vectors) with fixed $Z = (\frac{m_f}{a})^{\frac{2}{3}} = 0.8$ and fixed $\beta = 5$ and calculated the correlator only and without these zero-modes:



Although the zero-modes have a big importance with regard to the topological charge, we can observe only a small influence on the correlator at high fermion masses.

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