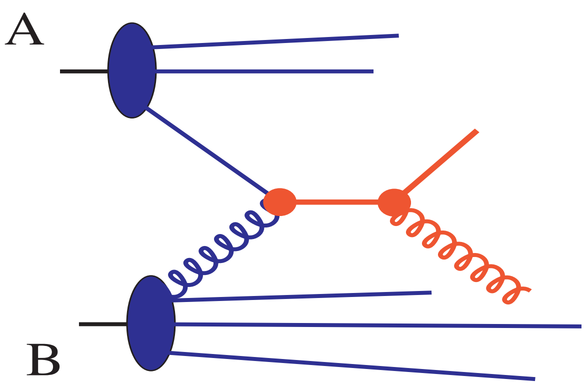
# Structure Functions on the Lattice: Where we stand and where we go

Karl Jansen

- **Introduction**
- **Dangerous Lattice Animals**
  - *discretization errors*
  - *finite volume effects*
- **Non-perturbative renormalization**
- **Some results and outlook**
- **Conclusions**

## Quark distribution functions

the probability to find a quark carrying a fraction $x_A$ of a hadron's momentum in the process




$$\text{hadron } A + \text{hadron } B \to \text{jet} + X$$

is

$$f_a(x_A)dx_A$$

where

$$f_a(x_A)$$

is the **"quark distribution function"**

(correspondingly a "gluon distribution function" may be defined)

## Moments of distribution functions

experiments can determine *Moments of distribution functions*

$$M_a^{(n)} = \int_0^1 dx x^{n-1} f_a(x) \quad , n = 1, 2, \cdots$$

of course

$$M_a^{(n)} = M_a^{(n)}(\mu)$$

$\rightarrow$ moments *run* with energy scale $\mu$

$\rightarrow$ need a scale dependend *renormalization*

renormalized average momentum (in a pion) at scale $\mu_0$

$$\langle x \rangle(\mu_0) = \langle \pi | \mathcal{O}^{\mathrm{ren}}(\mu_0) | \pi \rangle \equiv \langle \pi | \mathcal{O}^{\mathrm{bare}} / Z(\mu_0) | \pi \rangle$$

renormalized average momentum at a different scale

$$\langle x \rangle(\mu) = \langle x \rangle(\mu_0) \cdot \sigma_Z(\mu/\mu_0)$$

with

$$\sigma_Z(\mu/\mu_0) = \frac{Z(\mu_0)}{Z(\mu)}$$

the step scaling function

## Moments $\leftrightarrow$ local operators

the lattice can contribute because instead of the parton distribution functions themselves, moments of them can be considered

$$M_a^{(n)}(\mu) = \int_0^1 dx x^{n-1} f_a(x,\mu)$$

$$n = 1, 2, \cdots$$

and the moments can be related to expectation values of *local operators* $\mathcal{O}_a^n$

$$M_a^{(n)}(\mu) = \langle \mathcal{O}_a^n \rangle$$

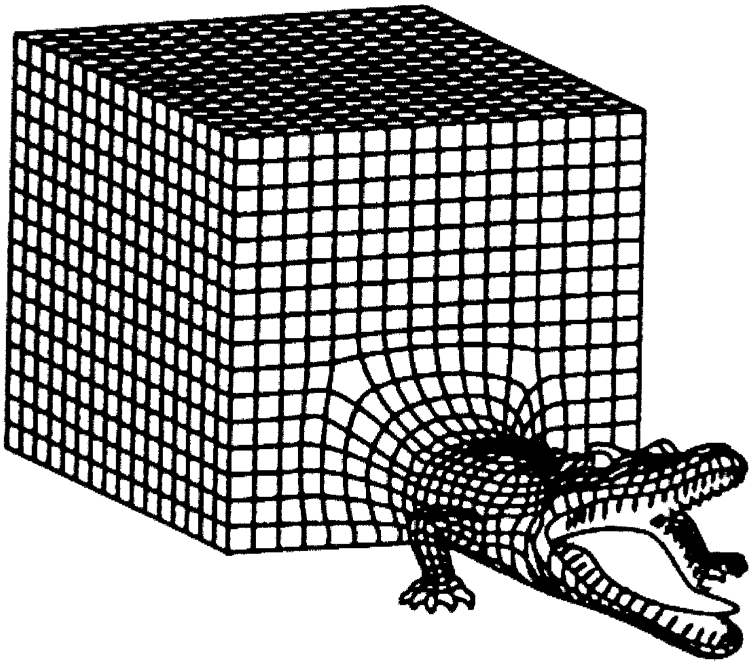
where

$$\{p^{\mu_1} \cdots p^{\mu_n}\}_{\mathrm{TS}} \langle \mathcal{O}_a^n \rangle =$$

$$\langle p|\bar{\psi}_a(x) \{\gamma^{\mu_1} iD^{\mu_2} \cdots iD^{\mu_n}\}_{\mathrm{TS}} \psi_a(x)|p\rangle$$

- $\langle p|$ state vector of a hadron with momentum $p$
- $D^\mu$ covariant derivative (gauge invariance)
- $\mathrm{TS}$ means to take traceless symmetric part

## There are dangerous animals on the lattice!



- finite physical boxlength $L$
- non-vanishing lattice spacing $a$

continuum limit:
keep *physical* length $L = Na = 1\mathrm{fm}$ fixed

$a = 0.1\mathrm{fm} \Rightarrow N = 10$

$a = 0.05\mathrm{fm} \Rightarrow N = 20$

$a = 0.01\mathrm{fm} \Rightarrow N = 100$

$\rightarrow$ easily running out of computertime and memory

solutions (?)

- keep $a \gg 0 \Rightarrow$ lattice artefacts
- keep $L < 1\mathrm{fm} \Rightarrow$ finite size effects

modern approach through theoretical advances

$\rightarrow$ accelerate continuum limit: improvement programme

$\rightarrow$ do not be afraid of finite size effects: make use of them

## Acceleration to the continuum limit

(old) standard lattice action of QCD is

$$S_{\mathrm{old}} = \underbrace{S_{\mathrm{G}}}_{\mathrm{O}(a^2)} + \underbrace{S_{\mathrm{wilson}}}_{\mathrm{O}(a)}$$

$\Rightarrow$ expectation values of physical observables

$$\langle O \rangle = \langle O \rangle_{\mathrm{cont}} + \mathrm{O}(a)$$

employing all lattice symmetries, equations of motions

$\Rightarrow$ only one more term in $\mathrm{O}(a)$ possible

(improved) standard lattice action

$$S_{\mathrm{new}} = S_{\mathrm{old}} + \underbrace{S_{\mathrm{sw}}}_{\mathrm{O}(a)}$$

$$S_{\mathrm{sw}} = a^5 \sum_x c_{\mathrm{sw}} \bar{\psi}(x) \tfrac{i}{4} \hat{F}_{\mu\nu}(x) \Psi(x)$$

with $c_{\mathrm{sw}}$ a *tunable* parameter

$\Rightarrow$ compute non-perturbatively $c_{\mathrm{sw}}$ such that $\mathrm{O}(a)$ cancel

$\Rightarrow$ (nota bene: if also the operator is improved)

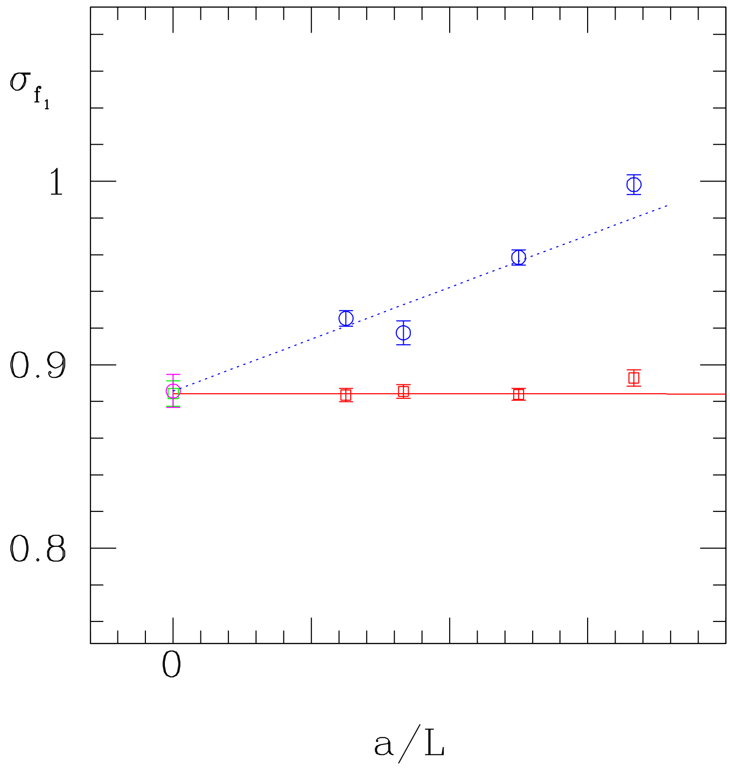
$$\langle O \rangle = \langle O \rangle_{\rm cont} + \mathrm{O}(a^2)$$

succesful *Symanzik improvement programme* of the

ALPHA Collaboration

Example of physical quantity derived from the action

$\Rightarrow$ no operator improvement neccessary



## Finite size scaling for
## *non-perturbative renormalization*

M. Guagnelli, K.J., R. Petronzio

in principle: need a box covering all energy scales

$$a \ll \underbrace{0.001\mathrm{fm}}_{\mathrm{M_W}} - \underbrace{1\mathrm{fm}}_{\mathrm{conf.scale}} \ll L$$

$\Rightarrow$ huge and unrealistic number of lattice points $N^4$

trick: *identify energy scale* $\mu$ *with inverse box length* $L^{-1}$

$$\mu = 1/L$$

$$\sigma_Z(\mu/\mu_0) \rightarrow \sigma_Z(L_0/L)$$

$\rightarrow$ split the problem: compute only small steps in the scale evolution of $\sigma_Z = Z(2L)/Z(L)$

$$\sigma_Z(L_0/L) = \sigma_Z(2L_0/L_0) \cdot \sigma_Z(4L_0/2L_0) \cdot \ldots \cdot \sigma_Z(2L/L)$$

$\Rightarrow$ idea can be realized in the
*Schrödinger functional renormalization scheme* that

- is a finite volume renormalization scheme

- can be implemented on the lattice

assume now that we can compute *non-perturbatively* the running matrix element $\langle x \rangle_{\mathrm{SF}}(\mu)$

$\Rightarrow$ for small enough coupling $\bar{g}$ , define the *renormalization group invariant matrix element*

$$O_{\mathrm{INV}}^{\mathrm{ren}} = O_{\mathrm{SF}}^{\mathrm{ren}}(\mu) \cdot f^{\mathrm{SF}}(\bar{g}^2(\mu))$$

$$f^{\mathrm{SF}}(\bar{g}^2(\mu)) = (\bar{g}^2(\mu))^{-\gamma_0/2b_0} \exp\left\{-\int_0^{\bar{g}(\mu)} dg \left[\frac{\gamma(g)}{\beta(g)} - \frac{\gamma_0}{b_0 g}\right]\right\}$$

$\gamma(g)$: anomalous dimension function
$\beta(g)$ : $\beta$-function
knowing $O^{\mathrm{ren}}$ we can get “easily” the matrixelement in a desired scheme

$$\langle x \rangle^{\mathrm{SF}} = O_{\mathrm{INV}}^{\mathrm{ren}} / f^{\mathrm{SF}}(\bar{g}^2(\mu))$$

$$\langle x \rangle^{\overline{\mathrm{MS}}} = O_{\mathrm{INV}}^{\mathrm{ren}} / f^{\overline{\mathrm{MS}}}(\bar{g}^2(\mu))$$

$$\langle x \rangle^{\mathrm{myprefered}} = O_{\mathrm{INV}}^{\mathrm{ren}} / f^{\mathrm{myprefered}}(\bar{g}^2(\mu))$$

$\Rightarrow$ make contact with analytical computations in perturbation theory!

## Excursion to the lattice

non-perturbative information from numerical simulations

$\rightarrow$ make $Z(\mu)$ dependend on the lattice spacing

$$Z(\mu) = Z(\mu, a)$$

$\Rightarrow$

the lattice step scaling function is then (step size s)

$$\Sigma_Z(a) = \frac{Z(sL,a)}{Z(L,a)}$$

s usually taken to be $2$

aim: continuum limit ( $\exists$ for $\Sigma_Z$!)

$$\sigma_Z = \lim_{a \to 0} \Sigma_Z(a)|_{\mu^{-1}=L \text{ fixed}}$$

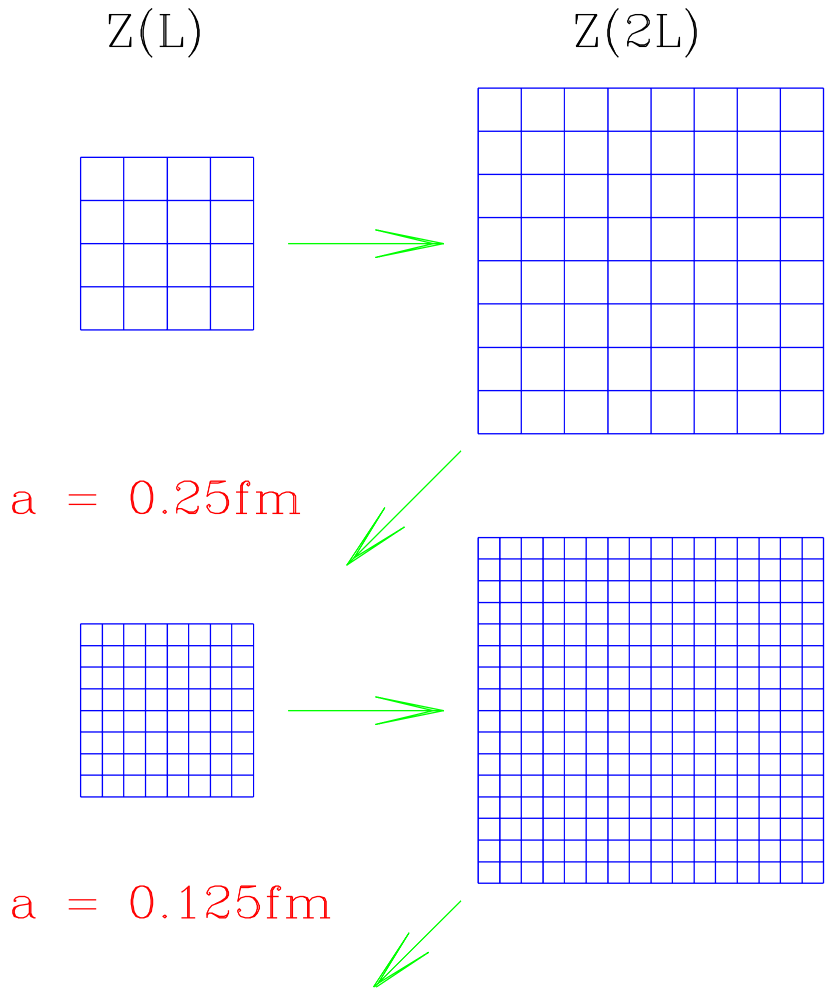
how do we fix the scale?

$\rightarrow$ similar to the moments, also the coupling constant *runs*

$$g = \bar{g}(\mu = 1/L)$$

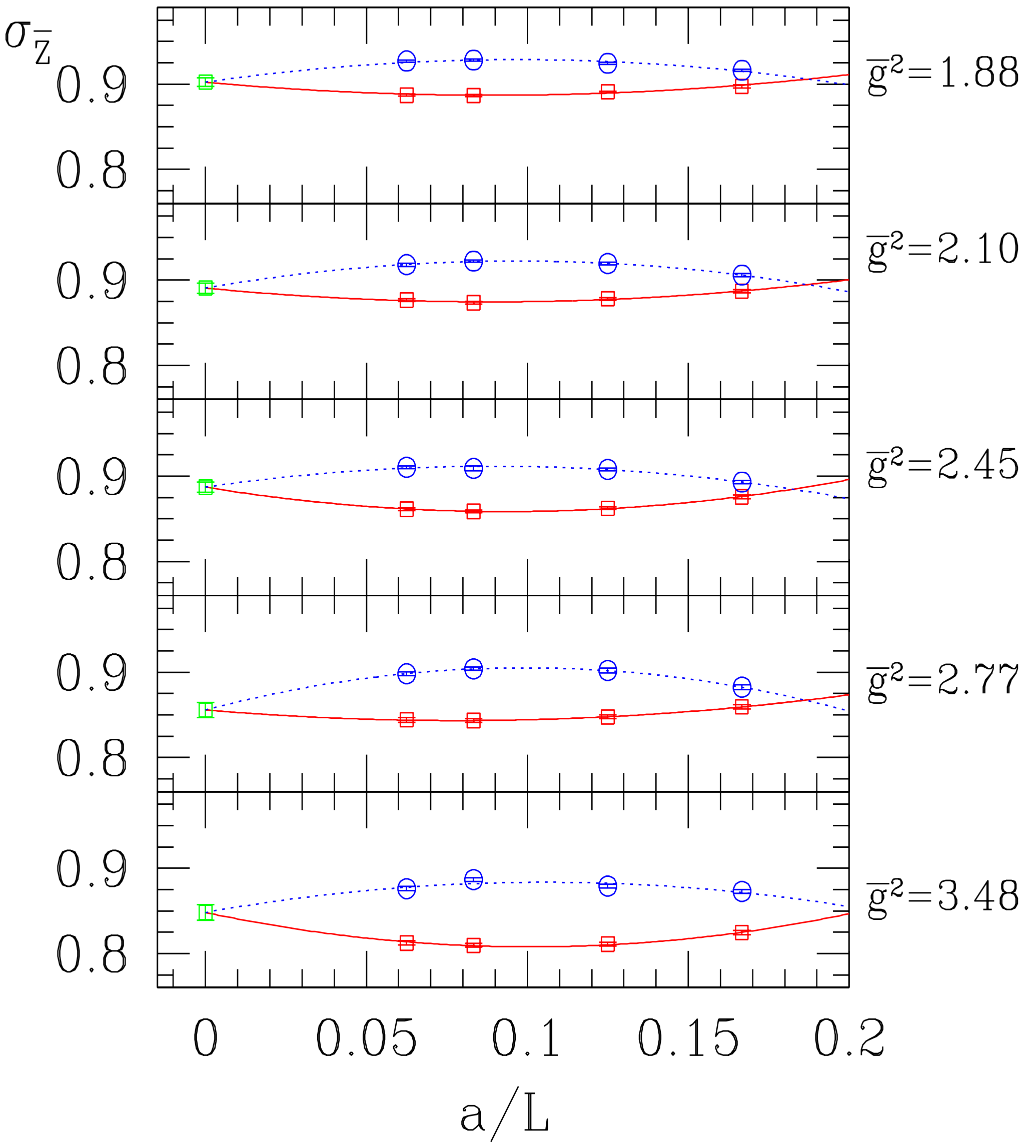
$\Rightarrow$ knowing $\bar{g}$ , means knowing $\mu$ and vice versa

full non-perturbative running of $\bar{g}$ $\rightarrow$ ALPHA Collaboration continuum limit is reached through a matching procedure

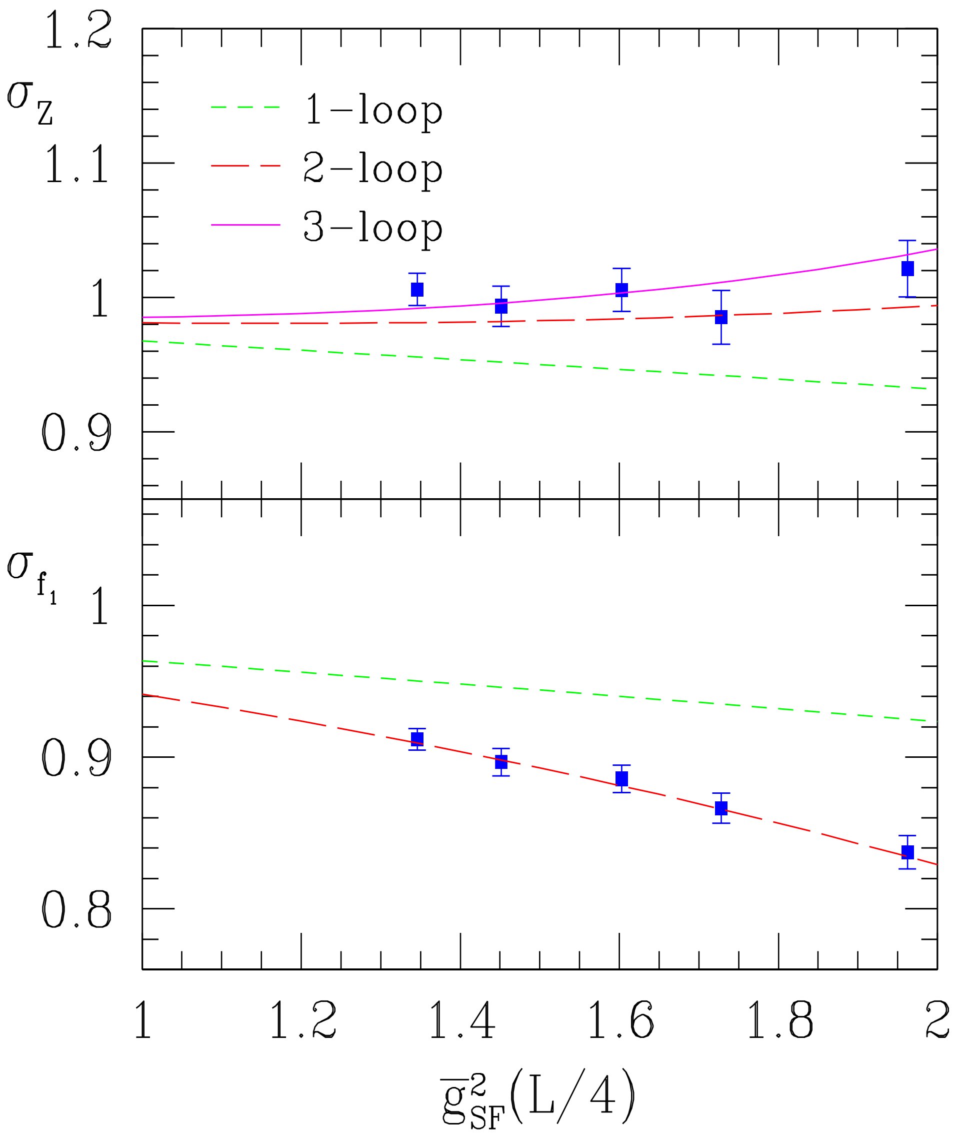



while keeping $\bar{g}^2(L)$ fixed

## Combined fit $\sigma_{\bar{Z}}$



## Continuum step scaling functions




choice of $\bar{g}^2_{\mathrm{SF}}(\mu^{-1} = L/4)$ from the definition of the operator taken at $x_0 = L/4$

## Continuum renormalized Matrixelement

$$\boxed{\lim_{a\to 0} \left.\frac{<\pi|\mathcal{O}|\pi>}{Z_O(L_0)}\right|_{m_q=0}}$$

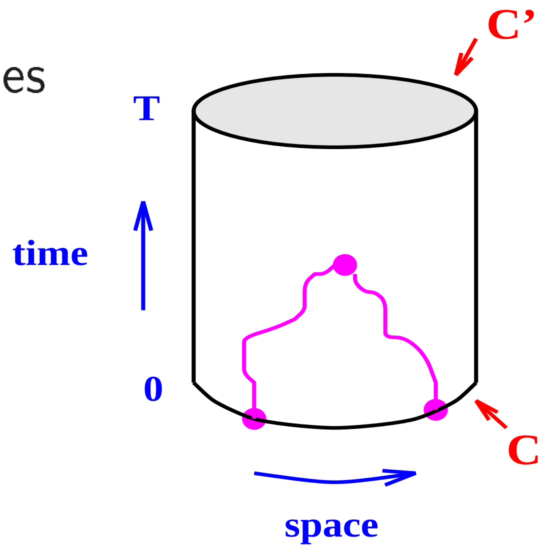
<u>usage of Schrödinger functional:</u>

correlation function $f_O$ of some operator $\mathcal{O}$
at a distance $x_0/a$

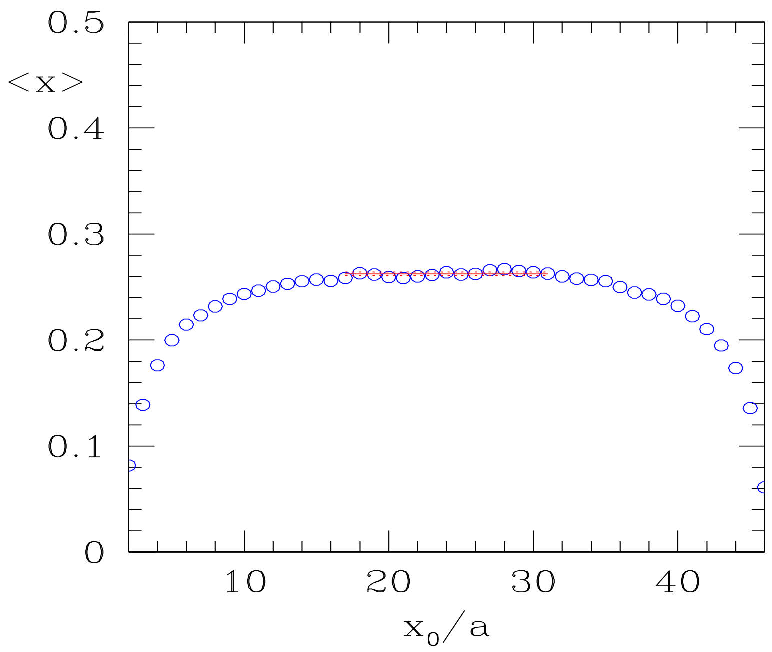
$$f_O(x_0/a) = \sum_{\mathbf{x},\mathbf{y},\mathbf{z}} < \mathcal{O}(x)\bar{\zeta}(y)\Gamma\tau^3\zeta(z) >$$

$\zeta$, $\bar{\zeta}$ are boundary quark fields at $x_0 = 0$

$\Gamma$ is a suitable set of $\gamma$ matrices



## Example of correlation function

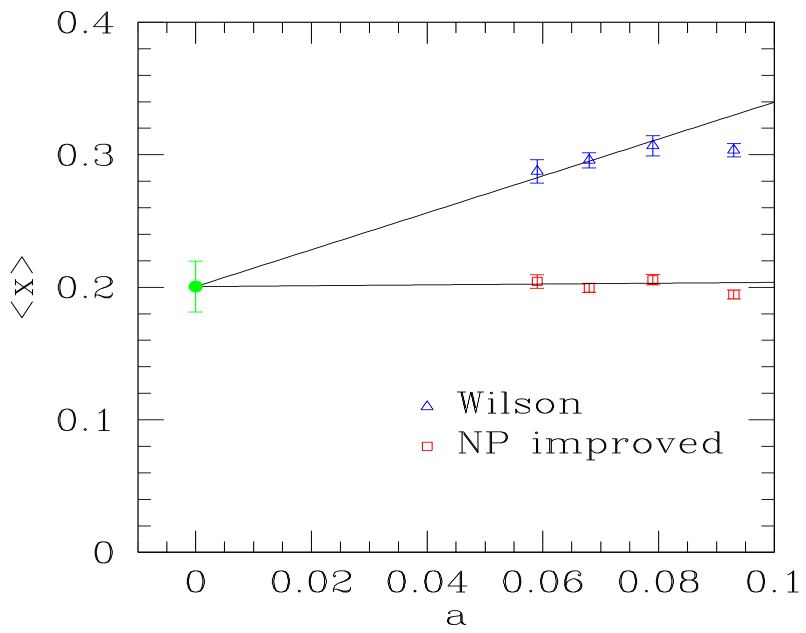



while sending $a \to 0$

- keep time extent $T = 3\text{fm}$ fixed
- keep time distance from boundary fixed to be $1\text{fm}$
  $\to$ sufficient to project on pion state
- keep plateau length for extracting the matrix element fixed to be $1\text{fm}$

## continuum limit of renormalized matrix element

- continuum limit while kepping $\mu_0^{-1} = L_0 = 0.7 \cdot r_0$ fixed $(r_0 = 0.5 fm)$
- use two action method to check for universal continuum limit




$$\langle x \rangle_{\mathrm{SF}}(\mu_0) = \lim_{a \to 0} \lim_{\mathrm{chiral}} \frac{\langle \pi | \mathcal{O}_{\mathrm{NS}}^{n=2} | \pi \rangle}{Z^{\mathrm{SF}}(1/\mu_0)} = 0.20(2)$$

- main result:
  *renormalization group invariant matrix element*

$$\boxed{O^{\mathrm{ren}}_{\mathrm{INV}} = O^{\mathrm{ren}}_{\mathrm{SF}}(\mu_0)\mathfrak{S}^{\mathrm{UV}}_{\mathrm{INV}}(\mu_0) = 0.20 \cdot 1.11}$$

$\uparrow$

$\mu_0$ dependence cancelled

- discretization errors:
  *sizable lattice artefacts*

  $\langle x\rangle(a = 0.093) = 0.30 \rightarrow \langle x\rangle(a = 0) = 0.20$

- renormalization:
  *non-perturbative renormalization* $\rightarrow$ 10-15% effect

- final numbers:
  putting it all together

$$\langle x\rangle^{\mathrm{experiment}}(\mu = 2.4\mathrm{GeV}) \quad = \quad 0.23 \pm 0.02$$

$$\langle x\rangle^{\mathrm{quenched}}_{\overline{\mathrm{MS}}}(\mu = 2.4\mathrm{GeV}) \quad = \quad 0.30(3)$$

## Where we stand (QUENCHED)

| obs. | cont. limit | non-pert. ren. | value |
|---|---|---|---|
| $\langle x \rangle_{\text{pion}}$ | X | X | $0.30(3)$ |
| $\langle x^2 \rangle_{\text{pion}}$ | – | – | $0.11(4)$ |
| $\langle x^3 \rangle_{\text{pion}}$ | – | – | $0.05(2)$ |
| $g_A$ | x | X | $1.14(4)$ |
| $\int_0^1 dx x^2 g_1(x)$ | – | X | $0.017(4)$ |
| $\int_0^1 dx x^2 g_2(x)$ | – | X | $-0.010(3)$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

$< x >$

Guagnelli, Jansen, Petronzio

Nucl.Phys. B542 (1999) 395, Phys.Lett. B493 (2000) 77

Jansen, hep-lat/0010038

QCDSF: (Göckeler, Horsley, Kürzinger, Pleiter, Rakow, Schäfer, Schierholz)

Phys.Rev. D63 (2001) 074506, Nucl.Phys.Proc.Suppl. 94 (2001) 299, hep-ph/9909253, Nucl.Phys.Proc.Suppl. 83 (2000) 232, Nucl.Phys.Proc.Suppl. 79 (1999) 548, hep-ph/9711245, Phys.Rev. D56 (1997) 2743

first unquenched adventure:

Dulgov et.al., Nucl.Phys.Proc.Suppl. 94 (2001) 303

## Where we *(want to)* go

assumption:

- continuum limit
- non-perturbative renormalization

| year | machines | errors | simulation |
|---|---|---|---|
| 2000 | 25 Gflops<br>APE100 | O(10%) | quenched |
| 2001 | 250 Gflops | O(1%) | quenched |
| 2001 | 250 Gflops<br>APEmille | first estimates | unquenched |
| 2003 | 2000 Gflops<br>apeNEXT<br>QCDOC | O(10%) | unquenched |
| $>$ 2006 | 20000 Gflops<br>??? | precision data | unquenched |

## Conclusion

★ modern lattice computations

$\rightarrow$ do not only want to have bigger computers

$\rightarrow$ work hard on algorithmic improvements

$\rightarrow$ incorporate theoretical progress:

- get rid of effects of finite physical boxlength $L$
  $\Leftarrow$ they use the finite extend of the box
  *Finite Size Scaling technique*

- continuum limit $a \rightarrow 0$
  $\leftarrow$ only acceleration of approach to the continuum limit
  $\leftarrow$ still sizable discretization errors

Future: *turn feasibility computations into physical results*

⋆ in the continuum

⋆ non-perturbatively renormalized

Future$^2$: do all that unquenched