Analytic progress
on
exact lattice chiral symmetry

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Three years since the re-discovery of the GW relation.
What are the remaining issues ?

A progress report of recent activities on theoretical issues

LATTICE '01, Berlin, August 19-24

Exact chiral symmetry on the lattice / G-W rel.

- GW (Dirac) fermions
  - Locality
  - Universality (Phase structure: weak vs. strong coupling)
    $\Longleftrightarrow$ Admissibility cond.
    ($\Longrightarrow$ Numerical applications)
  - Unitarity
  - Seek new Dirac operators
    with better local and chiral properties
    ($\Longrightarrow$ Numerical applications)
  - $\cdots$

- Construction of chiral gauge theories
  - Non-abelian chiral gauge theories
    - Classification of topological fields on the lattice
      $\Longleftrightarrow$ Admissibility cond.
    - Global anomalies
  - Practical construction ($\Longrightarrow$ Numerical tests)
  - $\cdots$

- Extensions of that nice idea to other fields
  - $\cdots$

Recent activities

- GW (Dirac) fermions
  - Locality   Horvath et al. [Non-Ultralocality]
  - Universality (Phase structure: weak vs. strong coupling)
    Brower et al. [Strong coupling exp.]
  - Unitarity
    Lüscher [Positivity]
    Creutz [Hamiltonian formalism]*
  - Seek new Dirac operators (new implementations) with better local and chiral properties
    Ishibashi-Fujikawa [Algebraic extension]*
    Holland et al. [Fixed point $\otimes$ Overlap]*
    Wenger et al. [Continued fraction expansion]*
    Fleming-Vranas [Staggered DWF]*
    Hofheinz [RG approach]*
  - Index theorem and GW rel.
    Yamada [RG approach]*,
    Chiu [Counter example]*
  - (Numerical) Applications
    Capitani et al [Perturbative renormalization]
    Alexandrou et al [Perturbative renormalization]
    Giusti et al [U(1) problem]
  - $\cdots$

- Construction of chiral gauge theories
  - (Non-) abelian chiral gauge theories
    Suzuki [Real rep.]
    Nakayama-Suzuki-YK [Electroweak $SU(2)\times U(1)$]*
  - Explicit construction ($\Longrightarrow$ Numerical tests)
    YK [Use of DWF]
  - $\cdots$

- Extensions of that nice idea to other fields
  - Odd dimensions and parity
    Bietenholz-Nishimura [GW fermions in odd dim.]
  - Application in the continuum theory
    So et al. [RG in the continuum and GW rel.]

In this talk

1. Chiral gauge theory with exact gauge invariance
   - Discrete formulation of integrability condition
   - Relation to Domain wall fermion

2. More on the G-W (Dirac) fermions
   - Unitarity
     - · · ·
   - Locality and Universality
   - Attempts to seek new Dirac operator
     - Algebraic extention of the GW rel.
     - · · ·
   - RG approach, the GW rel. and Index theorem
   - GW fermions in odd dimensions

Exact chiral symmetry on the lattice

the Ginsparg-Wilson relation:

Hasenfratz, Neuberger, Lüscher

$$\gamma_5 D + D\hat{\gamma}_5 = 0$$

$$\hat{\gamma}_5 \equiv \gamma_5 (1 - 2aD) \qquad \{\hat{\gamma}_5\}^2 = 1$$

admissibility condition:

$$\| 1 - U(p) \| < \epsilon \qquad \epsilon < \frac{1}{30}$$

$$U(p) = U(x, \mu) U(x + \hat{\mu}, \nu) U(x + \hat{\nu}, \mu)^{-1} U(x, \nu)^{-1}$$

$\Longrightarrow$ Local and differentiable with respect to gauge field

- overlap Dirac operator

  Neuberger, Neuberger & Narayanan
- rigorous proof of these properties

  Hernández, Jansen & Lüscher

Weyl fermion:

Hasenfratz & Niedermayer, Lüscher, Narayanan

$$\left(\frac{1 - \hat{\gamma}_5}{2}\right) \psi_L(x) = \psi_L(x), \quad \bar{\psi}_L(x) \left(\frac{1 + \gamma_5}{2}\right) = \bar{\psi}_L(x)$$

$\boxed{\text{Locality of } D}$ $\Rightarrow$ GW rel. $\Rightarrow$ Index of D

$$\gamma_5 D + D\gamma_5 = 2D\gamma_5 D$$

$\Downarrow$

$\searrow$

Index theorem on the lattice

$\Uparrow$

Local chiral anomaly

$$\sum_x -\mathrm{tr}\gamma_5 D(x,x)$$

$\searrow$ $\Uparrow$

Topological charge

$$Q = -\frac{1}{2}\sum_x \mathrm{tr}\gamma_5 D(x,x)$$

Admissibility cond.

$\|1 - U(p)\| < \epsilon$ $\Rightarrow$ $\boxed{\text{Topology of lattice gauge fields}}$

Laliena, Hasenfratz & Niedermayer

Narayanan & Neuberger (Topological charge)

Issues of constructing chiral gauge theories

Lüscher 1998, 1999

Path-Integral measure $\Longleftrightarrow$ Chiral basis $\{v_j(x)\}$

$$\psi_L(x) = \sum_j v_j(x)\, c_j, \quad \bar{\psi}_L(x) = \sum_k \bar{c}_k\, \bar{v}_k(x)$$

$$\hat{P}_- v_j(x) = v_j(x) \quad (j = \cdots), \quad \hat{P}_- = \left(\frac{1-\hat{\gamma}_5}{2}\right)$$

$$\mathcal{D}[\psi_L]\mathcal{D}[\bar{\psi}_L] \equiv \prod_i dc_i \prod_k d\bar{c}_k$$

- change of the chiral basis by a unitary transformation:

$$\tilde{v}_i(x) = v_j(x)\left(\mathcal{Q}^{-1}\right)_{ji}, \quad \tilde{c}_i = \mathcal{Q}_{ij} c_j$$

$$\mathcal{D}[\psi_L]\mathcal{D}[\bar{\psi}_L] \Longrightarrow \mathcal{D}[\psi_L]\mathcal{D}[\bar{\psi}_L]\, \det \mathcal{Q}$$

gauge-field dependent phase ambiguity

- to be fixed by requiring ...
  - Gauge invariance at finite lattice spacing
  - Local field equation
  - Smooth dependence on the gauge fields $\{U(x,\mu)\}$

- Gauge field dependence of the fermion measure

  Variation of effective action with respect to gauge field

$$\boldsymbol{\Gamma_{\mathrm{eff}} = \ln\det\left(\bar{v}_k, D v_j\right)}, \quad \boldsymbol{\delta_\eta U(x,\mu) = a\eta_\mu(x) U(x,\mu)}$$

$$\boldsymbol{\delta_\eta \Gamma_{\mathrm{eff}} = \mathrm{Tr}\left\{(\delta_\eta D)\hat{P}_- D^{-1} P_+\right\} + a^4 \sum_x v_j^\dagger(x)\delta_\eta v_j(x)}$$

- Measure term

  (Berry's connection   Neuberger 1998)

$$\mathsf{L}_\eta = \boldsymbol{i\sum_j (v_j, \delta_\eta v_j)}$$

$$\mathsf{L}_\eta \longrightarrow \mathsf{L}_\eta - \boldsymbol{i\delta_\eta \ln\det \mathcal{Q}}$$

  Geometrical interpretation:

$$\boldsymbol{v_j^a(x)} \in \boldsymbol{X}_a \qquad \boldsymbol{v_j^b(x)} \in \boldsymbol{X}_b$$

$$\boldsymbol{v_j^b(x) = \sum_l v_l^a(x) \mathcal{Q}_{lj}(a \rightarrow b)} \in \boldsymbol{X}_a \cap \boldsymbol{X}_b$$

$$\mathsf{V}_{ab} = \boldsymbol{\det \mathcal{Q}(a \rightarrow b)}$$

a U(1) bundle over the space of gauge fields

Smooth fermion measure $\Longleftrightarrow$ $\mathsf{V}_{ab} = 1 \in X_a \cap X_b$

Trivial U(1) bundle

- Measure term (U(1) bundle) as lattice U(1) gauge field

$$\cdots, \boldsymbol{U(x,\mu)}^{(n)}, \boldsymbol{U(x,\mu)}^{(n+1)}, \cdots$$

$$\mathsf{V}_{n,n+1} \equiv \frac{\mathbf{det}(\boldsymbol{v}_i^{(n+1)}, \boldsymbol{v}_j^{(n)})}{\left|\mathbf{det}(\boldsymbol{v}_i^{(n+1)}, \boldsymbol{v}_j^{(n)})\right|} \quad \in \ \boldsymbol{U(1)}$$

$$\mathsf{V}_{n,n+1} \longrightarrow \mathsf{G}_n \mathsf{V}_{n,n+1} \mathsf{G}_{n+1}^{-1} \quad \mathsf{G}_n = \mathbf{det}\, \boldsymbol{Q}^{(n)}$$

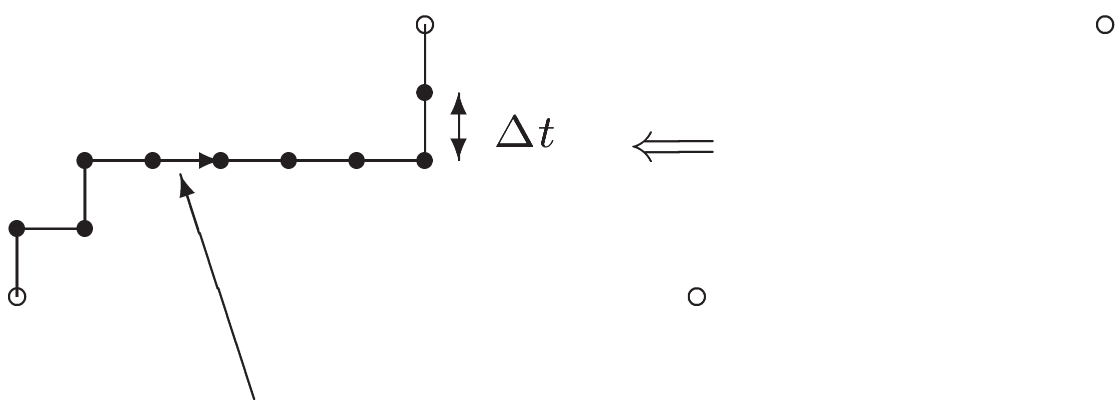
Global integrability condition

Change of phase of fermion measure
along smooth curve in the space of the gauge field

$$U_t(x,\mu) \qquad 0 \leq t \leq 1$$

$$W = \exp\left\{ i \int_0^1 dt\, \mathsf{L}_\eta \right\} \qquad \eta = \partial U_t\, U_t^{-1}$$

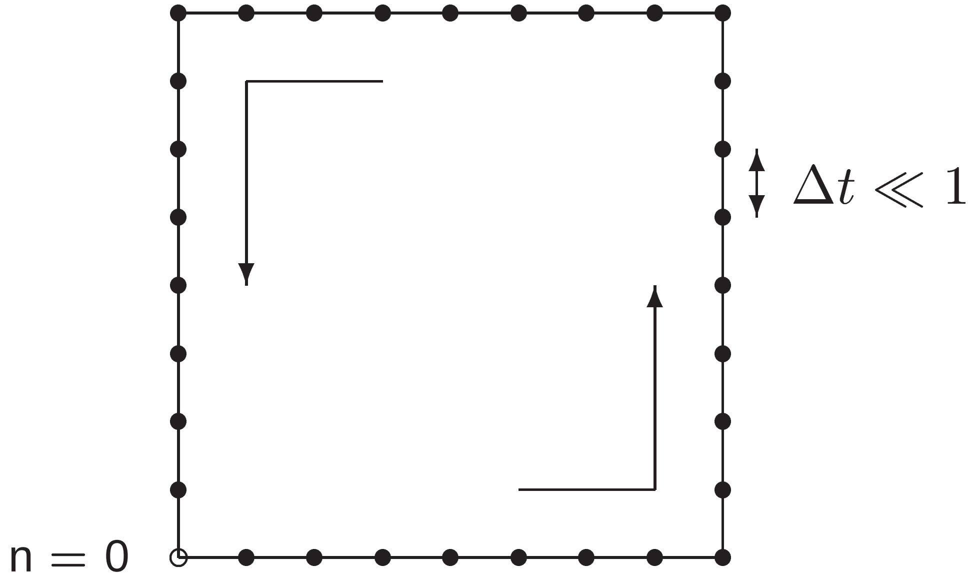
Discrete approach ! : $\quad t = n\Delta t \quad \Delta t \ll 1$



$$\mathsf{V}_{n,n+1} \equiv \frac{\det(v_i^{(n+1)}, v_j^{(n)})}{\left|\det(v_i^{(n+1)}, v_j^{(n)})\right|}$$

$$\mathsf{W} = \prod_{n=0}^{1/\Delta t - 1} \mathsf{V}_{n,n+1}$$

along a “smooth” closed curve

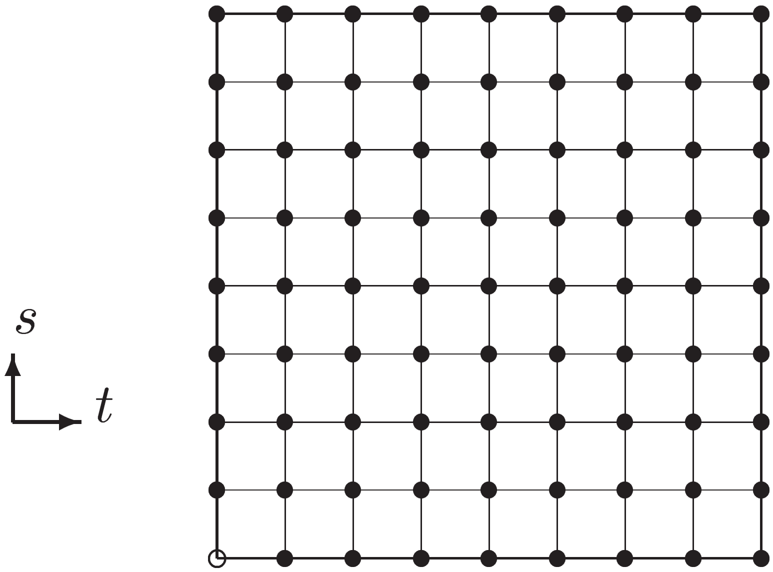


Global integrability condition

$$\mathrm{W} = \frac{\det\left(\mathbf{1} - \boldsymbol{P}^{(0)} + \boldsymbol{P}^{(0)} \prod_{n=1}^{1/\Delta t - 1} \boldsymbol{P}^{(n)}\right)}{\left|\det\left(\mathbf{1} - \boldsymbol{P}^{(0)} + \boldsymbol{P}^{(0)} \prod_{n=1}^{1/\Delta t - 1} \boldsymbol{P}^{(n)}\right)\right|}$$

$$\boldsymbol{P}^{(n)} = \hat{\boldsymbol{P}}_{-}^{(n)} = \sum_{j} \boldsymbol{v}_j^{(n)} \otimes \left(\boldsymbol{v}_j^{(n)}\right)^{\dagger}$$

$$\frac{\det(\boldsymbol{v}_i^{(n+2)}, \boldsymbol{v}_j^{(n+1)})}{\left|\det(\boldsymbol{v}_i^{(n+2)}, \boldsymbol{v}_j^{(n+1)})\right|} \frac{\det(\boldsymbol{v}_i^{(n+1)}, \boldsymbol{v}_j^{(n)})}{\left|\det(\boldsymbol{v}_i^{(n+1)}, \boldsymbol{v}_j^{(n)})\right|} = \frac{\det(\boldsymbol{v}_i^{(n+2)}, \boldsymbol{P}^{(n+1)}\boldsymbol{v}_j^{(n)})}{\left|\det(\boldsymbol{v}_i^{(n+2)}, \boldsymbol{P}^{(n+1)}\boldsymbol{v}_j^{(n)})\right|}$$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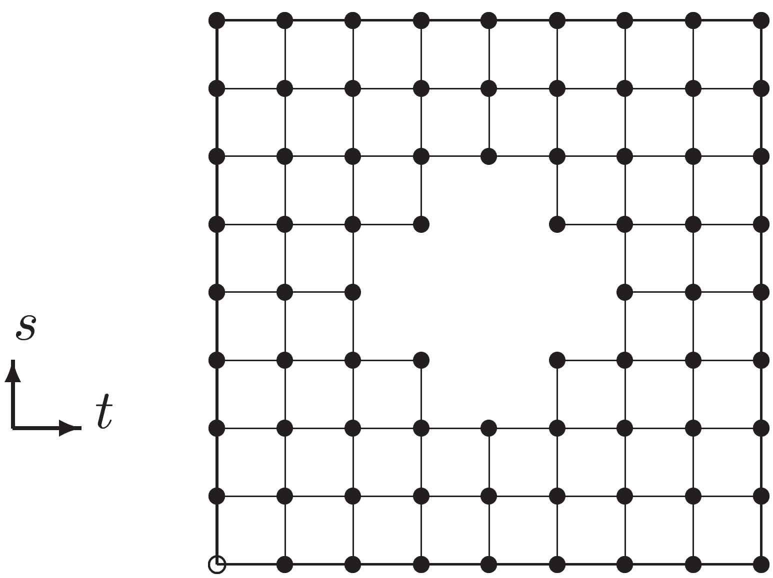
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Using Field strength (Plaquette variable):

$$\mathsf{U}(\boldsymbol{s},\boldsymbol{t}) = \mathsf{V}_t(\boldsymbol{s},\boldsymbol{t})\mathsf{V}_s(\boldsymbol{s},\boldsymbol{t}+\mathbf{1})\mathsf{V}_t(\boldsymbol{s}+\mathbf{1},\boldsymbol{t})^{-1}\mathsf{V}(\boldsymbol{s},\boldsymbol{t})^{-1}$$

$$\mathsf{W} = \prod_{s,t} \mathsf{U}(\boldsymbol{s},\boldsymbol{t}) \qquad \text{(contractible loop)}$$

Magnetic flux of U(1) lattice gauge field (Topological field)

$$\mathsf{F}_{s,t} = \frac{\mathbf{1}}{\mathbf{2}\boldsymbol{\pi i}} \ln \mathsf{U}(\boldsymbol{s},\boldsymbol{t}), \quad \sum_{s,t} \delta \mathsf{F}_{s,t} = \mathbf{0}$$

$\Longrightarrow$ topological field on 6 dim. lattice

$$\sum_{s,t} \mathsf{F}_{s,t} = \sum_{x,s,t} \mathsf{q}(\boldsymbol{x},\boldsymbol{s},\boldsymbol{t})$$

$$\mathsf{q}(\boldsymbol{x},\boldsymbol{s},\boldsymbol{t}) = \mathbf{ImLn}(\mathbf{1} - \boldsymbol{P}_{s,t} + \boldsymbol{P}_{s,t} \prod_{\square} \boldsymbol{P})(\boldsymbol{x},\boldsymbol{x})$$

$(\Delta t, \Delta s \to 0) \Longrightarrow$ Lüscher's topological field

on 4-dim. lattice + 2-dim. continuum space

1. Zero magnetic flux: $\sum_{s,t} \mathsf{F}_{s,t} = 0 \Longleftrightarrow$ No gauge anomaly

$$\mathsf{F}_{s,t} = \frac{1}{i} \ln \mathsf{U}(s,t) = \partial_s \mathsf{A}_t(s,t) - \partial_t \mathsf{A}_s(s,t)$$

2. Global integrability condition (non-contractible loops)

$$\prod_{n=0}^{1/\Delta t-1} e^{i\mathsf{A}_{n,n+1}} = \frac{\det\left(1-P^{(0)}+P^{(0)}\prod_{n=1}^{1/\Delta t-1} P^{(n)}\right)}{\left|\det\left(1-P^{(0)}+P^{(0)}\prod_{n=1}^{1/\Delta t-1} P^{(n)}\right)\right|}$$

$\Longrightarrow$

$$\mathsf{V}_{m,m+1} \times e^{-i\mathsf{A}_{m,m+1}} = \mathsf{H}_n \cdot \mathsf{H}_{n+1}^{-1}$$

Trivial U(1) gauge field

(Trivial U(1) bundle)

$\Longrightarrow$ Integrable & smooth measure

Fermion measure (basis) :

$$v_j^{(n)} = \begin{cases} w_1^{(n)} \dfrac{\det(w_k^{(n)}, \prod_{m=n-1}^{0} P^{(m)} w_l^{(0)})}{\left|\det(w_k^{(n)}, \prod_{m=n-1}^{0} P^{(m)} w_l^{(0)})\right|} e^{-i\sum_m \mathsf{A}_{m,m+1}} \\ w_j^{(n)} \qquad (j \neq 1) \end{cases}$$

- Interpolation from w_j^0 to $w_j^1 \Rightarrow$ phase change by Wilson line
- Add $\mathsf{A}_{m,m+1} \Rightarrow$ Path-independent measure

Integrability(Q1), Locality , Gauge invariance of $A_{m,m+1}$

$\Longrightarrow$ Local cohomology problem Lüscher

for topological field on 6-dim. space

$\Longleftarrow$ Require Gauge anomaly cancellation

- Need to explore the space of admissible lattice gauge fieds
- Need to go “higer dimensions”

Domain wall fermion for chiral gauge theory

Domain wall fermion   Kaplan, Shamir-Furman

$\boldsymbol{a = a_5 = 1, \; t \in [-N+1, N]}$

$$S_{\rm DW} = \sum_{t,x} \bar\psi(x,t)\,(D_{5\rm w} - m_0)\,\psi(x,t)$$

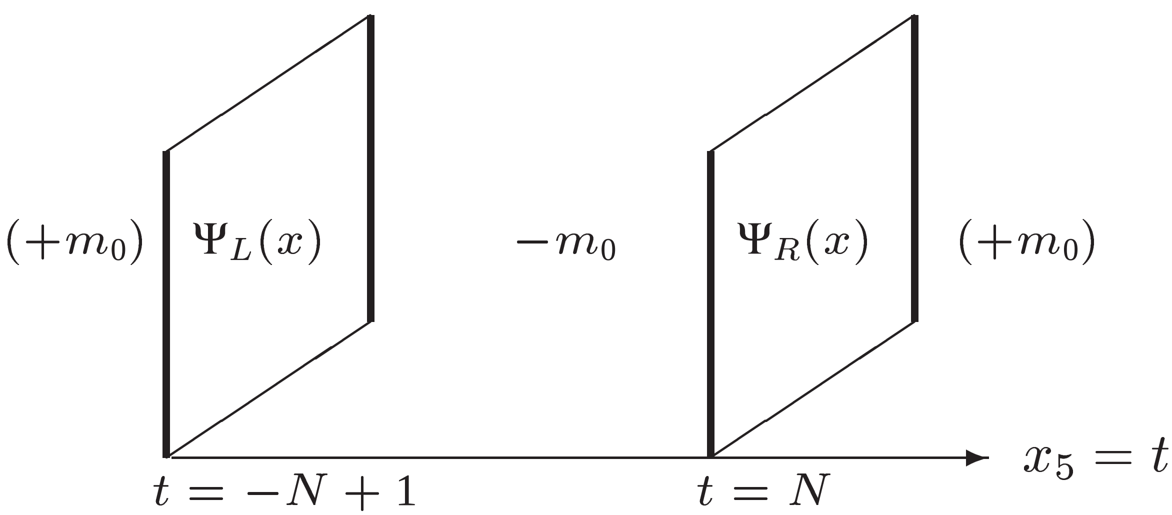
or the Dirichlet boundary condition

$$\psi_R(x,t)\,|_{t=-N} = 0, \qquad \psi_L(x,t)\,|_{t=(N+1)} = 0$$

single massless Dirac fermion in the limit $N \to \infty$!

- field variables on the walls:

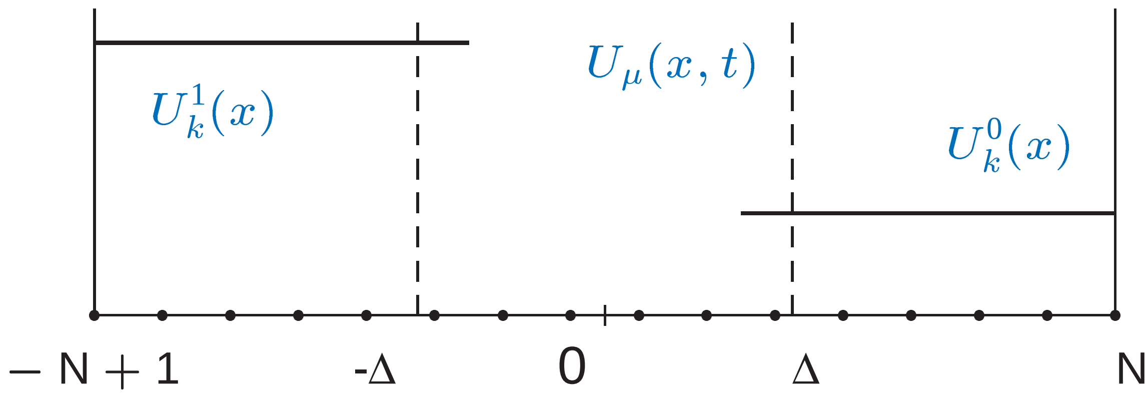
$$q(x) \equiv \psi_L\,(x, -N+1) + \psi_R\,(x, N)$$



- For vector-like theories:

$$U_k(x,t) = U_k(x)\;(k = 1,2,3,4), \quad U_5(x,t) = 1$$

- **Chiral gauge coupling by five-dimensional gauge field**



Take the limit $N \to \infty$ with Δ kept finite $(N \gg \Delta \simeq 1)$

5-dim. admissibility condition:

$$\|1 - P_{\mu\nu}(z)\| < \epsilon' \qquad \epsilon' < \frac{1}{50}\left\{1 - |1 - m_0|^2\right\}$$

$\Longrightarrow$ Smooth interpolation
$\Longrightarrow$ Locality of Chern-Simons current
cf. Golterman-Jansen-Kaplan 1993; Jansen 1996

Chiral coupling $\Longrightarrow$ 5-dim. dependence
How to obtain 4-dim. theory ? (dimensional reduction ?)
cf. Overlap $\Leftarrow$ Generalized PV (Slavnov-Frolov)
Waveguide, Creutz et al

Analysis of the path-dependence in DWF
$\Longleftrightarrow$ Analysis of the integrability condition

- Path-dependence of DWF partition function

$$\ln \det \left(D_{5\mathrm{w}} - m_0\right)\Big|_{\mathrm{Dir.}}^{c_2} - \ln \det \left(D_{5\mathrm{w}} - m_0\right)\Big|_{\mathrm{Dir.}}^{c_1}$$

Variation within $t \in [-\Delta, \Delta]$

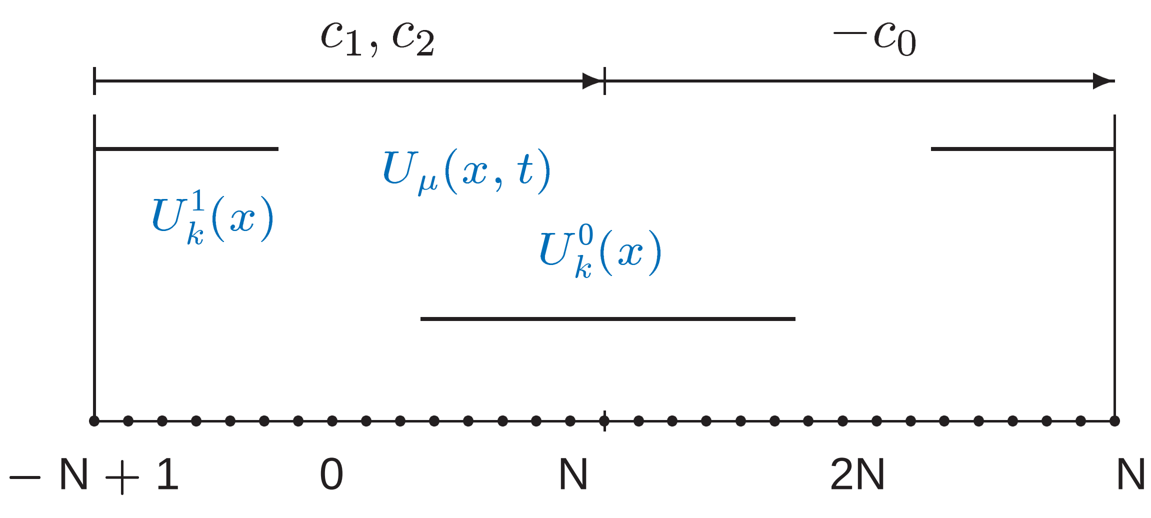
$$\delta \ln \det \left(D_{5\mathrm{w}} - m_0\right)\Big|_{\mathrm{Dir.}}^{c} = \sum_{x,t} \eta_\mu^a(z) J_\mu^a(z)\Big|_{\mathrm{Dir.}}$$

- In the limit $N \to \infty$

$$J_\mu^a(z)\big|_{\mathrm{Dir.}} \longrightarrow J_\mu^a(z) \qquad t \in [-\Delta, \Delta]$$

Independent of boundary conditions

- 5-dim. Wilson-Dirac fermion with anti-periodic b.c.
$t \in [-N+1, 3N]$



Variation within $t \in [-\Delta, \Delta]$

$$\delta \ln \det \left(D_{5\mathrm{w}} - m_0\right)\Big|_{\mathrm{AP}}^{c-c_0} = \sum_{x,t} \eta_\mu^a(z) J_\mu^a(z)\Big|_{\mathrm{AP}}$$

- Integrability condition for DWF ($N \to \infty$)

$$\ln\det(D_{5\mathrm{w}} - m_0)|^{c_2}_{\mathrm{Dir.}} - \ln\det(D_{5\mathrm{w}} - m_0)|^{c_1}_{\mathrm{Dir.}}$$

$$= \ln\det(D_{5\mathrm{w}} - m_0)|^{c_2-c_0}_{\mathrm{AP}} - \ln\det(D_{5\mathrm{w}} - m_0)|^{c_1-c_0}_{\mathrm{AP}}$$

$$= \frac{1}{2}\ln\det(D_{5\mathrm{w}} - m_0)|^{c_2-c_2}_{\mathrm{AP}} - \frac{1}{2}\ln\det(D_{5\mathrm{w}} - m_0)|^{c_1-c_1}_{\mathrm{AP}}$$

$$+i\,\mathrm{Im}\ln\det(D_{5\mathrm{w}} - m_0)|^{c_2-c_1}_{\mathrm{AP}}$$

$\Longrightarrow$ Chern-Simons term So, Coste-Lüscher

$$Q^{c_2-c_1}_{5\mathrm{w}} \equiv \lim_{N\to\infty} \mathrm{Im}\ln\det(D_{5\mathrm{w}} - m_0)|^{c_2-c_1}_{\mathrm{AP}}$$

for the loop $c_2 - c_1$

Path-dependence governed by lattice Chern-Simons term !

Refection : reversed interpolation = complex conjugation

$P : t \longrightarrow t' = -t + 1$

$$U_\mu(x,t) \longrightarrow U'_\mu(x,t) = \begin{cases} U'_k(x,t) = U_k(x,-t+1) \\ U'_5(x,t) = U_5(x,-t+1)^{-1} \end{cases}$$

$$D_{5\mathrm{w}} \longrightarrow D'_{5\mathrm{w}} = P\gamma_5 D^\dagger_{5\mathrm{w}}\gamma_5 P$$

- *If, for all possible loops $\boldsymbol{l}$ in the space of gauge fields,*

$$\boldsymbol{Q}^{l}_{5\mathrm{w}} = \lim_{N\to\infty} \sum_{t=-N+1}^{3N} \sum_{x} \boldsymbol{c}_{5\mathrm{w}}(\boldsymbol{x},\boldsymbol{t})$$

where $\boldsymbol{c}_{5\mathrm{w}}(\boldsymbol{x},\boldsymbol{t})$ is smooth, local, and gauge-invariant

1. *smooth, local and gauge-invariant counter term for each path $\boldsymbol{c}$:* $\quad \boldsymbol{l} \rightarrow \boldsymbol{c}_2 + (-\boldsymbol{c}_1)$

$$\boldsymbol{Q}^{c_2+(-c_1)}_{5\mathrm{w}} = \boldsymbol{C}^{c_2}_{5\mathrm{w}} - \boldsymbol{C}^{c_1}_{5\mathrm{w}}$$

2. *path-independent partition function ($N \to \infty$)*

$$\frac{\det\left(\boldsymbol{D}_{5\mathrm{w}} - \boldsymbol{m}_0\right)\Big|^{c}_{\mathrm{Dir.}}}{\left|\det\left(\boldsymbol{D}_{5\mathrm{w}} - \boldsymbol{m}_0\right)\Big|^{c+(-c)}_{\mathrm{AP}}\right|^{\frac{1}{2}}} \exp\left\{-\boldsymbol{i}\boldsymbol{C}^{c}_{5\mathrm{w}}\right\}$$

depends only on U^0_k and U^1_k $\Rightarrow$ reduction to 4-dim.
locality and gauge-invariance are OK

- local cohomology problem $\boldsymbol{z = (x, t), s \in [0, 1]}$

$$\boldsymbol{\delta_\eta Q^l_{\mathrm{5w}} = \sum_z \eta^a_\mu(z) J^a_\mu(z)}$$ local CS current

One-parameter family of $\boldsymbol{U_\mu(z)}$ $s \in [0,1]$

$$\boldsymbol{U_\mu(z, s), A_6(z, s); \quad \eta^a_\mu(z; s) = \left\{ D_s U_\mu(z, s) U^{-1}_\mu(z, s) \right\}^a}$$

$$\boldsymbol{q(z, s) = \eta^a_\mu(z; s) J^a_\mu(z; s), \quad \sum_z \int ds\, \delta q(z) = 0}$$

$$\boldsymbol{q(z, s) = \eta^a_\mu(z; s) J^a_\mu(z; s) \stackrel{?}{=} \partial_\mu K_\mu(z, s) + \partial_s K_6(z, s)}$$

$$\boldsymbol{\Longrightarrow K_6(z, s = 1) = c_{\mathrm{5w}}(z)}$$

- Effective action and fermion measure
  Using Master formula (Neuberger 1997)

$$T_t = \exp(-H_t)$$

$$\det\left(D_{5\mathrm{w}} - m_0\right)\big|_{\mathrm{Dir.}} = \det\left(1 - P_L + P_L \prod_{t=-N+1}^{N} T_t\right) \beta^{(N)}$$
$$\det\left(D_{5\mathrm{w}} - m_0\right)\big|_{\mathrm{AP}} = \det\left(1 + \prod_{t=-N+1}^{3N} T_t\right) \beta^{(3N)}$$

$$\beta^{(M)} = \prod_{t=-N+1}^{M} \alpha_t$$

($N \to \infty$)
Effective action:

$$\frac{\det\left(D_{5\mathrm{w}}-m_0\right)\big|_{\mathrm{Dir.}}^{c_1}}{\left|\det\left(D_{5\mathrm{w}}-m_0\right)\big|_{\mathrm{AP}}^{c_1+(-c_1)}\right|^{\frac{1}{2}}} e^{-iC_{5\mathrm{w}}(c_1)} = \det\left(\bar{v}_k, D^1 v_j^1\right) \det\left(\bar{v}_k, D^0 w_j^0\right)^*$$

cf. Alvarez-Gaumé et al (η-invariant)

Fermion measure (basis):

$$v_j^1(x) = \begin{cases} w_j^1(x)\; e^{i\phi(c_1)}\, e^{-iC_{5\mathrm{w}}(c_1)} & (j=1) \\ w_j^1(x) & (j \neq 1) \end{cases}$$

$$e^{i\phi(c_1)} = \frac{\det\left(w_i^1, \left\{U_{5,\Delta}^{-1} \prod_{t=-\Delta+1}^{\Delta} T_t U_{5,t-1}^{-1}\right\} w_j^0\right)}{\left|\det\left(w_i^1, \left\{U_{5,\Delta}^{-1} \prod_{t=-\Delta+1}^{\Delta} T_t U_{5,t-1}^{-1}\right\} w_j^0\right)\right|}$$

cf. WB phase choice

where

$$D = \frac{1}{2}\left(1 + \gamma_5 \frac{H}{\sqrt{H^2}}\right) \qquad \hat{P}_- = \frac{1}{2}\left(1 + \frac{H}{\sqrt{H^2}}\right)$$

- Chern-Simons term

$$\boldsymbol{T}^{N'} \propto \hat{\boldsymbol{P}}_{-} \quad (\boldsymbol{N}' \to \infty)$$

$$\begin{aligned} \exp\{i\boldsymbol{Q}_{5\mathrm{w}}\} \quad &\equiv \quad \frac{\mathbf{det}\left(\mathbf{1} + \prod_{t=-N+1}^{3N} \boldsymbol{T}_t\right)}{\left|\mathbf{det}\left(\mathbf{1} + \prod_{t=-N+1}^{3N} \boldsymbol{T}_t\right)\right|} \\ &\Longrightarrow \quad \frac{\mathbf{det}\left(\mathbf{1} - \boldsymbol{P}^{(0)} + \boldsymbol{P}^{(0)} \prod_{n=1}^{1/\Delta t-1} \boldsymbol{P}^{(n)}\right)}{\left|\mathbf{det}\left(\mathbf{1} - \boldsymbol{P}^{(0)} + \boldsymbol{P}^{(0)} \prod_{n=1}^{1/\Delta t-1} \boldsymbol{P}^{(n)}\right)\right|} \end{aligned}$$

in the Global integrability condition

Summary I

1. Descent relation through 6-dim, 5-dim, and 4-dim. lattices

$\boxed{\text{6 dim. lattice}}$		
Magnetic flux		
$\Downarrow$		
$\boxed{\text{5 dim. lattice}}$		Chiral DWF
Wilson loop	$\Longleftrightarrow$	Chern-Simons term from Wilson fermion
$\Downarrow$		
$\boxed{\text{4 dim. lattice}}$		$\boxed{\text{Weyl fermion / Overlap } D}$
Measure term as U(1) lattice gauge field		

2. Chiral DWF

   Formula of low energy effective action

   $\Longrightarrow$ fermion measure (basis)

3. Hints for practical implementation & theoretical analysis
   - abelian chiral gauge theory — Lüscher

     $\Longrightarrow$ Explicit construction (numerically) possible !?
     - 2D, 4D models

       cf. Narayanan, Neuberger & YK 1997

       cf. Bornyakov, Hoferichter & Schierholz 2000
   - analysis of non-abelian anomaly
     - Classification of topological fields on 6-dim. lattice
     - Global anomalies (for existence proof ?)

Unitarity

- Free Overlap Dirac fermion Lüscher 2000

  Spectral representation of free fermion propagator

$$aD = 1 - A(A^\dagger A)^{-1/2},\ A = 1 - aD_{\mathrm{w}}$$

$$D_{\mathrm{w}} = \frac{1}{2}\left\{\gamma_\mu\left(\partial_\mu^* + \partial_\mu\right) - a\partial_\mu^*\partial_\mu\right\}$$

$$\left\langle \psi(x)\bar{\psi}(y)\right\rangle_{x_0>y_0}$$

$$= \int_0^\infty dE \int_{-\pi/a}^{\pi/a} \frac{d^3\mathbf{p}}{(2\pi)^3}\,\rho(E,\mathbf{p})\,\mathrm{e}^{-E(x_0-y_0)+i\mathbf{p}(\mathbf{x}-\mathbf{y})}$$

such that

$$dE d^3\mathbf{p}\,\zeta^\dagger\,\rho(E,\mathbf{p})\,\zeta \geq 0$$

for all complex Dirac spinors.

Unitarity is OK for any value of a

$$\begin{aligned}\rho(\mathrm{E},\mathbf{p}) \quad = \quad & (\gamma_0 \sinh \mathrm{E} - \mathrm{i}\gamma_\mathrm{k}\bar{\mathrm{p}}) \\ & \times \left\{\delta(\mathrm{E}-\omega_\mathbf{p})\theta\left(\cosh \mathrm{E} - \frac{1}{2}\hat{\mathbf{p}}^2\right)\frac{\cosh \mathrm{E} - \frac{1}{2}\hat{\mathbf{p}}^2}{\sinh 2\mathrm{E}}\right. \\ & \left. + \frac{1}{2\pi}\theta(\mathrm{E}-\mathrm{E}_\mathbf{p})\frac{\left\{\hat{\mathbf{p}}^2\left(\cosh \mathrm{E} - \cosh \mathrm{E}_\mathbf{p}\right)\right\}^{1/2}}{\hat{\mathbf{p}}^2\left(\cosh \mathrm{E} - \cosh \mathrm{E}_\mathbf{p}\right) + \left(\cosh \mathrm{E} - \frac{1}{2}\hat{\mathbf{p}}^2\right)^2}\right\}\end{aligned}$$

Algebraic extension of the GW rel.

Fujikawa & Ishibashi, Chiu

$$\gamma_5 (\gamma_5 D) + (\gamma_5 D) \gamma_5 = 2a^{2k+1} (\gamma_5 D)^{2k+2}$$

$$\Gamma_5 \equiv \gamma_5 - (\gamma_5 a D)^{2k+2}$$

$$\Gamma_5 (\gamma_5 D) + (\gamma_5 D) \Gamma_5 = 0$$

- Index relation :

$$\mathrm{Tr}\Gamma_5 = n_+ - n_-$$

Eigenvalues and Eigenvectors of $\gamma_5 D$

1. Zero modes ($n_\pm$)

$$(\gamma_5 D) \phi_n = 0, \quad \gamma_5 \phi_n = \pm \phi_n$$

2. "Highest states" ($N_\pm$)

$$(\gamma_5 D) \phi_n = \pm \frac{1}{a} \phi_n, \quad \gamma_5 \phi_n = \pm \phi_n$$

$$\Gamma_5 \phi_n = 0$$

3. "Paired states"

$$0 < |\lambda_n| < \frac{1}{a}$$

$$(\gamma_5 D) \phi_n = \lambda_n \phi_n, \quad (\gamma_5 D) \Gamma_5 \phi_n = -\lambda_n \Gamma_5 \phi_n$$

- Explicit construction of solutions

$$H\gamma_5 + \gamma_5 H = 2H^{2k+2}, \quad H \equiv a\gamma_5 D$$

$$\gamma_5 H^2 = [\gamma_5 H + H\gamma_5]\, H - H\, [\gamma_5 H + H\gamma_5] + H^2\gamma_5 = H^2\gamma_5$$

$\Longrightarrow$

$$H^{2k+1}\gamma_5 + \gamma_5 H^{2k+1} = 2H^{2(2k+1)}$$
$$\gamma_5 H^2 - H^2\gamma_5 = 0$$

$H^{2k+1} \equiv H_{(2k+1)}$ satisfies the original GW rel. !
$\Rightarrow$ overlap construction

$$X_{(2k+1)} \equiv i(\gamma_\mu C_\mu)^{2k+1} + (B)^{2k+1} - \left(\frac{m_0}{a}\right)^{2k+1}$$
$$H_{(2k+1)} \equiv \gamma_5 \frac{1}{2}\left(1 + X_{(2k+1)}\left(X^\dagger_{(2k+1)} X_{(2k+1)}\right)^{-1/2}\right)$$

$$H = \left(H_{(2k+1)}\right)^{1/(2k+1)}$$

in the basis where $H_{(2k+1)}$ is diagonal

- Consistency checked
  - No species doublers
  - Chiral anomaly
  - Locality
    - Analytic in free theory
    - Locality of $\Gamma_5 \equiv \gamma_5 - (\gamma_5 a D)^{2k+2}$
    - Estimation of localization length $\propto$ 2k+1
  - Spectrum T.W. Chiu

GW fermions in odd dimensions

- GW relation in odd dimensions

Bietenholz & Nishimura 2000

3D Dirac fermion (two components)

$$S = a^3 \sum_x \bar{\psi}(x) D^{(3)} \psi(x)$$

GW rel. in 3D

$$D^{(3)} + D^{(3)\dagger} = a D^{(3)\dagger} D^{(3)}$$

or

$$a D^{(3)} = 1 - V, \qquad V^\dagger V = 1$$

$$\mathcal{R} : x \mapsto -x, \qquad U^P(x,\mu) = U(-x,\mu)^\dagger$$

$$D^{(3)}(U)^\dagger = \mathcal{R} D^{(3)}(U^P) \mathcal{R}$$

Action is invariant under the following parity transformation

$$U(x,\mu) \mapsto U^P(x,\mu)$$
$$\psi(x) \mapsto i \mathcal{R} V \psi(x)$$
$$\bar{\psi}(x) \mapsto i \bar{\psi}(x) \mathcal{R}$$

Fermion measure transforms non-trivially $\Rightarrow$ Parity anomaly

$$d\psi d\bar{\psi} \mapsto (\det V)^{-1} d\psi d\bar{\psi}$$

- Relation to 4D GW fermion

  Through dimensional reduction
  4D Dirac fermion $\Rightarrow$ a *doublet* of 3D Dirac fermion

  4D GW fermion $\Rightarrow$ a *doublet* of 3D GW fermion
  $How?$

- Overlap formalism in odd dimensions

Narayanan & Nishimura, Neuberger & YK

"In odd dimensions, a phase choice is possible so that the overlap formula is given by the determinant of a gauge-covariant operator D*"*

Neuberger & YK 1997

$$\boldsymbol{D = \frac{1}{2a}\left(1 + X(X^\dagger X)^{-1/2}\right)}$$

In 3D

$$\boldsymbol{X = D_{\mathrm{w}}^{(3)} - m_0/a \quad (0 < m_0 < 2)}$$
$$\boldsymbol{D_{\mathrm{w}}^{(3)} = \tfrac{1}{2}\sum_{\mu=1}^{3}\left\{\sigma_\mu(\nabla_\mu + \nabla_\mu^*) - a\nabla_\mu^*\nabla_\mu\right\}}$$

This gives a solution of the GW rel. in 3D

Parity anomaly:

$$\boldsymbol{\det V = \det\left(X(X^\dagger X)^{-1/2}\right) = \frac{\det(D_{\mathrm{w}} - m_0/a)}{|\det(D_{\mathrm{w}} - m_0/a)|}}$$

In the continuum limit, the Chern-Simons term is induced from Wilson-Dirac fermion with a negative mass

So, Coste-Lüscher

$\Leftarrow$ a natural definition of lattice CS term

Bietenholz & Nishimura

- Universality

  Heavy Dirac fermions in odd dimensions do not decouple

  Remnant as a CS term

Coefficient of the induced CS term

$$c_0 = c_\infty + \pi \qquad c_\infty = 2\pi n$$

n lavels the universality classes Coste-Lüscher

By choosing $D^{(3)}$ as

$$X = \left(D_{\rm w}^{(3)} - 1/a\right)\left(1 - 2aD_{\rm w}^{(3)}\right)^{2n}$$

$$2R = \frac{1}{4n+1}\delta_{x,y}$$

Cover all possible universality classes

Summary II

1. Unitarity
   QCD with Overlap Dirac fermions: OK
   $\Longleftarrow$ Domain wall fermion (Wilson fermion)

   - Hamiltonian formalism
   - Chiral gauge theories ?

2. Seek new Dirac operators
   with better local and chiral properties
   ($\Longrightarrow$ Numerical applications)

   Algebraic extension of GW rel.
   $\Longrightarrow$ Better chiral (scaling) properties

   - How to improve locality property in general ?
   - $\cdots$

3. GW fermions in odd dimensions
   Parity transformation
   Parity anomaly $\Longrightarrow$ CS term
   Non-universality

Conclusions

Construction of chiral gauge theories:

- Domain wall fermion can be used to construct chiral gauge theories
- It realizes the Descent relation over 6, 5, 4-dim. lattices
- Remaining issues
  - Classification of topological fields on the lattice
  - Global anomalies
  - $\cdots$

GW (Dirac) fermion:

- In QCD with Overlap Dirac fermions, unitarity is OK (?!)
- GW relation is also useful in odd dimensions
- new series of Dirac operators with better chiral properties
  (How to imporove locality in general ?)
  ($\Longrightarrow$ Numerical applications)