Lattice Matter

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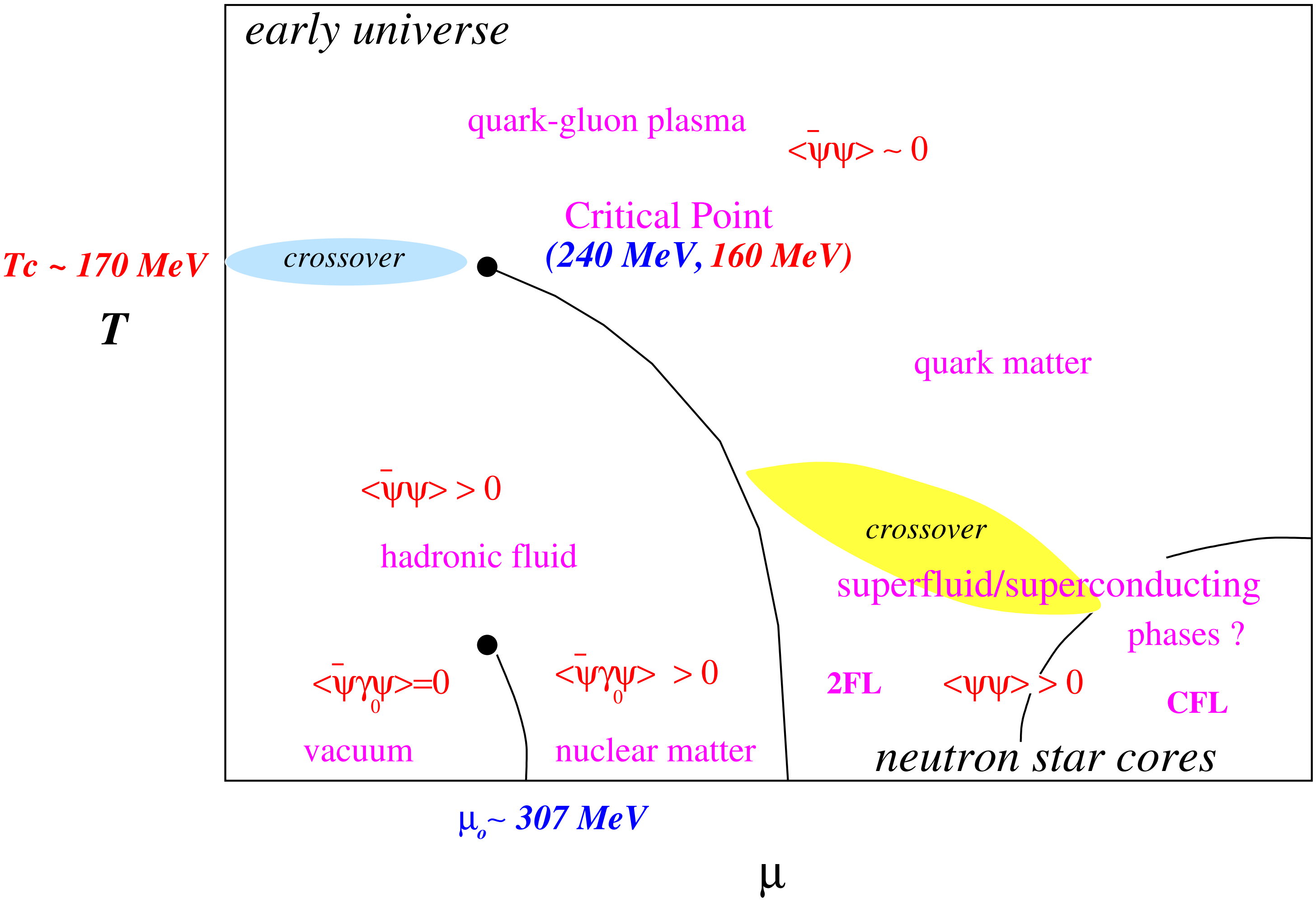
- Why $\mu \neq 0$ is difficult
- Progress at $T > 0$
- Two Color QCD
- Flatland NJL
- A conjecture about superconductivity



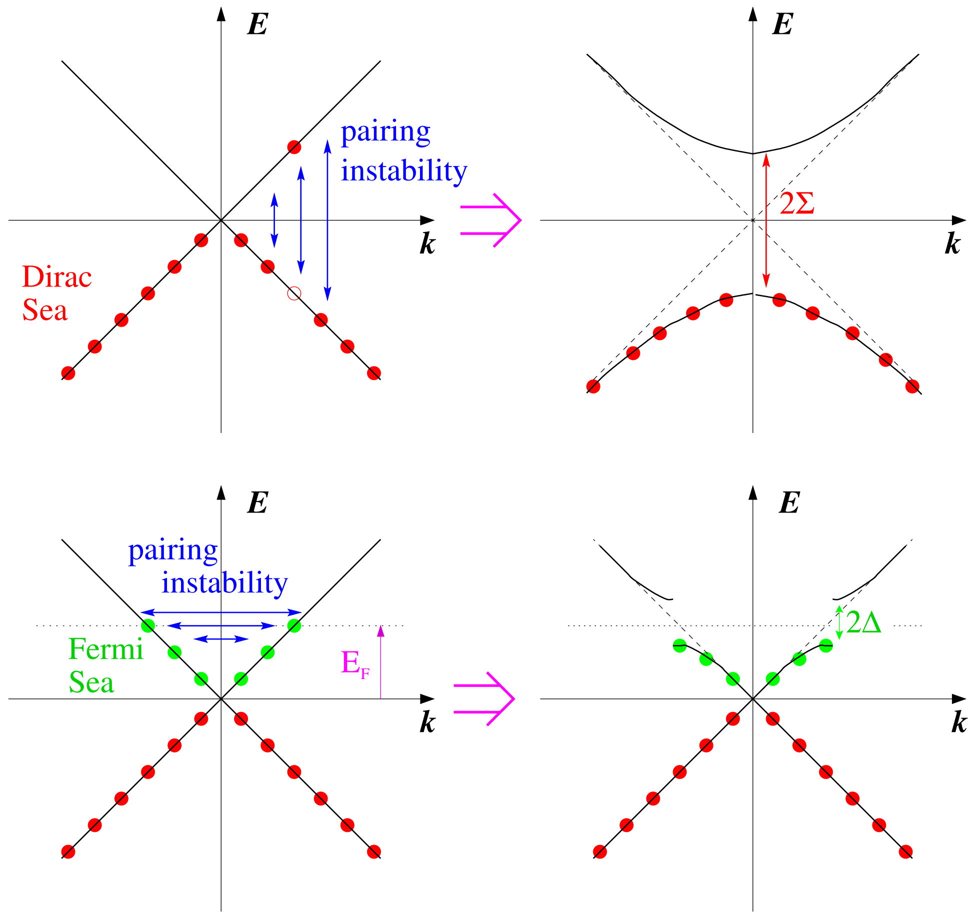
A possible QCD phase diagram

The equilibrium state minimises

$$\Omega(\mu, T) = E - TS - 3\mu N_B$$

This talk will focus on attempts to explore $\mu \neq 0$

Pairing Instabilities



Comparison between $\bar{q}q$ chiral condensation leading to a constituent quark mass Σ and non-conservation of axial charge (top) ...

... and qq diquark condensation leading to a BCS gap Δ implying superconductivity and/or superfluidity (bottom)

In QCD $\Sigma \simeq O(300)$MeV;
Model estimates suggest $\Delta \simeq O(100)$MeV.

[Berges & Rajagopal]

Why is it so difficult to simulate $\mu \neq 0$?

For a vectorlike gauge theory with fermions

$$\not{D}(\mu) \equiv \not{D}_0 + \mu\gamma_0 = \gamma_5 \not{D}^\dagger(-\mu)\gamma_5$$

implies eigenvalues of $\not{D}$ are not pure imaginary and hence not related by complex conjugation:

$$\mathrm{det}M(\mu) \neq \mathrm{det}M^*(\mu) = \mathrm{det}M(-\mu)$$

$\therefore$ the Euclidean functional measure is not positive definite and can't be used for importance sampling

"The Sign Problem"

$\Rightarrow$An exponentially large number of terms must be sampled

This situation is generic in quantum treatments of many-body systems.

Why is vacuum QCD so easy?

Two routes forward ...

Analytic Continuation from $\mu = 0$

• Taylor expansion – finite radius of convergence and no prospect of reaching critical point [QCDTARO;Ejiri]

• Reweighting – can go critical but problems with reaching thermodynamic limit [Barbour et al;Fodor & Katz]

Both approaches most effective for $T \neq 0$

Real Measure $\det MM^* = \det M(\mu)M(-\mu)$

This introduces *conjugate quarks* q^c carrying +ve baryon number in the conjugate representation of the gauge group. Possibility of light qq^c bound states radically altering the physics - eg. onset of nuclear matter at $\mu_o \approx m_\pi/2$, not $\Sigma \approx m_N/3$ [Goksch;Stephanov]

• Two Color QCD – baryonic Goldstones a feature, not a problem [Dagotto,Moreo,Wolff]

• QCD with non-zero isospin $\mu_I = \mu_u - \mu_d$ – leading to pion condensation [Son & Stephanov]

• NJL Model – qq^c states don't couple to the Goldstone mode [Barbour,SJH,Kogut,Lombardo & Morrison]

All three systems potentially superfluid

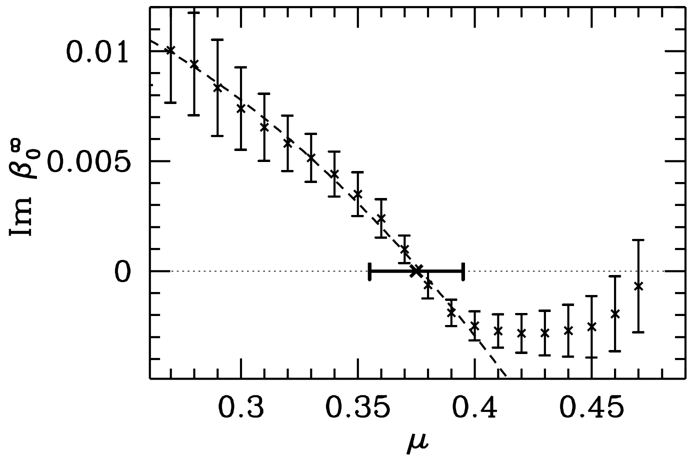
($\langle qq \rangle$ gauge invariant)

Progress in Reweighting

[Fodor & Katz]

$$Z[\alpha] = \int DU\, \mathrm{exp}(-S_{bos}[U;\alpha_0])\mathrm{det}M[U;\alpha_0] \times \left\{ \mathrm{exp}(-\Delta S_{bos}[U;\alpha,\alpha_0]) \frac{\mathrm{det}M[U;\alpha]}{\mathrm{det}M[U;\alpha_0]} \right\}$$

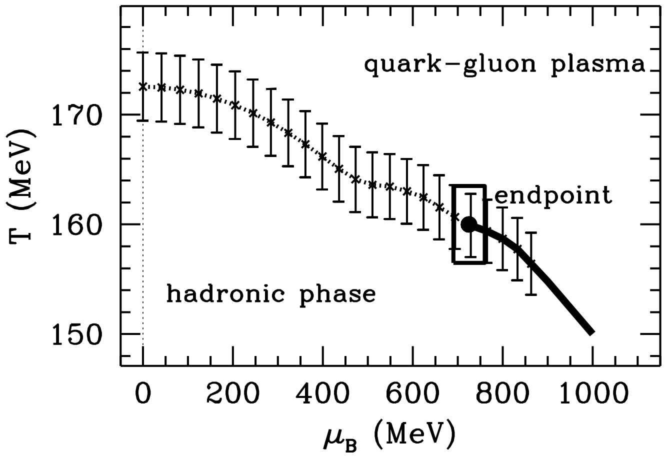
where the parameter set $\alpha = \{\beta, m, \mu\}$
Reweighting needs small $\Delta\alpha$ to maintain good overlap between trial and true ensembles. Effective along transition line with good overlap in both hadronic and QGP phases
$\Rightarrow$ reweight using both $\Delta\mu$ and $\Delta\beta$



Determine order of transition via volume scaling of lowest Lee-Yang zero β_0

$$\lim_{V\to\infty} \mathrm{Im}(\beta_0) \begin{cases} = 0 & \text{1st order} \\ \neq 0 & \text{2nd order} \end{cases}$$

Critical Point



$$N_f = 2+1,\ 4^3, 6^3, 8^3 \times 4,\ m_{u,d} = 0.025\ m_s = 0.2$$

Light quarks $\approx 4\times$ physical values

$$T_E = 160(4)\text{MeV}$$

$$\mu_E = 242(12)\text{MeV}$$

Taylor Expansion @ $\mu = 0$

- Quark number susceptibilities $\chi_{ij} = \frac{1}{V}\frac{\partial^2 \ln Z}{\partial\mu_i \partial\mu_j}$

[Gavai & Gupta]

$$\chi \begin{cases} \approx 0 & T < T_c \\ > 0 & T > T_c \end{cases} \qquad \chi_s \nearrow \chi_{u,d} \text{ for } T \gtrsim 2T_c$$

Results lie below free field prediction

- Response of screening masses [QCD-TARO]

$$\frac{\partial^2 M_\pi}{\partial \mu_B^2} > 0 \qquad \frac{\partial^2 M_\pi}{\partial \mu_I^2} < 0$$

large in QGP phase
pion no longer Goldstone

large in hadronic phase
pion condensation?

- Critical line $T_c(\mu) = T_c(0) + \frac{\mu^2}{2}\frac{\partial^2 T_c}{\partial \mu^2} + \cdots$ [Ejiri]

For $T \approx T_c$ sign problem under control on $16^3 \times 4$ for $\mu \lesssim 70\text{MeV}$, which includes **RHIC** regime

Two Color QCD

$\det M(\mu)$ real and positive

For N staggered fundamental or adjoint fermions

$$\bar{\chi}\not{D}\chi = \bar{X}_e \not{D} X_o \quad \text{with} \quad \bar{X}_e^{tr} = \begin{pmatrix} \bar{\chi}_e^{tr} \\ \tau_2 \chi_e \end{pmatrix},\; X_o = \begin{pmatrix} \chi_o \\ -\tau_2 \bar{\chi}_o^{tr} \end{pmatrix}$$

For $m = \mu = 0$ U(1)$\otimes$U(1)$_\varepsilon$ global symmetry is enhanced: $X \mapsto VX, \quad \bar{X} \mapsto \bar{X}V^\dagger \quad V \in$ U($2N$)
Chiral symmetry breaking alters this as follows:

Fundamental	Adjoint
U($2N$) $\to$ O($2N$)	U($2N$) $\to$ Sp($2N$)
$N(2N+1)$ Goldstones	$N(2N-1)$ Goldstones

Besides $q\bar{q}$ mesons, some of the Goldstones are qq or $\bar{q}\bar{q}$ baryons. U($2N$) rotations relate $\langle\bar{\chi}\chi\rangle$ to di-quark condensates

Fundamental	Adjoint ($N \geq 2$)
$\langle qq_{\mathbf{2}}\rangle = \frac{1}{2}\left\langle \chi^{tr}\tau_2\chi + \bar{\chi}\tau_2\bar{\chi}^{tr}\right\rangle$	$\langle qq_{\mathbf{3}}\rangle = \frac{i}{2}\left\langle \chi^{tr}\epsilon\chi + \bar{\chi}\epsilon\bar{\chi}^{tr}\right\rangle$

For $N = 1$ adjoint flavor $qq_{\mathbf{3}}$ forbidden
by the Exclusion Principle
No Goldstone baryons
$\det M(\mu)$ no longer positive definite

Chiral Perturbation Theory

[Kogut, Stephanov, Toublan, Verbaarschot, Zhitnitsky]

Can write effective theory in terms of $2N \times 2N$ matrix Σ with $N(2N \pm 1)$ independent components:

$$\mathcal{L}_{eff} = \frac{f_\pi^2}{2} \mathrm{ReTr}\Big[\partial_\nu \Sigma \partial_\nu \Sigma^\dagger - 2m_\pi^2 \begin{pmatrix} & \mathbb{1} \\ -\mathbb{1} & \end{pmatrix} \Sigma + 4\mu \begin{pmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{pmatrix} \Sigma^\dagger \partial_t \Sigma$$
$$- 2\mu^2 \left\{ \Sigma \begin{pmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{pmatrix} \Sigma^\dagger \begin{pmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{pmatrix} + \begin{pmatrix} \mathbb{1} & \\ & \mathbb{1} \end{pmatrix} \right\} \Big]$$

Gell-Mann-Oakes-Renner: $f_\pi^2 = \left. \frac{m\langle\bar{\chi}\chi\rangle}{2Nm_\pi^2} \right|_{\mu=0}$

Leading Order Prediction

$$\frac{\langle\bar{\chi}\chi\rangle}{\langle\bar{\chi}\chi\rangle_0} = \begin{cases} 1 \\ \frac{1}{x^2} \end{cases} ; \quad \tilde{n}_B = \begin{cases} 0 & x < 1 \\ \frac{x}{4}\left(1 - \frac{1}{x^4}\right) & x > 1 \end{cases}$$

with scaling variables $x = \frac{2\mu}{m_\pi}$, $\tilde{n}_B = \left(\frac{m_\pi}{8m\langle\bar{\chi}\chi\rangle_0} \right) n_B$

ie. a second order transition to a state of non-zero baryon density at $\mu = m_\pi/2$.

Diquark condensate:

$$\langle \chi^{tr} \begin{Bmatrix} \tau_2 \\ i\epsilon \end{Bmatrix} \chi \rangle^2 = \langle\bar{\chi}\chi\rangle_0^2 - \langle\bar{\chi}\chi\rangle^2$$

ie. a bosonic superfluid for $x > 1$ – Cf. ^{4}He

Measuring Diquark Condensates

Introduce diquark source terms via a *Gor'kov* basis

$$\mathcal{L}_{ferm} = (\bar\chi, \chi^{tr}) \begin{pmatrix} \bar{j}\tau_2 & \frac{1}{2}M \\ -\frac{1}{2}M^{tr} & j\tau_2 \end{pmatrix} \begin{pmatrix} \bar\chi^{tr} \\ \chi \end{pmatrix} \equiv \Psi^{tr}\mathcal{A}\Psi$$

whence

$$Z[j,\bar{j}] = \int DU \mathrm{Pf}(2\mathcal{A}[U,j,\bar{j}]) e^{-S_{bos}[U]}$$

The diquark condensate $\langle qq\rangle$ is then given by

$$\boxed{\langle qq\rangle = \frac{1}{V}\frac{\partial \ln Z}{\partial j} = \frac{1}{2V}\langle \mathrm{tr}\tau_2\mathcal{A}^{-1}\rangle}$$

Implement by

- Direct inversion of $\mathcal{A}(j)$ followed by $j \to 0$

  [SJH,Kogut,Morrison,Sinclair]

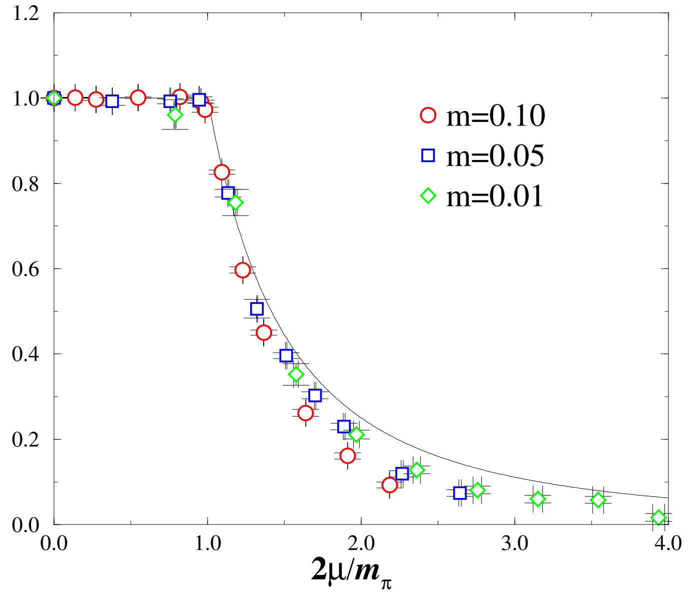
- Banks-Casher relation for $\tau_2\mathcal{A}$

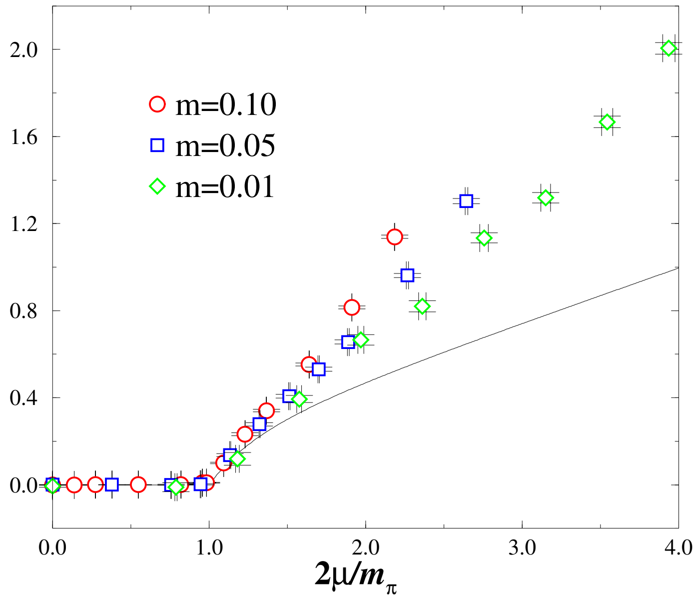
  [Bittner,Lombardo,Markum,Pullirsch]

- Probability distribution function for $\langle qq\rangle$

  [Aloisio,Azcoiti,Di Carlo,Galante,Grillo]

Two Color Highlights





$\langle \bar{\chi}\chi \rangle$ and n_B vs. μ for $\beta = 2.0$ on $4^3 \times 8$

[SJH,Montvay,Morrison,Oevers,Scorzato,Skullerud]

χPT works well over a decade of quark mass

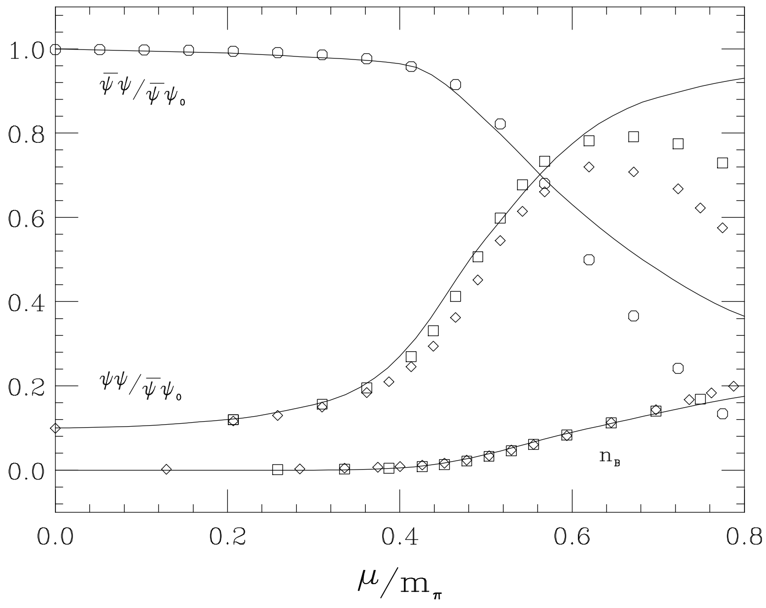
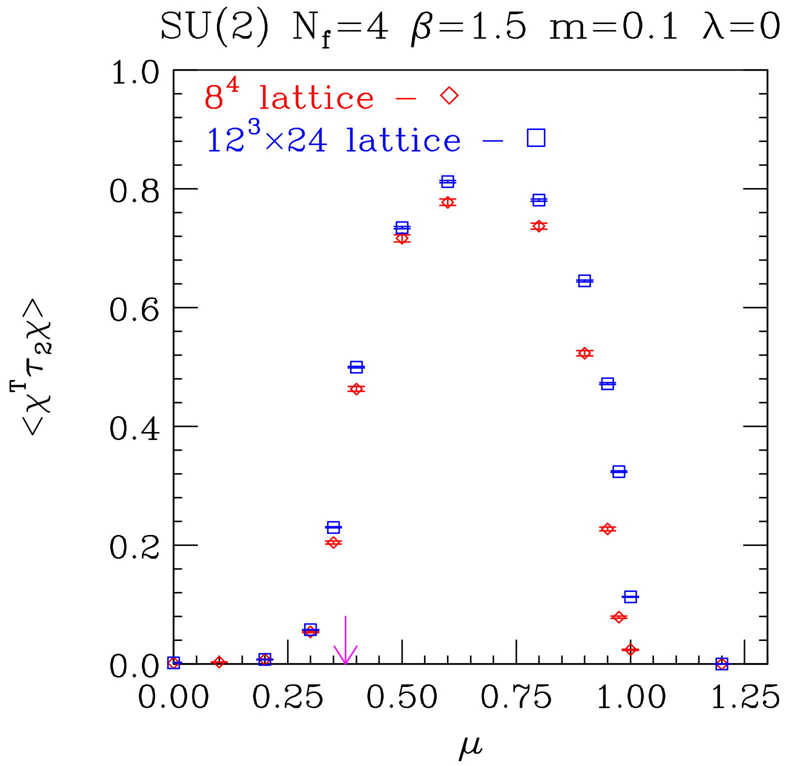
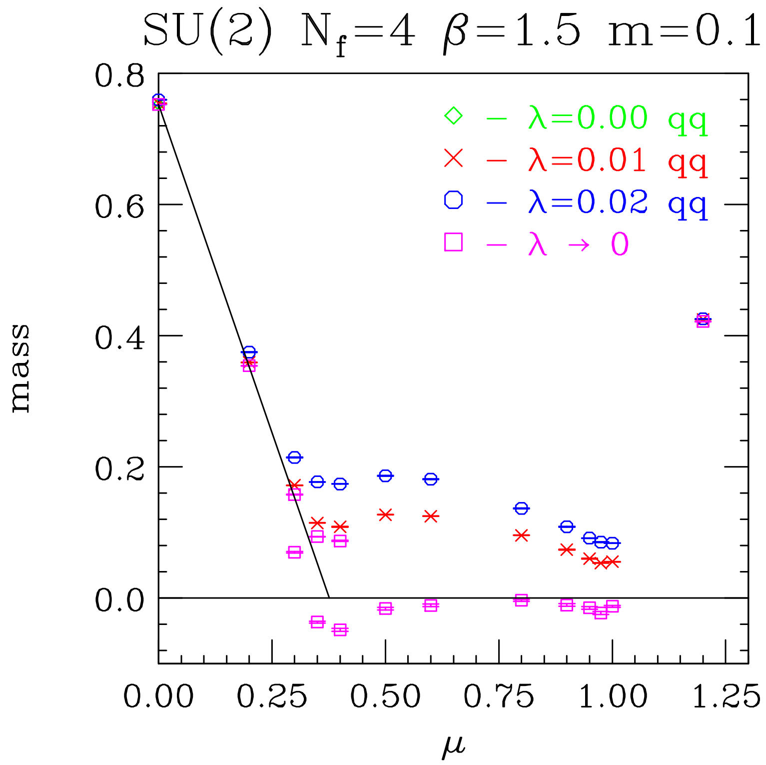


Fig. 4

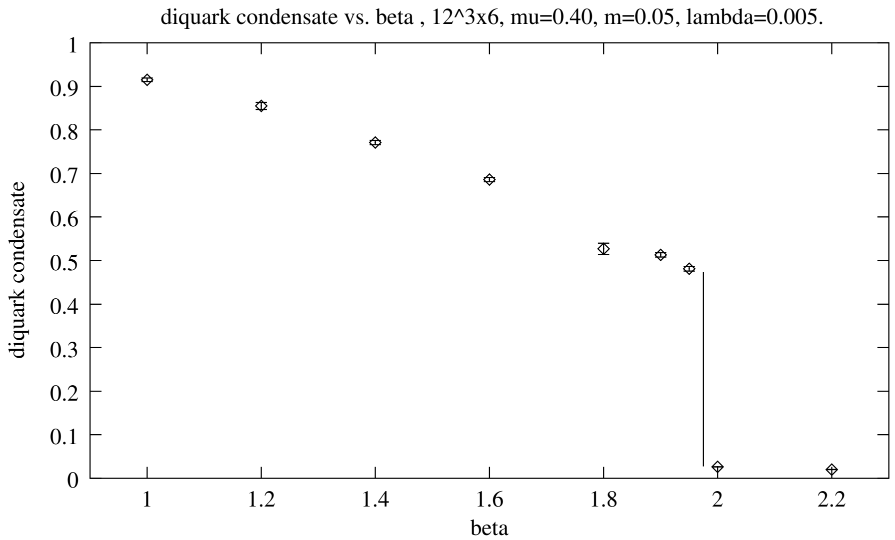
$\langle\bar{\chi}\chi\rangle$, $\langle qq\rangle$ and n_B vs. μ for $\beta = 0$, $m = 0.2$, $j/m = 0.1$ on 4^4 and 6^4 [Aloisio,Azcoiti,DiCarlo,Galante,Grillo]



$\langle qq\rangle$ vs. μ for $\beta = 1.5$, $m = 0.1$, $j \to 0$ on 8^4 and $12^3 \times 24$ [Kogut,Sinclair,SJH,Morrison]



Scalar diquark mass vs. μ for $\beta = 1.5$, $m = 0.1$ $12^3 \times 24$ [Kogut,Sinclair,SJH,Morrison]



First order transition to normal state for $\mu = 0.4$ on $12^3 \times 6$ [Kogut,Toublan,Sinclair]

The Sign Problem Revisited

For $N = 1$ adjoint flavor

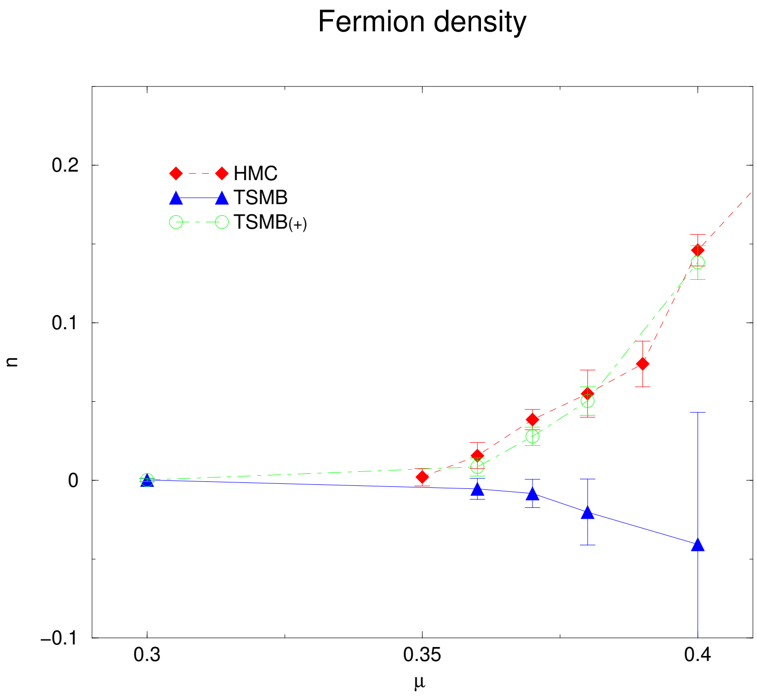
- χPT not expected to hold (no Goldstone baryons)
- simplest local diquark is superconducting

$$qq^i_{sc} = \frac{1}{2}\left[\chi^{tr} t^i \chi + \bar{\chi} t^i \bar{\chi}^{tr}\right] \in \mathbf{3} \text{ of } SU(2)$$

à la Georgi-Glashow

- $\det M(\mu)$ is real but not positive definite – use Multi-Bosonic algorithm and reweighting

[SJH,Montvay,Scorzato,Skullerud]



n_B vs. μ for $\beta = 2.0$, $m = 0.1$ on $4^3 \times 8$.
Average sign $\langle \mathrm{sgn}(\det) \rangle = 0.30(4)$ at $\mu = 0.38$

World's most expensive simulation of the vacuum?

The NJL Model in $d = 2+1$

$$\mathcal{L} = \bar{\psi}(\partial\!\!\!/ + \mu\gamma_0 + m)\psi - \frac{g^2}{2}\left[(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\vec{\tau}\psi)^2\right]$$

- $\text{SU}(2)_L \otimes \text{SU}(2)_R \otimes \text{U}(1)_B$ global symmetry spontaneously broken to $\text{SU}(2)_{isospin} \otimes \text{U}(1)_B$ for $g^2 > g_c^2 \approx 1.0a$, together with generation of a constituent quark mass $\Sigma = g^2\langle\bar{\psi}\psi\rangle$.

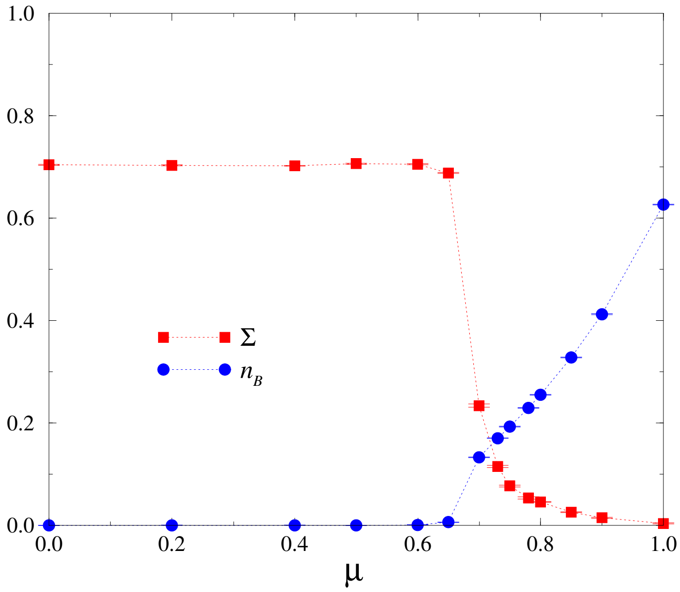
- Interacting continuum limit at $g^2 \to g_c^2$, $\Sigma a \to 0$

  [Rosenstein,Warr,Park]

- Strong first order transition restoring chiral symmetry at $\mu = \mu_c \approx \Sigma \gg m_\pi$ [SJH,Kim,Kogut]

- Baryon density $n_B = \langle\bar{\psi}\gamma_0\psi\rangle = 0$ for $\mu < \mu_c$, but increases as $n_B \propto \mu^2$ in the chirally restored phase.

Is $\text{U}(1)_B$ spontaneously broken by a diquark condensate for $\mu > \mu_c$ leading to superfluidity?



Since the lightest baryons are fermions, expect a Fermi surface. A BCS instability would yield superfluidity as in ^{3}He. We have investigated using a pfaffian simulation with scalar $\mathrm{SU}(2)_L \otimes \mathrm{SU}(2)_R$ singlet diquark source term

$$j_\pm(qq_\pm) \equiv j_\pm(\chi^{tr}\tau_2\chi \pm \bar{\chi}\tau_2\bar{\chi}^{tr})$$

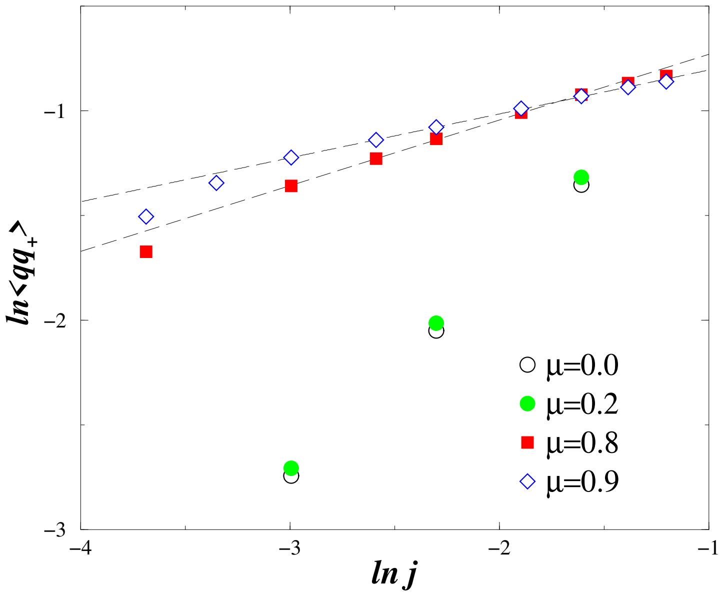
[SJH,Lucini,Morrison]

Investigate both condensate $\langle qq_+\rangle$
and susceptibilities $\chi_\pm = \sum_x \langle qq_\pm(0)qq_\pm(x)\rangle$.

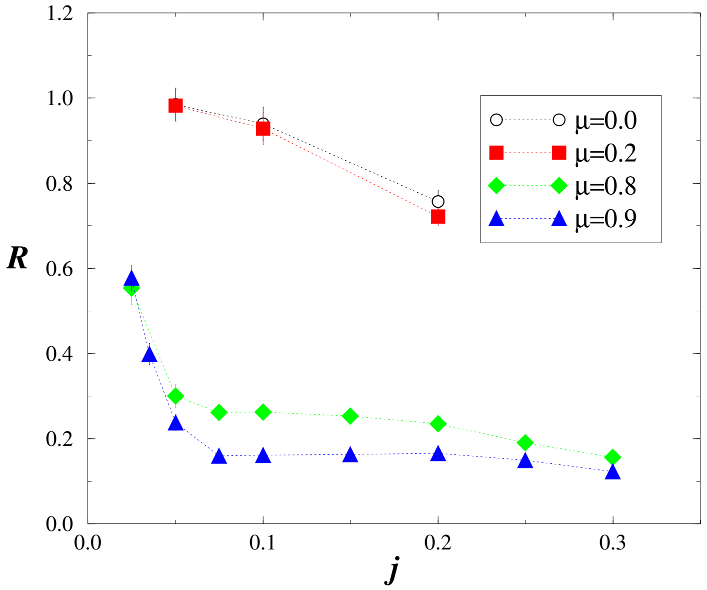
$+$ is "Higgs", $-$ is "Goldstone"

Ward identity:

$$\chi_-\Big|_{j_-=0} = \frac{\langle qq_+\rangle}{j_+}$$



The condensate scales as $\langle qq_+(j)\rangle \propto j^{\alpha_1}$



$$R = \left|\frac{\chi_+}{\chi_-}\right| = \frac{\partial \ln\langle qq_+\rangle}{\partial \ln j} = \alpha_2$$

Naively expect $\lim_{j\to 0} R = 0$ if $U(1)_B$ broken, 1 otherwise.

We find:

$\alpha_1 \approx \alpha_2 \approx 0.3$ $(\mu = 0.8)$, ≈ 0.2 $(\mu = 0.9)$

- This strongly suggests *critical behaviour* in the dense phase, with continuously varying exponents $\delta(\mu)$, $\eta(\mu)$ defined by

$$\langle qq\rangle \propto j^{\frac{1}{\delta}} \quad ; \quad \langle qq(0)qq(\vec{x})\rangle \propto \frac{1}{|\vec{x}|^\eta}.$$

Cf. the low temperature phase of the $2d$ XY model, with

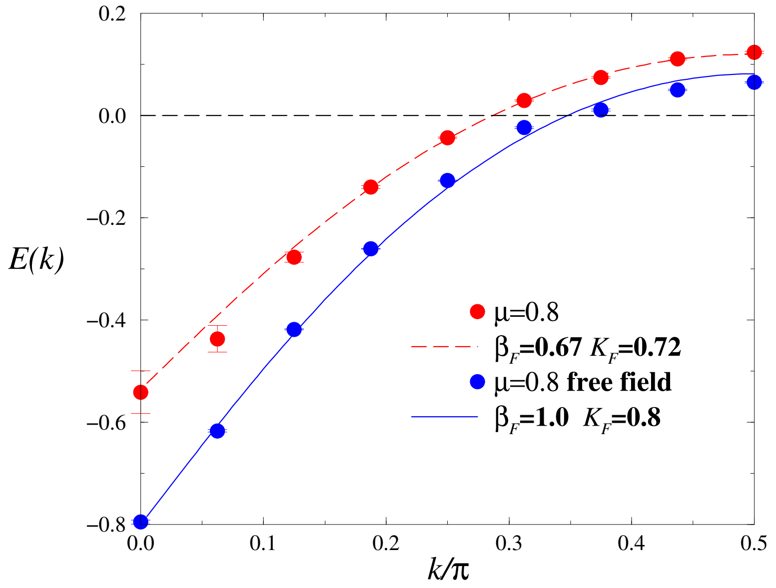
$$\delta(T) \geq 15 \quad ; \quad \eta(T) \leq \frac{1}{4}$$

- We conjecture a $2d$ critical system describing *thin film superfluidity*. The superfluid current is related to the phase of $qq(x) \simeq \phi_0 e^{i\theta(x)}$ via

$$\vec{J}_s = K_s \vec{\nabla}\theta.$$

Supercurrents are metastable thanks to *long range phase coherence*. [Kosterlitz & Thouless]

- NJL exponents are distinct from those of the XY model; dimensional reduction does not apply – a $2d$ description follows from the *static* nature of the phase fluctuations $\partial_t\theta \approx 0$.



Can also probe spin-$\frac{1}{2}$ sector via the Gor'kov propagator $\mathcal{G} = \mathcal{A}^{-1}$. Simple pole fits to the momentum-dependent timeslice propagator

$$\mathcal{G}(\vec{k}, t) = \sum_{\vec{x}} \mathcal{G}(\vec{0}, 0; \vec{x}, t) e^{-i\vec{k}.\vec{x}} = Ae^{-Et} + Be^{-E(L_t - t)}$$

yield the *quasiparticle dispersion relation* $E(k)$.

- Fermi momentum $k_F \lesssim \mu$
- Fermi velocity $v_F = \frac{\partial E}{\partial k}\Big|_{k=k_F} \approx 0.7c < c$
- No evidence for a BCS gap $\Delta \neq 0$

Characteristic of a *normal Fermi liquid* with repulsion between quasiparticles with parallel momenta.

$\Rightarrow$NJL$_{2+1}$ is a *relativistic gapless superfluid*.

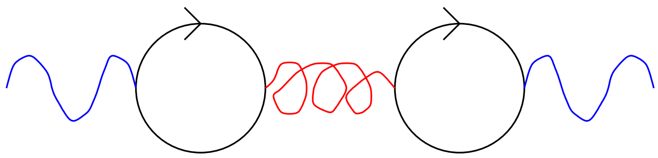
Other Sign/Phase Problems

• TCQCD with $N = 1$ adjoint staggered quark – superconductor at large μ?

• The Hubbard Model away from half-filling – models high-T_c superconductivity

• Technicolor – requires chiral fermions in complex representations of the gauge group

• "τ_3-QED" describes 2+1d superconductivity by giving the photon a mass via a mixed Chern-Simons term [Dorey & Mavromatos]



$\det M \neq \det M^*$ since $\{\gamma_5, \not{D}\} \neq 0$

• QCD itself?

Conjecture:
sign problem whenever
local symmetry broken by pairing

Summary

- Significant progress in QCD(μ) for $T \neq 0$. We have the first non-trivial LGT prediction in the (μ, T) plane. The **RHIC** regime is within reach. Expect much activity in coming year.

- At $T = 0$ models yield LGT's first contact with *ab initio* (relativistic) condensed matter physics

Two Color QCD	$\Leftrightarrow$	superfluid ^{4}He
NJL	$\Leftrightarrow$	superfluid ^{3}He

- NJL$_{3+1}$ will test model approaches to color superconductivity

- True superconductivity may *require* a sign problem