Topology and Fermionic Zero Modes

- Review recent results in the relation of fermionic zero modes and topology - will not cover topology in general
- Role of fermionic eigenmodes (including zero modes) important in 3 areas discussed here:
  - (Near) zero modes in spectrum
  - (Near) zero modes in global topology (e.g., chiral fermions)
  - (Near) zero modes affect implementation and meaning of chiral fermions
- Use fermion modes to probe for possible mechanism of chiral symmetry breaking in QCD
- Chiral fermions crucial in new studies

Eigenmodes in Spectrum

- Computation of the η′ mass is notoriously difficult – must compute disconnected term
- Consider spectral decomposition of propagator – use hermitian Dirac operator

$$H = \gamma_5 D_w , \quad H \left| \psi_i \right\rangle = \lambda_i \left| \psi_i \right\rangle , \quad H^{-1}(x, y) = \sum_i \frac{\left| \psi_i (x) \right\rangle \left\langle \psi_i (y) \right|}{\lambda_i}$$

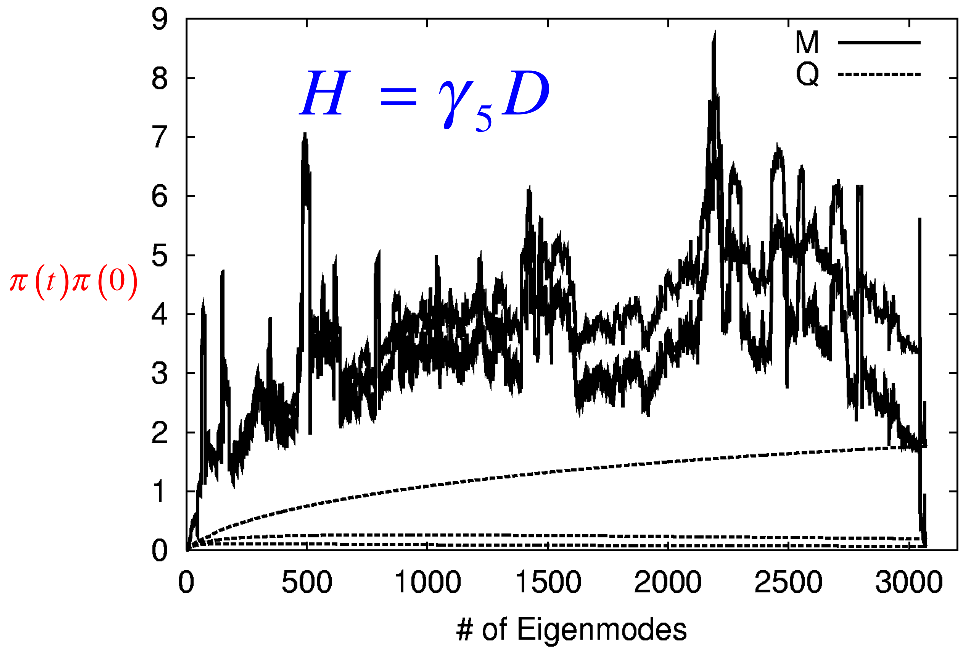
- Correlation function for η′

$$C_{\eta'}(\Delta t) = C_\pi(\Delta t) - N_f \left\langle \sum_t \mathrm{Tr}_{\vec{x}cs} H^{-1}(\vec{x}, t) \mathrm{Tr}_{\vec{x}cs} H^{-1}(\vec{x}, t + \Delta t) \right\rangle$$

- Typically use stochastic estimate of trace piece.
- Instead, truncate spectral some with lowest few eigenvectors (gives largest contribution) and stochastically estimate the remainder. Idea is $H = \Sigma_i H_i + H_\perp$
- For lowest modes, gives volume times more statistics

Spectral Decomposition

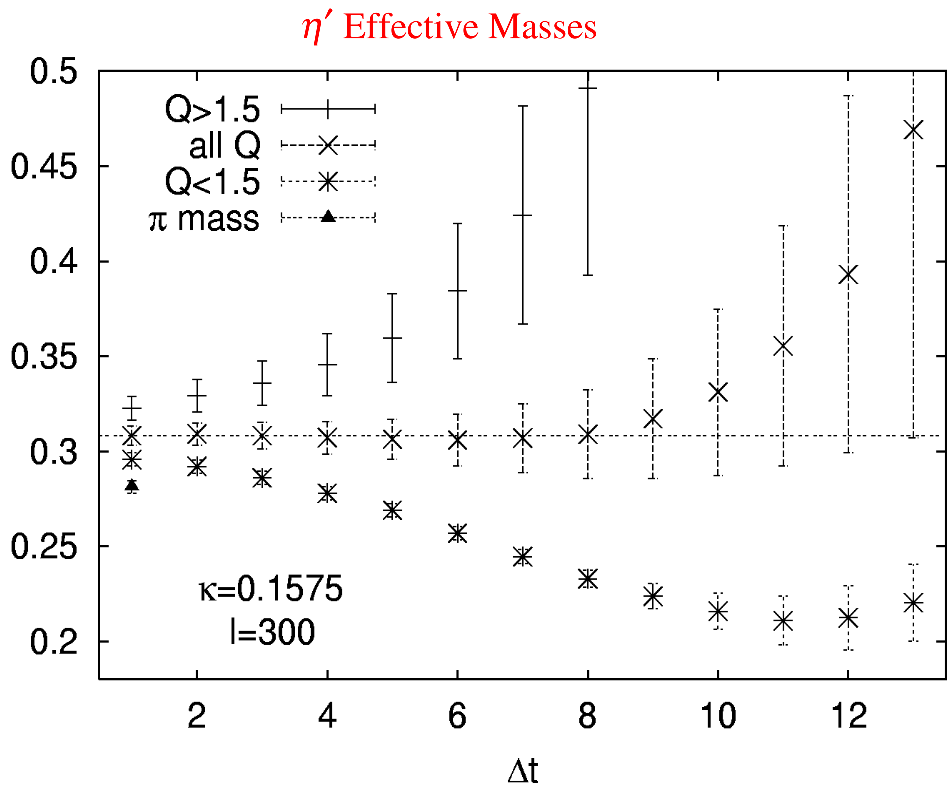
Correlation fn spectral decomp.



- Question: for Wilson fermions, is it better to use hermitian or non-hermitian operator?
- Comparison of different time slices of pion 2-pt correlation function as eigenmodes are added to (truncated) spectral decomposition
- Non-hermitian on top and hermitan on bottom
- Test config from quenched Wilson β=5.0, 4^4
- Non-hermitian approx. very unstable
- Note, for chiral fermions, choice is irrelevant

Neff, et.al, hep-lat/0106016

Mass dependence of η′



$$Q \leftarrow \mathrm{Tr}\, H^{-1} = \sum_i \frac{1}{\lambda_i}$$

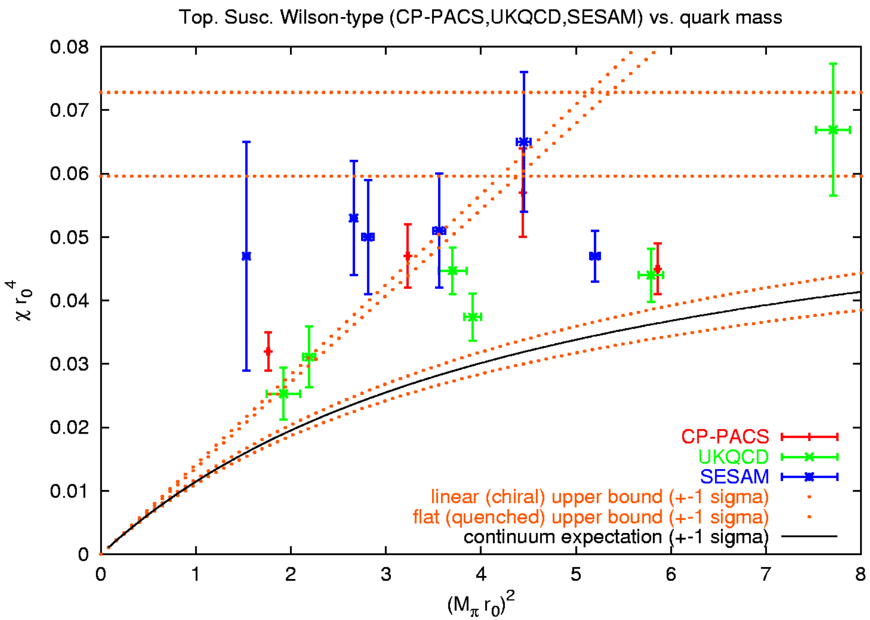
- Using suitable combinations of partial sums (positive and negative evs), an estimate of the global topology Q is obtained
- After binning configurations, effective masses show a Q dependence

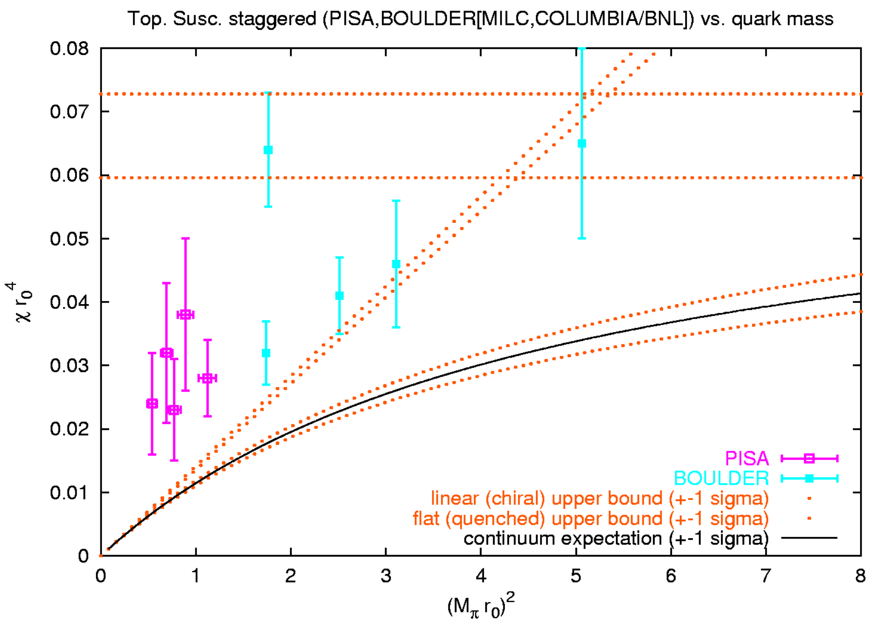
- New calc. of flavor singlet mesons by UKQCD – test of OZI rule (singlet – non-singlet mass splittings)

Neff, et.al., hep-lat/0106016

UKQCD, hep-lat/0006020, 0107003

Topological Susceptibility





- N_f=2 topological susceptibility (via gauge fields)
  - **CPPACS**: 24^3x48, RG-gauge, Clover with mean field c_{SW}
  - **UKQCD**: 16^3x32, Wilson gauge, non-pt Clover
  - **SESAM/TXL**: 16^3x32 & 24^3x40, Wilson gauge and Wilson fermion
  - **Thin-link staggered**: Pisa group and Boulder using MILC and Columbia configs

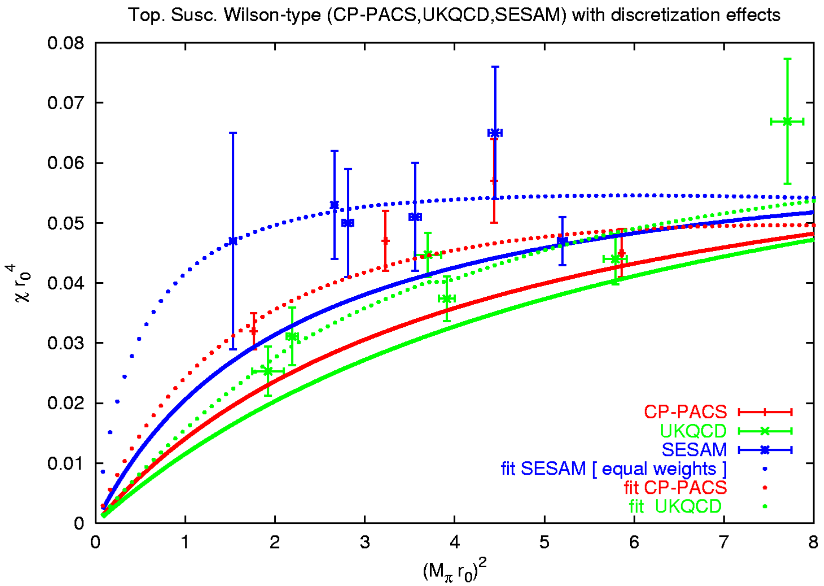
$$\chi(m)=\frac{\Sigma m}{N_f}\left(1+\mathrm{O}(m)\right)=\frac{m_\pi^2 F_\pi^2}{2N_f}\left(1+\mathrm{O}(m)\right)\equiv\chi_0\left(1+\mathrm{O}(m)\right)$$

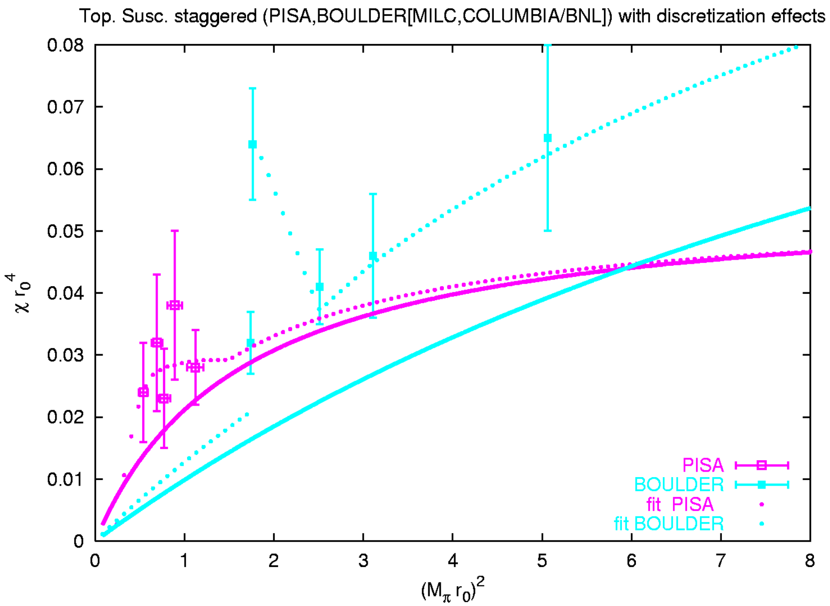
$$\chi(m)=1/\left(1/\chi_0+1/\chi_\infty\right)$$

- Naïve linear m_π^2 (fixing F_π) fit poor
- Suggested that discretization effects large. Also large quark masses

Durr, hep-lat/0108015. Data hep-lat/0106010, 0108006, 0102002, 0004020, 0104015

Topological Susceptibility





- Argued to extend fits to include lattice spacing and *intermediate* quark mass fits (combing both equations with additional O(a) term
- Wilson-type data qualitatively cleaner fits
- Staggered more complex – some finite-volume effected points.

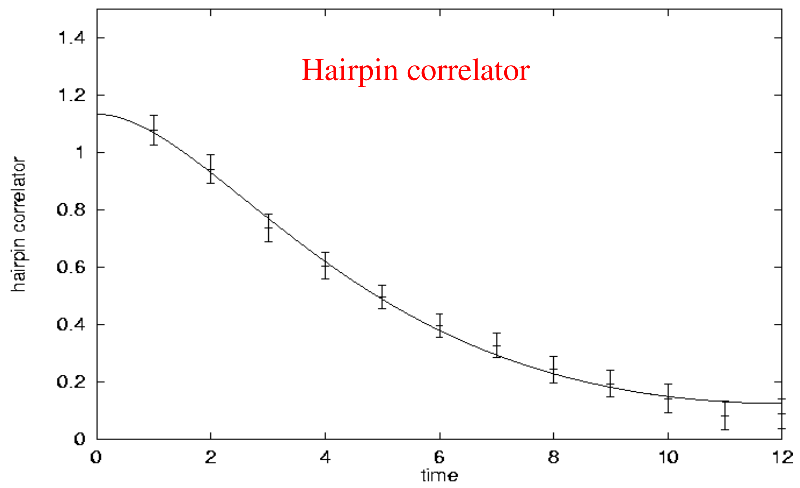
- Idea of using χPT theory to augment fits advocated by several groups (Adelaide)

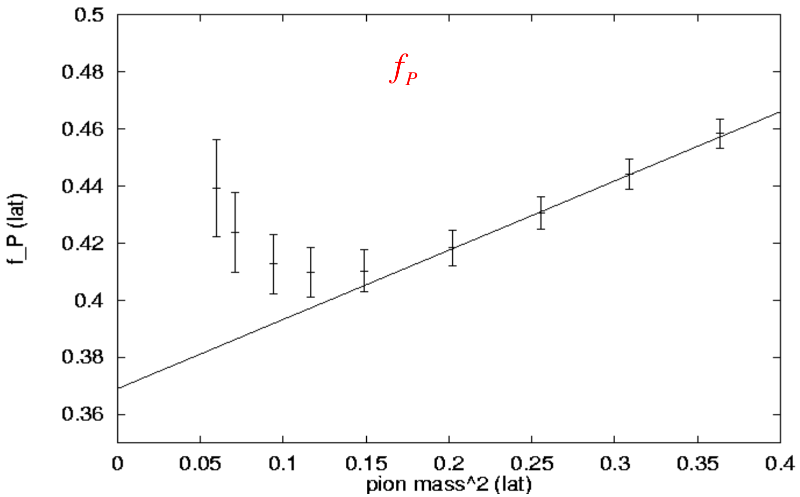
Quenched Pathologies in Hadron Spectrum

- How well is QCD described by an effective chiral theory of interacting particles (e.g., pions in chiral dynamics)?
- Suppressing fermion determinant leads to well known pathologies as studied in chiral pertubation theory – a particularly obvious place to look
- Manifested in η' propagator missing vacuum contributions

- New dimensionful parameter now introduced. Power counting rules changed leading to new chiral logs and powers terms.
- Studied extensively with Wilson fermions by CPPACS (LAT99)
- Recently studied with Wilson fermions in Modified Quenched Approximation (Bardeen, et.al.)
- Very recent calculation using Overlap (Kentucky)

Anomalous Chiral Behavior





- Compute η′ mass insertion from behavior in QχPT
- Hairpin correlator fit holding m_π fixed - well described by simple mass insertion

$$\langle \mathrm{Tr}\,\gamma_5 G(x,x)\,\mathrm{Tr}\,\gamma_5 G(0,0)\rangle$$

$$\to f_P \frac{1}{p^2+m_\pi^{\ 2}} m_0^{\ 2} \frac{1}{p^2+m_\pi^{\ 2}} f_P$$

$$f_P^{\ quenched} = \left(\frac{1}{m_\pi^2}\right)^\delta \tilde{f}_P$$

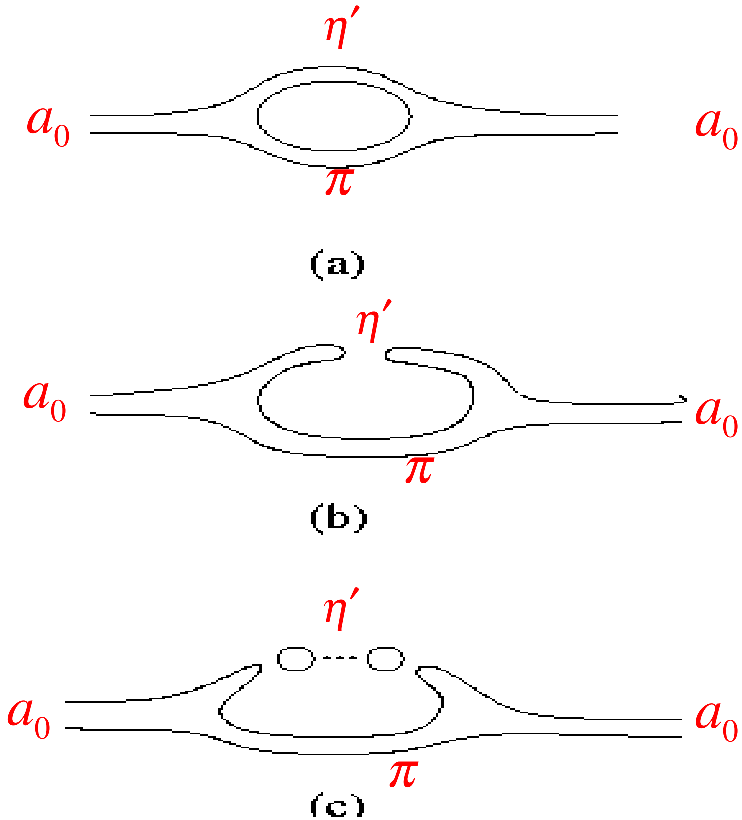
- f_P shows diverging term. Overall δ=0.059(15)

- Kentucky use Overlap 20^4, a=0.13fm, find similar behavior for f_P , δ=~0.2 – 0.3

Bardeen, et.al., hep-lat/0007010, 0106008

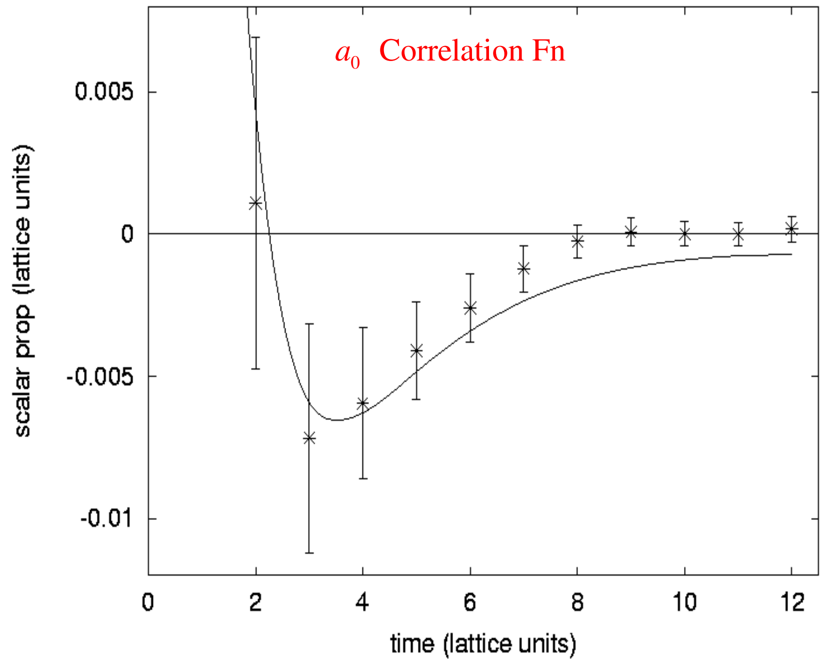
Dong, et.al., hep-lat/0108020

More Anomolous Behavior



- Dramatic behavior in Isotriplet scalar particle a_0 — η'-π intermediate state
- Can be described by 1 loop (bubble) term

- MILC has a new N_f=2+1 calc. See evidence of decay (S-wave decay)



Bardeen, et.al., hep-lat/0007010, 0106008

Chiral Condensate

- Several model calculations indicate the quenched chiral condensate diverges at T=0 (Sharan&Teper, Verbaarschot & Osborn, Damgaard)
- Damgaard (hep-lat/0105010), shows via QχPT that the first finite volume correction to the chiral condensate diverges logarithmically in the 4-volume
- Some relations for susceptibilities of pseudoscalar and scalar fields
  - Relations including and excluding global topology terms
  - a_o susceptibility is derivative of chiral condensate

$$\pi^a(x) = i\overline{\psi}(x)\gamma_5\tau^a\psi(x), \quad a_0(x) = -\overline{\psi}(x)\tau^a\psi(x)$$

$$\sum_x \left\langle \pi^a(x)\pi^a(0)\right\rangle = \frac{1}{m}\left\langle \overline{\psi}\psi\right\rangle, \quad \sum_x \left\langle a_0^a(x)a_0^a(0)\right\rangle = \left\langle \frac{d}{dm}\left\langle \overline{\psi}\psi\right\rangle_A\right\rangle$$

- Global topology term irrelevant in thermodynamic limit
- Recently, a method developed to determine non-PT the renormalization coefficients (hep-lat/0106011)

Chiral Condensate

- Banks-Casher result on a finite lattice

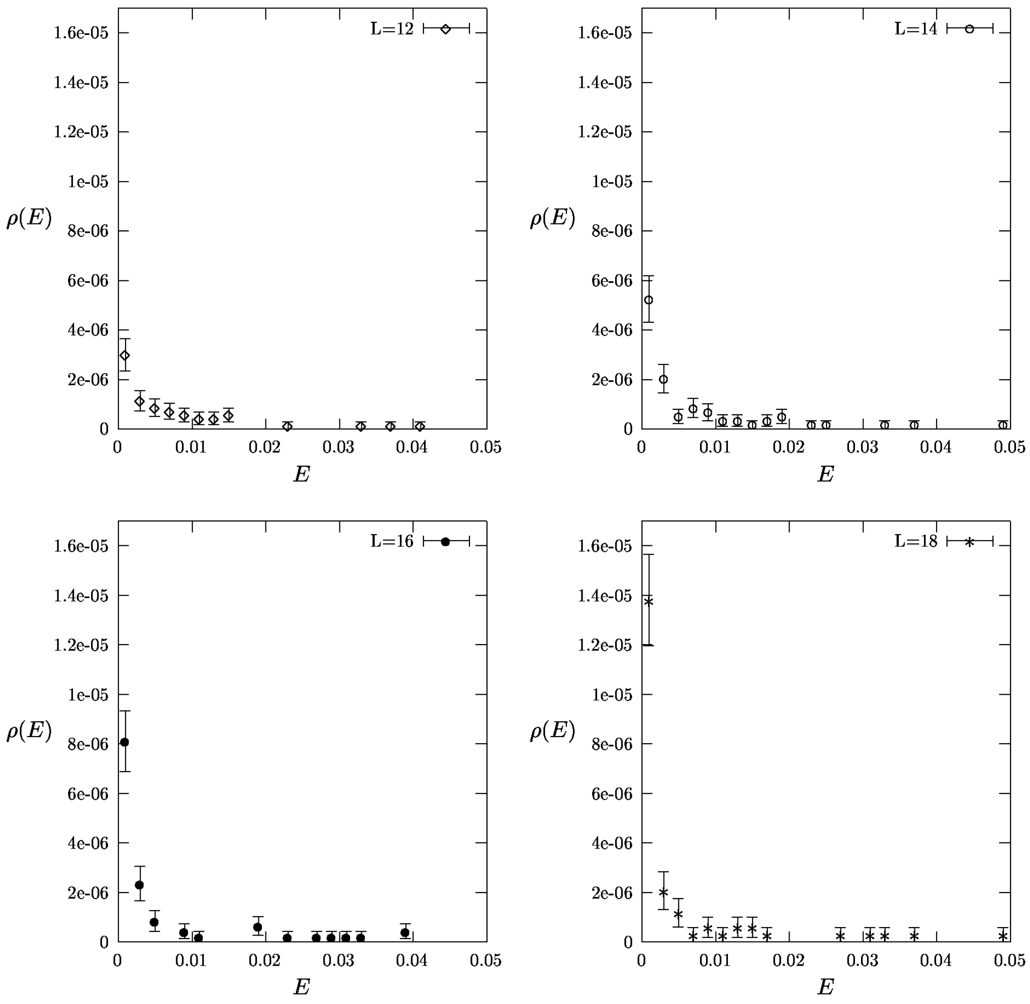
$$\frac{1}{V}\sum_x \left\langle \overline{\psi}(x)\psi(x)\right\rangle = \frac{|Q|}{mV} + \frac{1}{V}\sum_n f(\lambda_n, m), \quad \lim_{m\to 0}\lim_{V\to\infty}\frac{1}{V}\sum_x \left\langle \overline{\psi}(x)\psi(x)\right\rangle = \pi\rho\left(0^+\right)$$

- Susceptibility relations hold without topology terms

$$\sum_x \left\langle \pi^a(x)\pi^a(0)\right\rangle = \frac{1}{m}\left\langle \overline{\psi}\psi\right\rangle, \quad \sum_x \left\langle a_0^a(x)a_0^a(0)\right\rangle = \left\langle \frac{d}{dm}\left\langle \overline{\psi}\psi\right\rangle_A\right\rangle$$

- If chiral condensate diverges, a_0 susceptibility must be negative and diverge
- Require large enough physical volume to be apparent
  - Staggered mixes (would-be) zero and non-zero modes. Large finite lattice spacing effects
  - CPPACS found evidence with Wilson fermions
  - MQA study finds divergences; however, mixes topology and non-zero modes. Also contact terms in susceptibilities
  - Until recently, chiral fermion studies not on large enough lattices, e.g., random matrix model tests, spectrum tests, direct measurement tests

Quenched Pathologies in Thermodynamics



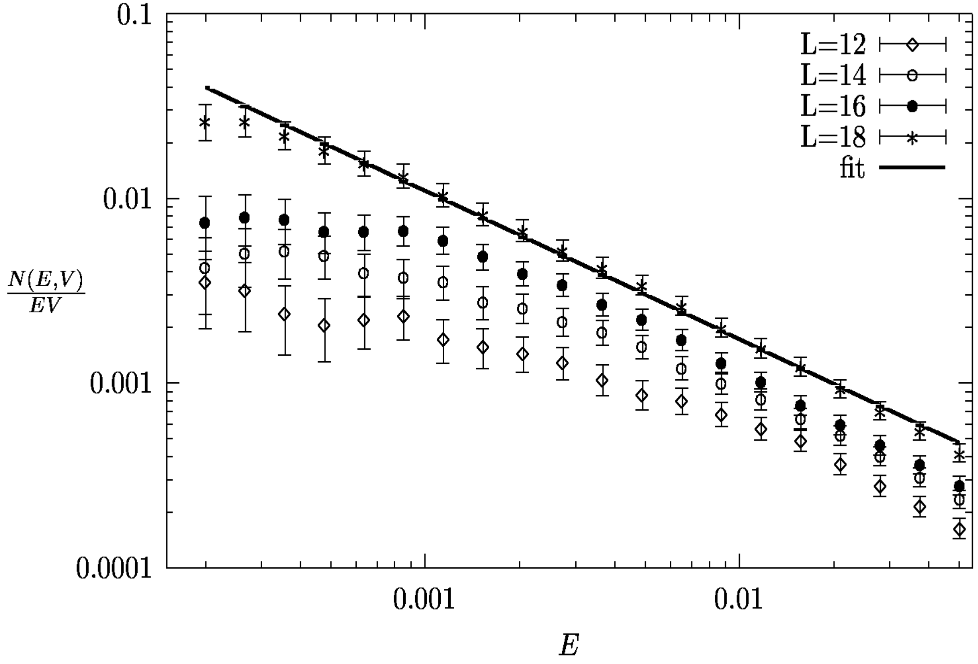
- Deconfined phase of SU(2) quenched gauge theory, L^3x4,
- β=2.4, above N_t=4 transition
- From study of build-up of density of eigenvalues near zero, ρ(E), indicates chiral condensate diverging

Kiskis & Narayanan, hep-lat/0106018

Quenched Pathologies in Thermodynamics

$$N(E,V) = \#(\lambda > 0) \text{ where } \lambda < E$$

$$\rho(E) = \frac{d}{dE} \lim_{V\to\infty} \frac{N(E,V)}{V}$$



- Define density from derivative of cumulative distribution
- Appears to continually rise and track line on log plot – hence derivative (condensate) diverges with increasing lattice size
- Spectral gap closed. However, decrease in top. susceptibility seen when crossing to $T > 0$
- Models predict change in vacuum structure crossing to deconfined and (supposedly) chirally restored phase

Kiskis & Narayanan, hep-lat/0106018

Nature of Debate – QCD Vacuum

- Generally accepted QCD characterized by strongly fluctuating gluon fields with clustered or lumpy distribution of topological charge and action density
- Confinement mechanisms typically ascribed to a dual-Meissner effect – condensation of singular gauge configurations such as monopoles or vortices
  - Instanton models provide $\chi-$symmetry breaking, but not confinement
  - Center vortices provide confinement and $\chi-$symmetry breaking
  - Composite nature of instanton (linked by monopoles - calorons) at $T_c>0$
- Singular gauge fields probably intrinsic to SU(3) (e.g., in gauge fixing)
  - Imposes boundary conditions on quark and gluon fluctuations – moderates action
  - E.g., instantons have locked chromo-electric and magnetic fields $\mathbf{E^a = \pm B^a}$ that decrease in strength in a certain way. If randomly orientation, still possible localization
- In a hot configuration expect huge contributions to action beyond such special type of field configurations
- Possibly could have regions or *domains* of (near) field locking. Sufficient to produce chiral symmetry breaking, and confinement (area law)

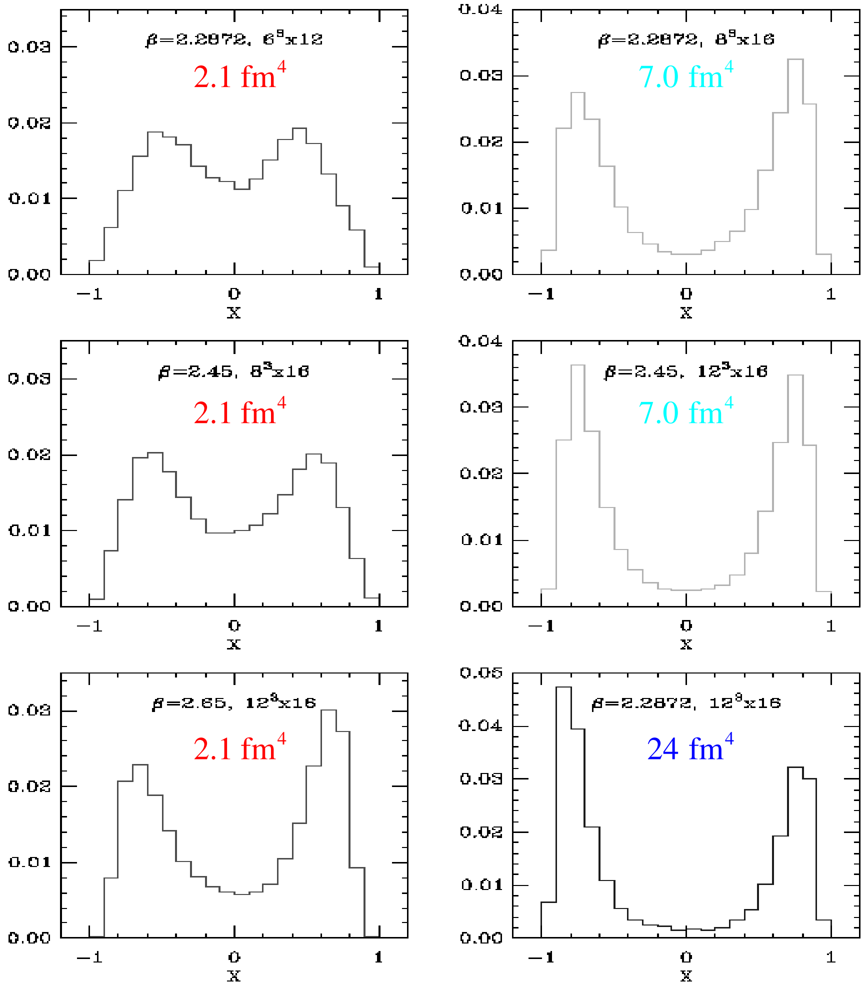
Lenz., hep-ph/0010099, hep-th/9803177; Kallloniatis, et.al., hep-ph/0108010; Van Baal, hep-ph/0008206; G.-Perez, Lat 2000

Instanton Dominance in QCD(?)

- Witten ('79)
  - Topological charge fluctuations clearly involved in solving $U_A(1)$ problem
  - Dynamics of η' mass need not be associated with semiclassical tunneling events
  - Large vacuum fluctuations from confinment also produce topological fluctuations
  - Large N_c incompatible with instanton based phenomology
    - Instantons produce η' mass that vanishes exponentially
    - Large N_c chiral dynamics suggest that η' mass squared ~ $1/ N_c$
  - Speculated η' mass comes from coupling of $U_A(1)$ anomaly to top. charge fluctuations and *not* instantons

Local Chirality

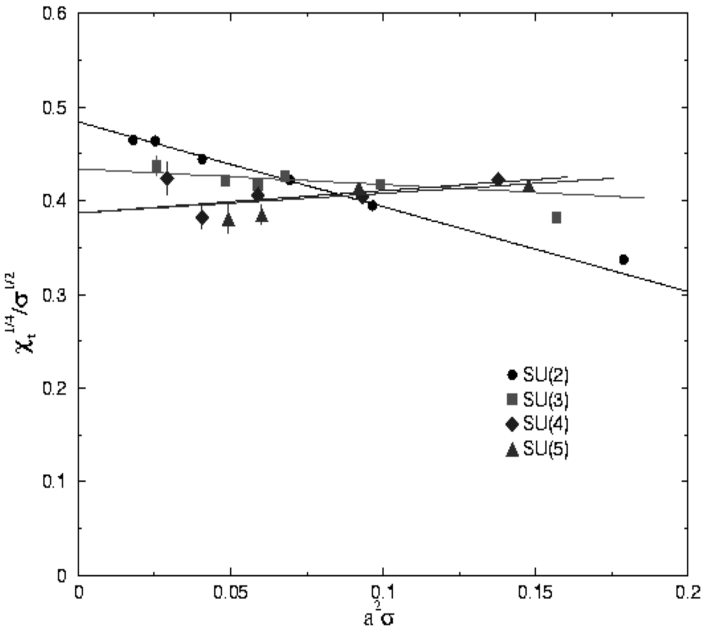
$$\tan\left(\frac{\pi}{4}\left(1+X(x)\right)\right)=\sqrt{\frac{\psi_L^+(x)\psi_L(x)}{\psi_R^+(x)\psi_R(x)}}$$

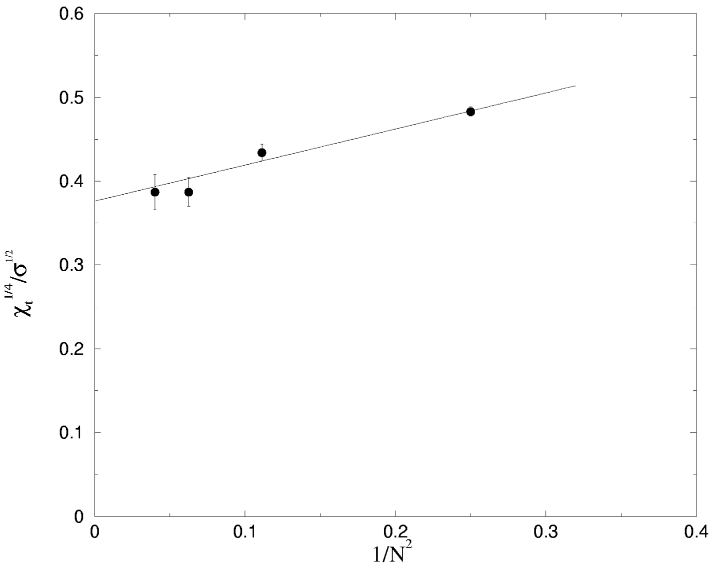


- Local measure of chirality of non-zero modes proposed in hep-lat/0102003
- Relative orientation of left and right handed components of eigenvectors
- Claimed chirality is random, hence no instanton dominance
- Flurry of papers using improved Wilson, Overlap and DWF
- Shown is the histogram of X for 2.5% sites with largest $\psi^+\psi$. Three physical volumes. Indications of finite *density* of such chiral peaked modes – survives continuum limit
- Mixing (trough) **not** related to dislocations
- No significant peaking in U(1) – still zero modes (Berg, et.al)
- Consistent with instanton phenomology. More generally, suitable regions of (nearly) locked **E** & **B** fields.

hep-lat/0103002, 0105001, 0105004, 0105006, 0107016, 0103022

Large N_c

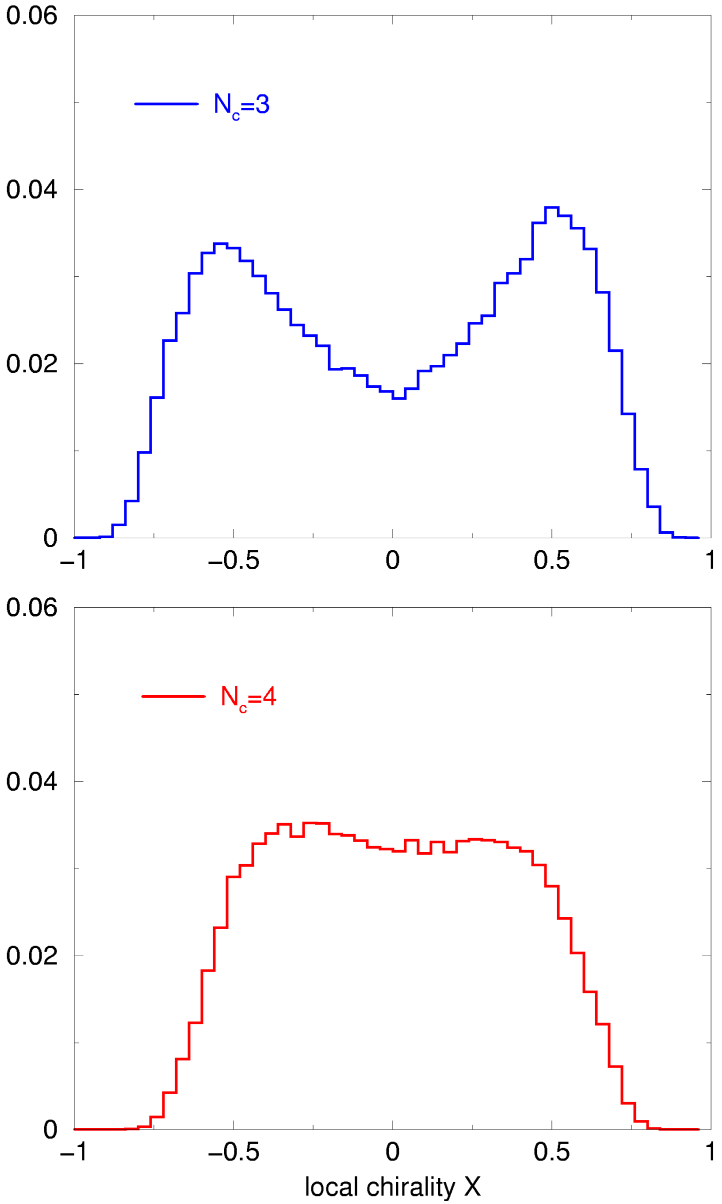




- Large N_c successful phenomenologically
  - E.g., basis for valence quark model and OZI rule, systematics of hadron spectra and matrix elements
  - Witten-Veneziano prediction for η' mass
- How do gauge theories approach the limit?
  - Prediction is that for a smooth limit, should keep a constant t'Hooft coupling, $\lambda = g^2 N$ as $N_c \to \infty$
  - Is the limit realized quickly?
- Study of pure glue top. susceptibility
  - Large N limit apparently realized quickly (seen more definitely in a 2+1 study)
  - Consistent with $1/N_c^2$ scaling
- Future tests should include fermionic observables (η' mass??)
- Recently, a new lattice derivation of Witten-Veneziano prediction (Giusti, et.al., hep-lat/0108009)

Lucini & Teper, hep-lat/0103027

Large N_c



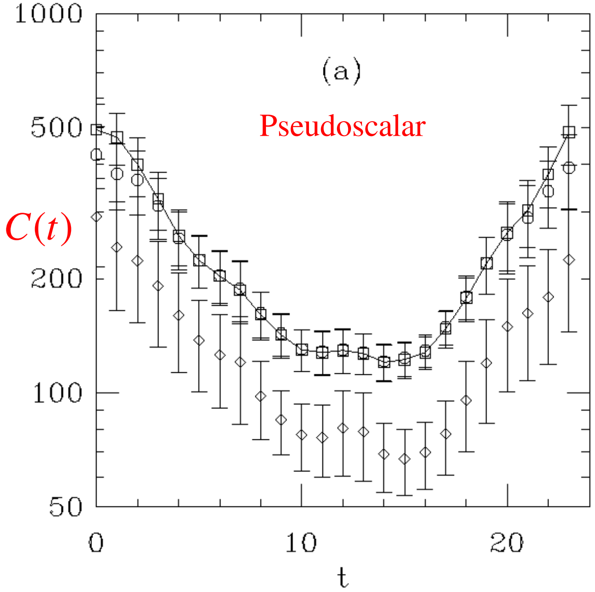
12^4 lattice
a = 0.12 fm
Im λ < 0.2
10% of latt. sites
20 conf.

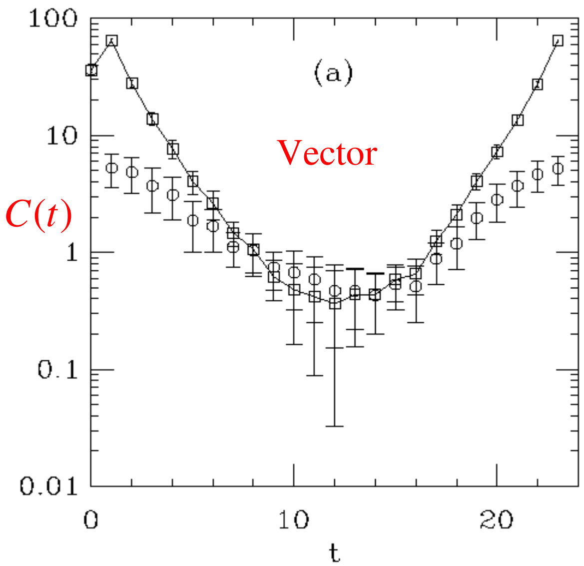
- Revisit chirality: chirality peaking decreases (at coupling fixed by string-tension) as N_c increases.
- Disagreement over interpretation?!
- Peaking disappearing consistent with large *instanton* modes disappearing, not small modes
- Witten predicts strong exponential suppression of instanton number density. Teper (1980) argues mitigating factors
- Looking like large N_c !!??

- Larger N_c interesting. Chiral fermions essential

Wenger, Teper, Cundy - preliminary

Eigenmode Dominance in Correlators





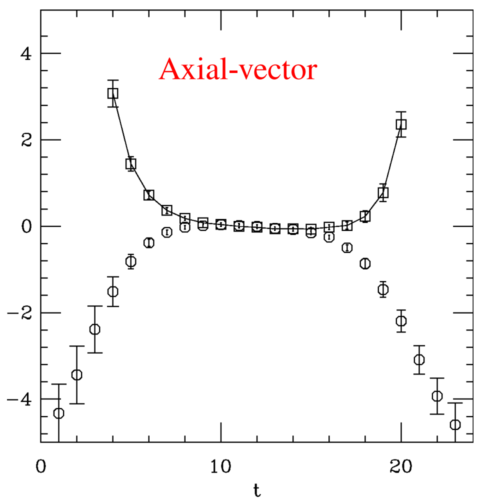
Saturation of correlators

- Full correlator
- Lowest 20 modes
- Zero modes

$12^3 \times 24,$

$m_q = 0.01 \rightarrow \pi/\rho \approx 0.34$

- How much are hadron correlators dominated by low modes?
- Comparisons of truncated and full spectral decomposition using Overlap. Compute lowest 20 modes (including zero modes)
  - Pseudoscalar well approximated
  - Vector not well approximated. Consistent with instanton phenomenology
  - Axial-vector badly approximated



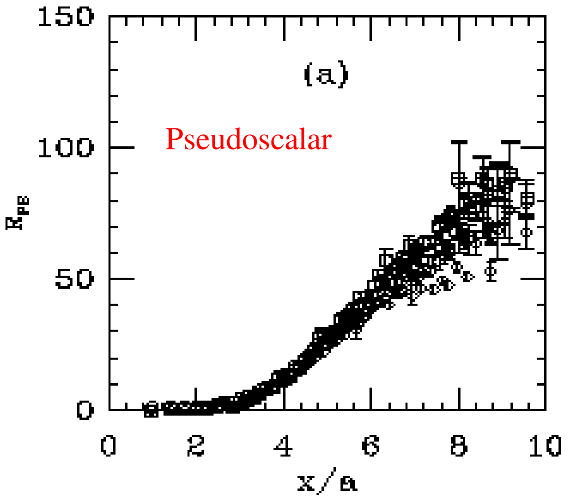
$\pi/\rho \approx 0.61$

DeGrand & Hasenfratz, hep-lat/0012021,0106001

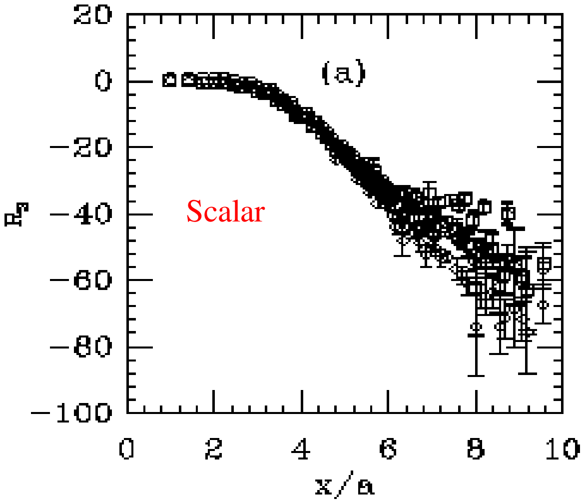
Short Distance Current Correlators

$$R_i(x) = \Pi_i(x)/\Pi_i^0(x), \quad \Pi_i(x) = \mathrm{Tr}\langle J_i^a(x) J_i^a(0)\rangle,$$
$$J_i^a(x) = \overline{\psi}(x)\tau^a\Gamma(i)\psi(x)$$

R_{PS}



R_S

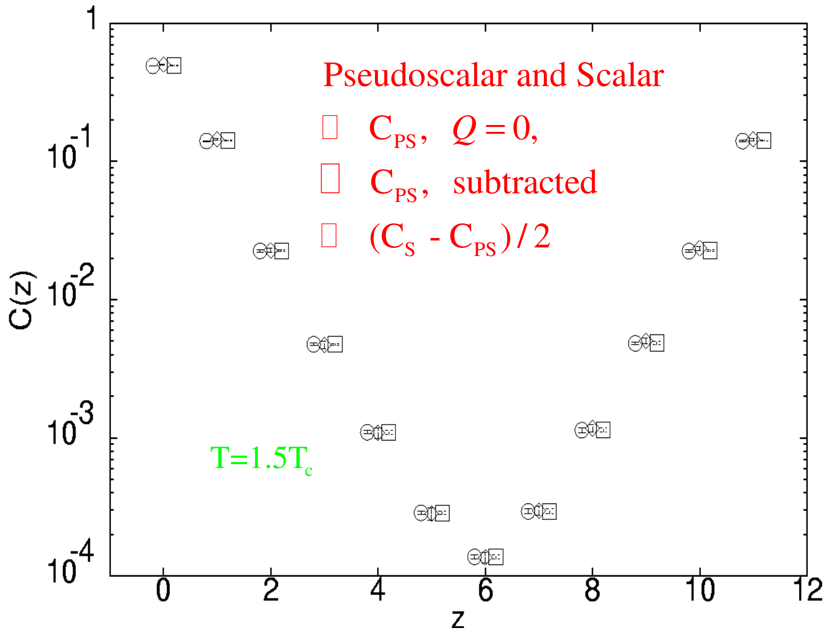


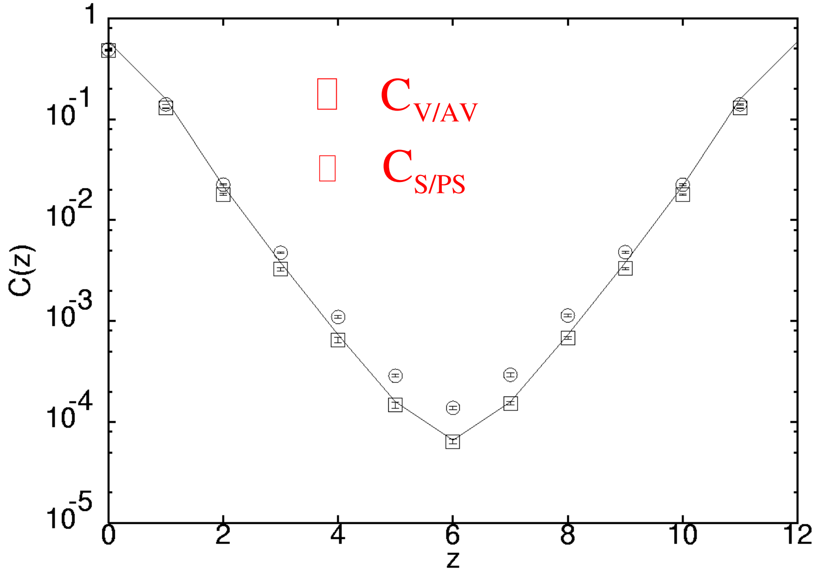
$m_q = 0.01 \rightarrow \pi/\rho \approx 0.34$

- QCD sum rule approach parameterizes short distance correlators via OPE and long dist. by condensates
- Large non-pertubative physics in non-singlet pseudo-scalar and scalar channels
- Studied years ago by MIT group - now use χ-fermions!
- Truncated spectral sum for pt-pt propagator shows appropriate attractive and repulsive channels
- Saturation requires few modes
- Caveat – using smearing

DeGrand, hep-lat/0106001; DeGrand & Hasenfratz, hep-lat/0012021

Screening Correlators with Chiral Fermions





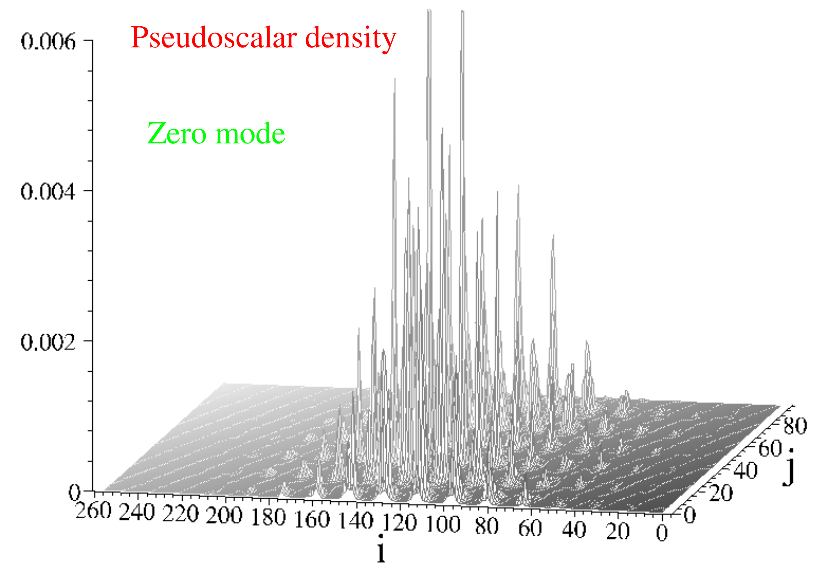
- Overlap: SU(3) (Wilson) gauge theory, N_t=4, 12^3x4
- Expect in chirally symmetric phase as $m_q a \to 0$ equivalence of (isotriplet) screening correlators:

$$C_S(z) = -C_{PS}(z), \quad C_V(z) = C_{AV}(z)$$

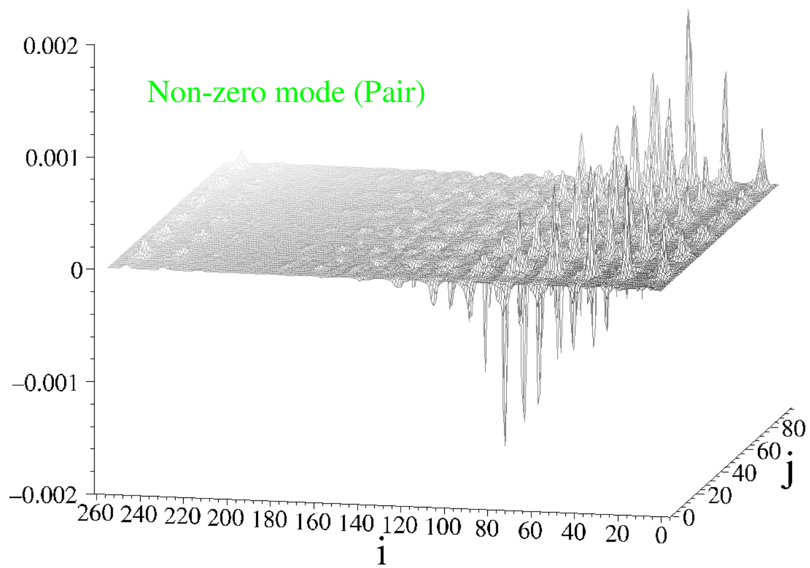
- Previous N_f=0 & 2 calculations show agreement in vector (V) and axial-vector (AV), but not in scalar (S) and pseudoscalar (PS)
- Have zero mode contributions: look at Q=0, subtract zero-mode, or compare differences
- Parity doubling apparently seen
- Disagreements with other calc. On density of near-zero modes. Volume?

Gavai, et.al., hep-lat/0107022

Thermodynamics - Localization of Eigenstates



$i = x + 16y, \quad j = z + 16t, \quad 16^3 \times 6$ lattice



- SU(3) gauge theory: No cooling or smearing
- Chiral fermion: in deconfined phase of N_t=6 transition, see spatial but not temporal localization of state
- Also seen with Staggered fermions
- More quantitatively, participation ratio shows change crossing transition
- Consistent with caloron-anti-caloron pair (molecule)

Gattringer, et.al., hep-lat/0105023; Göckeler, et.al., hep-lat/0103021

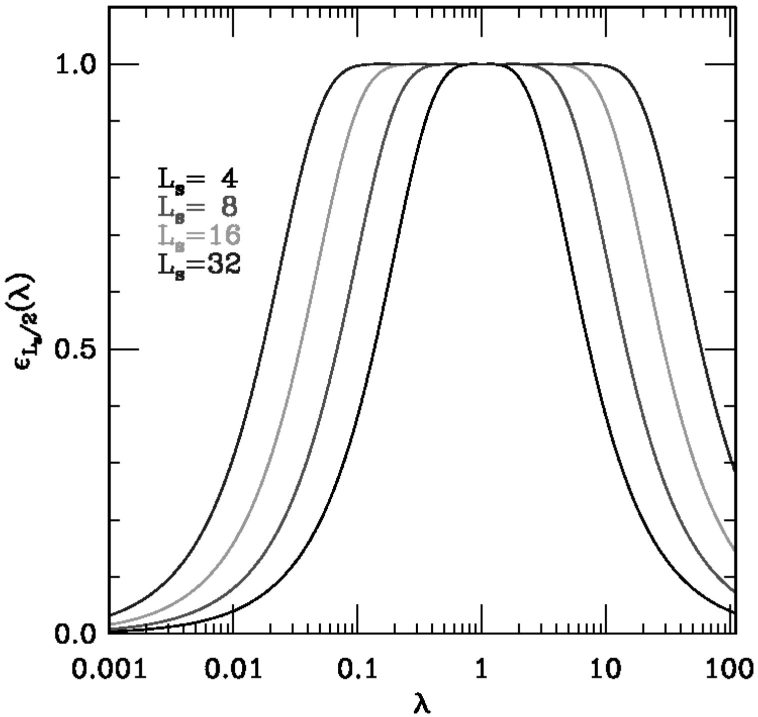
Chiral Fermions

Chiral fermions for vector gauge theories (Overlap/DWF)

- Many ways to implement (See talk by Hernandez; Vranas, Lat2000)
  - 4D (Overlap), 5D (DWF) which is equivalent to a 4D Overlap
  - 4D Overlap variants recasted into 5D (but not of domain wall form)
  - Approx. solutions to GW relation
- Implementations affected by (near) zero modes in underlying operator kernel (e.g., super-critical hermitian Wilson)
  - Induced quark mass in quenched extensively studied in DWF (Columbia/BNL, CPPACS) – implies fifth dimension extent dependence on coupling
  - For 4D and 5D variants, can eliminate induced mass breaking with projection – in principle for both quenched and dynamical cases (Vranas Lat2000)
  - No free lunch theorem – projection becomes more expensive at stronger couplings. One alternative: with no projection go to weak coupling and live with induced breaking

Implementation of a Chiral Fermion

$$D_{\text{overlap}}(0) = \frac{1}{2}\left[1 + \gamma_5 \varepsilon_{L_s/2}\left(H(-M)\right)\right]$$



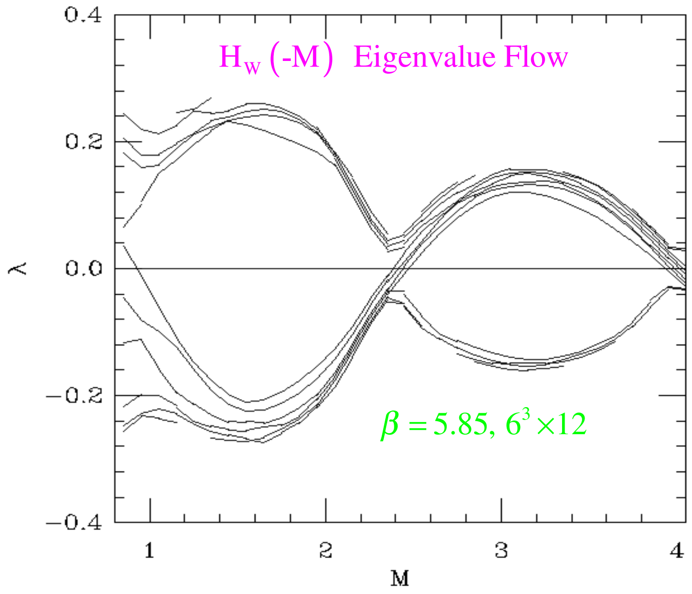
- Overlap-Dirac operator defined over a kernel H(-M). E.g., hermitian Wilson-Dirac operator. Approximation to a sign-function projects eigenvalues to ±1
- DWF (with 5D extent L_s) operator equivalent after suitable projection to 4D
- Chiral symmetry recovered as $L_s \to \infty$
- (Near) zero eigenvalues of H(-M) outside approximation break chiral symmetry
- Straightforward to fix by projection – use lowest few eigenvectors to move eigenvalues of kernel to ±1. Also, works for 5D variants

Neuberger, 1997, Edwards, et.al., hep-lat/9905028, 0005002,
Narayanan&Neuberger, hep-lat/0005004, Hernandez, et.al., hep-lat/0007015

Spectral Flow

$$D_{\text{overlap}}(0) = \frac{1}{2}\left[1+\gamma_5 \varepsilon\left(H_W(-M)\right)\right]$$

$$Q = \mathrm{Tr}\left(\gamma_5 D_{\text{overlap}}(0)\right) = \frac{1}{2}\mathrm{Tr}\,\varepsilon\left(H_W(-M)\right)$$

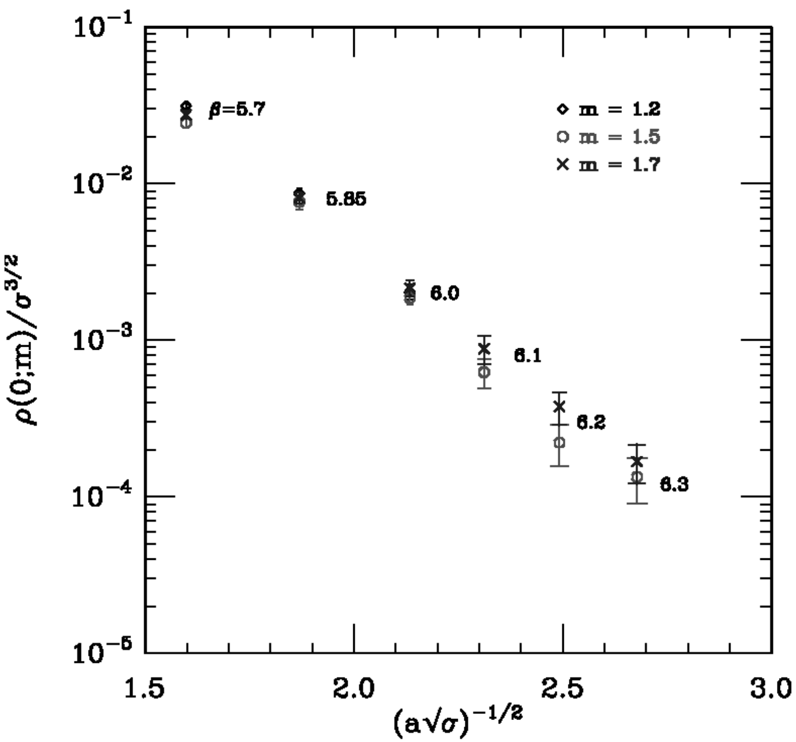


- One way to compute index Q is to determine number of zero modes in a background configuration
- Spectral flow is a way to compute Q which measures deficit of states of (Wilson) H
- Flow shows for a background config how Q changes as a function of *regulator* parameter M in doubler regions. Here Q goes from –1 to 3=4-1 to –3 = 3-6

- No multiplicative renormalization of resulting susceptibility (Giusti, et.al., hep-lat/0108009)

Narayanan, Lat 98; Fujiwara, hep-lat/0012007

Density of Zero Eigenvalues

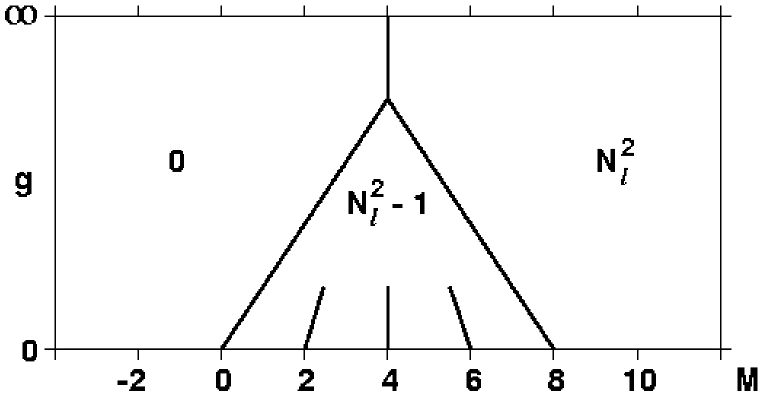


- Non-zero density of H(-M) observed
- Class of configs exist that induce small-size zero-modes of H(-M), so exist at all non-zero gauge coupling – at least for quenched gauge (Wilson-like) theories; called dislocations
- In 5D, corresponds to tunneling between walls where chiral pieces live
- **NOT** related to (near) zero-eigenvalues of chiral fermion operators accumulating to produce a diverging chiral condensate
- Can be significantly reduced by changing gauge action. Ideal limit (??) is RG fixed point action – wipes out dislocations. Also restricts change of topology
- Possibly finite (localized) states – do not contribute in thermodynamic limit?

Edwards, et.al., hep-lat/9901015, Berrutto, et.al., hep-lat/0006030, Ali Khan, et.al., hep-lat/0011032; Orginos, Taniguchi, Lat01

Chiral Fermions at Strong Coupling

G&S Proposed Goldstone phases



$$D_{\text{overlap}}(0) = \frac{1}{2}\left[1+\gamma_5 \varepsilon\left(H_W(-M)\right)\right]$$

$$Q = \mathrm{Tr}\left(\gamma_5 D_{\text{overlap}}(0)\right) = \frac{1}{2}\mathrm{Tr}\,\varepsilon\left(H_W(-M)\right)$$

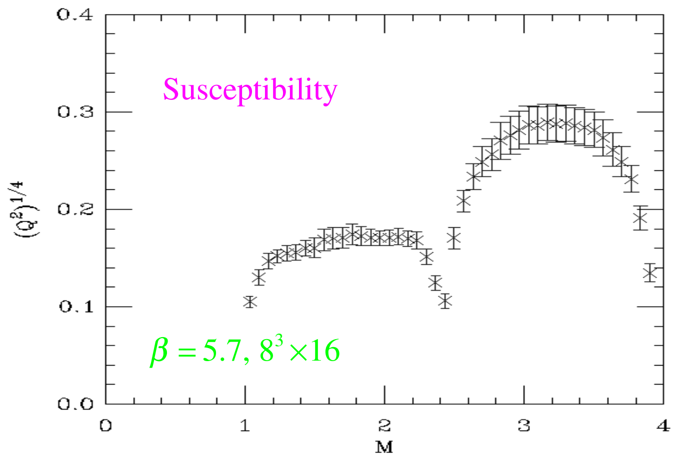
- Recent calculations disagree over fate of chiral fermions in strong coupling limit
- Do chiral fermions become massive as coupling increases? (Berrutto, et.al.)
- And/or do they mix with doubler modes and replicate? (Golterman&Shamir, Ichinose&Nagao)
- Concern is if there is a phase transition from doubled phase to a single flavor phase (e.g., into the region M=0 to 2)

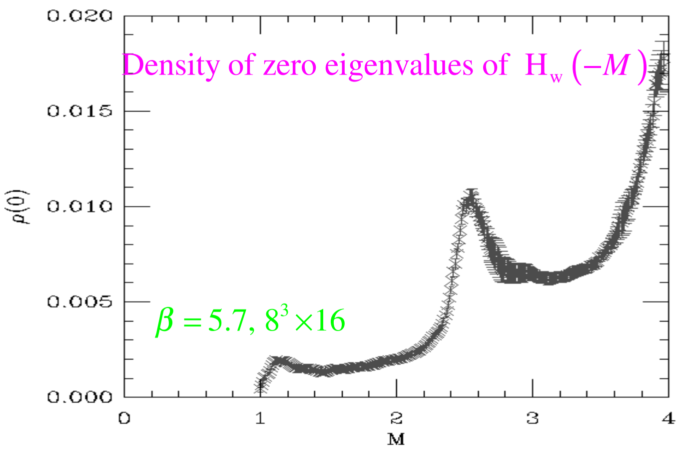
- Can study using spectral flow to determine topological susceptibility

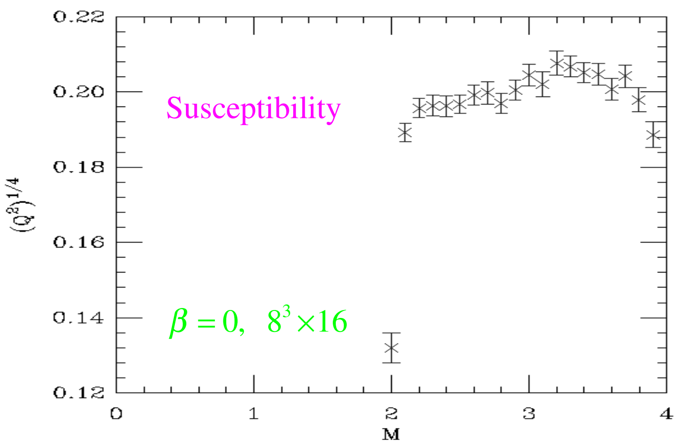
Golterman & Shamir, hep-lat/0007021; Berrutto, et.al., hep-lat/0105016; Ichinose & Nagao, hep-lat/0008002

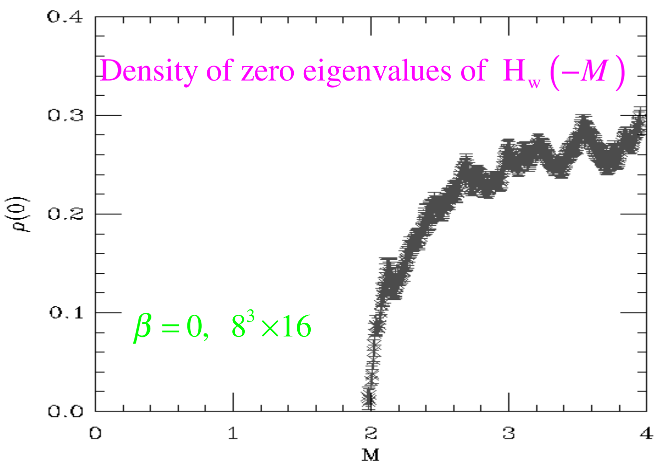
Mixing with Doublers

- As coupling increases, regions of distinct topological susceptibility merge
- Apparent mixing of all doubler regions









Conclusions

- No surprise – eigenmodes provide powerful probe of vacuum
- Technical uses: some examples of how eigenmodes can be used to improve statistics – spectral sum methods
- Chiral fermions:
  - Many studies using fermionic modes in quenched theories
  - Obviously need studies with dynamical fermions