THE MICROSCOPIC DIRAC OPERATOR SPECTRUM

– EXACT RESULTS –

- CHIRAL SYMMETRY BREAKING, GAUGE FIELD TOPOLOGY

- ANALYTICAL TOOLS:

  * RANDOM MATRIX THEORY

  * "SUPERSYMMETRY"

  * REPLICA METHOD

- LATTICE RESULTS

- BEYOND RANDOM MATRIX THEORY ...

GAUGE THEORY OF GAUGE GROUP G

N_f FERMIONS CARRYING ANY REPRESENTATION r

ASSUME SPONTANEOUS BREAKING OF CHIRAL SYMMETRY

$\Rightarrow$ (PSEUDO) GOLDSTONE BOSONS, MASS m_π

CHIRAL CONDENSATE

$$\Sigma \equiv \lim_{m \to 0} \lim_{V \to \infty} \langle \bar{\psi} \psi \rangle$$

FINITE VOLUME $L \gg \frac{1}{\Lambda_{QCD}}$

EUCLIDEAN PARTITION FUNCTION DOMINATED BY GOLDSTONE BOSONS

NOW LET $V \to \infty$ SO THAT $L \ll \frac{1}{m_\pi}$

WHILE KEEPING

$$\mu_i \equiv m_i \Sigma V$$

FIXED

m_i: FERMION MASSES

CHIRAL SYMMETRY BREAKING VS. RANDOM MATRIX THEORY :

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r **COMPLEX:** [EX.: $G = SU(N_c \geq 3)$, r = FUND. REP.]

$$SU_L(N_f) \times SU_R(N_f) \rightarrow SU(N_f)$$

RMT : chUE ("CHIRAL UNITARY ENSEMBLE"), $\beta = 2$.

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r **PSEUDO-REAL:** [EX.: $G = SU(2)$, r = FUND. REP.]

$$SU(2N_f) \rightarrow Sp(2N_f)$$

RMT : chOE ("CHIRAL ORTHOGONAL ENSEMBLE"), $\beta = 1$

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r **REAL:** [EX.: $G = SU(N_c \geq 2)$, r = ADJ. REP.]

$$SU(2N_f) \rightarrow SO(2N_f)$$

RMT : chSE ("CHIRAL SYMPLECTIC ENSEMBLE"), $\beta = 4$

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* ALL PATTERNS CONSISTENT WITH VAFA-WITTEN THRI

* RELATION TO RMT: ZIRNBAUER '92

→ VERBAARSCHOT '95 ←

TOPOLOGICAL CHARGE $\nu \equiv \frac{1}{32\pi^2}\int d^4x\, F_{\mu\nu}\tilde{F}^{\mu\nu}$

PARTITION FUNCTION

$$Z = \sum_{\nu=-\infty}^{\infty} e^{i\nu\theta}\int [dA]_\nu [d\bar{\psi}\, d\psi]\, e^{-S[A,\bar{\psi},\psi]}$$

$$\equiv \sum_{\nu=-\infty}^{\infty} e^{i\nu\theta} Z_\nu$$

$$\Rightarrow \quad Z_\nu = \frac{1}{2\pi}\int_0^{2\pi} d\theta\, e^{-i\nu\theta} Z$$

AS $m_\pi \to 0$ GOLDSTONE MODES DOMINATE Z

TO AS HIGH ACCURACY WE LIKE

$$Z = \int_{G/H} dU\, \exp\left[V\Sigma\, \mathrm{Re}\left[e^{i\theta/N_f}\, \mathrm{Tr}[MU^\dagger]\right]\right]$$

FOR QCD, $G/H = SU(N_f)$, AND

$$\boxed{Z_\nu = \int_{U(N_f)} dU\, (\det U)^\nu \exp\left[V\Sigma\, \mathrm{Re}\,\mathrm{Tr}[MU^\dagger]\right]}$$

THIS IS ONLY A FUNCTION OF $\mu_i = m_i \Sigma V$!

LEUTWYLER-SMILGA '93

SURPRISE : RANDOM MATRIX THEORY

CHIRAL RANDOM MATRIX THEORY:

$$\tilde{Z}_\nu \equiv \int dW \prod_{f=1}^{N_f} \det(iM + \tilde{m}_f)\, e^{-N\,\mathrm{Tr}\,V(M^2)}$$

- $\int dW$: COMPLEX $N \times (N+\nu)$ MATRIX
- M: $M \equiv \begin{pmatrix} 0 & W \\ W^\dagger & 0 \end{pmatrix}$
- $e^{-N\,\mathrm{Tr}\,V(M^2)}$: POTENTIAL, UNSPECIFIED!

SHURYAK-VERBAARSCHOT '9

$$M\tilde{\phi}_n = \tilde{\lambda}_n \tilde{\phi}_n$$

($N+\nu$, N)

$$\tilde{\gamma}^5 \equiv \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}: \qquad \{M, \tilde{\gamma}^5\} = 0 \Rightarrow \pm\tilde{\lambda}_n$$

$+\ \nu$ ZERO MODES

C.F. DIRAC OPERATOR EIGENVALUES:

$$\not{D}\phi_n = \lambda_n \phi_n$$

$$\{\not{D}, \gamma^5\} = 0 \Rightarrow \pm\lambda_n$$

BUT: A PRIORI NO CONNECTION TO RANDOM MATRIX THEORY!

MICROSCOPIC LIMIT AND UNIVERSALITY:

DENSITY OF EIGENVALUES OF RANDOM MATRIX: $\tilde{\rho}(\tilde{\lambda})$

$\tilde{\rho}(0) \neq 0$: LET $\tilde{\Sigma} \equiv \pi \tilde{\rho}(0)$

> CF. BANKS-CASHER RELATION
> FOR DIRAC OPERATOR: $\Sigma = \pi\rho(0)$

$\mu_i \equiv \tilde{\Sigma}(2N)\,\tilde{m}_i$ FIXED AS $N \to \infty$

> CF. $\mu_i \equiv \Sigma V m_i$ FIXED AS $V \to \infty$
>
> ANALOGY: $\tilde{\Sigma} \leftrightarrow \Sigma$
>
> $2N \leftrightarrow V$
>
> $\tilde{m}_i \leftrightarrow m_i$

THEN:

$$\tilde{Z}_\nu(\mu_1, \ldots, \mu_{N_f}) = Z_\nu(\mu_1, \ldots, \mu_{N_f})$$

SHURYAK-VERBAARSCHOT '94

$\therefore$ RANDOM MATRIX THEORY: EXACT REPRESENTATION OF QCD PARTITION FUNCTION IN FINITE-VOLUME SCALING REGIME

$$\frac{1}{\Lambda_{QCD}} \ll L \ll \frac{1}{m_\pi}$$

UNIVERSALITY THRM.: INDEPENDENT OF $V(M^2)$

AKEMANN, D., MAGNEA, NISHIGAKI '97

EIGENVALUES: BLOW UP THE SCALE

$$\zeta \equiv \tilde{\Sigma} \cdot 2N \cdot \tilde{\lambda} \quad \underline{\text{FIXED}} \text{ AS } N \to \infty$$

CF. $\xi \equiv \Sigma V \lambda$ FIXED AS $V \to \infty$ IN QCD
Σ: DIRAC OP. EIGENV.

ALL SPECTRAL CORRELATION FUNCTIONS PROVEN TO BE UNIVERSAL.

AKEMANN, D., MAGNEA, NISHIGAKI '97

MASTER FORMULA:

$$K(\zeta,\zeta';\{\mu_f\}) = \sqrt{\zeta\zeta'}\,\prod_f \sqrt{(\zeta^2+\mu_f^2)(\zeta'^2+\mu_f^2)}\;\frac{Z_\nu^{(N_f+2)}(\{\mu_f\},i\zeta,i\zeta')}{Z_\nu^{(N_f)}(\{\mu_f\})}$$

D. '98, AKEMANN, D. '98

$$\tilde{\rho}(\zeta_1,\ldots,\zeta_k;\{\mu_f\}) = \det_{a,b} K(\zeta_a,\zeta_b;\{\mu_f\})$$

SPECIAL CASE: SPECTRAL DENSITY:

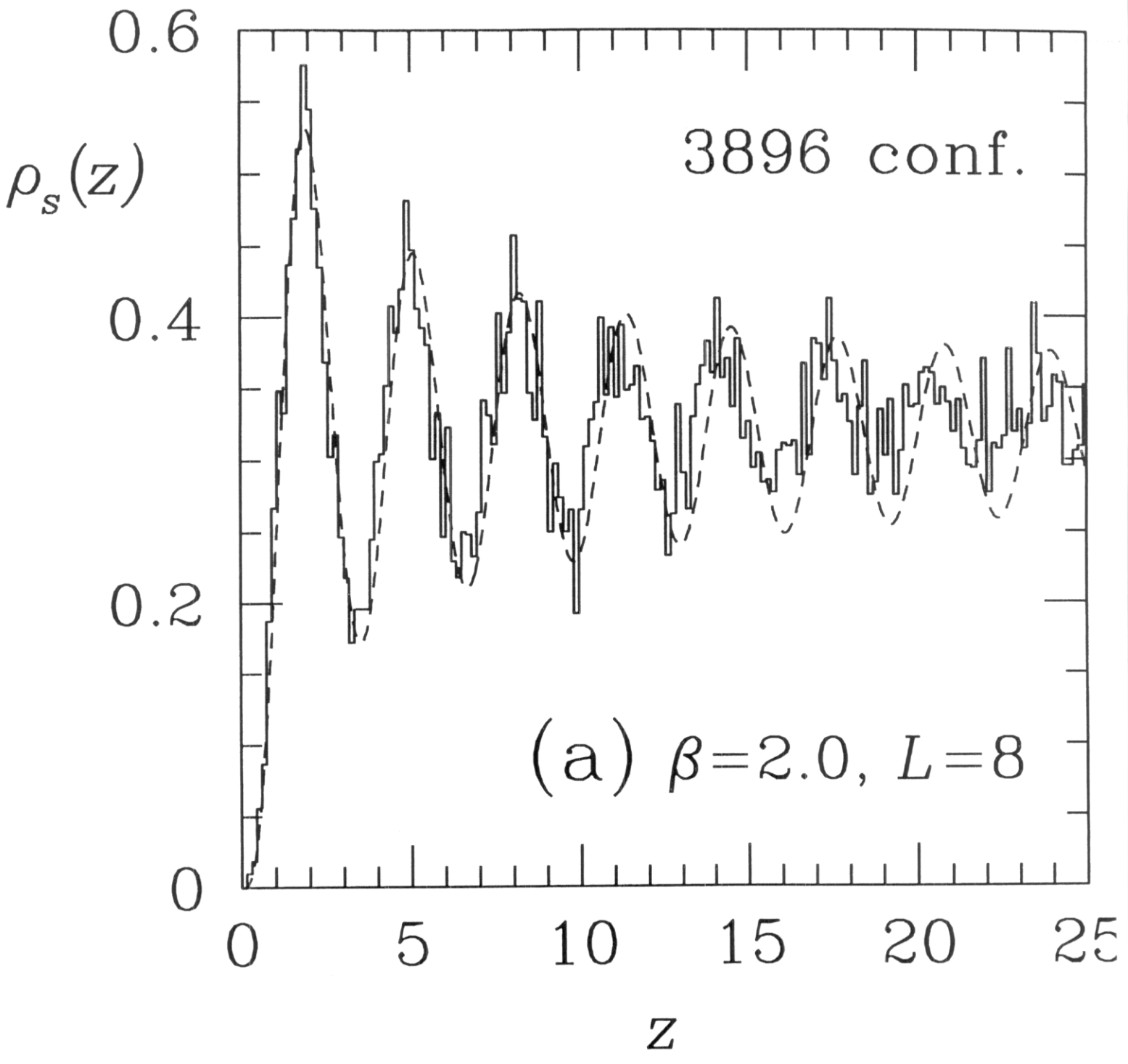
$$\tilde{\rho}(\zeta;\{\mu_f\}) = K(\zeta,\zeta;\{\mu_f\})$$

EXAMPLE: N_f MASSLESS FERMIONS:

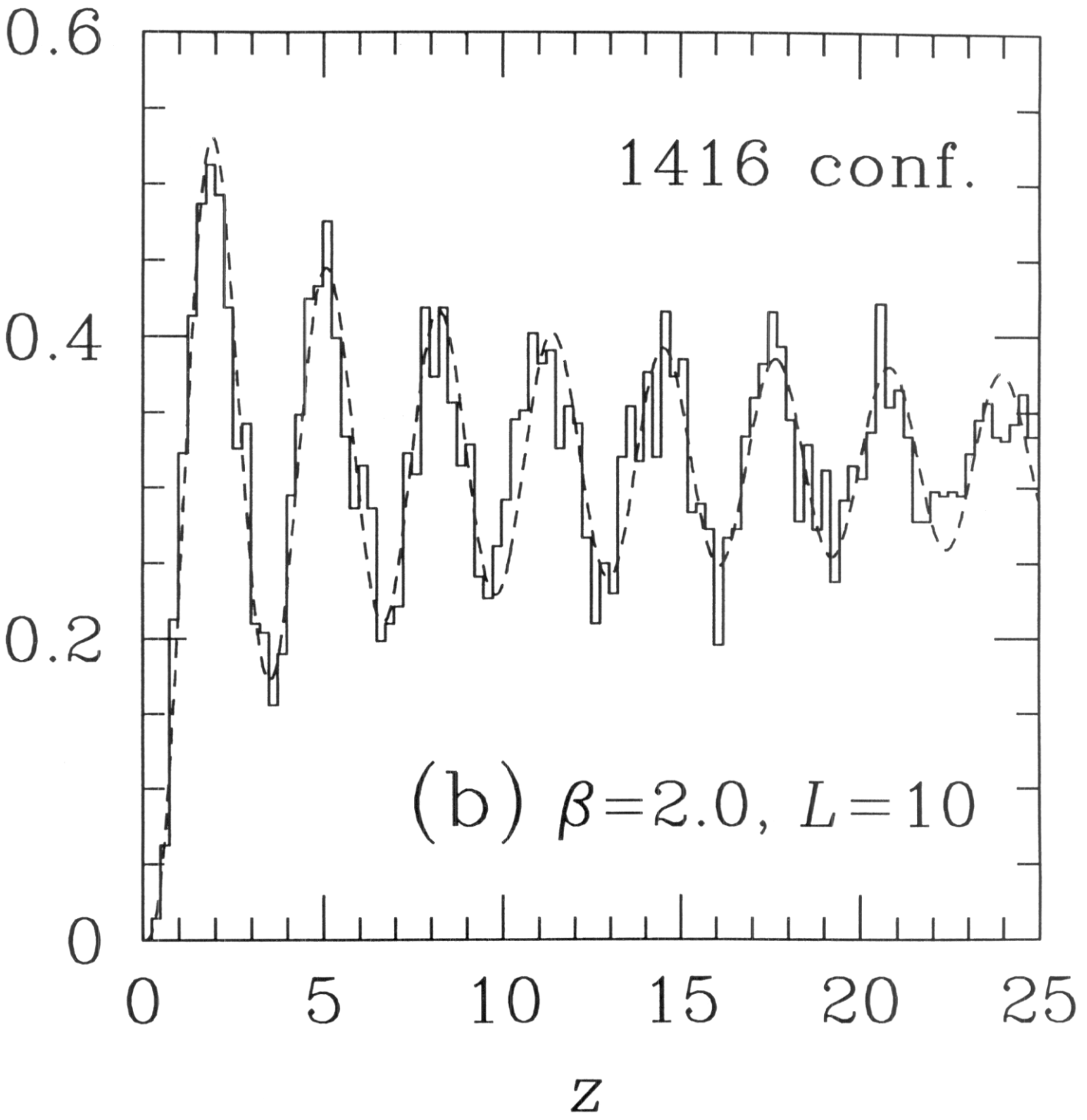
$$\tilde{\rho}(\zeta) = \tfrac{1}{2}\zeta\left[J_{N_f+\nu}(\zeta)^2 - J_{N_f+\nu+1}(\zeta)J_{N_f+\nu-1}(\zeta)\right]$$

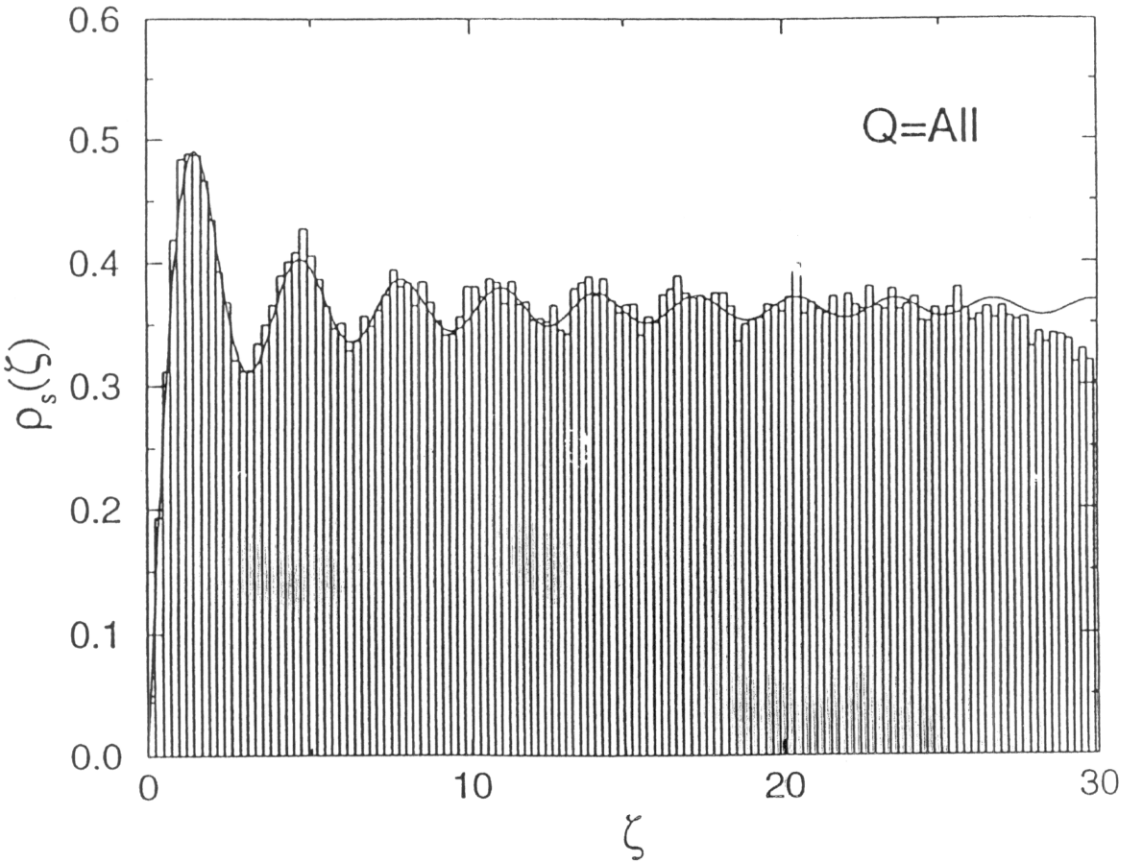
VERBAARSCHOT-ZAHED '95

SU(2) QUENCHED (STAGGERED):



BERBENNI-BITSCH, MEYER, SCHÄFER, VERBAARSCHOT, WETTIG '97


0.6
0.4
0.2
0
1416 conf.
(b) β=2.0, L=10
0
5
10
15
20
25
z

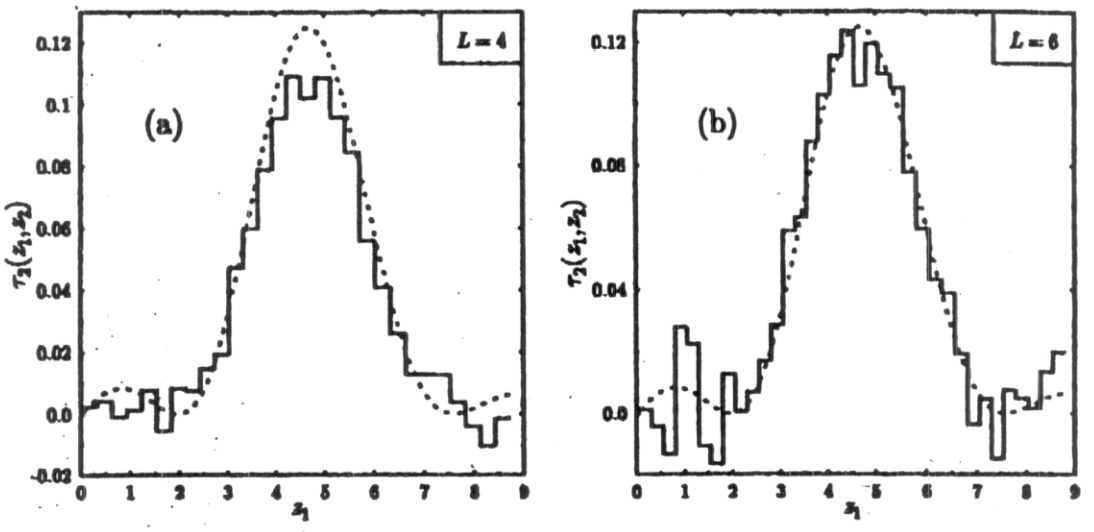


D., HELLER, NICLASEN, RUMMUKAINEN '99

SU(3), STAGGERED, QUENCHED

$g_s^c(\xi_1, \xi_2)$

SU(3) QUENCHED:

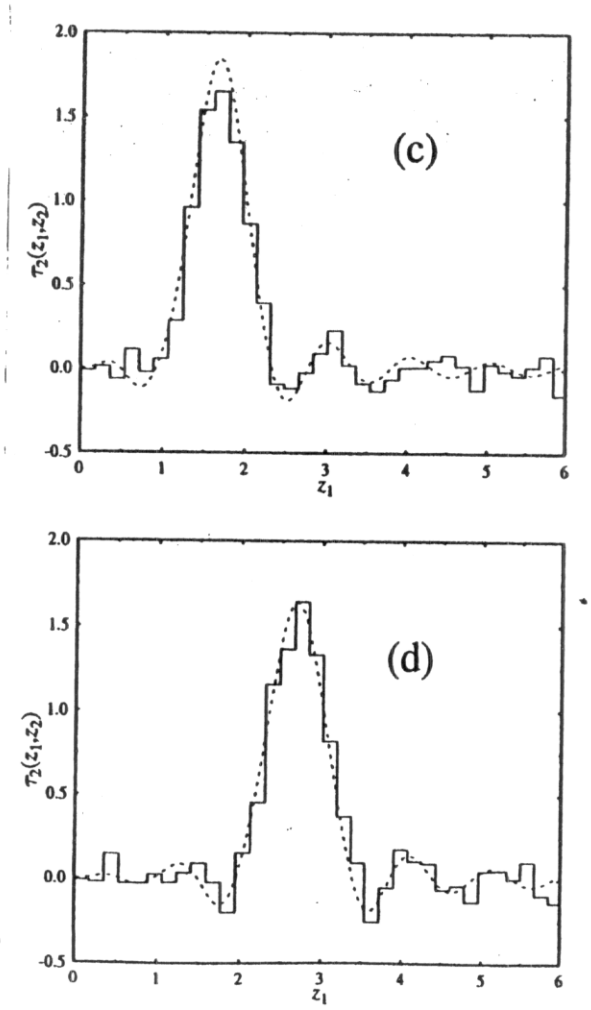


$\beta = 5.4$

GÖCKELER, HEHL, RAKOW, SCHÄFER, WETTIG '98

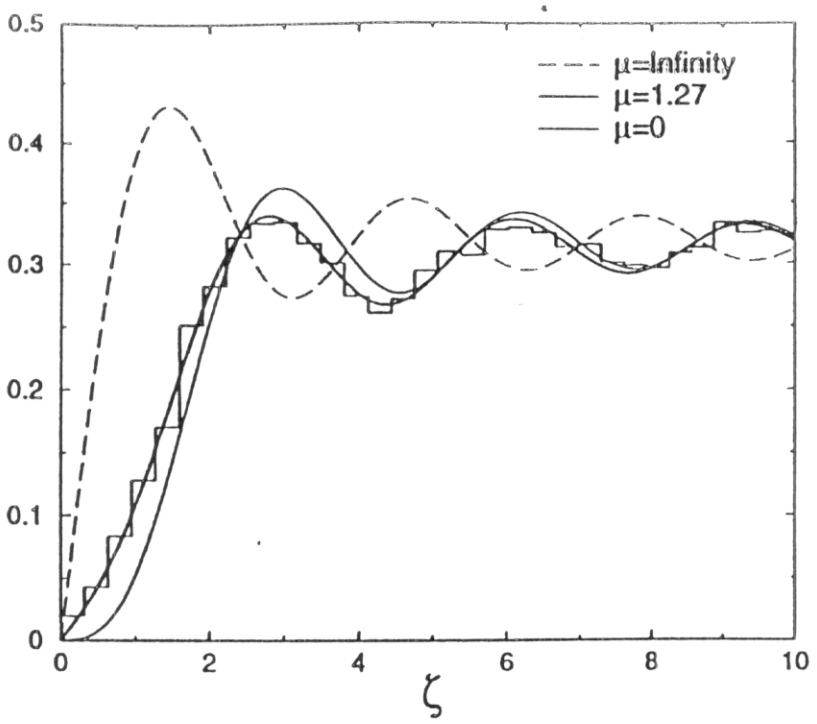
SU(2) QUENCHED (STAGGERED):

8^4, $\beta = 2.0$

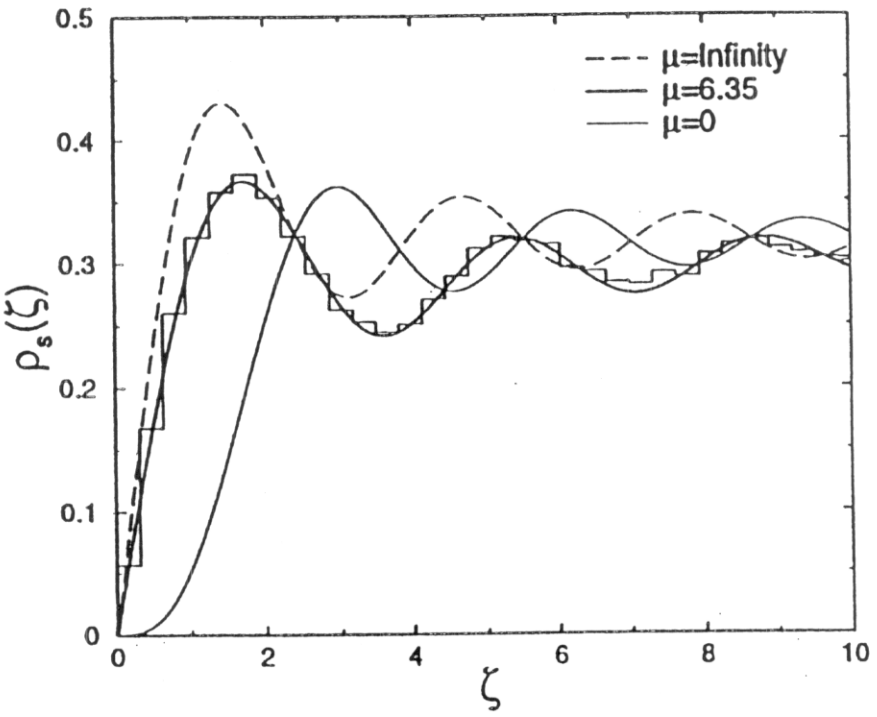


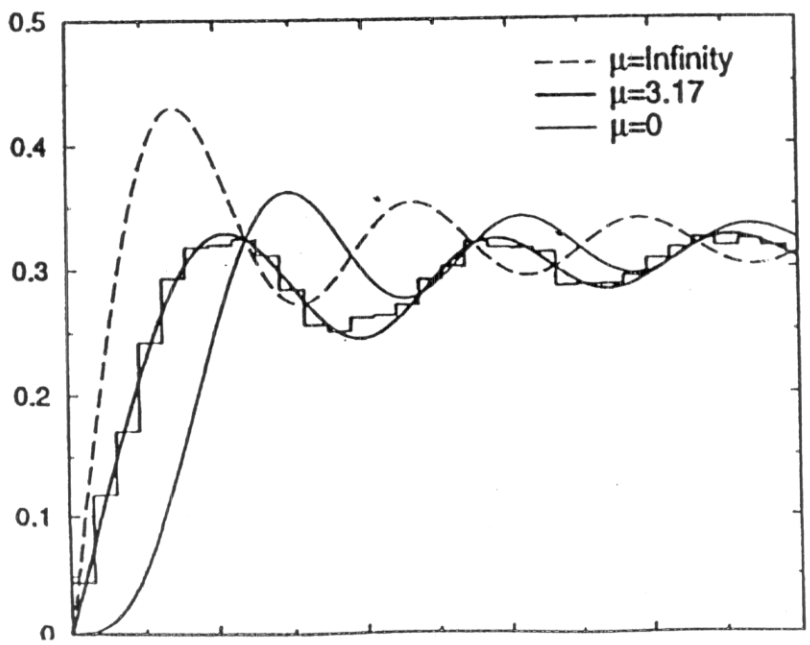
MA, GUHR, WETTIG '98

SU(3)
STAGGERED
~~ANISOTROPIC~~



D., HELLER, NICLASE
RUMMUKAINEN '0





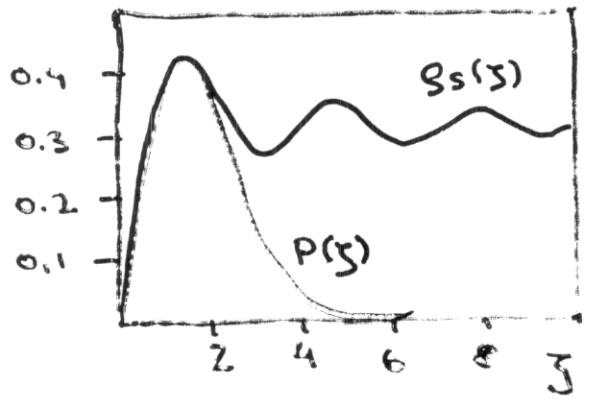
QUESTION: CAN WE ALSO COMPUTE INDIVIDUAL EIGENVALUE DISTRIBUTIONS OF THE DIRAC OPERATOR?

1ST EIGENVALUE:

$$P(\zeta, \{\mu\}) = -\frac{\partial}{\partial \zeta}\left[e^{-\zeta^2/4} \; \frac{Z(\{\sqrt{\mu^2+\zeta^2}\})}{Z(\{\mu\})} \right] \qquad (\beta=2)$$

WILKE, GUHR, WETTIG '98

NISHIGAKI, D., WETTIG '98



PEAKS IN $\rho_s(\zeta)$ SHOULD CORRESPOND TO INDIVIDUAL EIGENVALUES!

- TURNS OUT TO BE TRUE...

CAN BE GENERALIZED:

- ANY PATTERN OF CHIRAL SYMMETRY BREAKING ($\beta = 1, 2, 4$)
- ANY TOPOLOGICAL CHARGE ν
- ANY NUMBER OF MASSIVE/MASSLESS FERMIONS N_f
- EXACT DISTRIBUTION OF k^{th} DIRAC OPERATOR EIGENVALUE

TRICK: USE RMT REPRESENTATION

$W(\zeta_1, \ldots, \zeta_k)$ = JOINT PROBABILITY DENSITY OF k SMALLEST EIGENVALUES $\{0 \le \zeta_1 \le \ldots \le \zeta_k\}$

$$= e^{-\beta\zeta_k^2/8}\, \zeta_k^{\beta\left(\frac{\nu+1}{2}\right)-\nu-1+\frac{2}{\beta}} \left(\prod_{j=1}^{N_f} \mu_j^{\nu} \left(\mu_j^2+\zeta_k^2\right)^{\frac{1}{2}-\frac{1}{\beta}}\right) \times$$

$$\times \prod_{i=1}^{k-1} \left(\zeta_i^{\beta(\nu+1)-1} \left(\zeta_k^2-\zeta_i^2\right)^{\frac{\beta}{2}-1} \prod_{j=1}^{N_f}\left(\zeta_i^2+\mu_j^2\right)\right) \prod_{i>j}^{k} \left(\zeta_i^2-\zeta_j^2\right)^{\beta} \times$$

$$\times Z_{1+2/\beta}\left(\{\sqrt{\mu_i^2+\zeta_k^2}\}, \{\sqrt{\zeta_k^2-\zeta_1^2}\}, \ldots, \{\sqrt{\zeta_k^2-\zeta_{k-1}^2}\}, \{\zeta_k\}\right)/Z_\nu$$

(N_f, β, $\beta(\nu+1)/2-1$)

D., NISHIGAKI '00

DISTRIBUTION OF k^{th} SMALLEST EIGENVALUE:

$$P_k(\zeta_k) = \frac{1}{(k-1)!} \int_0^{\zeta_k} d\zeta_1 \cdots d\zeta_{k-1}\, W(\zeta_1, \ldots, \zeta_k)$$

- EXPLICIT ANALYTICAL FORMULAS

EXAMPLE: SMALLEST EIGENVALUE DISTR.

$\beta=2$ UNIVERSALITY CLASS [E.G. SU(3), N_f FUND.]:

QUENCHED:

$$P_1(\xi) = \frac{1}{2}\xi\, e^{-\xi^2/4} \det[I_{2+i-j}(\xi)]$$

MATRIX SIZE $\nu\times\nu$

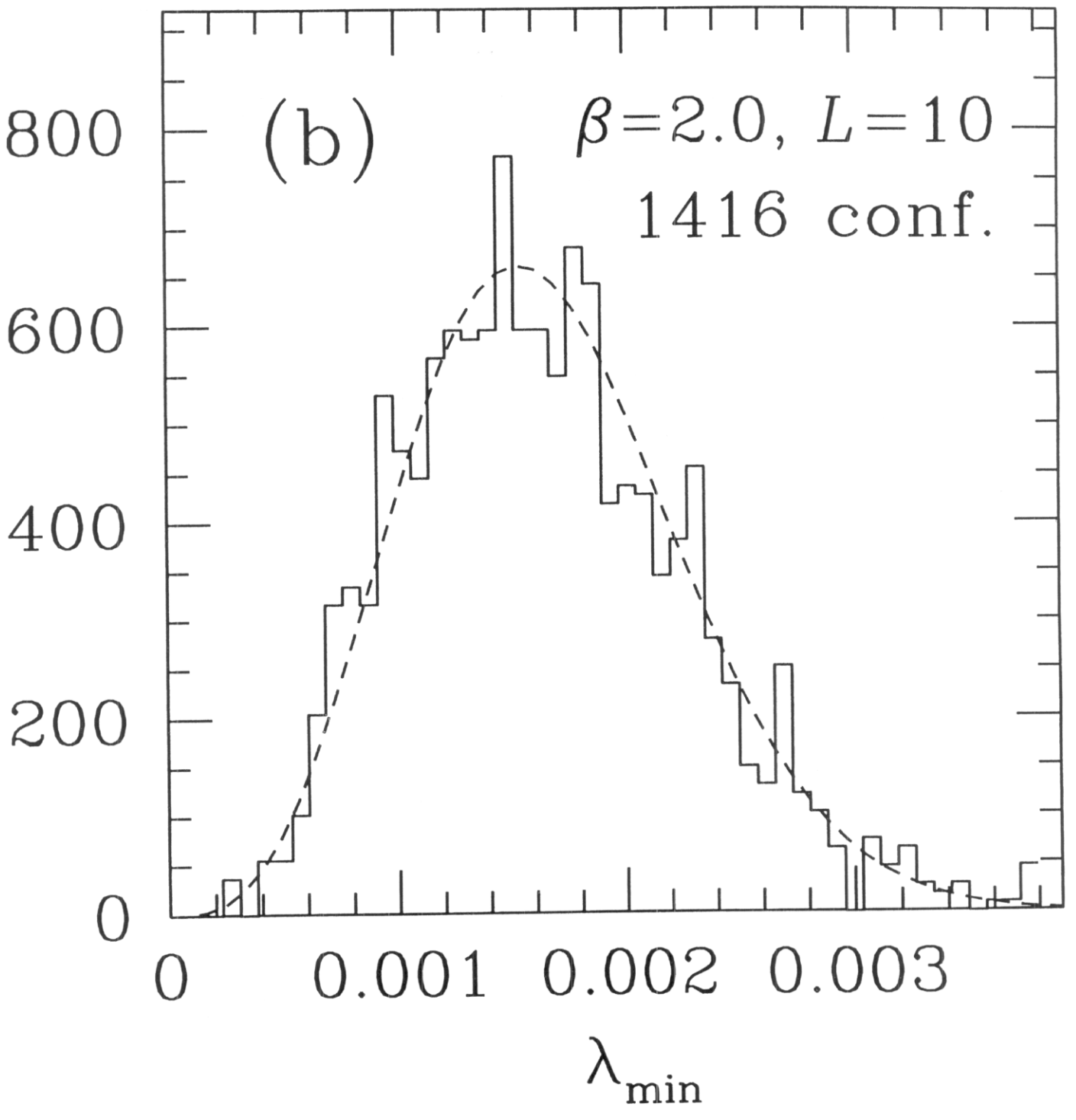
$\beta=1$ UNIVERSALITY CLASS [E.G. SU(2), N_f FUND.]:

QUENCHED:

$$P_1(\xi) = \xi^{\frac{3-\nu}{2}} e^{-\xi^2/8} \mathrm{Pf}[(i-j) I_{i+j+3}(\xi)]$$

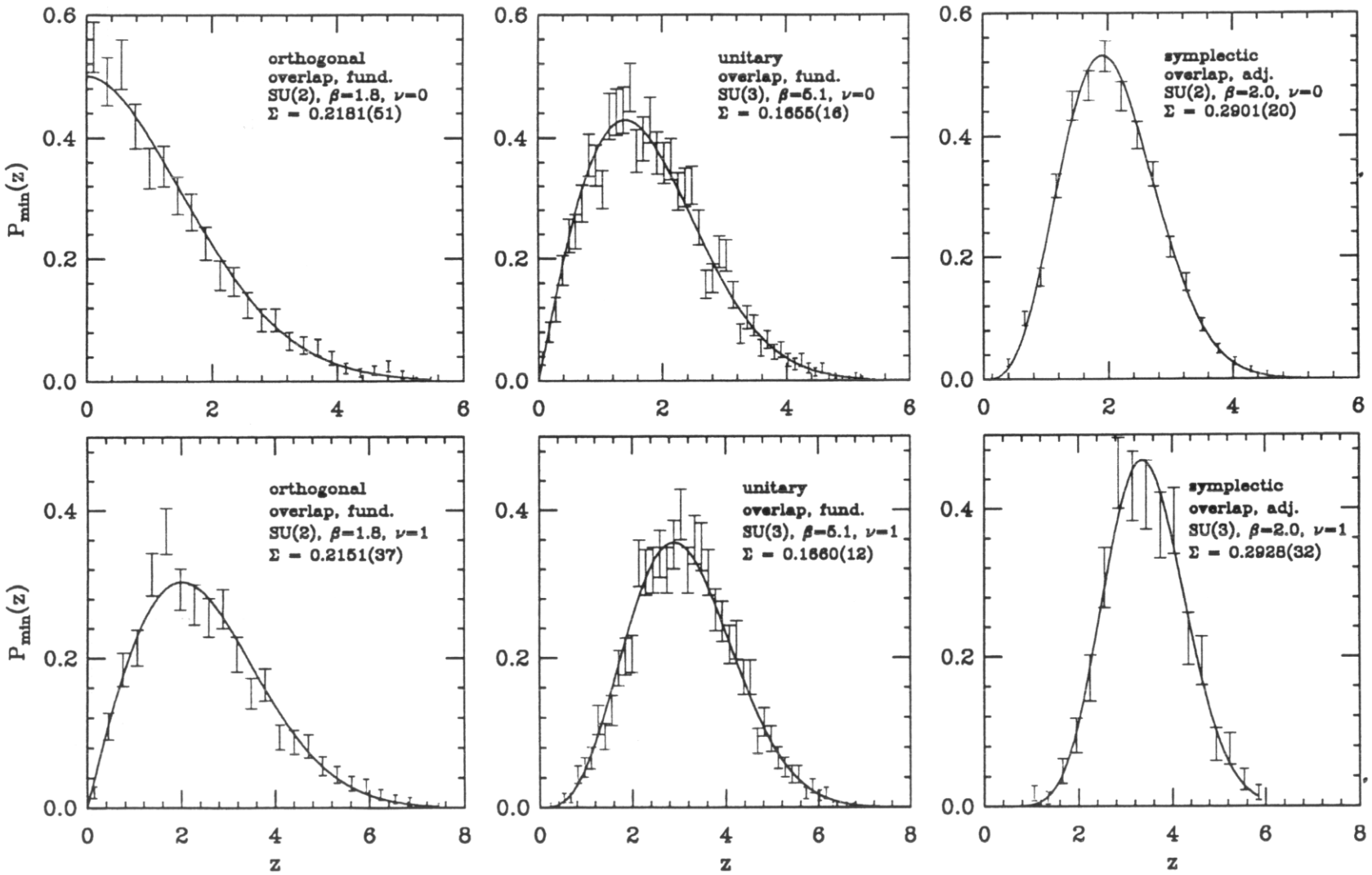
MATRIX SIZE $(\nu+1)\times(\nu+1)$

SU(2) STAGGERED (QUENCHED)



BERBENNI-BITSCH ET AL., '97

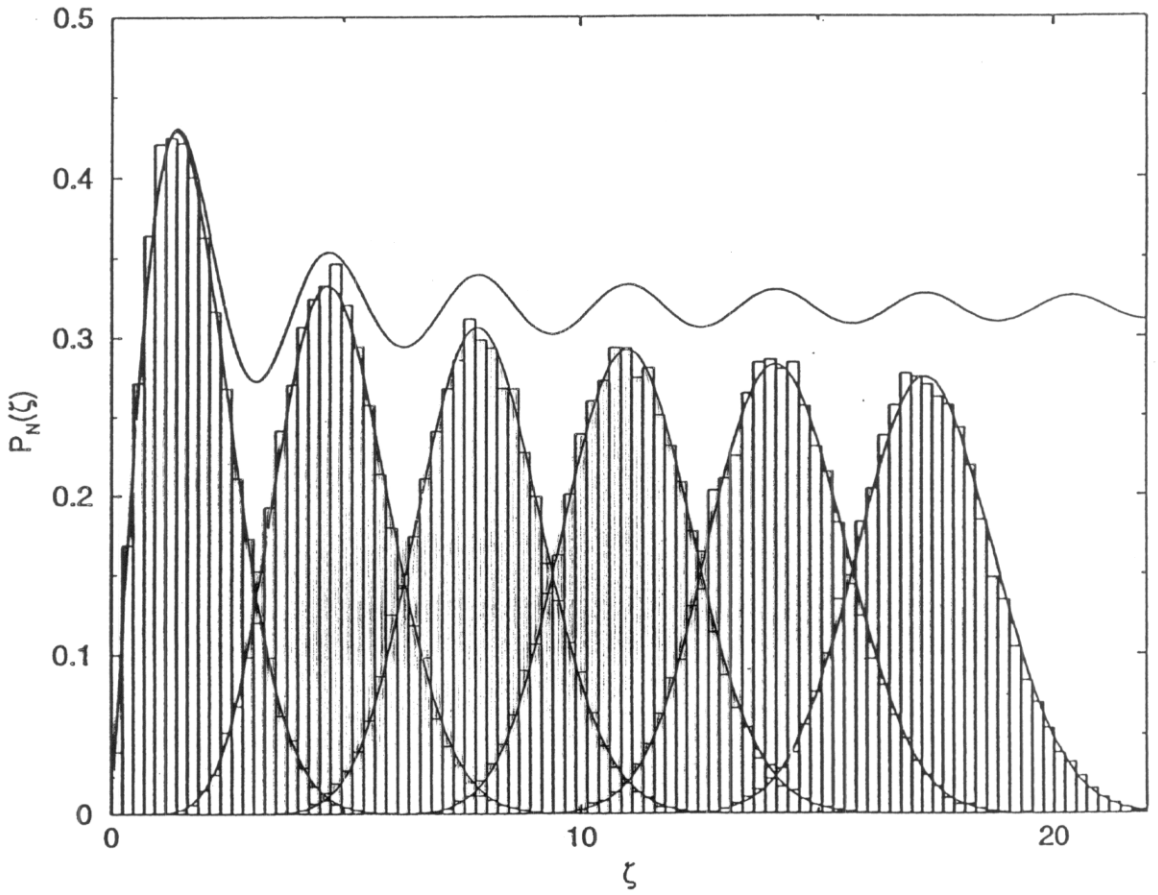
OVERLAP FERMIONS:



EDWARDS, HELLER, KISKIS, NARAYANAN '99

INDIVIDUAL EIGENVALUES:

SU(3) QUENCHED (STAGGERED)



D., HELLER, NICLASEN, RUMMUKAINEN, '00

OTHER OBSERVABLES:

EXAMPLE: SU(3) GAUGE THEORY

N_f MASSLESS FERMIONS, FUND. REP.

$$\rho_s(\xi) = \frac{\xi}{2}\left[J_{N_f+\nu}(\xi)^2 - J_{N_f+1+\nu}(\xi)\, J_{N_f+\nu-1}(\xi)\right]$$

QUENCHED CHIRAL CONDENSATE:

$$\Sigma_\nu(\mu) = 2\mu \int_0^\infty d\xi \, \frac{\rho_s(\xi)}{\xi^2+\mu^2} + \frac{\nu}{\mu}$$

$$= \mu\left[I_\nu(\mu) K_\nu(\mu) + I_{\nu+1}(\mu) K_{\nu-1}(\mu)\right] + \frac{\nu}{\mu}$$

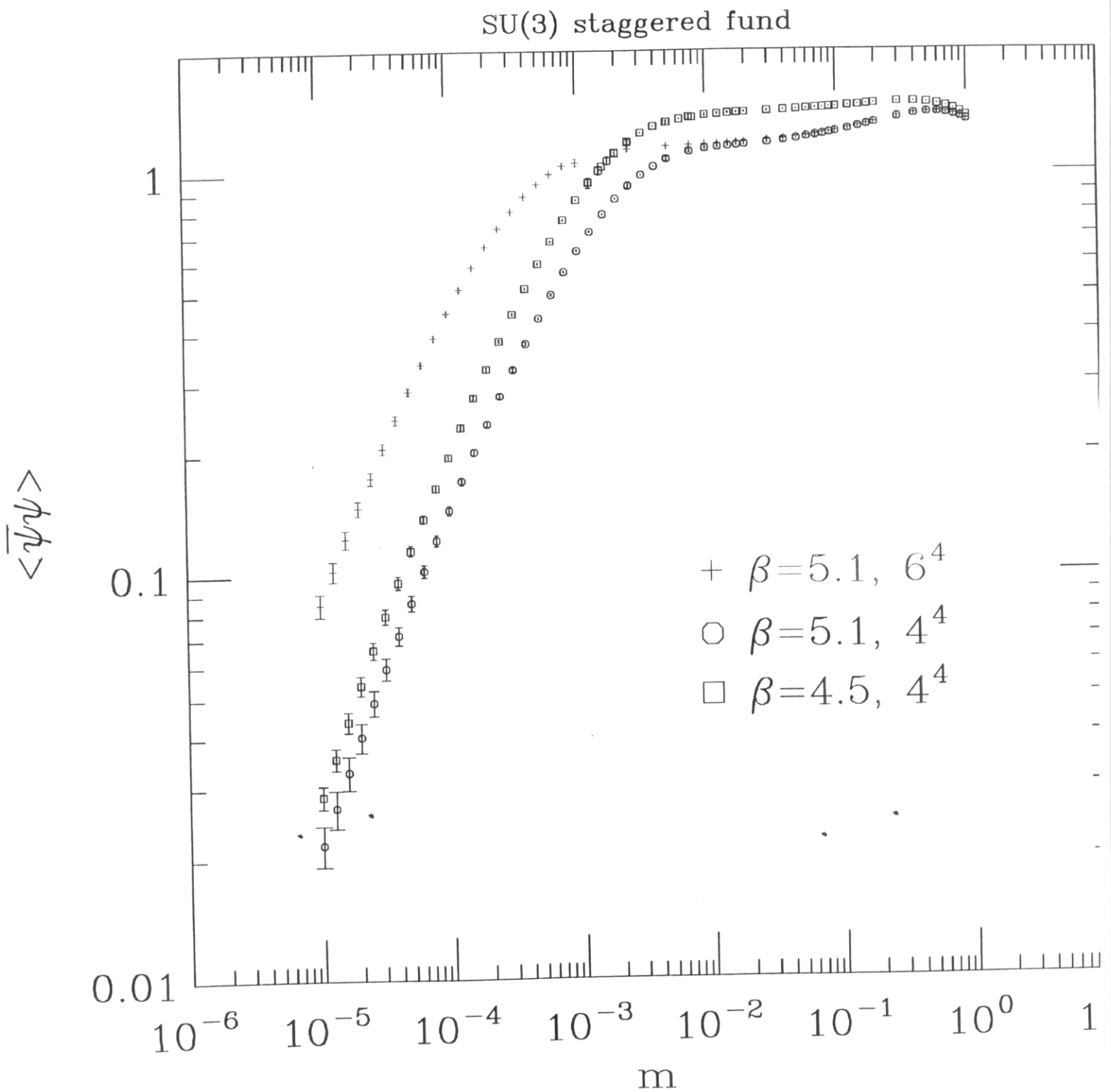
QUENCHED CHIRAL SUSCEPTIBILITY:

$$\frac{\omega_\nu(\mu)}{V} = 4\mu^2 \int_0^\infty d\xi \, \frac{\rho_s(\xi)}{(\xi^2+\mu^2)^2} + \frac{2\nu}{\mu^2}$$

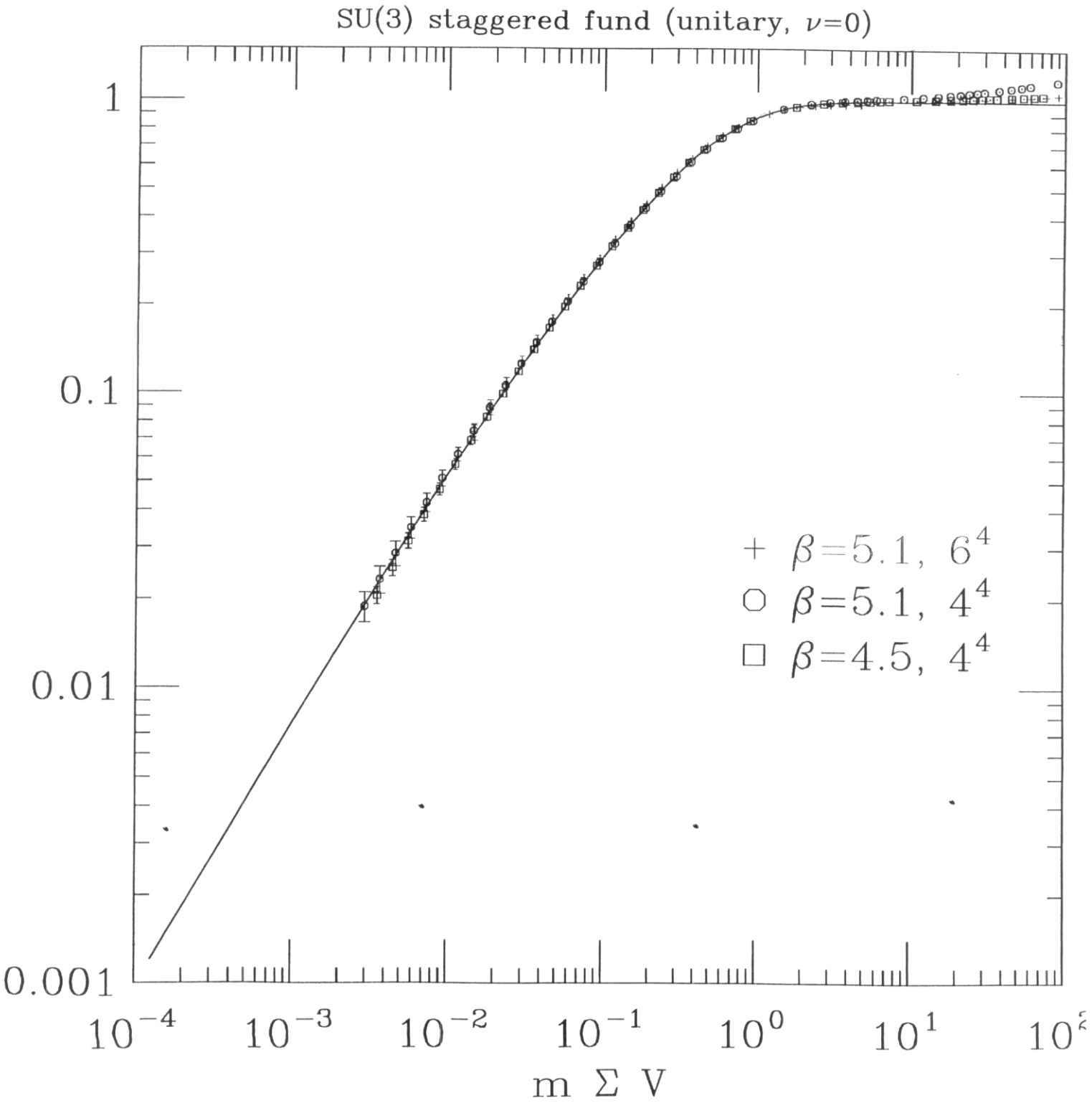
$$= 2\mu\, I_{\nu+1}(\mu) K_{\nu-1}(\mu) + \frac{2\nu}{\mu^2}$$

$$\left(= \frac{\Sigma_\nu(\mu)}{\mu} - \Sigma_\nu'(\mu) \right)$$

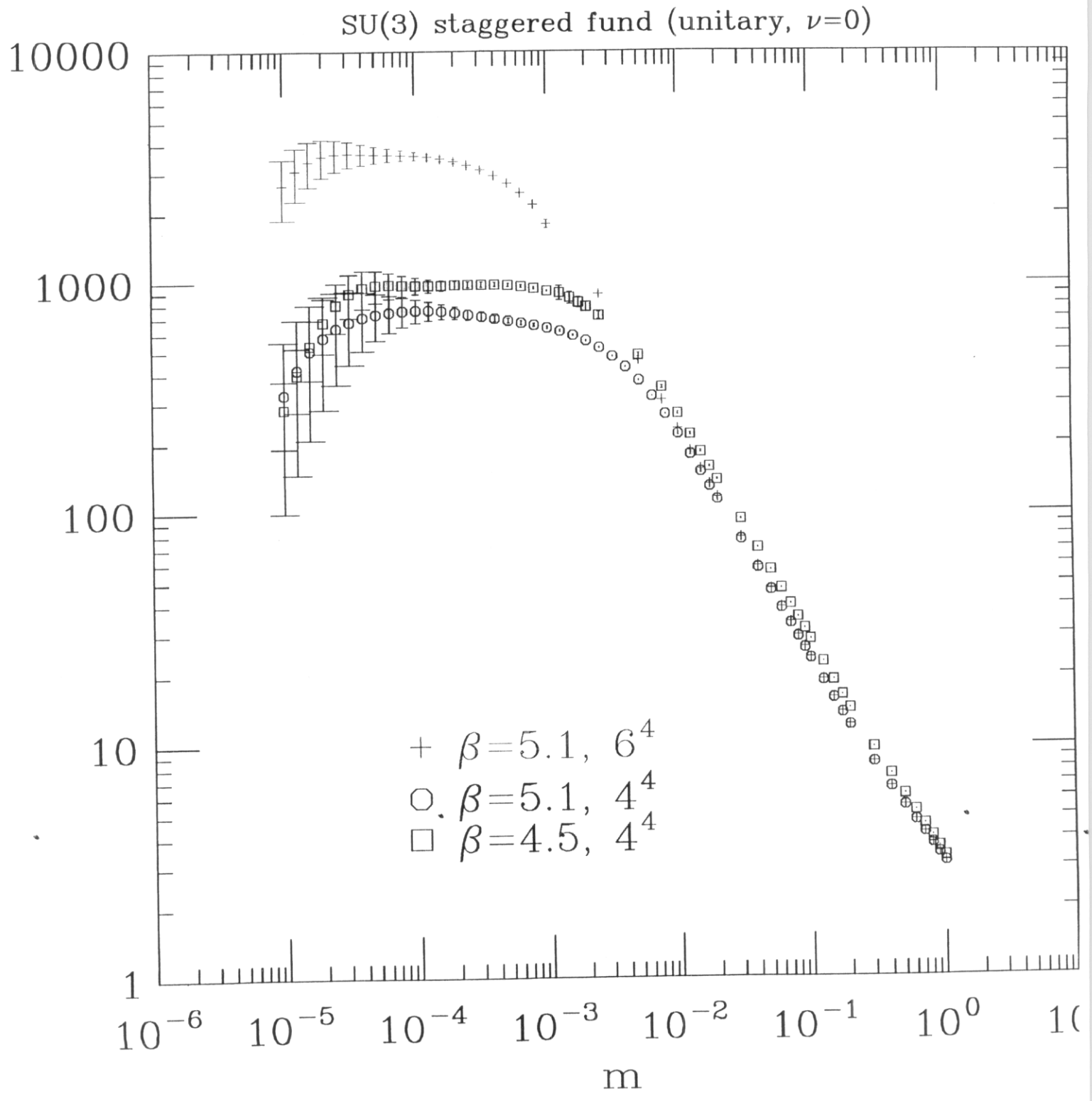
D., EDWARDS, HELLER, NARAYANAN '99

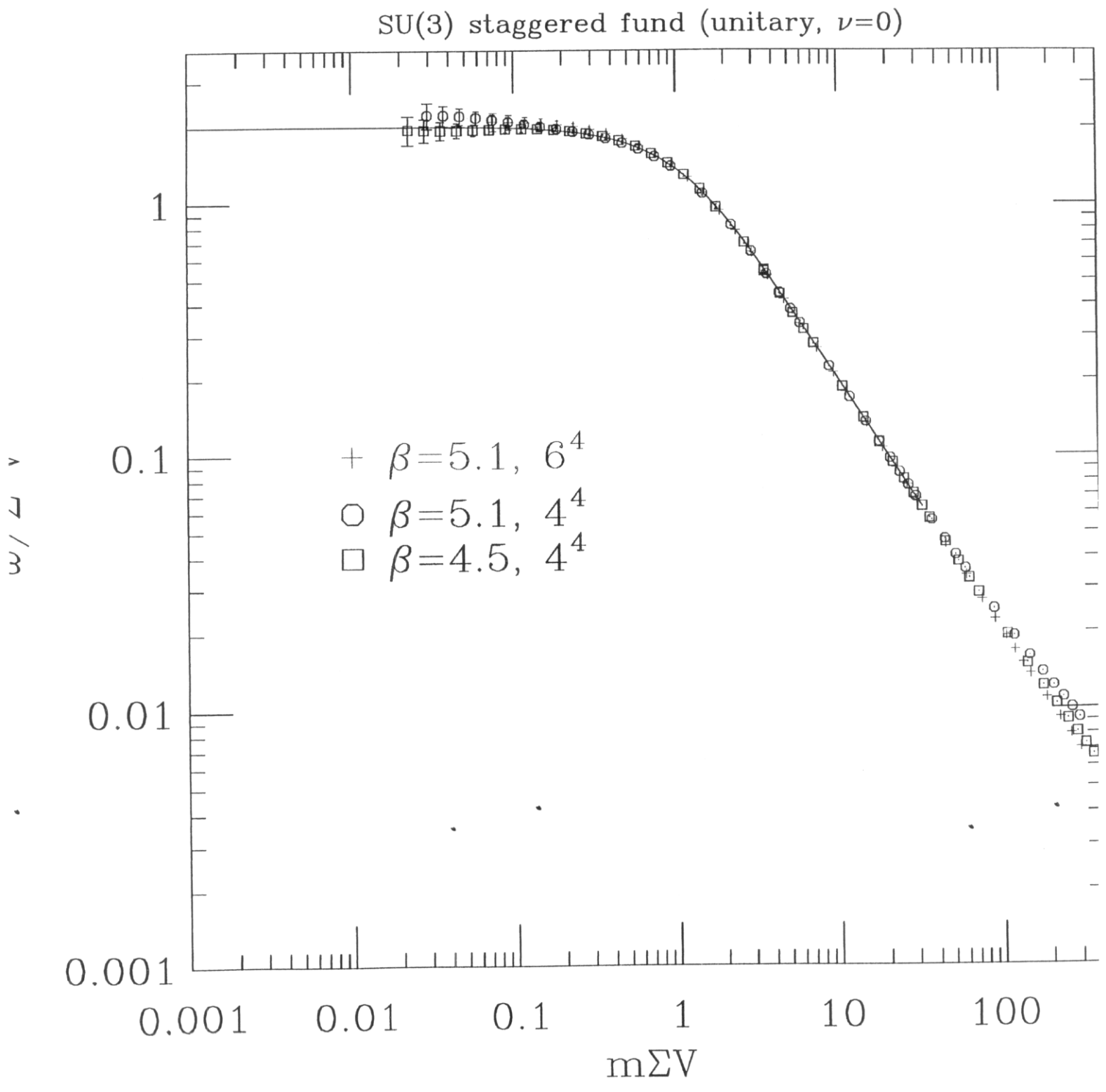


$\frac{\langle \bar{\psi}\psi \rangle}{\Sigma}$



D., EDWARDS, HELLER, NARAYANAN '99




SU(3) staggered fund (unitary, ν=0)
1
0.1
0.01
0.001
Σ / T^3
+ β=5.1, 6^4
○ β=5.1, 4^4
□ β=4.5, 4^4
0.001
0.01
0.1
1
10
100
mΣV

CAREFUL:

$$\Sigma_\nu(\mu) = \mu\left[I_\nu(\mu) K_\nu(\mu) + I_{\nu+1}(\mu) K_{\nu-1}(\mu)\right] + \frac{\nu}{\mu} \quad ,$$

QUENCHED

SMALL-MASS EXP.:

$$\Sigma_\nu(\mu) = \frac{\nu}{\mu} + \frac{1}{2\nu}\mu + \dots$$

LARGE-MASS EXP.:

$$\Sigma_\nu(\mu) = 1 + \frac{4\nu^2-1}{8}\frac{1}{\mu^2} + \dots \qquad (\text{NO } 1/\mu \text{ TERM!})$$

UV CUT-OFF:

ADDITIONAL TERMS $\quad Am\Lambda^2 + Bm^3 \ln\Lambda$

$$= \left(\frac{A}{\Sigma}\right)\frac{\mu}{V}\Lambda^2 + \left(\frac{B}{\Sigma^3}\right)\frac{\mu^3}{V^3}\ln\Lambda$$

$$\to 0 \quad , V\to\infty \, , \; \Lambda = \text{FIXED}$$

FINALLY:

WE CAN PROVE IT DIRECTLY!

OUTLINE: ADD FERMION + BOSONIC PARTNER:

$$Z_\nu = \left(\prod_f m_f^\nu\right)\left(\frac{m^F}{m^B}\right)^\nu \int [dA]_\nu \frac{\det{}'(i\not{D}-m^F)}{\det{}'(i\not{D}-m^B)} \prod_f \det(i\not{D}-m_f)\, e^{-S_{YM}[A]}$$

$m^F = m^B$ ⇝ USUAL QCD

"SUPERSYMMETR

FIND QUENCHED CONDENSATE:

$$\Sigma_\nu(m) \equiv \frac{\partial}{\partial m^F} \ln Z_\nu \Big|_{m^F = m^B}$$

TAKE MICROSCOPIC LIMIT.

INVERT TO GET SPECTRAL DENSITY:

$$\rho(\zeta) = \frac{1}{2\pi} \operatorname{Disc} \Sigma(\mu)\Big|_{\mu = i\zeta}$$

OSBORN, TOUBLAN, VERBAARSCHO '98

EXAMPLE: N_f MASSLESS FERMIONS ($\nu = 0$):

$$\Sigma_{N_f}(\mu) = \mu\left[I_{N_f}(\mu)K_{N_f}(\mu) + I_{N_f+1}(\mu)K_{N_f+1}(\mu)\right]$$

⇛

$$\rho_S(\zeta) = \frac{1}{2\pi} \operatorname{Disc} \Sigma = \frac{\zeta}{2}\left[J_{N_f}(\zeta)^2 - J_{N_f+1}(\zeta)J_{N_f-1}(\zeta)\right]$$

D., OSBORN, TOUBLAN, VERBAARSCHO '98

EXACTLY THE SAME AS FROM RMT!

THE DIRAC SPECTRUM FROM REPLICA SYMMETRY:

ADD N_ν FERMIONS TO QCD, AND LET $N_\nu \to 0$

$$SU(N_f+N_\nu) \times SU(N_f+N_\nu) \to SU(N_f+N_\nu)$$

$$Z_\nu = \int [dA]_\nu \Big[\prod_f \det(i\not{D} - m_f)\Big] \big(\det(i\not{D} - m_\nu)\big)^{N_\nu} e^{-S_{YM}[A]}$$

EFFECTIVE THEORY:

$$Z_\nu = \int_{U(N_f+N_\nu)} dU \, (\det U)^\nu \exp\Big[\frac{V\Sigma}{2} \mathrm{Re}\,\mathrm{Tr}[M U^\dagger]\Big]$$

$$\Sigma_i(\mu_\nu) = \lim_{N_\nu \to 0} \frac{1}{N_\nu} \frac{\partial}{\partial \mu_\nu} \ln Z_\nu$$

$$\vdots \quad \text{(SERIES EXPANSION)}$$

$$= \mu_\nu \big[I_{N_f}(\mu_\nu) K_{N_f}(\mu_\nu) + I_{N_f+1}(\mu_\nu) K_{N_f-1}(\mu_\nu)\big]$$

D., SPLITTORFF '00, D. '00

DALMAZI, VERBAARSCHOT '00

EXACTLY THE SAME RESULT AS FROM

- RANDOM MATRIX THEORY
- THE "SUPERSYMMETRIC" METHOD

BEYOND RANDOM MATRIX THEORY:

EXACT RESULTS FOR $V \to \infty$

COMPUTE $1/V$ CORRECTIONS:

EXPAND $U(x) = U_0 e^{i\xi(x)}$ ← NON-ZERO MOM. MODE

U_0 ↑ ZERO MOM. MODE

CORRECTIONS TO SCALING:

$$\Sigma \to \Sigma_{eff} = \Sigma\left(1 + \frac{N_f^2-1}{N_f}\frac{1}{F_\pi^2} g_1(L) + \dots\right)$$

$$g_1(L) = \frac{\beta_1}{L^2}$$

GASSER-LEUTWYLER
NEUBERGER
HASENFRATZ-LEUTWYLER

FROM CORRECTIONS TO SCALING ONE DETERMINES F_π

QUENCHED:

$$\Sigma \to \Sigma_{eff} = \Sigma\left(1 + \frac{1}{3F_\pi^2}\left[\frac{m_0^2}{8\pi^2}\ln(L) - \frac{\alpha\beta_1}{L^2}\right] + \dots\right)$$

D, '01 "QUENCHED FINITE VOLUME LOGARITHM"

FOR CORRELATION FUNCTIONS: POSTER BY C. DIAMANTIN

STAGGERED FERMIONS: WHY ONLY $\nu = 0$?

EMPIRICALLY:

AT INTERMEDIATE COUPLINGS NO ZERO MODES (SMIT-VINK)

$\Rightarrow$ EFFECTIVELY $\nu = 0$.

THE GOLDSTONE MANIFOLD:

AWAY FROM CONTINUUM

$$U(1) \times U(1) \to U(1) \qquad [N_f = 1]$$

PROJECTION ON $\nu = 0$ SECTOR

$$Z_{\nu=0} = \frac{1}{2\pi}\int_0^{2\pi} d\theta\, Z(\theta) = \int_{U(1)} dU\, e^{\mu \cos\theta} = Z \; !$$

[DIFFERENCES SHOW UP BEYOND LEADING ORDER]

CONCLUSIONS

- THE POWER OF EFFECTIVE FIELD THEORY

- EXACT ANALYTICAL RESULTS FOR DIRAC OPERATOR SPECTRUM FROM

  * RANDOM MATRIX THEORY
  * THE "SUPERSYMMETRIC" METHOD
  * THE REPLICA METHOD

  } IDENTICAL RESULTS

- CHOOSE UNPHYSICAL REGIMES TO EXTRACT PHYSICAL RESULTS:

  * $1/M_\pi \gg L$ ("PION DOESN'T FIT INSIDE BOX")
  * FIXED TOPOLOGY (OPTIONAL)

- EXPLICIT ANALYTIC RESULTS FOR OBSERVABLES IN QUARK MASS RANGE $m\Sigma V \sim \mathcal{O}(1)$

- CAN MEASURE Σ AND F_π FROM JUST ONE SINGLE DIRAC OPERATOR EIGENVALUE DISTRIBUTION