THE MICROSCOPIC DIRAC OPERATOR SPECTRUM
– EXACT RESULTS –

- CHIRAL SYMMETRY BREAKING,
  GAUGE FIELD TOPOLOGY
- ANALYTICAL TOOLS:
  * RANDOM MATRIX THEORY
  * "SUPERSYMMETRY"
  * REPLICA METHOD
- LATTICE RESULTS
- BEYOND RANDOM MATRIX THEORY ...

GAUGE THEORY OF GAUGE GROUP G

N_f FERMIONS CARRYING ANY REPRESENTATION r

$$\boxed{\text{ASSUME SPONTANEOUS BREAKING OF CHIRAL SYMMETRY}}$$

$\Rightarrow$ (PSEUDO) GOLDSTONE BOSONS, MASS m_π

CHIRAL CONDENSATE

$$\boxed{\Sigma \equiv \lim_{m \to 0} \lim_{V \to \infty} \langle \bar{\psi} \psi \rangle}$$

FINITE VOLUME $L \gg \frac{1}{\Lambda_{QCD}}$

EUCLIDEAN PARTITION FUNCTION DOMINATED BY GOLDSTONE BOSONS

NOW LET $V \to \infty$ SO THAT $L \ll \frac{1}{m_\pi}$

WHILE KEEPING

$$\boxed{\mu_i \equiv m_i \Sigma V} \qquad \underline{\text{FIXED}}$$

FERMION MASSES

CHIRAL SYMMETRY BREAKING VS. RANDOM MATRIX THEORY:

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r COMPLEX: [EX.: $G = SU(N_c \geq 3)$, r = FUND. REP.]

$$SU_L(N_f) \times SU_R(N_f) \to SU(N_f)$$

RMT : chUE ("CHIRAL UNITARY ENSEMBLE"), $\beta = 2$

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r PSEUDO-REAL: [EX.: $G = SU(2)$, r = FUND. REP.]

$$SU(2N_f) \to Sp(2N_f)$$

RMT : chOE ("CHIRAL ORTHOGONAL ENSEMBLE"), $\beta = 1$

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r REAL : [EX.: $G = SU(N_c \geq 2)$, r = ADJ. REP.]

$$SU(2N_f) \to SO(2N_f)$$

RMT : chSE ("CHIRAL SYMPLECTIC ENSEMBLE"), $\beta = 4$

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* ALL PATTERNS CONSISTENT WITH VAFA-WITTEN THR

* RELATION TO RMT: ZIRNBAUER '92

  → VERBAARSCHOT '95 ←

TOPOLOGICAL CHARGE $\nu \equiv \frac{1}{32\pi^2}\int d^4x\, F_{\mu\nu}\tilde{F}^{\mu\nu}$

PARTITION FUNCTION

$$Z = \sum_{\nu=-\infty}^{\infty} e^{i\nu\theta}\int [dA]_\nu [d\bar{\psi}\, d\psi]\, e^{-S[A,\bar{\psi},\psi]}$$

$$\equiv \sum_{\nu=-\infty}^{\infty} e^{i\nu\theta} Z_\nu$$

$$\Rightarrow \qquad Z_\nu = \frac{1}{2\pi}\int_0^{2\pi} d\theta\, e^{-i\nu\theta} Z$$

AS $m_\pi \to 0$ GOLDSTONE MODES DOMINATE Z

TO AS HIGH ACCURACY WE LIKE

$$Z = \int_{G/H} dU\, \exp\left[V\Sigma\, \mathrm{Re}\left[e^{i\theta/N_f}\,\mathrm{Tr}[\mathcal{M}U^\dagger]\right]\right]$$

FOR QCD, $G/H = SU(N_f)$, AND

$$\boxed{Z_\nu = \int_{U(N_f)} dU\, (\det U)^\nu \exp\left[V\Sigma\, \mathrm{Re}\,\mathrm{Tr}[\mathcal{M}U^\dagger]\right]}$$

<u>THIS IS ONLY A FUNCTION OF</u> <u>$\mu_i = m_i \Sigma V$!</u>

LEUTWYLER-SMILGA '93

SURPRISE: RANDOM MATRIX THEORY

CHIRAL RANDOM MATRIX THEORY:

$$\tilde{Z}_\nu \equiv \int dW \prod_{f=1}^{N_f} \det(iM + \tilde{m}_f)\, e^{-N \operatorname{Tr} V(M^2)}$$

COMPLEX $N \times (N+\nu)$ MATRIX

POTENTIAL, UNSPECIFIED!

$$M \equiv \begin{pmatrix} 0 & W \\ W^\dagger & 0 \end{pmatrix}$$

SHURYAK-VERBAARSCHOT '9

$$M \tilde{\phi}_n = \tilde{\lambda}_n \tilde{\phi}_n$$

$$\tilde{\gamma}^5 \equiv \begin{array}{c} \begin{array}{cc} \overbrace{\quad}^{N+\nu} & \overbrace{\quad}^{N} \end{array} \\ \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \end{array} : \qquad \{M, \tilde{\gamma}^5\} = 0 \Rightarrow \pm \tilde{\lambda}_n$$

$+\ \nu$ ZERO MODES

C.F. <u>DIRAC</u> <u>OPERATOR</u> <u>EIGENVALUES</u>:

$$\not{D} \phi_n = \lambda_n \phi_n$$

$$\{\not{D}, \gamma^5\} = 0 \Rightarrow \pm \lambda_n$$

BUT: A PRIORI NO CONNECTION TO RANDOM MATRIX THEORY!

MICROSCOPIC LIMIT AND UNIVERSALITY:

DENSITY OF EIGENVALUES OF RANDOM MATRIX: $\tilde{\rho}(\tilde{\lambda})$

$\tilde{\rho}(0) \neq 0$: LET $\tilde{\Sigma} \equiv \pi \tilde{\rho}(0)$

> CF. BANKS-CASHER RELATION
> FOR DIRAC OPERATOR: $\Sigma = \pi \rho(0)$

$\mu_i \equiv \tilde{\Sigma}\,(2N)\,\tilde{m}_i$ FIXED AS $N \to \infty$

> CF. $\mu_i \equiv \Sigma V m_i$ FIXED AS $V \to \infty$
>
> ANALOGY: $\tilde{\Sigma} \leftrightarrow \Sigma$
>
> $2N \leftrightarrow V$
>
> $\tilde{m}_i \leftrightarrow m_i$

THEN:

$$\boxed{\tilde{Z}_\nu(\mu_1, \ldots, \mu_{N_f}) = Z_\nu(\mu_1, \ldots, \mu_{N_f})}$$

SHURYAK-VERBAARSCHOT '94

$\therefore$ RANDOM MATRIX THEORY: EXACT REPRESENTATION OF QCD PARTITION FUNCTION IN FINITE-VOLUME SCALING REGIME

$$\frac{1}{\Lambda_{QCD}} \ll L \ll \frac{1}{m_\pi}$$

> UNIVERSALITY THRM.: <u>INDEPENDENT</u> OF $V(M^2)$

AKEMANN, D., MAGNEA, NISHIGAKI '97

EIGENVALUES: BLOW UP THE SCALE

$$\zeta \equiv \tilde{\Sigma} \cdot 2N \cdot \tilde{\lambda} \quad \underline{\text{FIXED}} \text{ AS } N \to \infty$$

$$\boxed{\text{CF.} \quad \zeta \equiv \Sigma V \lambda \quad \text{FIXED AS } V \to \infty \text{ IN QCD}}$$

DIRAC. OP. EIGENV.

ALL SPECTRAL CORRELATION FUNCTIONS PROVEN TO BE UNIVERSAL.

AKEMANN, D., MAGNEA, NISHIGAKI '97

MASTER FORMULA:

$$\boxed{K(\zeta, \zeta'; \{\mu\}) = \sqrt{\zeta \zeta'} \prod_f \sqrt{(\zeta^2 + \mu_f^2)(\zeta'^2 + \mu_f^2)} \, \frac{\mathcal{Z}_\nu^{(N_f+2)}(\{\mu\}, i\zeta, i\zeta')}{\mathcal{Z}_\nu^{(N_f)}(\{\mu\})}}$$

D. '98, AKEMANN, D. '98

$$\boxed{\tilde{\rho}(\zeta_1, \ldots, \zeta_k; \{\mu\}) = \det_{a,b} K(\zeta_a, \zeta_b; \{\mu\})}$$

SPECIAL CASE: SPECTRAL DENSITY:

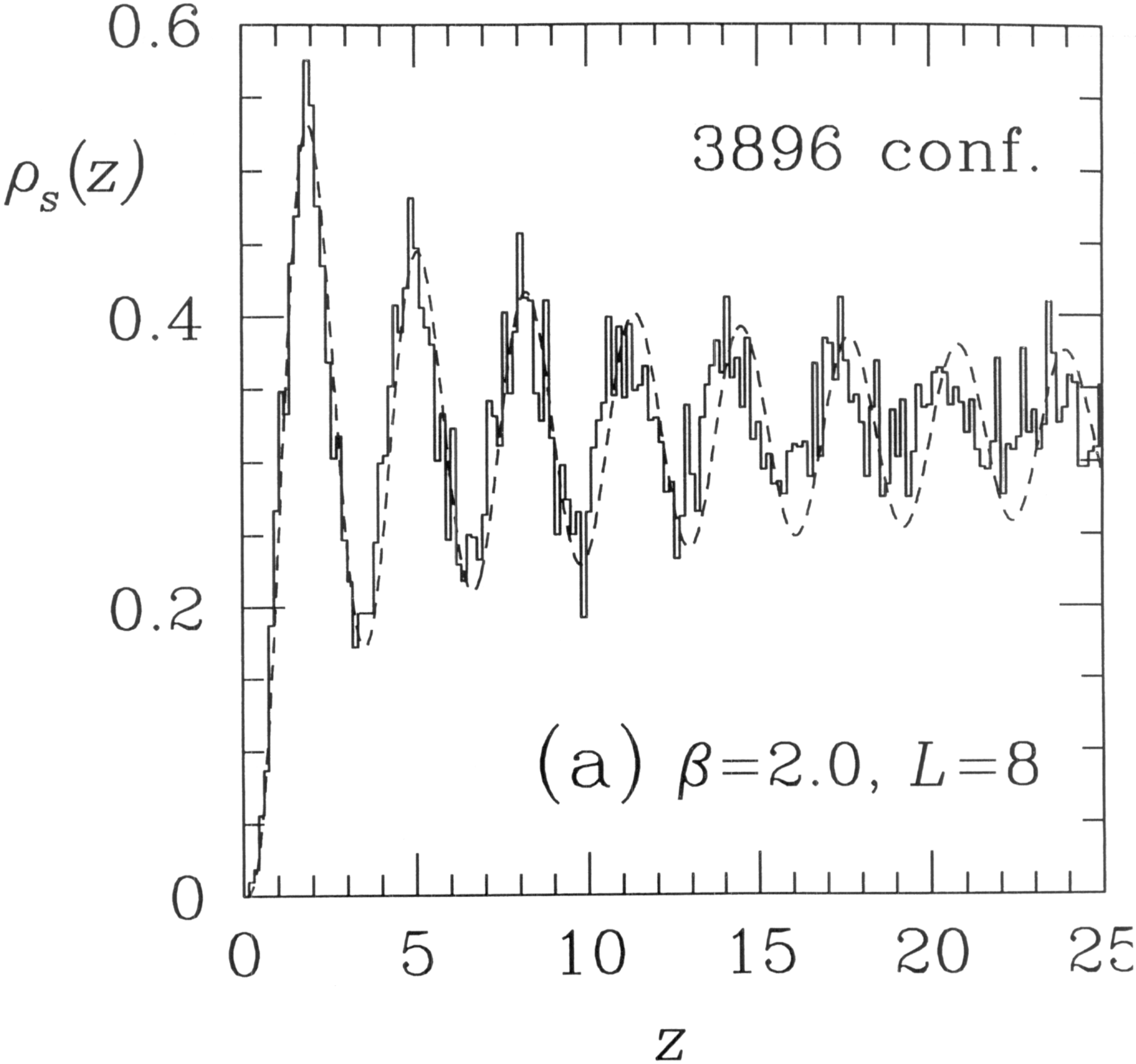
$$\tilde{\rho}(\zeta; \{\mu\}) = K(\zeta, \zeta; \{\mu\})$$

EXAMPLE: N_f MASSLESS FERMIONS:

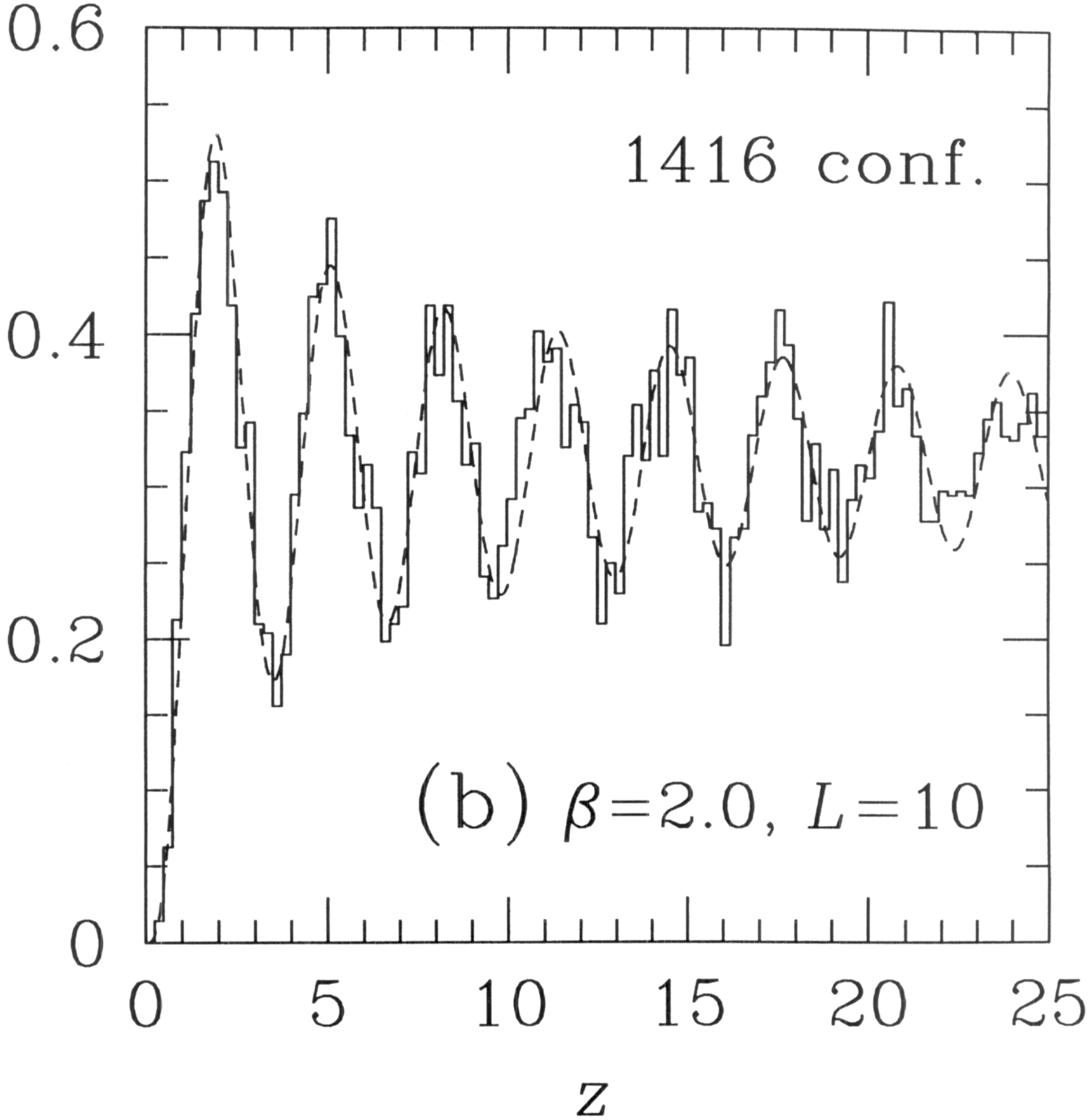
$$\boxed{\tilde{\rho}(\zeta) = \frac{1}{2} \zeta \left[J_{N_f+\nu}(\zeta)^2 - J_{N_f+\nu+1}(\zeta) J_{N_f+\nu-1}(\zeta) \right]}$$

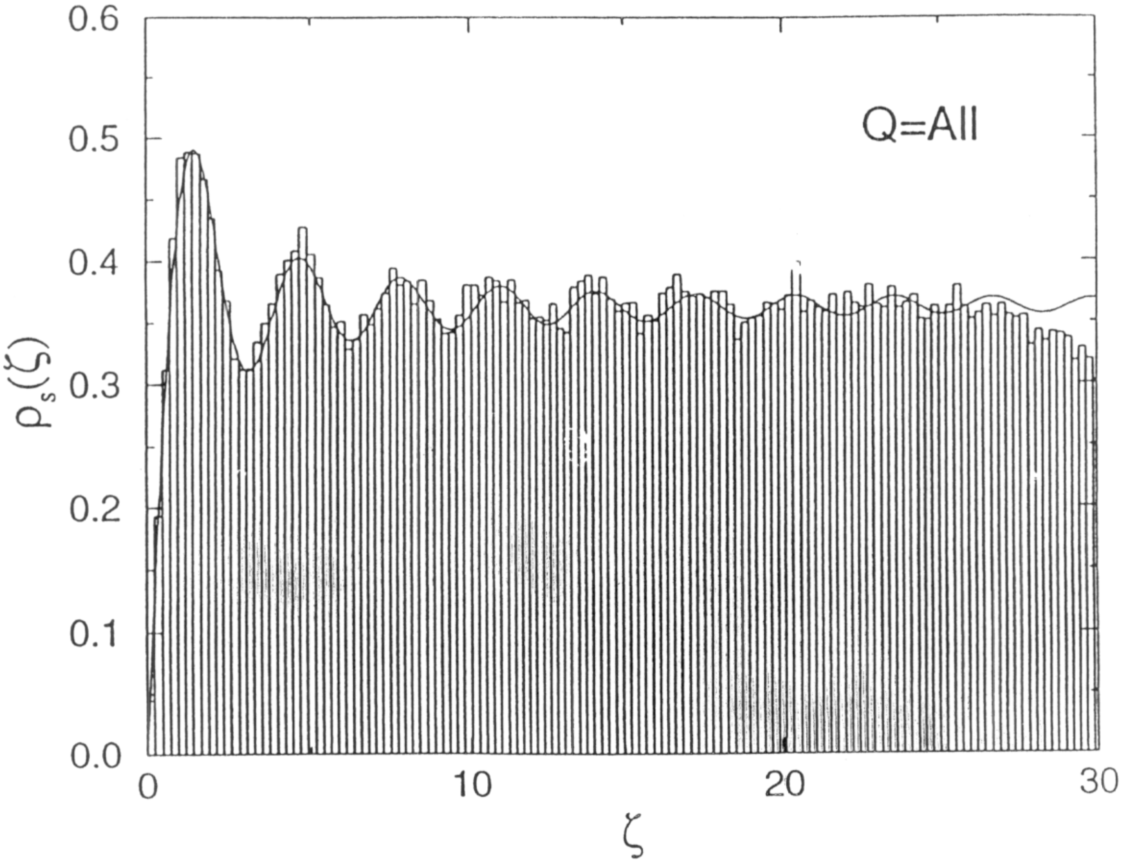
VERBAARSCHOT - ZAHED '95

SU(2) QUENCHED (STAGGERED):



BERBENNI-BITSCH, MEYER, SCHÄFER, VERBAARSCHOT, WETTIG '97


0.6
0.4
0.2
0
1416 conf.
(b) β=2.0, L=10
0
5
10
15
20
25
z

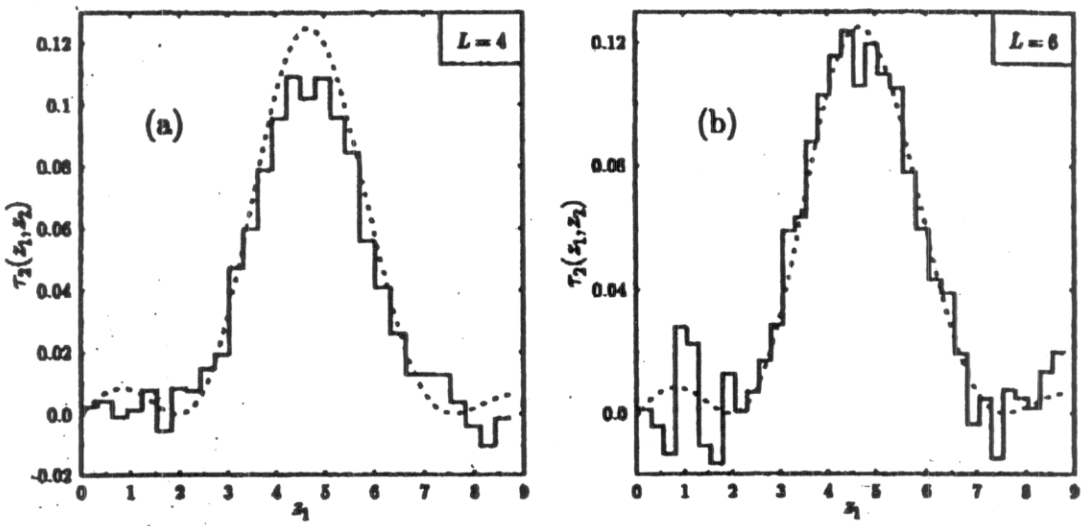


D., HELLER, NICLASEN, RUMMUKAINEN '99

SU(3), STAGGERED, QUENCHED

$\rho_S^c(\xi_1, \xi_2)$

SU(3) QUENCHED:

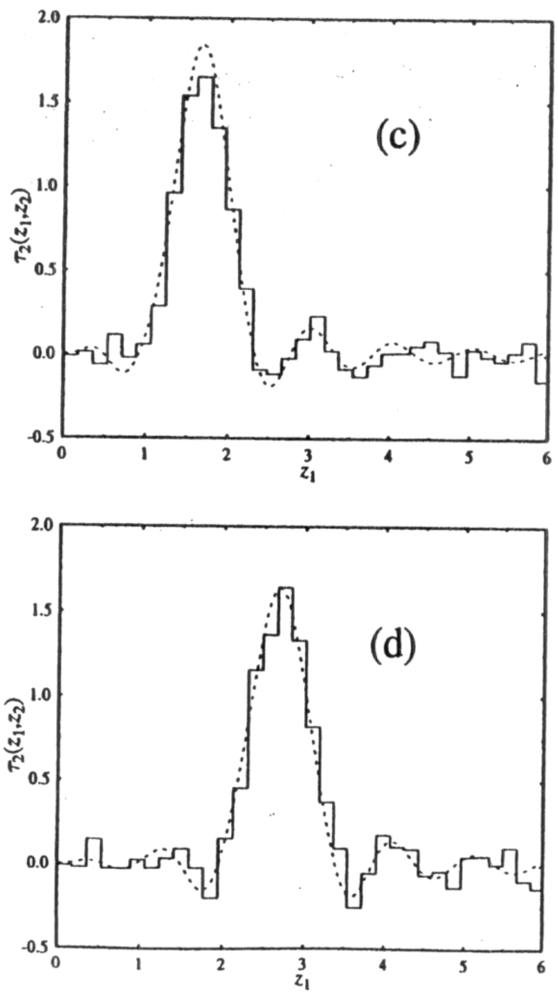


$\beta = 5.4$

GÖCKELER, HEHL, RAKOW, SCHÄFER, WETTIG '98

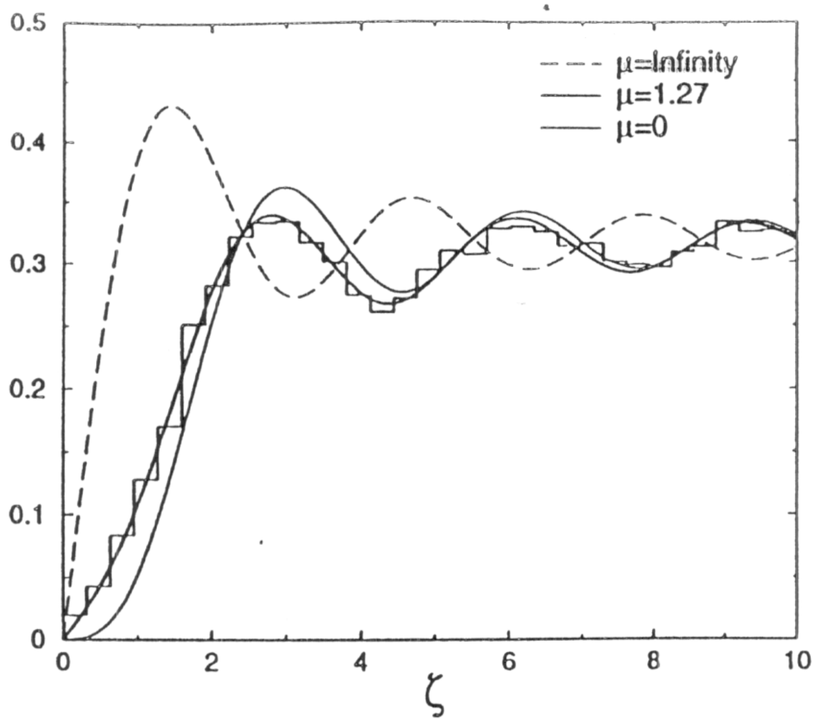
SU(2) QUENCHED (STAGGERED):

8^4, $\beta = 2.0$

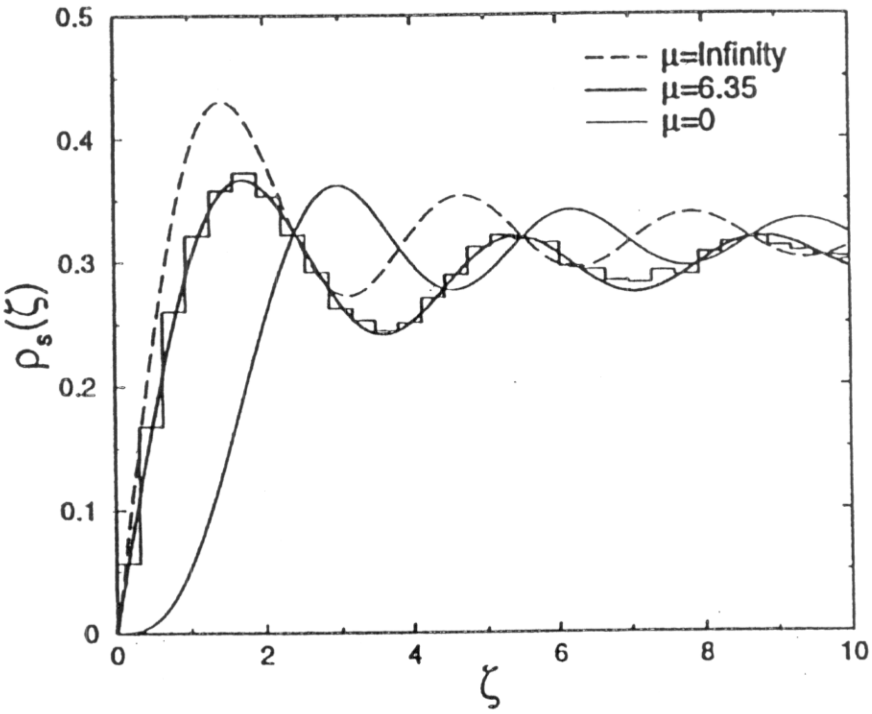


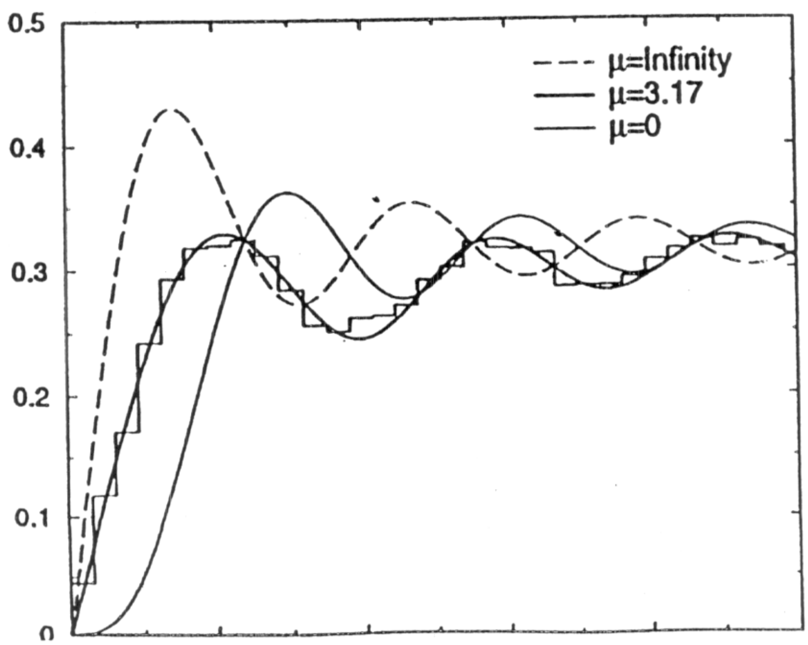
MA, GUHR, WETTIG '98

SU(3)
STAGGERED



D., HELLER, NICLASE
RUMMUKAINEN '0.





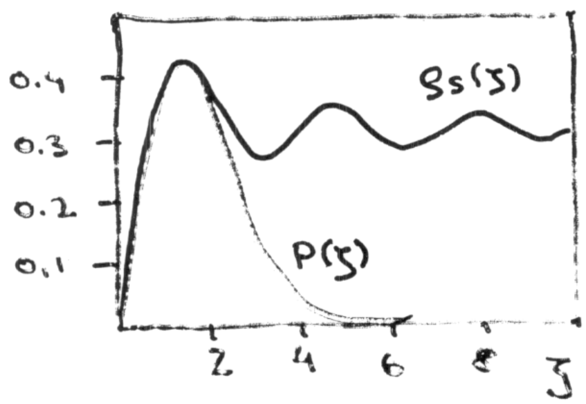
QUESTION: CAN WE ALSO COMPUTE INDIVIDUAL EIGENVALUE DISTRIBUTIONS OF THE DIRAC OPERATOR?

1ST EIGENVALUE:

$$P(\zeta, \{\mu\}) = -\frac{\partial}{\partial \zeta}\left[e^{-\zeta^2/4} \frac{Z(\{\sqrt{\mu^2+\zeta^2}\})}{Z(\{\mu\})} \right] \qquad (\beta=2)$$

WILKE, GUHR, WETTIG '98

NISHIGAKI, D., WETTIG '98



PEAKS IN $\rho_s(\zeta)$ SHOULD CORRESPOND TO INDIVIDUAL EIGENVALUES!

- TURNS OUT TO BE TRUE...

CAN BE GENERALIZED:

- ANY PATTERN OF CHIRAL SYMMETRY BREAKING ($\beta = 1, 2, 4$
- ANY TOPOLOGICAL CHARGE ν [illegible]
- ANY NUMBER OF MASSIVE/MASSLESS FERMIONS N_f
- EXACT DISTRIBUTION OF k^{th} DIRAC OPERATOR EIGENVALUE

TRICK: USE RMT REPRESENTATION

$\omega(\zeta_1, \ldots, \zeta_k)$ = JOINT PROBABILITY DENSITY OF k SMALLEST EIGENVALUES $\{0 \le \zeta_1 \le \ldots \le \zeta_k\}$

$$= e^{-\beta \zeta_k^2/8} \, \zeta_k^{\beta\left(\frac{\nu+1}{2}\right) - \nu - 1 + \frac{2}{\beta}} \left(\prod_{j=1}^{N_f} \mu_j^{\nu} (\mu_j^2 + \zeta_k^2)^{\frac{1}{2} - \frac{1}{\beta}} \right) \times$$

$$\times \prod_{i=1}^{k-1} \left(\zeta_i^{\beta(\nu+1)-1} (\zeta_k^2 - \zeta_i^2)^{\frac{\beta}{2}-1} \prod_{j=1}^{N_f} (\zeta_i^2 + \mu_j^2) \right) \prod_{i>j}^{k} (\zeta_i^2 - \zeta_j^2)^{\beta} \times$$

$$\times Z_{1+2/\beta}\Big(\overbrace{\{\sqrt{\mu_i^2 + \zeta_k^2}\}}^{N_f}, \overbrace{\{\sqrt{\zeta_k^2 - \zeta_1^2}\}, \ldots, \{\sqrt{\zeta_k^2 - \zeta_{k-1}^2}\}}^{\beta}, \overbrace{\{\zeta_k\}}^{\beta(\nu+1)/2-1} \Big) / Z_{\nu}$$

D., NISHIGAKI '00

DISTRIBUTION OF k^{th} SMALLEST EIGENVALUE:

$$\boxed{P_k(\zeta_k) = \frac{1}{(k-1)!} \int_0^{\zeta_k} d\zeta_1 \cdots d\zeta_{k-1} \, \omega(\zeta_1, \ldots, \zeta_k)}$$

- EXPLICIT ANALYTICAL FORMULAS

EXAMPLE: SMALLEST EIGENVALUE DISTR.

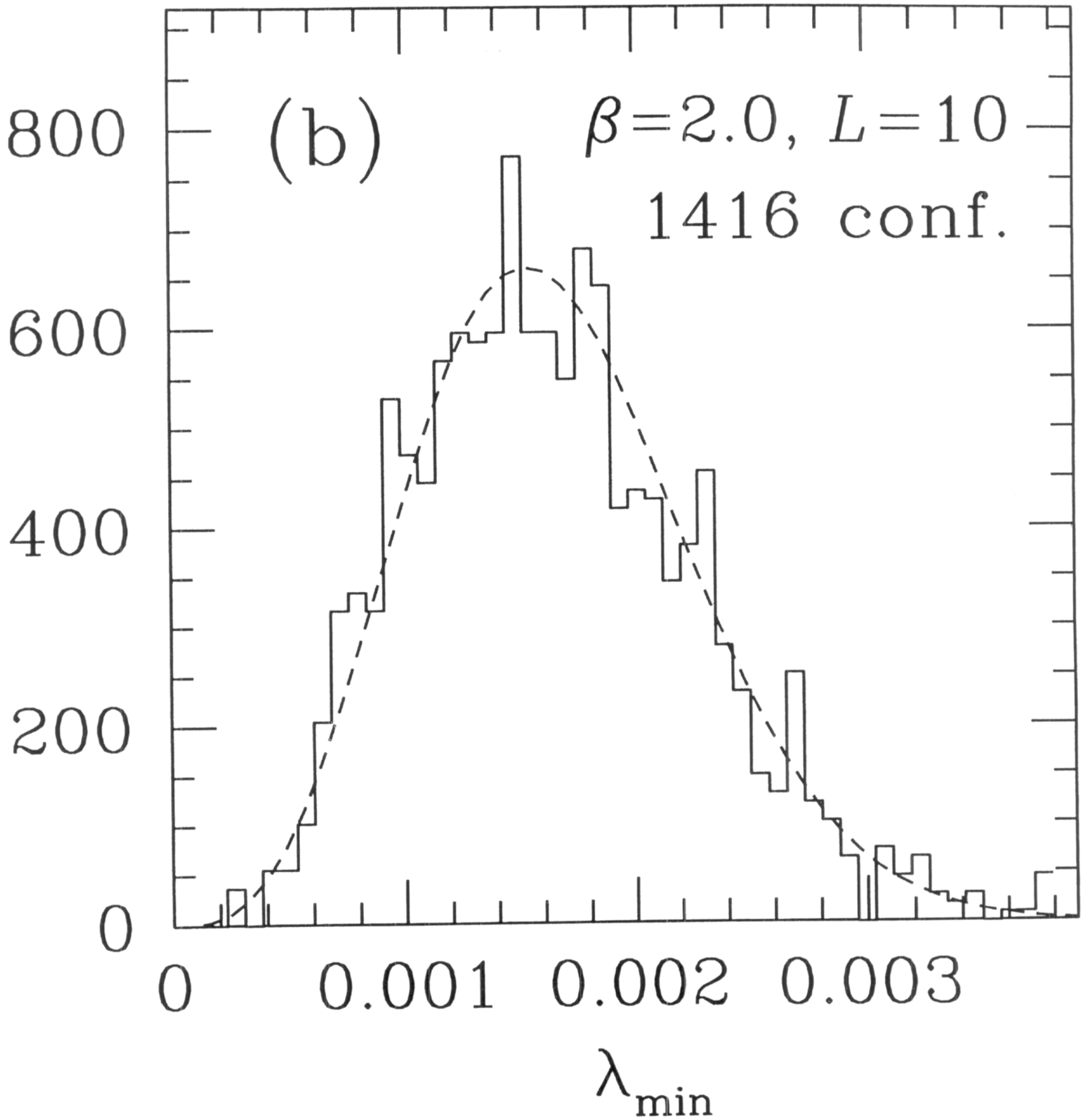
$\beta = 2$ UNIVERSALITY CLASS [E.G. SU(3), N_f FUND.]:

QUENCHED: $$\boxed{P_1(\zeta) = \frac{1}{2}\zeta\, e^{-\zeta^2/4} \det[\underbrace{I_{2+i-j}(\zeta)]}_{\text{MATRIX SIZE } \nu\times\nu}}$$

$\beta = 1$ UNIVERSALITY CLASS [E.G. SU(2), N_f FUND.]:

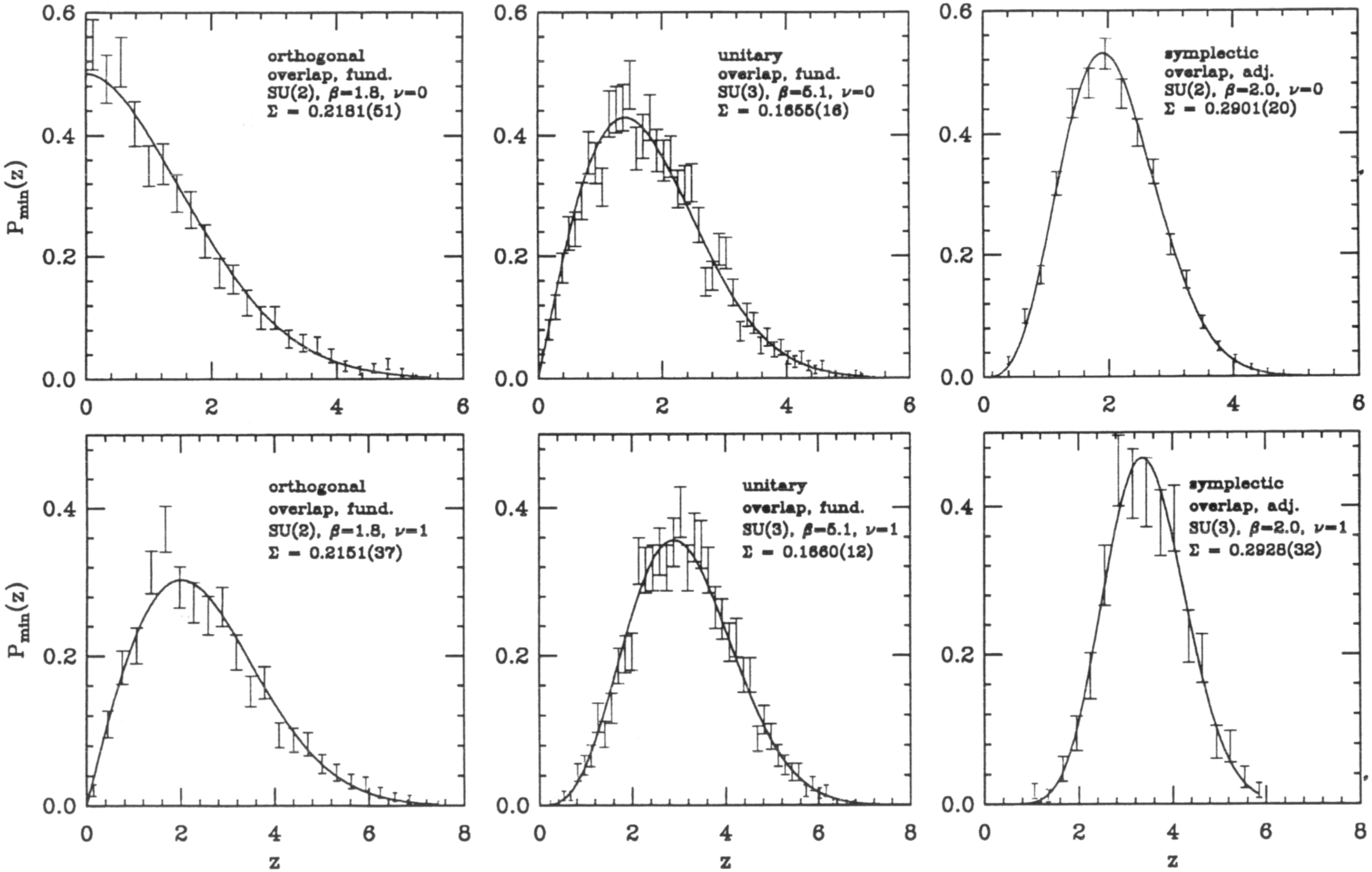
QUENCHED: $$\boxed{P_1(\zeta) = \zeta^{\frac{3-\nu}{2}} e^{-\zeta^2/8}\, \underbrace{\mathrm{Pf}[(i-j) I_{i+j+3}(\zeta)]}_{\text{MATRIX SIZE } (\nu+1)\times(\nu+1)}}$$

SU(2) STAGGERED (QUENCHED)



BERBENNI-BITSCH ET AL., '97

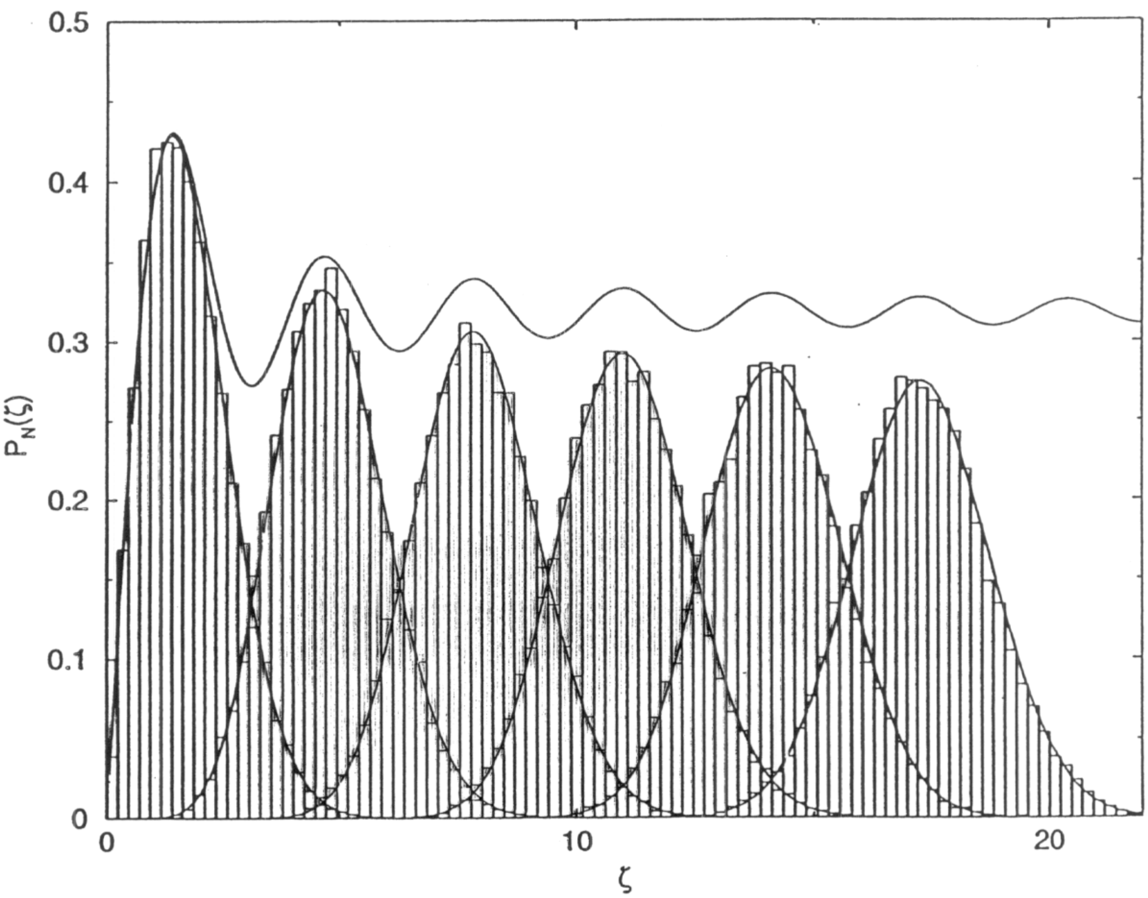
OVERLAP FERMIONS:



EDWARDS, HELLER, KISKIS, NARAYANAN '99

INDIVIDUAL EIGENVALUES:

SU(3) QUENCHED (STAGGERED)



D., HELLER, NICLASEN, RUMMUKAINEN, '00

OTHER OBSERVABLES:

EXAMPLE: SU(3) GAUGE THEORY

N_f MASSLESS FERMIONS, FUND. REP.

$$\rho_s(\zeta) = \frac{\zeta}{2}\left[J_{N_f+\nu}(\zeta)^2 - J_{N_f+1+\nu}(\zeta)\, J_{N_f+\nu-1}(\zeta)\right]$$

QUENCHED CHIRAL CONDENSATE:

$$\Sigma_\nu(\mu) = 2\mu \int_0^\infty d\zeta \, \frac{\rho_s(\zeta)}{\zeta^2+\mu^2} + \frac{\nu}{\mu}$$

$$= \mu\left[I_\nu(\mu) K_\nu(\mu) + I_{\nu+1}(\mu) K_{\nu-1}(\mu)\right] + \frac{\nu}{\mu}$$

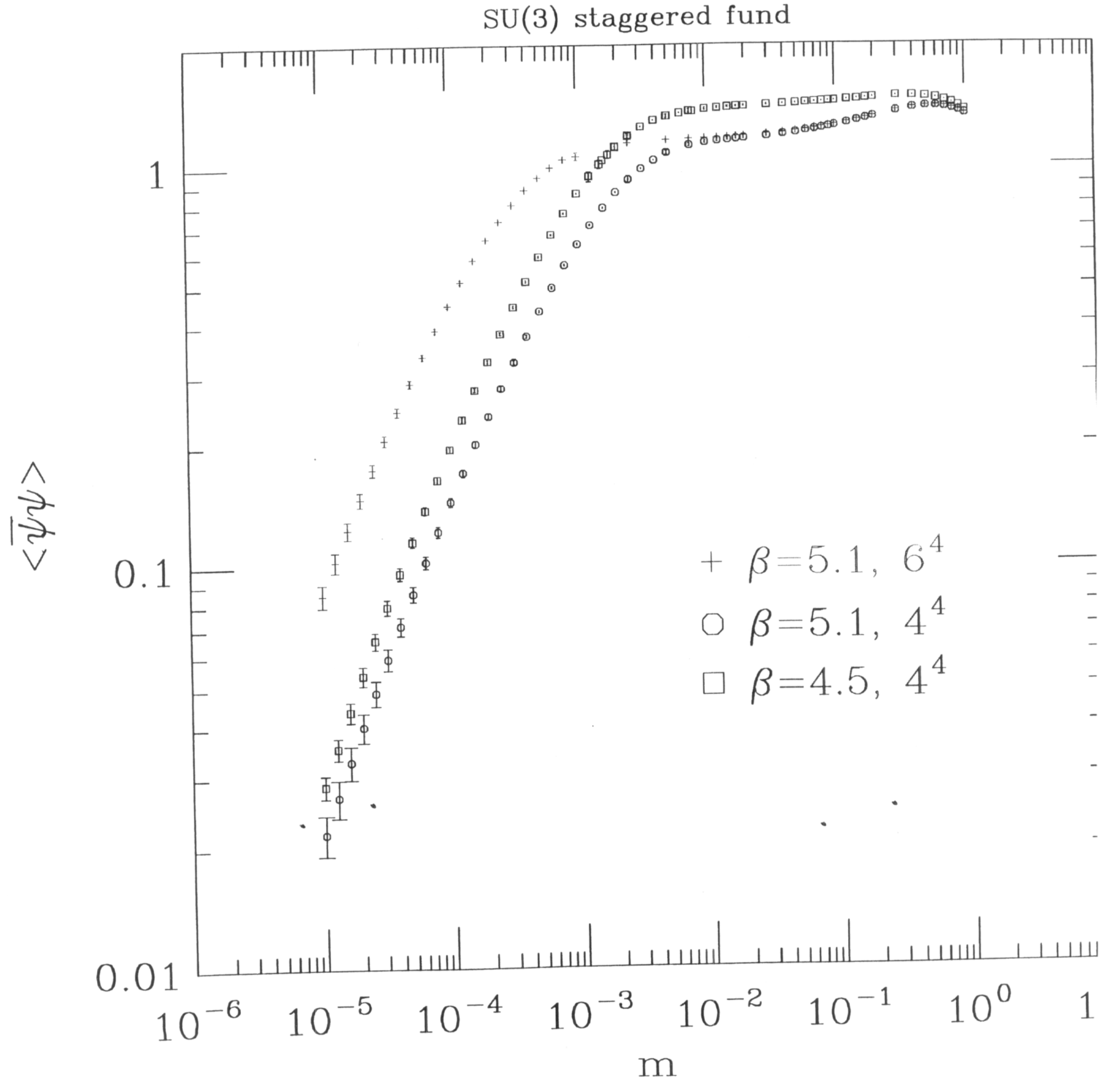
QUENCHED CHIRAL SUSCEPTIBILITY:

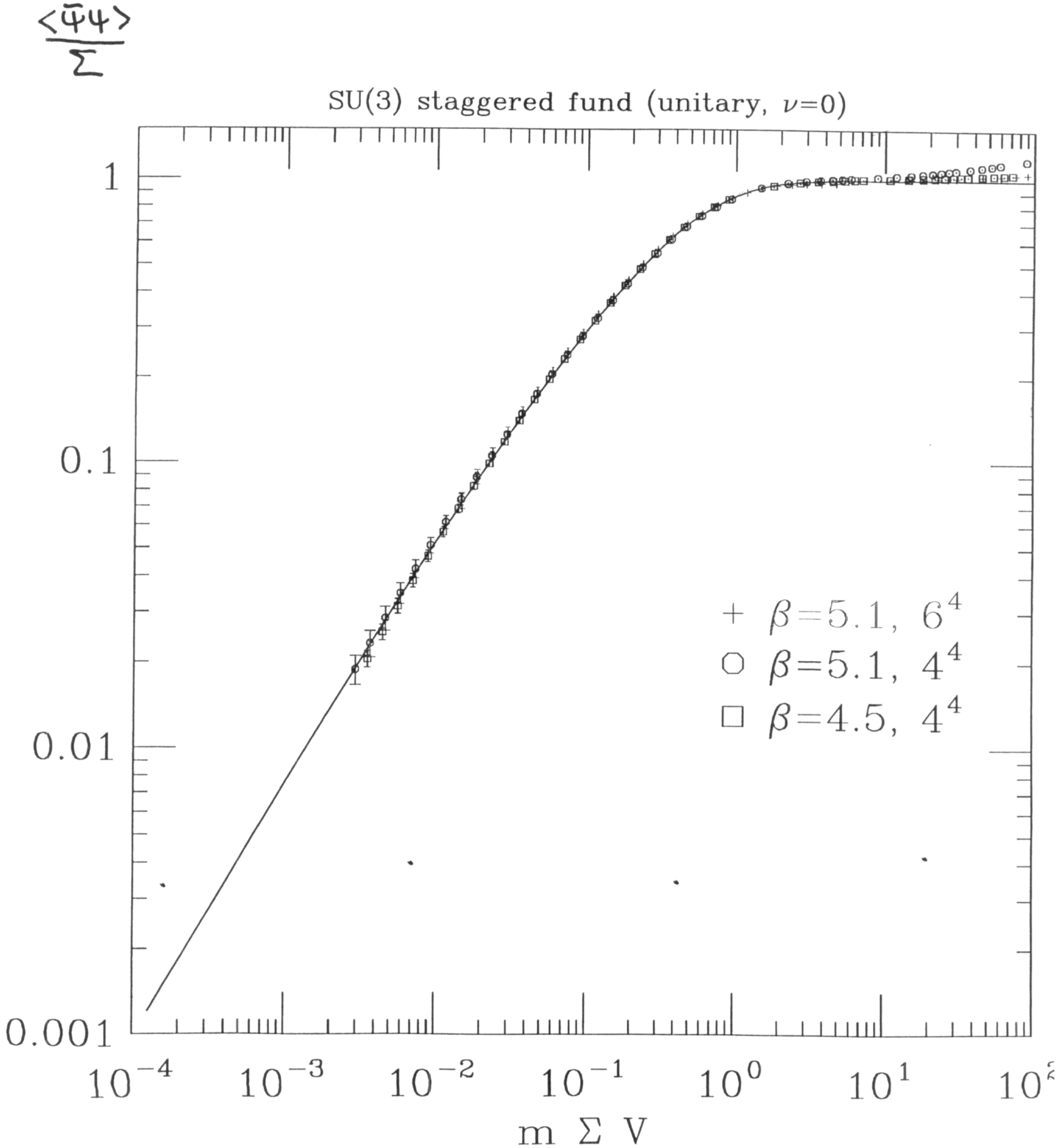
$$\frac{\omega_\nu(\mu)}{V} = 4\mu^2 \int_0^\infty d\zeta \, \frac{\rho_s(\zeta)}{(\zeta^2+\mu^2)^2} + \frac{2\nu}{\mu^2}$$

$$= 2\mu\, I_{\nu+1}(\mu) K_{\nu-1}(\mu) + \frac{2\nu}{\mu^2}$$

$$\left(= \frac{\Sigma_\nu(\mu)}{\mu} - \Sigma_\nu'(\mu) \right)$$

D., EDWARDS, HELLER, NARAYANAN '99

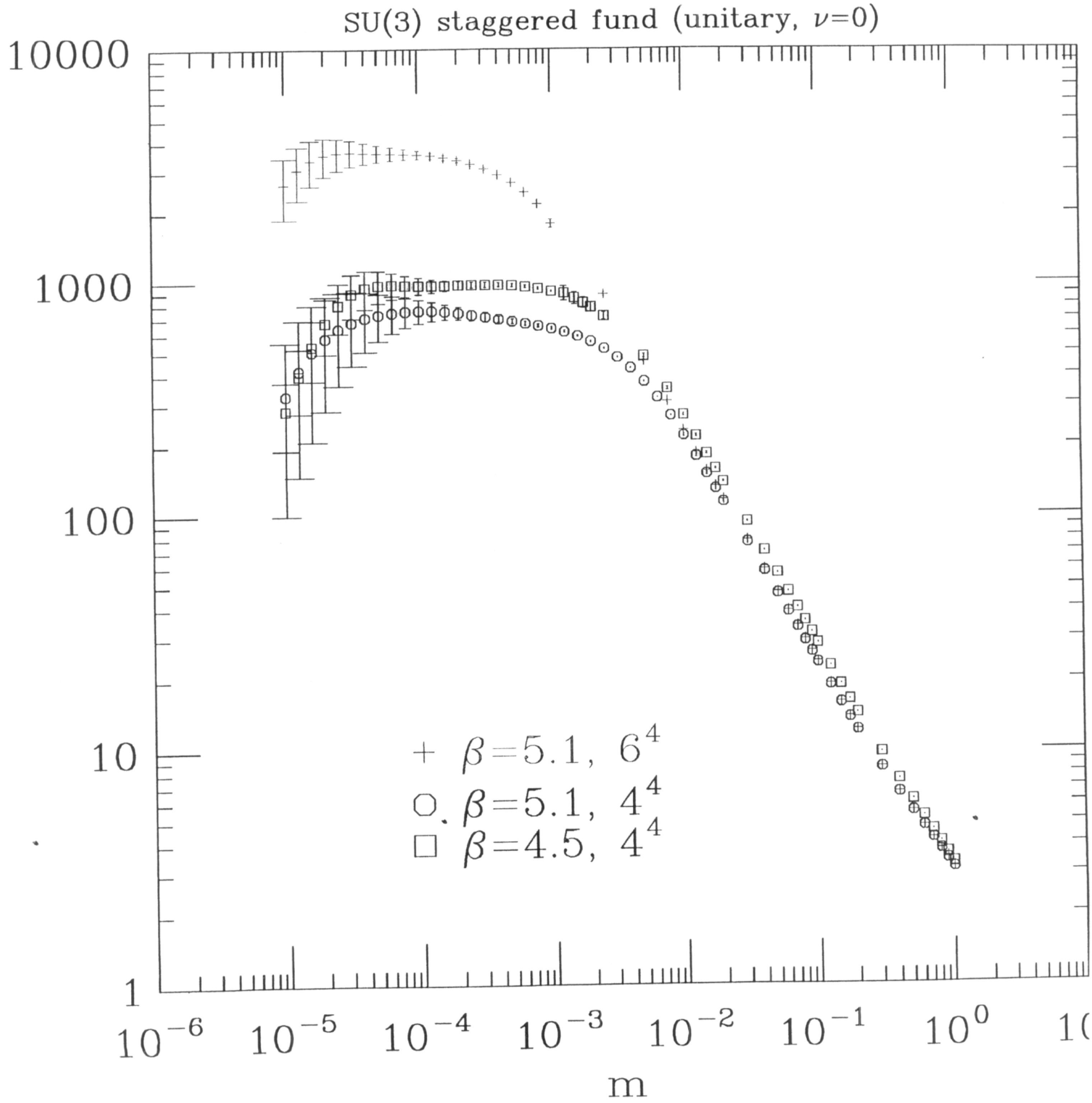


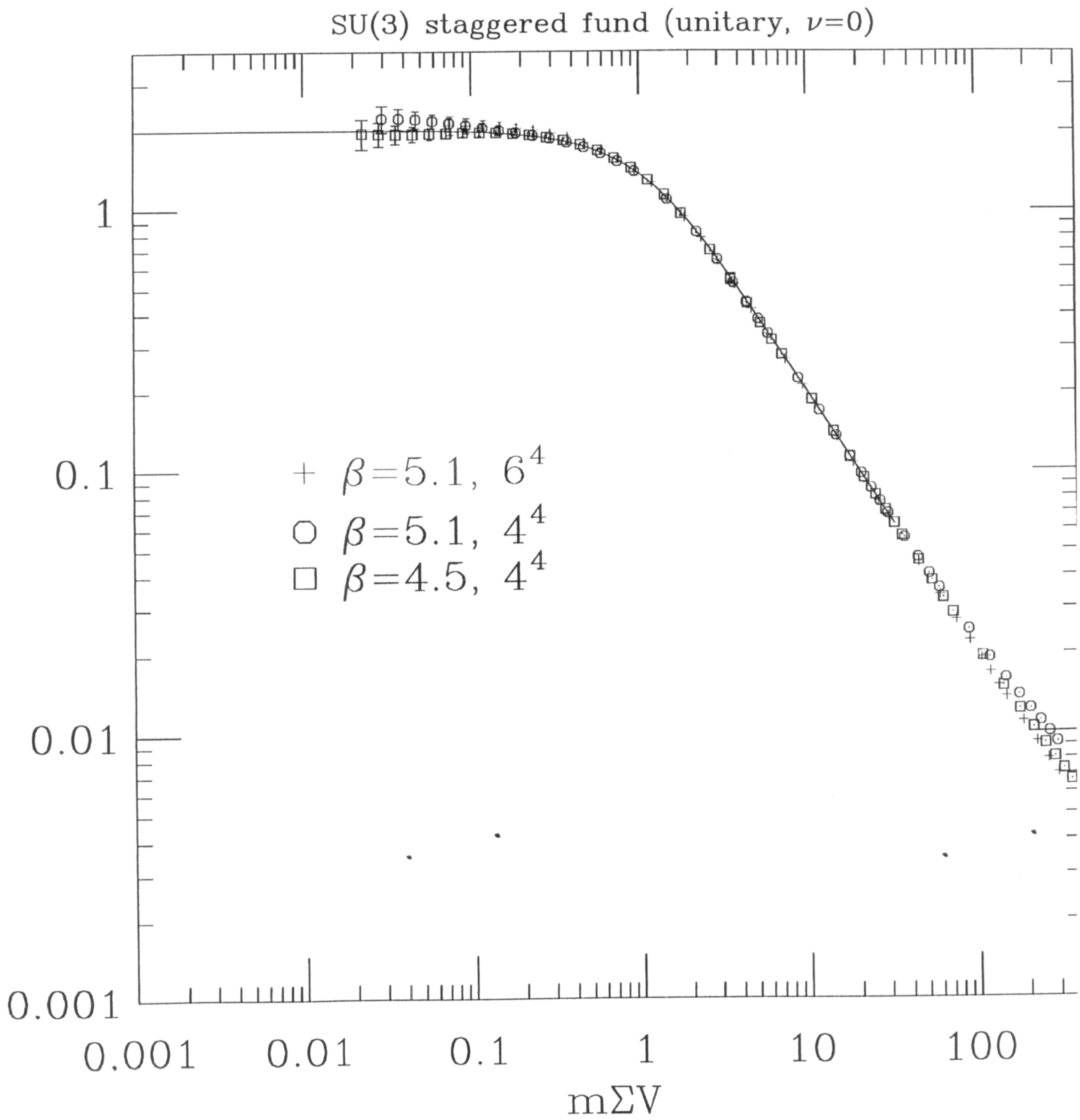


$\frac{\langle\bar{\Psi}\Psi\rangle}{\Sigma}$
SU(3) staggered fund (unitary, ν=0)
1
0.1
0.01
0.001
10^{-4} 10^{-3} 10^{-2} 10^{-1} 10^{0} 10^{1} 10^{2}
m Σ V
+ β=5.1, 6^4
○ β=5.1, 4^4
□ β=4.5, 4^4

D., EDWARDS, HELLER, NARAYANAN '99

ω





SU(3) staggered fund (unitary, ν=0)
1
0.1
0.01
0.001
0.001
0.01
0.1
1
10
100
mΣV
+ β=5.1, 6^4
○ β=5.1, 4^4
□ β=4.5, 4^4

CAREFUL :

$$\Sigma_\nu(\mu) = \mu\left[I_\nu(\mu) K_\nu(\mu) + I_{\nu+1}(\mu) K_{\nu-1}(\mu) \right] + \frac{\nu}{\mu} ,$$

QUENCHED

SMALL-MASS EXP. :

$$\Sigma_\nu(\mu) = \frac{\nu}{\mu} + \frac{1}{2\nu}\mu + \ldots$$

LARGE-MASS EXP. :

$$\Sigma_\nu(\mu) = 1 + \frac{4\nu^2-1}{8}\frac{1}{\mu^2} + \ldots \qquad \text{(NO } 1/\mu \text{ TERM!)}$$

UV CUT-OFF :

ADDITIONAL TERMS $\quad A m \Lambda^2 + B m^3 \ln \Lambda$

$$= \left(\frac{A}{\Sigma}\right)\frac{\mu}{V}\Lambda^2 + \left(\frac{B}{\Sigma^3}\right)\frac{\mu^3}{V^3}\ln \Lambda$$

$$\to 0 \quad , \; V \to \infty \, , \; \Lambda = \text{FIXED}$$

FINALLY:

WE CAN PROVE IT DIRECTLY!

OUTLINE: ADD FERMION + BOSONIC PARTNER:

$$Z_\nu = \left(\prod_f m_f^\nu\right)\left(\frac{m^F}{m^B}\right)^\nu \int [dA]_\nu \underbrace{\frac{\det'(i\not{D} - m^F)}{\det'(i\not{D} - m^B)} \prod_f \det'(i\not{D} - m_f)}\, e^{-S_{YM}[A]}$$

$m^F = m^B \rightarrow$ USUAL QCD

"SUPERSYMMETR

FIND QUENCHED CONDENSATE:

$$\Sigma_\nu(m) \equiv \frac{\partial}{\partial m^F} \ln Z_\nu \Big|_{m^F = m^B}$$

TAKE MICROSCOPIC LIMIT.

INVERT TO GET SPECTRAL DENSITY:

$$\rho(\zeta) = \frac{1}{2\pi} \operatorname{Disc} \Sigma(\mu)\Big|_{\mu = i\zeta}$$

OSBORN, TOUBLAN, VERBAARSCHO '98

EXAMPLE: N_f MASSLESS FERMIONS ($\nu = 0$):

$$\Sigma_{N_f}(\mu) = \mu\left[I_{N_f}(\mu) K_{N_f}(\mu) + I_{N_f+1}(\mu) K_{N_f+1}(\mu)\right]$$

$\Rightarrow$

$$\boxed{\begin{aligned} \rho_S(\zeta) &= \frac{1}{2\pi} \operatorname{Disc} \Sigma \\ &= \frac{\zeta}{2}\left[J_{N_f}(\zeta)^2 - J_{N_f+1}(\zeta) J_{N_f-1}(\zeta)\right] \end{aligned}}$$

D., OSBORN, TOUBLAN, VERBAARSCHO '98

EXACTLY THE SAME AS FROM RMT!

THE DIRAC SPECTRUM FROM REPLICA SYMMETRY:

ADD N_v FERMIONS TO QCD, AND LET $N_v \to 0$

$$\boxed{SU(N_f+N_v)\times SU(N_f+N_v) \to SU(N_f+N_v)}$$

$$Z_v = \int [dA]_v \left[\prod_f \det(i\not{D}-m_f)\right]\left(\det(i\not{D}-m_v)\right)^{N_v} e^{-S_{YM}[A]}$$

EFFECTIVE THEORY:

$$Z_v = \int_{U(N_f+N_v)} dU\,(\det U)^{\nu} \exp\left[V\Sigma\,\mathrm{Re}\,\mathrm{Tr}[\mathcal{M}U^\dagger]\right]$$

$$\Sigma(\mu_v) = \lim_{N_v\to 0}\frac{1}{N_v}\frac{\partial}{\partial\mu_v}\ln Z_v$$

⋮ (SERIES EXPANSIONS)

$$= \mu_v\left[I_{N_f}(\mu_v)K_{N_f}(\mu_v) + I_{N_f+1}(\mu_v)K_{N_f-1}(\mu_v)\right]$$

D., SPLITTORFF '00, D. '00

DALMAZI, VERBAARSCHOT '00

EXACTLY THE SAME RESULT AS FROM

- RANDOM MATRIX THEORY
- THE "SUPERSYMMETRIC" METHOD

BEYOND RANDOM MATRIX THEORY:

EXACT RESULTS FOR $V \to \infty$

COMPUTE $1/V$ CORRECTIONS:

EXPAND $\quad U(x) = U_0 \, e^{i\xi(x)}$ NON-ZERO MOM. MODE

ZERO MOM. MODE

CORRECTIONS TO SCALING:

$$\Sigma \to \Sigma_{eff} = \Sigma \left(1 + \frac{N_f^2 - 1}{N_f} \frac{1}{F_\pi^2} g_1(L) + \ldots \right)$$

$$g_1(L) = \frac{\beta_1}{L^2}$$

GASSER-LEUTWYLER
NEUBERGER
HASENFRATZ-LEUTWYLER

$\boxed{\text{FROM CORRECTIONS TO SCALING ONE DETERMINES } F_\pi}$

QUENCHED:

$$\Sigma \to \Sigma_{eff} = \Sigma \left(1 + \frac{1}{3F_\pi^2}\left[\frac{m_0^2}{8\pi^2} \ln(L) - \frac{\alpha \beta_1}{L^2}\right] + \ldots \right)$$

D, '01 "QUENCHED FINITE VOLUME LOGARITHM"

FOR CORRELATION FUNCTIONS: POSTER BY C. DIAMANTINI

$$\boxed{\text{STAGGERED FERMIONS: WHY ONLY } \nu = 0\text{ ?}}$$

<u>EMPIRICALLY</u>:

AT INTERMEDIATE COUPLINGS NO ZERO MODES

(SMIT-VINK)

$\Rightarrow$ EFFECTIVELY $\nu = 0$.

<u>THE GOLDSTONE MANIFOLD</u>:

AWAY FROM CONTINUUM

$$\boxed{U(1) \times U(1) \to U(1)} \qquad [N_f = 1]$$

PROJECTION ON $\nu = 0$ SECTOR

$$Z_{\nu=0} = \frac{1}{2\pi}\int_0^{2\pi} d\theta \, Z(\theta) = \int_{U(1)} dU \, e^{\mu \cos\theta} = Z \; !$$

[DIFFERENCES SHOW UP BEYOND LEADING ORDER]

CONCLUSIONS

- THE POWER OF <u>EFFECTIVE</u> <u>FIELD</u> <u>THEORY</u>

- <u>EXACT</u> <u>ANALYTICAL</u> <u>RESULTS</u> FOR DIRAC OPERATOR SPECTRUM FROM

  * RANDOM MATRIX THEORY
  * THE "SUPERSYMMETRIC" METHOD
  * THE REPLICA METHOD

  } <u>IDENTICAL</u> <u>RESULTS</u>

- CHOOSE <u>UNPHYSICAL</u> <u>REGIMES</u> TO EXTRACT <u>PHYSICAL</u> <u>RESULTS</u>:

  * $1/M_\pi \gg L$ ("PION DOESN'T FIT INSIDE BOX")
  * FIXED TOPOLOGY (OPTIONAL)

- <u>EXPLICIT</u> <u>ANALYTIC</u> <u>RESULTS</u> FOR OBSERVABLES IN QUARK MASS RANGE $m\Sigma V \sim \mathcal{O}(1)$

- CAN MEASURE Σ AND F_π FROM JUST <u>ONE</u> <u>SINGLE</u> <u>DIRAC</u> <u>OPERATOR</u> <u>EIGENVALUE</u> DISTRIBUTION