Chiral perturbation theory, dispersion relations and final state interactions in $K \to \pi\pi$

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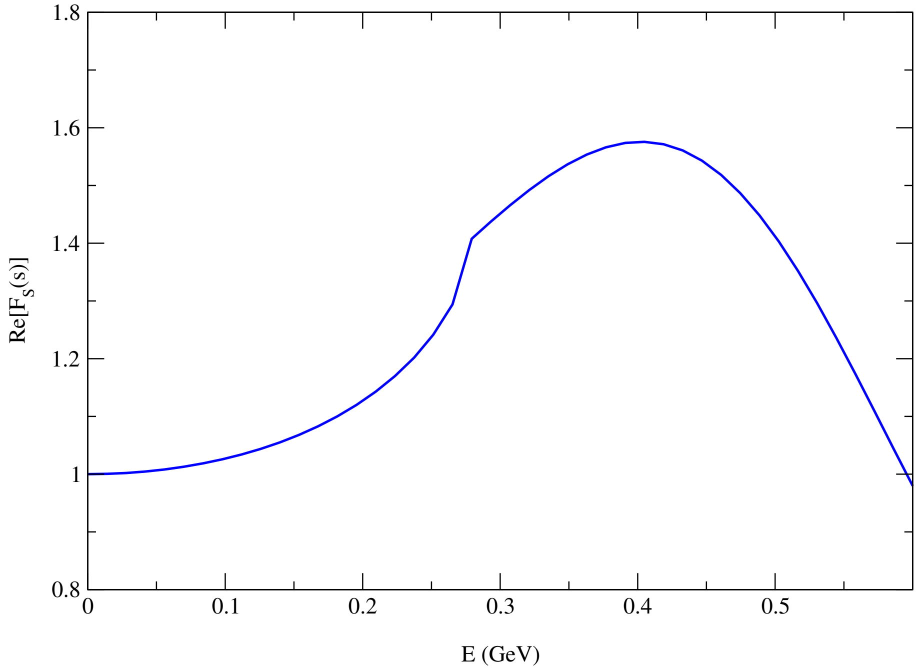
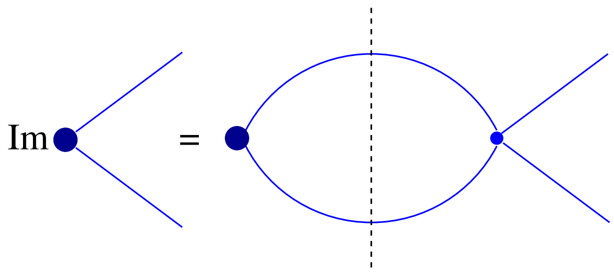
Outline

- Final state interactions in $K \to \pi\pi$
  - FSI in CHPT
  - Dispersive treatment:
    1. kaon off-shell
    2. Hamiltonian carrying momentum
- Large corrections in CHPT
  - $\pi\pi$ scattering lengths
  - Masses and decay constants in SU(2)
  - The SU(3) chiral expansion
- Summary and discussion

The scalar form factor: an example of large FSI



$$F_S(s) \quad = \quad \mathcal{N}\langle\pi(p_1)\pi(p_2)|\bar{u}u + \bar{d}d|0\rangle$$
$$F_S(0) \quad := \quad 1\ , \qquad s = (p_1 + p_2)^2$$



FSI in $K \to \pi\pi$

At "leading order" in different approaches FSI were neglected:

- Bardeen, Buras, Gerard (86);
- Bernard, et al. (85);

Various authors have stressed their importance:

- Truong (88);
- Kambor, Missimer and Wyler (91);
- Bertolini, Eeg, Fabbrichesi (98);
- Hambye et al. (99);
- Pallante and Pich (00);
- Bijnens and Prades (00);
- Büchler, G.C., Kambor, Orellana (01)

Blue $\Rightarrow$ Dispersive treatment, Green $\Rightarrow$ CHPT

On the lattice FSI represent a very special problem, because of the Maiani–Testa (90) theorem. See, however:

- Lellouch and Lüscher (00);
- Lin, Martinelli, Sachrajda and Testa (01).

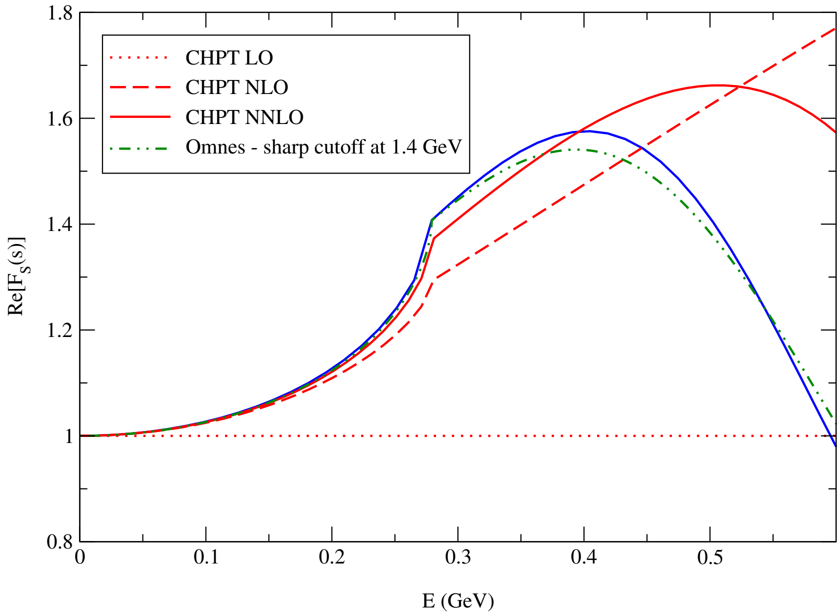
How to treat FSI in $K \to \pi\pi$

1. Lattice: finite volume methods (Lellouch–Lüscher).

2. CHPT: push the calculations to $O(p^4)$ or $O(p^6)$ (Hambye et al.) – that should account for most of the effect.

3. Dispersive methods:
   - apply these to the $K \to \pi\pi$ amplitude with the kaon off-shell (Truong); $\Rightarrow$ back to CHPT.
   - apply these to the $K \to \pi\pi$ amplitude with the weak Hamiltonian carrying momentum (Zurich).

CHPT treatment of FSI



At $s = M_K^2$ the series converges slowly:

$$
\begin{array}{rcl}
\text{LO} & \to & 1 \\
\text{NLO} & \to & 1.62 \qquad \text{vs.} \qquad \text{Full} \to 1.42 \\
\text{NNLO} & \to & 1.67
\end{array}
$$

The bending down of the real part is a NNLO effect.

$$
\begin{aligned}
\text{Re}\, F_S(s) &\sim \cos \delta_0^0(s) \sim 1 - \frac{\delta_0^0(s)^2}{2} + \ldots \\
\delta_0^0(s) &\sim O(p^2)
\end{aligned}
$$

Dispersive treatment for the scalar form factor

Assuming that there are no zeros in the form factor, the solution of its dispersion relation is remarkably simple (Omnès (58)):

$$\begin{aligned} F_S(s) &= \Omega(s) \\ \Omega(s) &= \exp\left\{ \frac{s}{\pi} \int_{4M_\pi^2}^{\infty} ds' \frac{\delta(s')}{s'(s'-s)} \right\} \end{aligned}$$

where δ is the phase of the form factor.

The normalization at $s = 0$ is given by: $\frac{1}{\mathcal{N}} = \frac{\partial M_\pi^2}{\partial \hat{m}}$

Below the inelastic threshold:

$$\delta(s) = \delta_0^0(s)$$

the $\pi\pi$ phase shift (S-wave and $I = 0$).

A simple way to understand the form factor in the elastic region is to neglect inelastic channels (cut off the dispersive integral somewhere above 1 GeV), and put in the $\pi\pi$ phase shift.

CHPT vs. dispersive treatment

CHPT respects automatically analiticity and unitarity, but only perturbatively.

In the chiral counting: $\delta_0^0 \sim O(p^2)$.

$$\Rightarrow \qquad \Omega(s) = 1 + \frac{s}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\delta^{(2)}(s')}{s'(s'-s)} + O(p^4)$$

However, since $\delta^{(2)}(s) \sim 2s - M_\pi^2$

$\Rightarrow$ the dispersive integral does not converge!

Indeed in the chiral representation one gets:

$$F_S(s) = 1 + cs + \frac{s^2}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\delta^{(2)}(s')}{s'^2(s'-s)} + O(p^4)$$

Notice that c contains a chiral log:

$$c \sim \ln \frac{M_\pi^2}{\Lambda}$$

CHPT vs. dispersive treatment: summary

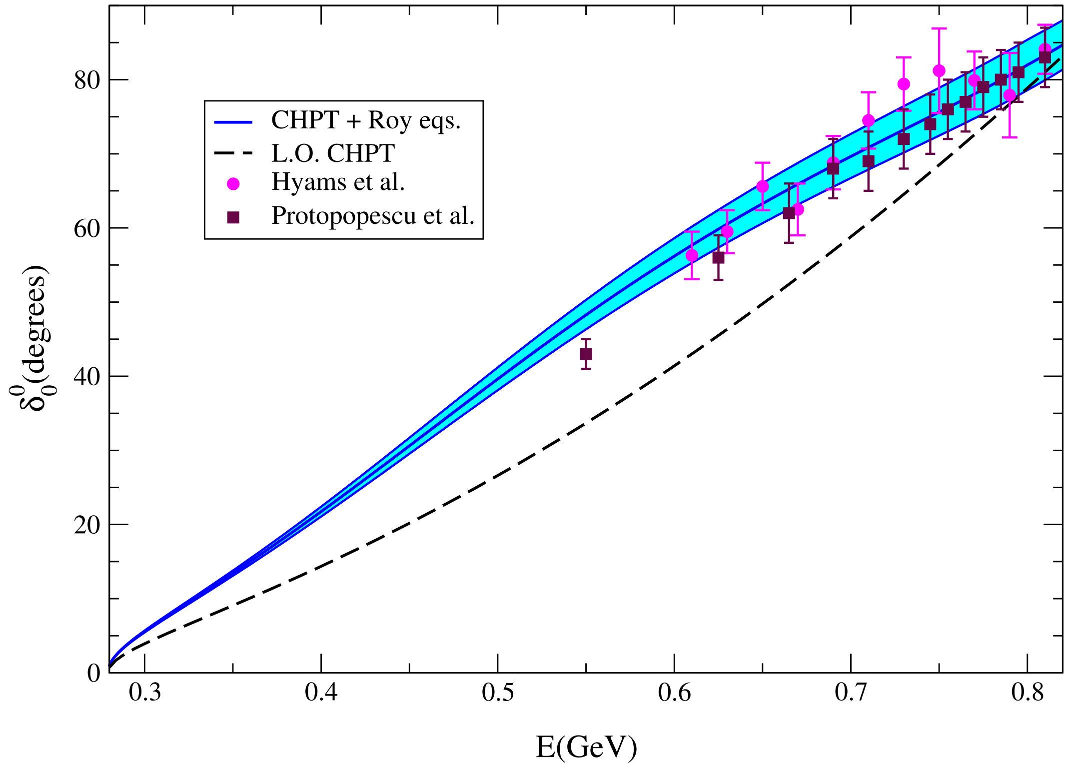
Input needed in the dispersive treatment
(Donoghue, Gasser and Leutwyler (90)):

- subtraction constants (for the form factor, one needs only the normalization);
- accurate knowledge of the $\pi\pi$ phase shift in the elastic region (G.C., Gasser and Leutwyler (01));
- less accurate knowledge of the contribution from the inelastic channels (πK scattering).

Input needed in CHPT:

- all the low energy constants needed at the required perturbative order (1 to one loop (Gasser and Leutwyler (84)), 2 more to two loops (Bijnens, G.C. and Talavera (98)).

For the form factor, the dispersive treatment is more economical and gives a more accurate description in a larger energy range.



G.C., J. Gasser and H. Leutwyler (01)

Dispersive treatment of $K \to \pi\pi$: K off-shell

Do the same as for the form factor for the amplitude with the kaon off-shell (Truong (88), Pallante and Pich (00)).

Main difference: chiral symmetry implies a low-energy zero in the amplitude

$$\text{“}\mathcal{A}^{(2)}(s) = A(s - M_\pi^2)\text{”}$$

In the presence of a zero the Omnès solution is:

$$\text{“}\mathcal{A}(s) = A(s - M_\pi^2)\Omega(s)\text{”}$$

Problem: where does one get the constant A from?

(Ciuchini et al. (00))

In the framework of CHPT A is defined as the coefficient at leading order of $M_K^2 - M_\pi^2$ (Cabibbo and Gell-Mann (64)) for the on-shell amplitude.

Does it make sense to go off-shell?

- "the amplitude with the kaon off-shell reads:"

$$\mathcal{A}^{(2)}(s) = A(s - M_\pi^2)$$

  is not a physically meaningful statement;

- one can use various interpolating fields to go off-shell with the kaon – while all choices lead to the same on-shell amplitude, they all differ off-shell;

- I am not aware of any calculational scheme which does not make explicit reference to CHPT, which allows one to calculate the subtraction constants for the off-shell amplitude.

Büchler et al. (01)

Kaon off-shell

$$G_X(s) = \frac{1}{iN_X}\int dx e^{ikx}\langle \pi(p_1)\pi(p_2)_{\rm out}|T\mathcal{H}_W(0)X^K(x)|0\rangle$$

$s = k^2$, and X^K stands for a generic interpolating field for the kaon: $A^K_\mu = \bar{s}\gamma_\mu\gamma_5 d$, $P^K = \bar{s}\gamma_5 d$.

Dispersion relation for $G_X(s)$:

$$\begin{aligned} G_X(s) &= G_X(s_0) + \frac{(s-s_0)\mathcal{A}}{(M_K^2-s_0)(s-M_K^2)} \\ &+ (s-s_0)\int_{4M_\pi^2}^\infty ds' \frac{\mathrm{disc}[\mathrm{G_X(s')}]}{(\mathrm{s'}-\mathrm{s_0})(\mathrm{s'}-\mathrm{s})} \end{aligned}$$

The solution again involves the Omnès function:

$$G_X(s) = \left[G_X(s_0) + \frac{(s-s_0)\mathcal{A}}{(M_K^2-s_0)(s-M_K^2)\Omega(M^2,s_0)}\right]\Omega(s,s_0)$$

$\mathcal{A}$ is part of the input, and cannot be obtained from the solution of the dispersion relation.

Define:

$$F_X(s) = (s - M_K^2) G_X(s)$$

Dispersion relation:

$$\begin{aligned} F_X(s) &= F_X(s_0) + (s - s_0) F'_X(s_0) \\ &+ (s - s_0)^2 \int_{4M_\pi^2}^{\infty} ds' \frac{\mathrm{disc}[\mathrm{F_X(s')}]}{\mathrm{(s' - s_0)^2 (s' - s)}} \end{aligned}$$

And its solution:

$$\begin{aligned} F_X(s) &= \{F_X(s_0) \\ &+ (s - s_0) \left[F'_X(s_0) - F_X(s_0)\Omega'(s_0, s_0)\right]\} \, \Omega(s, s_0) \end{aligned}$$

If one can solve the dispersion relation for F_X one can obtain the amplitude from:

$$\mathcal{A} = \left\{F_X(s_0) + (M_K^2 - s_0)\left[F'_X(s_0) - F_X(s_0)\Omega'(s_0, s_0)\right]\right\} \Omega(M_K^2, s_0)$$

Combining the dispersion relation and CHPT

$$
\begin{aligned}
G_X(s_0) &= G_X^{(0)}(s_0) + G_X^{(2)}(s_0) + O(p^4) \\
\mathcal{A} &= \mathcal{A}^{(2)} + \mathcal{A}^{(4)} + O(p^6)
\end{aligned}
$$

If we use the LO input from CHPT and insert it in the solution of the dispersion relation, we get:

$$
\mathcal{A} = \left[\mathcal{A}^{(2)} - G_X^{(0)}(s_0)(M_K^2 - s_0)^2 \Omega'(s_0, s_0)\right] \Omega(M_K^2, s_0)
$$

The result depends on the choice of the interpolationg field!

For example [normalizing to $\mathcal{A}^{(2)} = 2c_2(M_K^2 - M_\pi^2)$]:

$$
G_P^{(0)} = 2c_2 - \frac{4}{3}c_5\left(1 + \frac{M_\pi^2}{2M_K^2}\right) \qquad G_A^{(0)} = c_2 \quad ,
$$

$$
\begin{aligned}
\mathcal{A}_A^{DR} &= \mathcal{A}^{(2)}\left[1 - \frac{(M_K^2 - s_0)^2}{2(M_K^2 - M_\pi^2)}\Omega'(s_0, s_0)\right] \Omega(M_K^2, s_0) \\
&= \mathcal{A}^{(2)}\left[1 - 0.29\right] \Omega(M_K^2, M_\pi^2)
\end{aligned}
$$

Back to CHPT

The only way to make sense of this approach is to explicitly refer to CHPT for the on-shell amplitude:

- at tree level:

$$A(M_K^2 - M_\pi^2) \quad \Rightarrow \quad A(M_K^2 - M_\pi^2)\Omega(M_K^2)$$

- at the one loop level:

$$\begin{aligned} & A(M_K^2 - M_\pi^2)\left(1 + \Delta^{(2)}\right) \\ \Rightarrow \quad & A(M_K^2 - M_\pi^2)\left(1 + \Delta^{(2)} - \omega^{(2)}(M_K^2)\right)\Omega(M_K^2) \end{aligned}$$

  where

$$\Omega(s) = 1 + \omega^{(2)}(s) + O(p^4)$$

- at the two loop level ...

This kind of approach was introduced by Gasser and Meißner (91) for the scalar form factor.

Soft-pion theorem

$K \to \pi\pi$ amplitude:

$$_{I=0}\langle \pi(p_1)\pi(p_2)|\mathcal{H}_W^{1/2}(0)|K(q_1)\rangle =: T^+(s,t,u)$$

$$s = (p_1+p_2)^2, \quad t = (q_1-p_1)^2, \quad u = (q_1-p_2)^2 \quad ,$$
$$s+t+u = 2M_\pi^2 + M_K^2 + q_2^2$$

q_2 is the momentum carried by the weak Hamiltonian.

Physical amplitude:

$$\mathcal{A}(K \to \pi\pi) = T^+(M_K^2, M_\pi^2, M_\pi^2)$$

Soft-pion theorem:

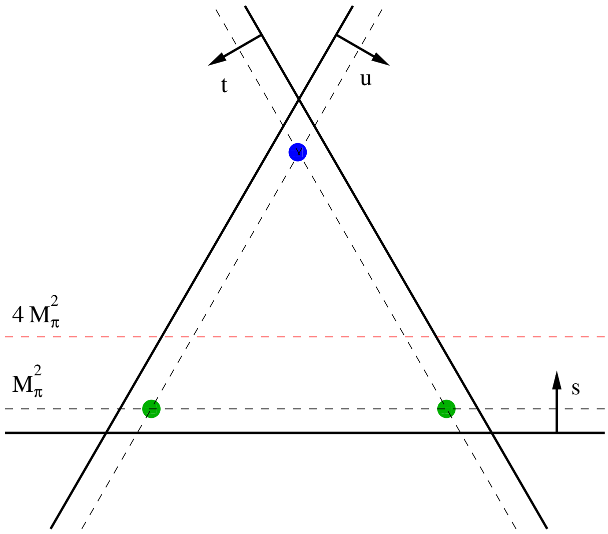
$$T^+(M_\pi^2, M_K^2, q_2^2) = \frac{-1}{2F_\pi}\underbrace{\langle \pi(p_2)|\mathcal{H}_W^{1/2}(0)|K(q_1)\rangle}_{F_W(q_2^2)} + \mathcal{O}(M_\pi^2)$$

From now on $q_2^2 = 0$.

The theorem is based on:

- the chiral symmetry $SU(2)_L \times SU(2)_R$
- the approximation $M_\pi = 0$ for $\pi(p_1)$

$\Rightarrow$ expect corrections of order $M_\pi^2/(1\,\mathrm{GeV})^2 \sim 1\%$



In practice, to get the physical amplitude people use

$$T^+(M_K^2, M_\pi^2, M_\pi^2) = 4T^+(M_\pi^2, M_K^2, M_\pi^2) = \frac{-2}{F_\pi} F_W \left(M_\pi^2\right)$$

The relation is valid to leading order in chiral perturbation theory $(SU(3)_L \times SU(3)_R)$

$\Rightarrow$ expect corrections of order $M_K^2/(1\,\mathrm{GeV})^2 \sim 25\%$.

Decomposition of $T^+(s,t,u)$

$$T^+(s,t,u) = M_0(s) + \left\{ \frac{1}{3}\left[N_0(t) + 2R_0(t)\right] + \frac{1}{2}\left(s-u-\frac{M_\pi^2(M_K^2-M_\pi^2)}{t}\right) N_1(t) \right\} + \{t \leftrightarrow u\}$$

The functions $M_0(s)$, $N_{0,1}(t)$ and $R_0(t)$ are defined to have only a right-hand cut:

$$\mathrm{disc} M_0(s) = \sin\delta_0^0(s) e^{-i\delta_0^0}\left[M_0(s) + \hat{M}_0(s)\right]$$

$$M_0(s) = \Omega_0^0(s,s_0)\left\{ a + b(s-s_0) + \frac{(s-s_0)^2}{\pi}\int_{4M_\pi^2}^{\Lambda_1^2} \frac{\sin\delta_0^0(s')\hat{M}_0(s')ds'}{|\Omega_0^0(s',s_0)|(s'-s)(s'-s_0)^2} \right\}$$

and similarly for $N_{0,1}(t)$ and $R_0(t)$.

$$\Omega_0^0(s,s_0) = \exp\left\{ \frac{(s-s_0)}{\pi}\int_{4M_\pi^2}^{\tilde{\Lambda}_1^2} ds' \frac{\delta_0^0(s')}{(s'-s_0)(s'-s)} \right\}$$

Determination of the subtraction constants

Choosing $s_0 = M_\pi^2$ as subtraction point for $M_0(s)$ one can get a from the soft-pion theorem:

$$-\frac{1}{2F_\pi^2}F_W(0) = a + \frac{1}{3}\left[N_0(M_K^2) + 2R_0(M_K^2)\right] + O(M_\pi^2)$$

The other subtraction constant b can be related to the derivative of the amplitude at the soft-pion point:

$$b = \frac{\partial}{\partial s}T^+(s, \Sigma - s, M_\pi^2)_{|s=M_\pi^2} + \dots$$

There is a Ward identity for this derivative:

$$\frac{\partial}{\partial s}T^+(s, \Sigma - s, M_\pi^2)_{|s=M_\pi^2} = \frac{1}{2}C_{|\mathrm{SPP}} + O(M_\pi^2)$$

where

$$\frac{i}{F_\pi}\int dx e^{ip_1 x}\langle\pi(p_2)|T\left(\mathcal{H}_W^{1/2}(0)A^\mu(x)\right)|K(q_1)\rangle =$$

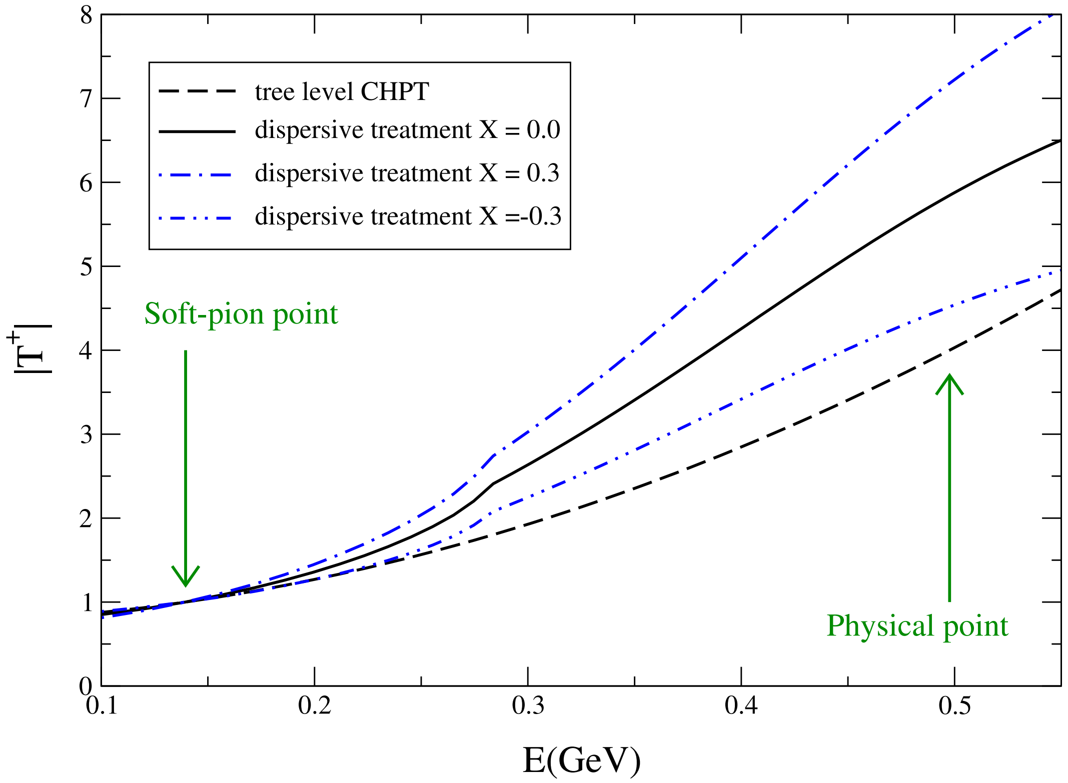
$$ip_1^\mu B + iq_1^\mu C + iq_2^\mu D$$

Numerical study of the dispersion relation

We use the following CHPT relation between a and b:

$$b = \frac{3a}{M_K^2 - M_\pi^2}\Big(1 + X + O(M_K^4)\Big)$$

With $X = \pm 0.3$ $(X \sim \mathcal{O}(M_K^2/1\,\mathrm{GeV}^2))$



$$\frac{|\mathcal{A}(K \to \pi\pi)|}{|\mathcal{A}^{\text{LO CHPT}}(K \to \pi\pi)|} = 1.5\,(1 + 0.76\,X)$$

The weak mass term

$$T^{+}_{c_5}(s,t,u) = \frac{-ic_5\Delta}{F_\pi^2}\left[\frac{s}{q_2^2 - M_K^2} + 1\right]$$

$$\Rightarrow T^{+}_{c_5}(M_K^2, M_\pi^2, M_\pi^2) = 0$$

Using LO CHPT to go from $\mathcal{A}(K \to \pi)$ to $\mathcal{A}(K \to \pi\pi)$, the pole term is a problem:
The standard recipe requires the calculation of $\mathcal{A}(K \to 0)$ on the lattice, to remove the spurious contribution of c_5 to $\mathcal{A}(K \to \pi\pi)$.

If one uses two subtraction constants, and determines also the derivative of the amplitude at the soft-pion point, this *ad hoc* subtraction is not necessary.

FSI in $K \to \pi\pi$: discussion

- In order to treat FSI there are two possible routes: rely on the chiral expansion, or do a full-fledged dispersive treatment. The example of the scalar form factor clearly showed the advantages of the latter.

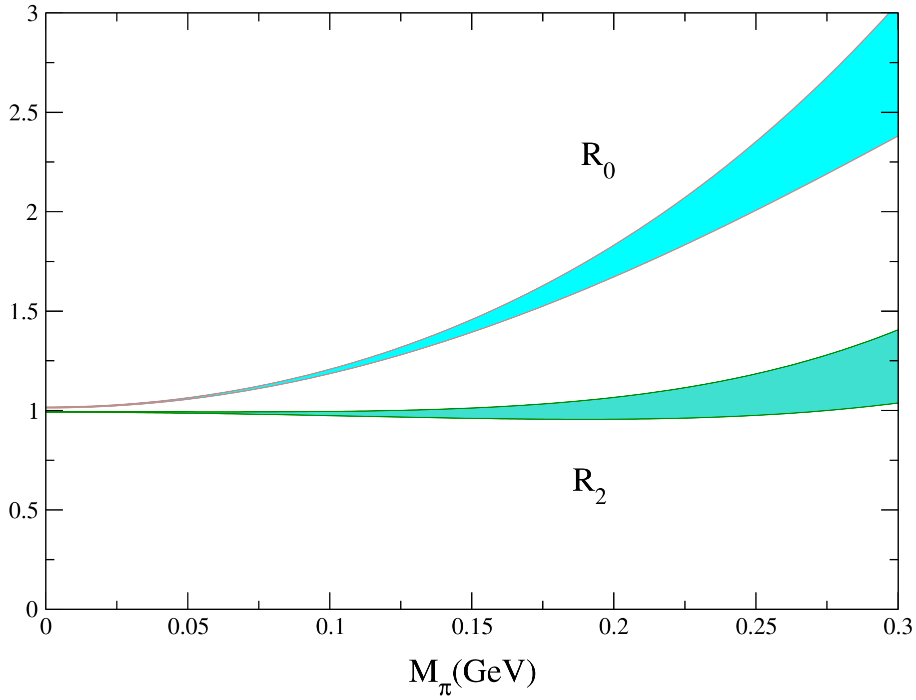
- I have critically reviewed the approach in which the dispersive treatment is applied to the amplitude with the kaon off-shell, and shown that this has to rely inevitably on the chiral expansion.

- If a dispersive treatment is applied to the $K \to \pi\pi$ amplitude in which the weak Hamiltonian carries momentum, reference to the chiral expansion is not needed: one can use chiral Ward identities to obtain the subtraction constants from amplitudes which are FSI–free (only 1 pion in the final state).

- Alternatively, one could rely only on lattice calculations (*à la* Lellouch-Lüscher), to get the $K \to \pi\pi$ amplitude including FSI.

Do we have a good understanding of the $\pi\pi$ interaction on the lattice?

Quark mass dependence of the scattering lengths to $O(p^6)$



$$R_I := \frac{a_0^I}{a_0^{I\ Weinberg}}$$

G.C., J. Gasser and H. Leutwyler (01)

$\pi\pi$ scattering to $O(p^6)$

This calculation of the scattering lengths is based on:

- The CHPT calculation to $O(p^6)$ of the amplitude in the unphysical region, below threshold (Bijnens et al. (95));

- A numerical treatment of the dispersion relations for $\pi\pi$ scattering (Roy equations) in the region below 0.8 GeV (Ananthanarayan et al. (00)).

$$a_0^0 = \frac{7M_\pi^2}{32\pi F_\pi^2} C_0 + M_\pi^4 \alpha_0 + O(M_\pi^8)$$

CHPT $\Rightarrow$ C_0 dispersive treatment $\Rightarrow$ α_0

Outcome: sharp prediction for the S–wave scattering lengths:

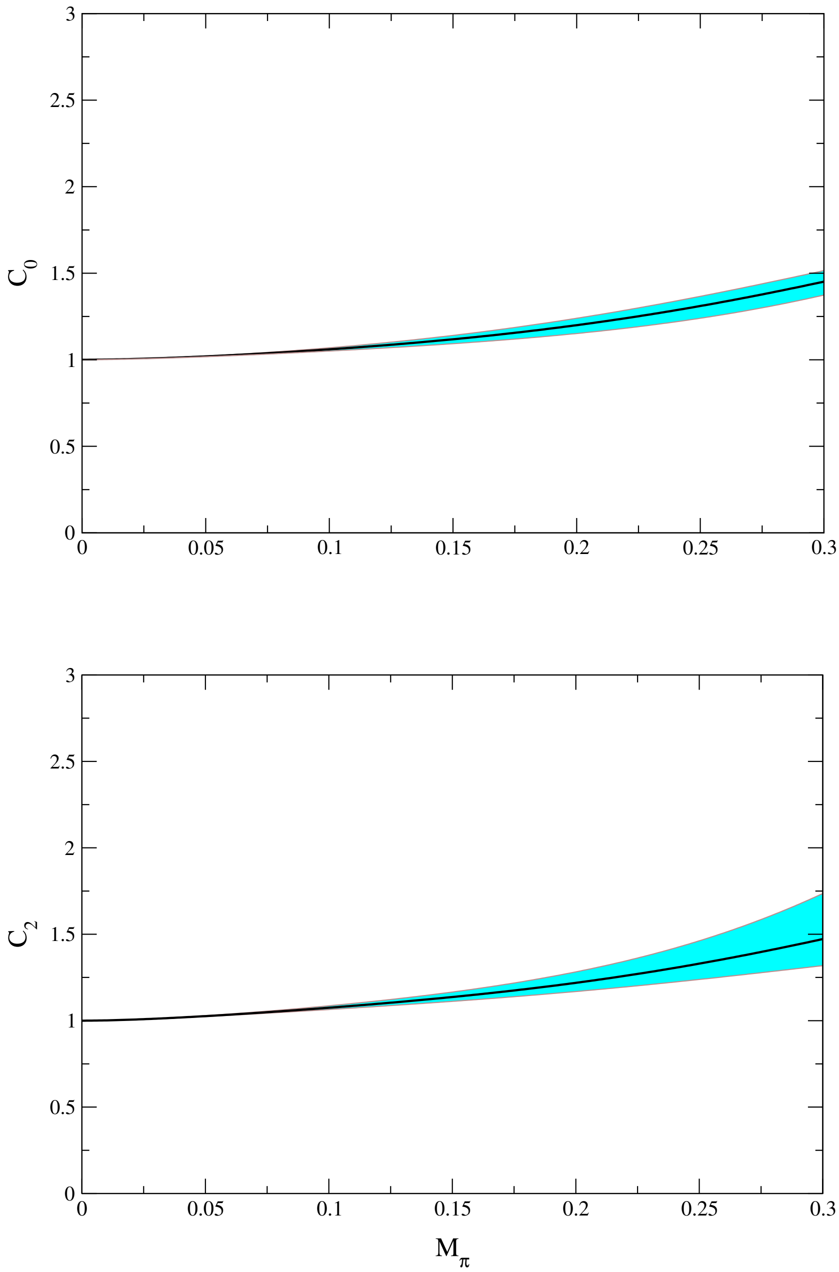
$$a_0^0 = 0.220 \pm 0.005 \quad a_0^0 = -0.0444 \pm 0.0010$$

Beautifully confirmed by the E865 experiment at BNL (01):

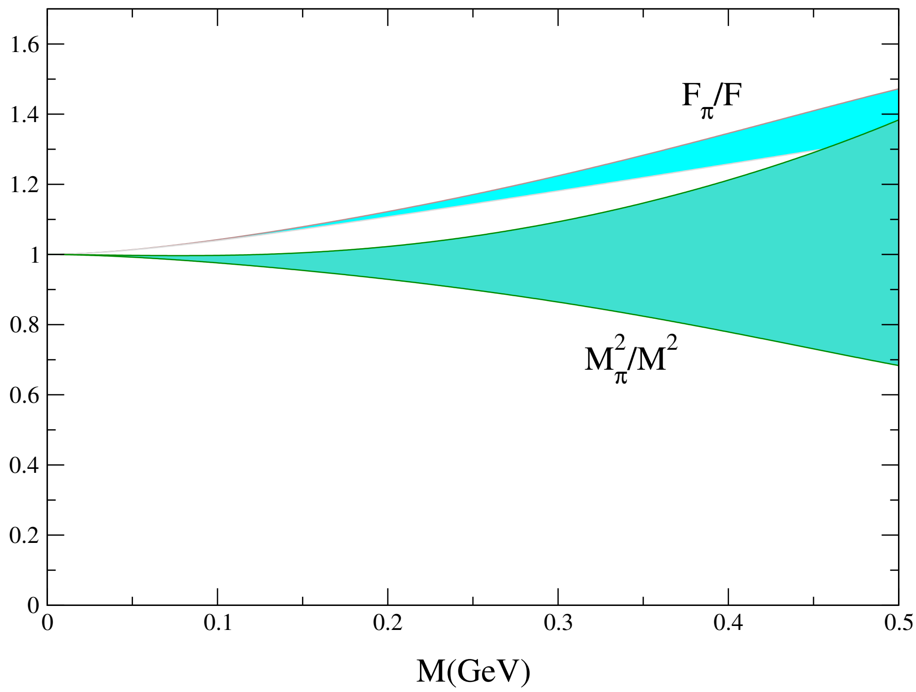
$$a_0^0 = 0.221 \pm 0.026 \quad (95\%\text{C.L.})$$

Future tests from: DIRAC, KLOE and NA48.

Quark mass dependence of the subthreshold amplitude



Quark mass dependence of F_π and M_π



$$M_\pi^2 = M^2\{1 - \tfrac{1}{2}\,x\,\hat{\ell}_3 + \tfrac{17}{8}\,x^2\hat{\ell}_M^{\,2} + x^2 k_M + O(x^3)\}$$

$$F_\pi = F\{1 + x\,\hat{\ell}_4 - \tfrac{5}{4}\,x^2\hat{\ell}_F^{\,2} + x^2 k_F + O(x^3)\}$$

$$x := \frac{M^2}{16\,\pi^2 F^2}, \quad \hat{\ell}_n := \ln\frac{\Lambda_n^{\,2}}{M^2}$$

$$0.2\text{GeV} < \Lambda_3 < 2\text{GeV}, \quad \Lambda_4 = 1.26 \pm 0.14\text{GeV}$$

Dependence on the strange quark mass

Pushing SU(3) to $O(p^6)$ is technically much more demanding. Predictions are more difficult for the larger number of constants.

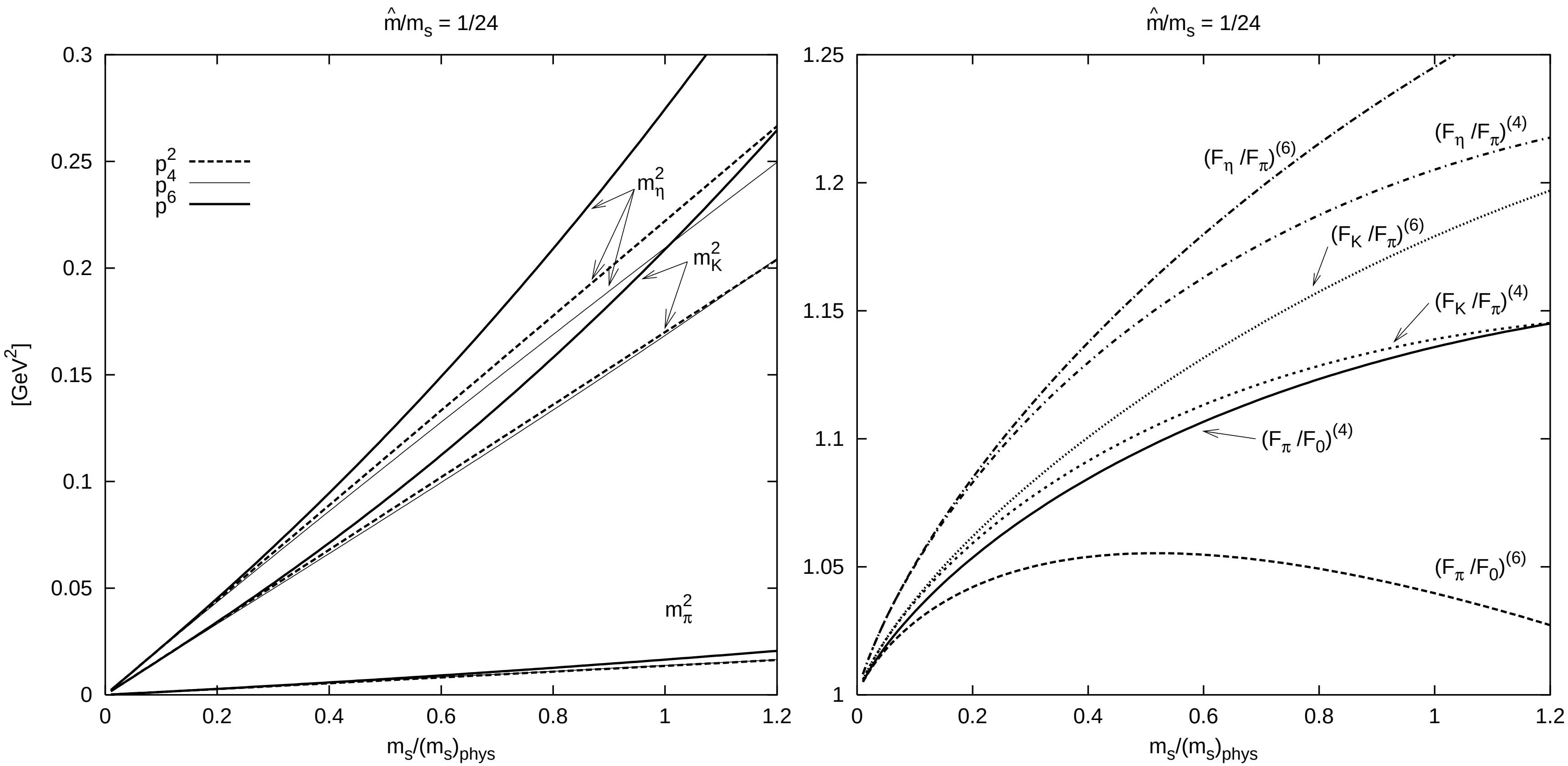
Bijnens, Amoros and Talavera have studied masses, decay constants and K_{e4} decays.

- VMD estimate for the $O(p^6)$ constants;
- fit to K_{e4} data for the $O(p^4)$ constants.

Results for masses and decay constants:

$$
\begin{aligned}
M_\pi^2 &= M_{\pi\,phys}^2\,(0.746 + 0.007 + 0.247) \\
M_K^2 &= M_{K\,phys}^2\,(0.695 + 0.019 + 0.286) \\
F_\pi &= F_0\,(1 + 0.136 - 0.076) \\
F_K &= F_\pi\,(1 + 0.134 + 0.086)
\end{aligned}
$$

Dependence on the strange quark mass



Summary

- I have reviewed different treatments of FSI interactions in $K \to \pi\pi$, and shown the advantage of a dispersive approach *vs* the chiral expansion

- The dispersive approach needs as input two subtraction constants that lattice calculations could provide (e.g. $\mathcal{A}(K \to \pi)$)

- These treatments are alternative to those purely based on lattice calculations (finite volume methods)

- I have suggested as preliminary test for lattice calculations a study of the $\pi\pi$ scattering amplitude (semileptonic K decays would also be an interesting intermediate step!)

- The $\pi\pi$ scattering amplitude has been thoroughly studied by using CHPT to $O(p^6)$ and dispersion relations. Predictions have a few percent accuracy, and have been confirmed by experiments (with more experimental tests coming)

- CHPT also predicts a very strong quark mass dependence of the scattering lengths: what will the lattice calculations see?